Classifying finite populations by their achievements

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Abstract

Many international organizations categorize populations with respect to their success on several qualities. In this spirit, the United Nations perform country classification relating to their accomplishment on Human Development. For that purpose, the UN combines each country's indicators of Income, Education Level and Life Expectancy into an overall index that represents its achieved Human Development. The idea behind Human Development is enriching people's choices and improving human capabilities. Therefore, the overall index objective is to provide governments with benchmarks for efficient development and motivate them to sustain Human Development by growth on every component. However, the methodology used by the UN allows countries to use trade-offs between different attributes and hence obtain higher level of Development by focusing for instance, on Income growth only. The paper reviews the classification techniques (rules) used when performing population categorization according to different attributes. We extend the standard classification procedures known in the literature and within this new family we characterize those rules that meet our axiom of no trade-offs among different attributes.

Keywords: No Trade-Offs, Population Classification, Multiple Attributes

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1 Introduction

In everyday life we see plenty of examples where the members of a finite population (countries, people, restaurants...) are grouped into categories. Given the information about the population achievements in a set of attributes, a jury, usually represented by an international organization or association, classifies the individuals following some classification rule. Here are some examples:

Every year the *United Nations* issues a report where countries are classified into three categories: Developed, In Development and Undeveloped. The classification combines indicators of life expectancy, educational level and income. In our paper we will frequently refer to this example.

The World Bank (WB) classifies the countries according to their Gross National Income (GNI) per capita. Based on this, the resulting categories are High Income, Middle Income (further divided into lower-middle and higher-middle) and Low Income Countries.

The World Trade Organization (WTO), uses the same categories as the UN, but classifies them according to a completely different approach. When a new member enters the organization, the country itself decides in which of the three categories it belongs. However, the already existing members can challenge the classification proposed by the new country.

Standard & Poor's is an agency that provides country credit rankings. Creditworthiness includes likelihood of default, payment priority, recovery and credit stability. After normalizing, these multivariate data on these characteristics are aggregated into one-dimensional scale. Standard & Poor's ratings result 17 categories, from highest AAA, for countries that have strong capacity to meet their financial commitments, to CC, for countries that are currently highly vulnerable to meet their commitments.
Similar country rating are also made by *Fitch and Moody's.*

Freedom House generates the Freedom in the World survey that measures the freedom of a country according to two broad categories: political rights and civil liberties. Each country or territory is rated on a seven-point scale for both political rights and civil liberties and then given a category status of Free, Partly Free, or Not Free.

Freedom House also provides a classification on Freedom of the press. It is based on a survey which measures the degree to which each country permits the free flow of news and information, and determines the classification of its media as Free, Partly Free or Not Free. The criteria used for such judgments are The legal environment, political environment and economic environment for the media.

Transparency International (TA) is an organization that calculates the corruption index and then categorizes the countries into classes of corruption, from the lowest to the highest. In order to construct the index, TA uses thirteen different indicators gathered from ten international institutions.

Some *Universities* provide Full, Partial or no Scholarship to their admitted students. In other words, they classify the students into three categories. Different institutions consider different components that enter in the evaluation. As an example, the rating of the high school that the student used to study and his grades are two attributes that are often considered.

Restaurant guides and reviewers often use stars in restaurant ratings. The Michelin system reserves stars for exceptional restaurants, and gives up to three; the vast majority of recommended restaurants have no star at all. Other guides now use up to four or five stars, with one star being the lowest rating. The Michelin system remains the best known star system. A single star denotes "a very good restaurant in its category", two stars "excellent cooking, worth a detour", and three stars, "exceptional cuisine, worth a special journey".
Different restaurant classifications are obtained by combining different criteria. The most common ones are the quality of food, service, ambiance, and even noise level.

The classification examples above are done in order to provide a summary measure of individuals achievements. By this we have a basis to claim that one has performed better than another. Moreover, the classifications that are carried out periodically evaluate the advance or retreat of an individual during this period.

Notice that, with the exception of WTO and WB, in the examples above the classification is done using the data from more than one attribute. Moreover, in each case, the methodology consists of transforming the initial data, aggregating them into a scalar and deciding the threshold lines.
The multi-variate data are aggregated to obtain a composite index that will represent the overall individual achievement. Such indexes are useful in focusing attention and simplifying the problem, as argued by Streeten (1994)[13]. They have a stronger impact on the mind and draw public attention more powerfully than a long list of many indicators combined with a qualitative discussion. They are eye-catching. Having a scalar for each individual allows for easy pairwise comparison and complete population ordering.

The composite indexes are also used in the literature that deals with measuring population inequality in well-being. Some authors construct a composite index of well-being and then measure the inequality in that composite index, e.g. see Becker at al (2005)[5]. On the other hand there is a literature that looks at the inequality of the individual dimensions separately and refrain from constructing any composite index of well-being. But, getting an overall conclusion without constructing an index is difficult. See World Development Report (2006)[16], Decancq et al (2010)[7], (2009)[6].

In this paper we will not address the problem of the choice of the attributes. We shall not challenge, for instance whether GDP, education level and Life expectancy should be the only attributes in the HDI that determine the human development. Instead, we will focus on the classification methodology. In particular, we will argue that the construction of composite index, except in one case, gives the possibility of trade-offs. This notion gives the opportunity of an individual to reach a certain level of overall development, without focusing on the growth on every component. In the next paragraph we give an example that motivates the prevention of trade-offs. Then we will generalize and extend the common classification techniques and characterize the ones that will not allow trade-offs.

The question of allowing or not for trade-offs between components when achieving an overall objective is equivalent to the question of using the utilitarian or Rawlsian approach. These two methodologies express very different views about how should an individual needs to be organized in order to achieve its global well-being. The first emphasizes total well-being and the second emphasizing the needs of the poorest.

In a statement taken from the HDR 1995[14], we find:

> *Losses in human welfare linked to life expectancy, for example, cannot be compensated for by gains in other areas such as income or education.*

We find the reason for which the UN make this statement in the work of Anand & Sen (1994)[2] and Anand & Sen (2000)[3] where they argue that the income growth is considered as a necessary, but not sufficient condition when providing people with resources for a decent life. Therefore, increasing the income of an individual that lives in a conflict area with a low life expectancy, according to Anand & Sen (2000)[3] should not be considered as an improvement in human development. Nevertheless, averaging the three components used for the measurement of the Human Development by the UN disagrees with the quoted statement. In addition, Decancq (2009)[6] implicitly calculates the rates on which the human development losses of a country, caused by a decline in an attribute could be compensated by improvement in another.

Further comments concerning the human development compensation we find in Fukuda-Parr (2001)[8]. There we also find that people have many choices to make their lives meaningful and capabilities that allow them to make those choices. These choices and capabilities are not substitutable. People want to live long and healthy lives but also be educated and develop their intellectual and creative potential. Human development is a multidimensional concept and it cannot be reduced to one dimension. Sager and Najam (1998)[11] argue that the additive structure on the HDI contradicts the fact that the different attributes are not comparable. The authors of the HDI, Anand & Sen (2000)[3] said that although humans are the means and the ends of the development, those two sides of development are related but quite distinct. Decancq et al (2009)[6] said that if one accepts that the different dimensions of well-being are incommensurable, one has to limit oneself to an analysis of the evolution of inequality for each of the dimensions separately. Hopkins (1991)[10], furthermore, argues that additive aggregation implies that one can measure apples and oranges individually and then aggregate them into some meaningful index.

There is some literature that tries to seize the notion of trade-offs, in the spirit of the preceding remarks. In the paper of Herrero at al[9] the composite index of a country is calculated as the geometric mean of the three components. By doing this, the country with zero achievement in some attribute cannot recover by growth on the others. This methodology has been integrated in the official HDR 2010[15].

Other authors (Aleskerov at al (2010)[1]) require that if at least one agent evaluates an alternative as bad, then, no matter how many good grades it admits, in the social ranking this alternative is ranked lower than any alternative evaluated as average by all agents.

Following the presented arguments, in this work we will focus on the classification rules that will satisfy a property that we call No Trade-Offs. It will demand that the fall of an individual in a lower category, caused by a drop in a specific attribute, cannot be compensated by any gain in the others. This property should be demanded in the classification rules whose goal is individual development in every dimension (e.g. HDR).

Let us here make a brief review of the standard classification techniques. In a finite population, every individual is represented by its achievements in a set of components. A classification is an ordered partition of the set of population in which we say that the individuals belonging in a higher ordered partition set are better classified. In order to construct and then order the partition sets, we use a composite index that is created using the given components. Then two individuals belong in the same partition set if their value of the index is the same. On this way, we have made the partition set and in order to sort them, we use the index value of the individuals in each partition. Higher value means higher order set.

In the procedure explained above, the source where the trade-offs emerge is the construction of composite index. As we will show, except in one case the composite index always permits trade-offs. Therefore, we ask if we could something more in order to avoid the step of constructing the index, but still maintain the explained classification procedure. In other words, we want to extend the existing classification algorithm.

For that purpose, we need to find a new basis on which we could perform the pairwise comparison between individuals. Consider a binary relation defined on real vectors, whose coordinates represent the individual accomplishments in the different components. Then the individuals whose information vectors are not dominated by any other according to the chosen relation are classified as highest. Removing them from the population set, the new set of non-dominated elements will be classified as second best and so on.

Using this algorithm, we actually make an ordered partition of the population set. The different sets of non-dominated elements represent the different categories. The first one will contain the individuals classified as highest. Then similar as before, we say that one individual is better classified than another if the partition set where he belongs is higher classified. On this way, we define a partial order on the population set.

The classification procedure that we just explained depends and it is governed by the binary relation chosen. Therefore, by $R-$method we will call the classification rule that is regulated by the binary relation R.

We note that the Human Development classification performed using the HDI uses the binary relation *"greater or equal"* defined on the real numbers.

The properties that the $R-$method satisfies depend on the binary relation R. In this paper we perform a characterization of the $R-$methods that do not allow Trade-Offs.

In the next section, we provide the notation that we are going to use. In section 3 we formalize the classification rules, and in particular we focus on the $R-$methods for which we give some examples. Section 4 contains our characterization result that is followed by some comments on our work. We prove our theorem in the appendix.

2 Notation

Consider a set X of k individuals, $X = \{1, 2, \ldots, k\}$, and m states of well being $M = \{1, 2, \ldots, m\}$, $(m \geq 2)$ that we call attributes. We have information about individual i's achievement in every attribute, represented by the vector $x_i = (x_{i1}, x_{i2}, \ldots, x_{im})$.

The matrix $A = [x_{ij}]$ contains the complete population information, with x_{ij} standing for the achievement of individual i in the j^{th} attribute. We call the matrix A an information matrix or state of the world. By A^j and A_i we denote the j^{th} column and the i^{th} row of A respectively. Let $\mathcal{A}_{k\times m}$ be the space of all information matrices.

Given the information matrix, we face a problem to classify or categorize the population into classes. Formally,

Definition 2.1 *A* ***classification problem*** *is a matrix A, whose rows represent individuals achievements.*

Definition 2.2 *A* ***classification*** *is a set $\mathcal{U}$ of ordered disjoint subsets of X, $\mathcal{U} = \{U_j\}_j$, $\cup_j U_j = X$.*

Let $\mathscr{U}$ be the space of all classifications $\mathcal{U}$.

We see that by the ordering of the elements of $\mathcal{U}$, the individuals belonging U_i are higher classified than the ones in U_j when $i < j$.

We note that $\mathcal{U}$ is not a partition of X, because we allow existence of one or more empty sets as an elements of $\mathcal{U}$. Moreover, by ordering its elements we want to distinguish between $U_i = \emptyset$ and $U_j = \emptyset$, for $i \neq j$, as two different categories. Examples of this kind can be seen when the classification of an individual in a category is conditioned by achieving a particular level of well being. For instance, in the classifications performed by Standard & Poor's there could be no country that is able to withstand an extreme level of crisis and still meet its financial obligations, hence the highest rating category will be empty.

At this point we have all the necessary ingredients to move to the question of how to define the set of rules that will determine the process for solving the classification problems. We will call this procedure a classification rule, whose definition we will provide in the following section.

3 Classification Rules

Given a classification problem A, to obtain a classification $\mathcal{U}$, we need to specify the rule that will perform the classification. We begin by defining what a classification rule is, and then concentrate on the ones that will satisfy a set of properties.

Definition 3.1 *A* ***classification rule*** *is a function $F : \mathcal{A}_{k\times m} \to \mathscr{U}$ that assigns to each classification problem A a classification $\mathcal{U}$.*

We see that the function F maps the individuals information into the classification classes which are defined as subsets of X. However, since the individual information is fixed and invariable, we will abuse notation by writing a classification for a subsets that contain the individual information vectors instead of the individuals themselves.

Let $\mathcal{F}$ be the space of all classification rules $F : \mathcal{A}_{k\times m} \to \mathscr{U}$. For convenience we will use another notational abuse by writing $F(x_i)$ for the order of the category where x_i belongs, i.e. $F(x_i) = j$ where $U_j \ni x_i$.

The space of classification rules that we defined is very broad and it contains all the possible classification rules that one can imagine. However, we will concentrate our work to a subset of $\mathcal{F}$ that contains only the classification rules satisfying the following properties.

Definition 3.2 *We say that the classification rule $F \in \mathcal{F}$ is **anonymous** if for every classification problem A and for every individuals ordering permutation π we have $F(A) = F(A^\star)$, where $A^\star_j = A_{\pi(j)}$.*

With anonymity we demand that the assignment of the individuals into the categories U_i should not be affected by population reordering, but instead, the evaluation should be performed only with respect to their achievements.

Definition 3.3 *We say that the classification rule $F \in \mathcal{F}$ is **symmetric** if for every classification problem A and for every attribute permutation σ on M, we have $F(A) = F(A^*)$, where $A^{j*} = A^{\sigma(j)}$.*

Similar as the anonymity, by symmetric classification rule we mean a rule for which the labeling of the attributes should not matter.

Definition 3.4 *Let $i, j \in X$ be such that $x_i \geq x_j$. The classification rule $F \in \mathcal{F}$ is **monotone** if $F(x_i) < F(x_j)$.*

The monotonicity property says that when the individual i is at least as good as individual j, and sometimes better, then i is higher classified than j.

We consider that the previous three classification rule properties provide some basic rationality conditions. In what follows, we focus on the classification rules that meet them.

Following the arguments given in the introduction, we are interested in classification rules that do not allow trade-offs. The following axiom embodies this property.

Axiom 3.1 (No T-O) *F does not allow **trade-offs** between different attributes if*

$$F(x_{l1}, \ldots, x_{li} - \Delta, \ldots, x_{lm}) = F(x_{l1}, \ldots, x_{lm}) + 1$$

$$\Rightarrow F(x_{l1}, \ldots, x_{lj} + \delta_j, \ldots, x_{li} - \Delta, \ldots, x_{lm}) = F(x_{l1}, \ldots, x_{lm}) + 1,\ \forall j \neq i, \delta_j > 0$$

We perceive the preceding axiom as literal translation of the UN statement given before. The definition of the axioms states that the individual losses in an attribute, whereas the rest of the world is not varying, cannot be recovered by gains in other component. However, let us imagine a dynamic setting where the population members sustain the different attributes in every period. In this case, the axiom would demand that a category fall because of leaving an attribute constant, cannot be restored by rigorous growth on another.

In other words, the growth on every component is mandatory as for achieving better classification as for maintaining the existing one.

As we wrote in the introduction, Herrero at al[9] improve the full substitutability between all the HDI attributes. The full substitutability means that for instance, no matter how bad the life expectancy is, it can be compensated with additional income at a constant rate. Therefore, they impose no trade-offs between different attributes when an individual is at the worst level in some component. This is allowed by considering the geometrical mean of the achievement vector as an index representing the individual. That is to say, if there is an i such that $x_{is'} = 0$ for some s', then any other individual j with $x_{js} \neq 0\ \forall s$ $x_j R x_i$, since

$$\prod_{s=1}^{m} x_{js} > \prod_{s=1}^{m} x_{is} = 0$$

However, the trade-offs out of this context are still possible.

Similar no trade-offs notion has been introduced by Aleskerov at al (2010)[1]. In their paper, they consider a problem of social ranking over population. Agents evaluate every population member using a scale of m grades, from the worst to the best. In their no trade-offs axiom, named as The Non-compensatory Threshold, they demand that an individual that if at least one agent evaluates an individual as the worst, then, no matter how many "good" grades it admits, in the social ranking this individual is ranked lower than any population member evaluated as "average" by all agents.

Our axiom can be comprehended as a complement to the no trade-offs notions introduced by Herrero at al[9] and Aleskerov at al (2010)[1]. The full substitutability allowed by the averaged index is followed by partial no trade-offs, that are limited to the individuals that experienced the bottom level. We introduce a full no trade-offs property, that refers to any individual at any achievement level.

3.1 $R-$classification Rules

As discussed in the introduction, we will extend the known classification methodologies by introducing an individual pairwise comparison that is not only based on the index representing individual achievement.

Let R be a strict partial order on the set $\{x_i\}_i$, i.e. R is irreflexive, antisymmetric and transitive. By $\mathcal{R}$ we denote the space of all such binary relations. We note that R is defined on the rows of the information matrix A that represent the vectors of individual information.

For instance, in the case when the averaged composite index is used to classify the population, the binary relation R is defined as

$$x_i R x_j \Leftrightarrow \frac{1}{m}\sum_{l=1}^{m} x_{il} > \frac{1}{m}\sum_{l=1}^{m} x_{jl}, \tag{1}$$

and in particular, for the case of the Human Development classification by the UN, the number of components m is three. $(m = 3)$

The individuals in X whose achievement vectors are not dominated with respect to $R \in \mathcal{R}$ by any of the vectors of the other population, are classified the highest. We take away the highest category and we continue the search of the set of non-dominated elements on the rest of the population. We classify the individuals whose vectors belong to this set in the second best category. This procedure finishes in a finite number of steps, which is actually the number of categories obtained. Using the notation employed above, we have

$$U_1 = \{i \in X | \nexists j \in X, x_j R x_i\}$$

representing the set that contains the highest classified countries. We remove U_1 and we perform again the search for the set of non-dominated elements, that are classified as second best and so on. In general, the l^{th} category U_l contains of

$$U_l = \{i \in X \setminus (U_1 \cup \ldots \cup U_{l-1}) | \nexists j \in X \setminus (U_1 \cup \ldots \cup U_{l-1})\, x_j R x_i\}. \tag{2}$$

Formally, we provide a formal definition of this extended classification procedure:

Definition 3.5 *Let $R \in \mathcal{R}$ and let $\mathcal{U} \in \mathscr{U}$ be such that its elements are obtained by (2). We call the classification rule $F_R : \mathcal{A}_{k\times m} \to \mathscr{U}$ an $R-$method if $F_R(x_i) = j$ when $i \in U_j$.*

We refer to the example of the UN classification, according to which, the countries with the greatest indexes are classified the highest. Adopting their definition to our setting we have that the highest classified countries are the ones whose achievement vector is not dominated by the achievement vector of any other, following the binary relation (1).

In the next subsection we provide examples of some binary relations of our interest, and then compare the evaluations between two countries carried out by two $R-$methods.

3.2 Examples

We start by defining three binary relations defined on real vectors with equal dimension. We formally state their definition since they will be part of out final results. Furthermore we give some examples of classification rules that employ some of them.

3.2.1 Binary Relations

For the real vector x_i and the real number a we define the set

$$J(a, x_i) = \{s \in M | x_{is} \leq a\}$$

Then using this set we define the following binary relations

Definition 3.6 *The binary relation R is the Maximin if*

$$x_i R x_j \Leftrightarrow \exists a,\ s.t.\ J(a, x_i) = \emptyset,\ \ J(a, x_j) \neq \emptyset$$

Definition 3.7 *The binary relation R is the Leximin if*

$$x_i R x_j \Leftrightarrow \exists a,\ s.t.\ |J(a, x_i)| < |J(a, x_j)| \text{ and } \forall b < a$$

$$|J(b, x_i)| = |J(b, x_j)|$$

Definition 3.8 *The binary relation R is the Protective Criterion if*

$$x_i R x_j \Leftrightarrow \exists a,\ s.t.\ J(a, x_i) \subset J(a, x_j) \text{ and } \forall b < a$$

$$J(b, x_i) = J(b, x_j)$$

In the following subsection we give some examples of classification rules inspired by these binary relations and compare the results with the corresponding one that arise from the average composite index.

3.2.2 Classification Rules

Consider a simple example where we want to decide which of two countries is better, evaluating them on the same components as the UN: Life Expectancy (L), Education Level (E) and Income Level (I). We consider the data of Equatorial Guinea (1) and Egypt (2) used in the HDR 2000. Adopting to the notation introduced, we have $X = \{1, 2\}$ and three states of well being (m=3). The achievements of each individual are given by the vectors x_1 and x_2 in $\mathbb{R}^3$. The classification problem A is given by:

	L	E	I
x_1	0.42	0.76	0.48
x_2	0.69	0.60	0.57

We present the evaluation results according to different rules.

***R*−method 1, year 2000**

Let R be the averaging composite index. In this case we have that $x_2 R x_1$ since

$$\frac{1}{3}(0.69 + 0.60 + 0.57) > \frac{1}{3}(0.42 + 0.76 + 0.48)$$

and hence $U_1 = \{2\}$ and $U_2 = \{1\}$.

Now let us consider an $R-$method where R is one of the above defined relations, e.g. the Leximin[2]

$R-$method 2, year 2000

Let the binary relation R be the Leximin. Then for $a = 0.42$ we have $J(0.42, x_1) = \{1\}$ and $J(0.42, x_2) = \emptyset$ and $\forall\, b < a$ $J(b, x_1) = J(b, x_2) = \emptyset$.

According to the definition of the Leximin $x_2 R x_1$ and hence we obtain $U_1 = \{2\}$ and $U_2 = \{1\}$, i.e. we obtain the same classification as in the previous case.

Let us now consider the evolution of the same two countries on the same three attributes during nine years. Here we present their data in a form of classification problem, used in the HDR 2009

	L	E	I
x_1	0.415	0.787	0.955
x_2	0.749	0.697	0.664

and perform the classification using the same two methods as before.

$R-$method 1, year 2009

Let R be averaging composite index. In this case we have that $x_1 R x_2$ since

$$\frac{1}{3}(0.415 + 0.787 + 0.955) > \frac{1}{3}(0.749 + 0.697 + 0.664).$$

We have a change in the classification result, $U_1 = \{1\}$ and $U_2 = \{2\}$.

$R-$method 2, year 2009

Let the binary relation R be the Leximin. Then for $a = 0.415$ we have $J(0.415, x_1) = \{1\}$ and $J(0.42, x_2) = \emptyset$ and $\forall\, b < a$ $J(b, x_1) = J(b, x_2) = \emptyset$.

According to the definition of the Leximin $x_2 R x_1$ and hence we obtain $U_1 = \{2\}$ and $U_2 = \{1\}$, i.e. there is no change in the classification.

From the data and the classification of these two countries by $R-$method 1,

[2] The results that we will present do not change if we take any of the Maximin, Leximin or the Protective Criterion

we see that the Life Expectancy losses that E. Guinea experienced in 2009, are recovered by the income increase. Moreover, this trade-off is used to go past Egypt, that maintain its three attributes during this period.

The reason that allowed E. Guinea to achieve vast increase on income level is the oil funding. Unfortunately, the income benefits that had arise, are used for the improvement of human capabilities. Instead, as the organization of Human Rights Watch state, the "the dictatorship has used an oil boom to entrench and enrich itself further at the expense of the country's people".

To conclude the example, we see that we have detected a Human Development growth in the classification made using the averaged composite index. At the same time, the above facts indicate that a decline on the same.

We note that this classification change does not take place when we considered the $R-$method inspired by the Leximin.

4 Results and Comments

Here we present our main result, in which we fully characterize all $R-$methods that satisfy the No Trade-offs axiom. As indicated in the examples above, the outcome involves the three binary relation defined. We provide the proof in the appendix.

Theorem 4.1 *The $R-$method F_R does not allow trade-offs iff R is the Maximin, Leximin or the Protective Criterion.*

We have mentioned in the introduction that in only one case the $R-$method does not allow trade-offs and it is a composite index (R is "greater or equal" defined on real numbers). That is the case when R is the Maximin. In the other two cases the binary relation representation by a real number is not possible.

The three binary relations that appear in our results differ in their evaluation only when the lowest attribute value of two individuals is the same. In that case, Maximin stops the evaluation without claiming dominance of one of them, while the Leximin and the Protective Criterion continue comparing the second worst coordinate. The further difference between the Leximin and the Protective Criterion is that the first compares different values from different components. On the other hand the evaluation that one individual dominates another made by the Protective Criterion is done only if the comparing coordinates belong in a same attribute. As we mentioned in our example, the evaluations on E. Guinea and Egypt by the three resulting relations are the same, since they do not have a coordinates with same value.

To conclude this paper here we would like to give some general comments towards the classification problems and the rules chosen to solve them.

The provided characterization results seem to be very restrictive concerning the imposed demands on the binary relations. However, the binary relation chosen will only direct the individuals how to improve their classification, of course in a comparable sense with the other population members.

The institution or the judges that are in charge of designing the solution of a classification problem, in reality face several type of question that we do not address in this work.

What about the attributes that should enter in the procedure of individuals' classification. This is a very difficult question, and unless the goal of the development is not exactly precisely defined, it may not have an answer. An example of this problem is the country classification performed by the UN. There has been some literature criticizing their choice of attributes. However, as we mentioned, there is no exact definition of what actually human development is, and hence the selection of the three attributes is an agreement.

Some classification problems are conditioned on the number of individuals that could or should be positioned in a category. For instance, the number of students that will be given a full scholarship is usually bounded by an upper limit. The university budget and the amount of the scholarship establish the maximal number of students that can get it.

Other classification problems require obtaining fixed number of classes with the possibility of allowing or not an empty categories. An example of the later is the classification performed by Standard and Poor's in which the number of classes is settled in advance and as we explained before, there exists the option for a category having no countries classified in.

At the end, briefly we give some comments on the choice of the classification rules we considered in this paper. We can say that there are classifications of two types. The first contains the classifications performed only once. For instance, the same group of students that were accepted at university is classified only once to decide on who will get a scholarship. So maybe in cases of this type, we should not care so much about the trade-offs, but concentrate on the magnitude and the importance of each component.

The second type of classifications are the ones performed periodically, which include the classification apropos to Human Development. When including different components in the periodical classification, we should ensure their growth in each period. Therefore, imposing the no trade-offs property would guarantee this growth.

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5 Appendix

Proof of Theorem 4.1 Let $A \in \mathcal{A}_{k\times n}$ and $R \in \mathcal{R}$.

$\Longrightarrow$ Let the classification rule F_R does not allow trade-offs. Since F_R is symmetric, for any two individuals i and j and for every attribute permutation σ on M, we have that

$$x_i R x_j \Leftrightarrow x_{\sigma(i)} R x_{\sigma(j)}. \tag{3}$$

Because of monotonicity of F_R for every two individuals i and j such that $x_i \geqq x_j$, we have that i is better classified than j. In other words

$$x_i \geqq x_j \Leftrightarrow x_i R x_j \tag{4}$$

Let x_i, x_j and x_s be three achievements vectors such that $x_i P x_s$ and $x_j P x_s$. Assume that $x_s R z_i$.

Let $z_i = \frac{1}{2}(x_i + x_j)$ and y_i be such that $y_{il} = \min\{x_{sl}, z_{sl}\}$. Then since all coordinates of z_i are at least good as the coordinates of y_i and sometimes better, we have that $z_i P y_i$. With other words, by drop of the value in the coordinates $M' \subset M$ of x_s, where $x_{st} > z_{it}$ for all $t \in M'$, is has fallen a category down. Moreover, the values of x_{st} for $t \in M'$ are equal to the corresponding ones from z_i.

Then with sufficient increase to the coordinates of y_i (representing the modified x_s) that belong to $M \setminus M'$ he would again rise to the previous level ($x_s R z_i$). This contradicts the no trade-offs assumption. Hence $z_i P x_s$.

Hence we have that

$$x_i P x_s \text{ and } x_j P x_s \Rightarrow \frac{1}{2}(x_i + x_j) P x_s. \tag{5}$$

We have proved that a binary relation $R \in \mathcal{R}$ determining the classification rule F_R and does not allow trade-offs satisfies the equations (3), (4) and (5).

Hence by one of the results from Barberà & Jackson (1988)[4], R must be the Maximin, Leximin or the Protective Criterion.

$\Longleftarrow$ If R is one of the Maximin, Leximin and Protective Criterion, than as proven in , Barberà & Jackson (1988)[4], it satisfies (3) and (4), i.e. the classification rule will be symmetric and monotone. We only need to check if it allows trade-offs.

Let the individual i be such that $F_R(x_i) = l$, i.e. i is classified in the l^{th} category following F_R. Assume that after having suffered a loss on its s coordinate, individual i experiences category fall. That is to say

$$F_R((x_{i1}, \ldots, x_{is} - \Delta, \ldots, x_{im})) = l + 1.$$

Suppose that the reason for this fall is the individual j, i.e. the individual j is such that

$$(x_{j1}, \ldots, x_{jm})R(x_{i1}, \ldots, x_{is} - \Delta, \ldots, x_{im})$$

Then an increase on any coordinate s' different than s and $\delta_{s'} > 0$ by the definition of either Maximin, Leximin or Protective Criterion will not change the set $J(a, x_i)$, where $a = x_{is} - \Delta$ and therefore

$$(x_{j1}, \ldots, x_{jm})R(x_{i1}, \ldots, x_{is} - \Delta, \ldots, x_{is'} + \delta_s, \ldots, x_{im}).$$

Hence i will not be able to eliminate the dominance of the individuals by increase on the other components. Hence there will be no trade-offs in the final classification result.

$\blacksquare$