

Monetary policy with Heterogeneous Collateralized Borrowing*

José Suárez-Lledó[†]
Universidad Autónoma of Barcelona

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Abstract

This paper presents a model with money and credit with endogenous borrowing constraints and heterogeneous borrowing capacity. The analysis focuses on economies where collateralized credit plays an essential role. The model explores the effects of changes in the aggregate value of collateral on the number of agents that are borrowing constrained, real allocations, interest rates, and the liquidity premium of the asset that secures the loans. The results apply to a wide range of economies as long as inflation is not too low.

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Keywords: collateral, liquidity, money, monetary policy, asset pricing.

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[†] Departamento de Economía e Historia Económica. Universidad Autónoma de Barcelona. Building B, Bellaterra (Barcelona) 08193. Tel. (+34) 606202610. Contact: jose.suarezlledo@uab.es

Monetary Policy and Heterogeneous Collateral

Corresponding Author: *José Suárez-Lledó*. Departamento de Economía e Historia Económica. Universidad Autónoma de Barcelona. Building B, Bellaterra (Barcelona) 08193. Tel. (+34) 606202610. Contact: jose.suarezlledo@uab.es

Abstract

This paper presents a model with money and credit with endogenous borrowing constraints and heterogeneous borrowing capacity. The analysis focuses on economies where collateralized credit plays an essential role. The model explores the effects of changes in the aggregate value of collateral on the number of agents that are borrowing constrained, real allocations, interest rates, and the liquidity premium of the asset that secures the loans. The results apply to a wide range of economies as long as inflation is not too low.

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1 Introduction

In the last years we have witnessed a significant development of credit markets.¹ On the other side of this remarkable credit extension, information and repayment issues have often arisen. Thus, credit contracts have incorporated collateral requirements to alleviate those informational and incentive problems. In fact secured debt has become a vastly common form of borrowing contract and collateral is widely required in most credit market and money market transactions.²

A wide variety of assets can serve as collateral, among which durable goods seem to be a predominant instrument. In part because their price is supposed to be less volatile, since it is subject to longer term consumption or investment decisions. Interestingly, however, many agents also hold low-yield assets as collateral in a variety of transactions. Central Banks consistently accept low-return securities as collateral in their operations with commercial banks; private agents also use bonds and other low-yield low-risk securities in their investment operations.

In this paper I build a model where agents can invest in an asset that may be pledged as collateral to obtain a loan. The value of loans will always be bounded by the return of the asset used as collateral. In equilibrium agents will value assets not only for their specific expected return, but mainly for the value that can be generated as collateral in the case the agent wants to borrow. The latter is referred to as the liquidity premium of the asset and, along with the

specific return, will determine the equilibrium price of the asset. In line with what I mentioned before, I want to abstract from possible specific effects of the price of collateral on the equilibrium dynamics in order to isolate the role of endogenous constraints and different borrowing capacity. Therefore, I analyze investment in low-return assets that can be used as collateral, which would optimally not be purchased otherwise. This modeling approach will allow me to endogenously determine the money growth rate in response to changes in the value of collateral, something that other models do not have.

The key feature in this model though is that the return of the asset is idiosyncratic and random to each agent. A very interesting implication is that there will always be a fraction agents that will be borrowing constrained and a remaining fraction of agents that will not. The number of borrowing constrained agents will be endogenously determined in equilibrium and will prove itself a crucial element in the model. As I will detail in following sections, such heterogeneity generates relevant differences in the real allocations and interest rate with respect to situations in which borrowers are homogeneous.

Very much in relation to that, the second important qualitative result is that when the aggregate value of collateral falls more people are constrained, which raises the liquidity premium of the asset, and contrary to some intuition interest rates increase. The latter is achieved by implementing a money growth rate that lowers real money balances. One might expect lower interest rates to help relax tightening borrowing constraints. However, there are three forces pushing interest rates in the opposite direction. First, monetary authority discourages

inefficient investment in assets whose already low return is even lower. Second, there are more people borrowing constrained and the credit markets adjusts interest rates to prevent an excessive loan supply. Third, since the liquidity premium of the asset is increasing in the interest rate the asset can recover value through its liquidity premium. Thus, a new feature generated in this model is that interest rates, consumption, and liquidity premia depend on the fraction of constrained agents and on the aggregate value of collateral. These results apply to a wide range of economies (those with relative risk aversion greater than one) in which inflation is not too low.

In line with Iacoviello (2005) and Kiyotaki and Moore (1997), borrowing constraints are tied to the value of the asset used as collateral. In equilibria where both money and collateralized credit exist, the interest rate is related to the aggregate value of collateral. However, in my paper interest rates and monetary policy also depend on the number of borrowing constrained agents. Iacoviello (2005) is closer to Bernanke et al. (1996, 1999) where a financial accelerator mechanism is described by which a shock to a firm's net worth leads to a change in the value of its collateral assets, and this to an adjustment in production. Instead, I here describe a world in which changes in the asset valuation may affect final consumption through collateral constraints. Geanakoplos and Zame (2007) also show how the requirement to hold collateral distorts allocations. However, in their world the main distortion comes from the asset being held by the agents who value the asset less, but still acquire it because of collateral requirements. This issue does not arise in my environment.

On a different branch of the literature, Kiyotaki and Moore (1997) present an asset that is a factor of a standard non-linear production function and also serves as collateral. In my model the collateral asset also produces an amount of a good, but with a linear technology. Therefore, the relationship between asset prices, credit limits, and borrowing constraints in their paper is framed in a very different way. A more closely related paper is Ferraris and Watanabe (2008). They focus on the interaction between money and capital accumulation. Capital is used both as a factor of production (again according to a non-linear technology) and as collateral. When borrowers are constrained capital exhibits a liquidity premium and is overaccumulated. The liquidity premium of the collateral asset is also present in my pricing equation. Agents may also be said to overinvest in it, since it is too costly to store, but they are still willing to do it in order to be able to borrow. More importantly, one key difference is that borrowers in their paper are homogeneous: they are either constrained or not, depending on parameterizations. My world has both constrained and unconstrained agents at all periods, with the fraction of them changing with the asset valuation. This heterogeneity, which is a new element in models with collateral constraints, allows me to study how monetary policy acts on interest rates in reaction to changes in the value of collateral.

Berentsen and Monnet (2008) study interest rate spreads in a channel system of monetary policy. In their framework agents (commercial banks) also store a low-return asset that serves as collateral in transactions with a Central Bank. But again borrowers are homogeneous and they focus on how a channel

for interest rates is determined in relation to the value of the collateral used. Berentsen, Camera and Waller (2007) explore the welfare implications of credit in the money search framework. They do not have collateral per se, but they introduce borrowing constraints that would enforce contracts. It is also the case, however, that they only consider homogeneous borrowers. On a different but still related dimension, Geromichalos, Licari, and Suárez-Lledó (2010) model money in an investment market with frictions where securities are derived from another financial asset, which serves as collateral for the promises on the return of the securities. They show how the creation and portfolio re-balancing of this securities increases welfare, and monetary policy can act to enlarge such welfare gains.

Even though collateral constraints have been treated to a certain extent in the literature, they are often exogenously imposed. On the other hand, the introduction of collateral in search-theoretic models of money is a very recent development. In any case, heterogeneous borrowing capacity is a feature that has not been explicitly considered before. Besides believing this kind of heterogeneity may be innovative, I will also try to show that it bears interesting insights for the analysis of monetary policy.

The rest of the paper is structured as follows. Section 2 describes the main aspects of the model. Section 3 discusses the features of the equilibrium and main results, and section 4 concludes.

2 The Model

The structure of the environment is similar to that in Berentsen et al. (2007), which itself builds on Lagos and Wright (2005). Time is discrete and there is a $[0, 1]$ continuum of agents that live forever and discount future at rate $\beta \in (0, 1)$. In every period there are three markets that open sequentially in the following order: first, a credit market; second, a decentralized trade market where agents meet bilaterally; and finally a centralized Walrasian market. The commodities traded in the second and third market are perfectly divisible and of different types. There is a technology available to all agents such that some positive amount of the general good in the centralized market, a , can be stored for the next period. This technology will allow agents to recover $\delta_{t+1}a_{t+1}$ in the next centralized market. At the beginning of every period, and before entering the first market, uncertainty is resolved and agents learn whether they are going to produce or consume in the second market of that period. With probability $1 - n$ an agent will only produce and sell the good in the decentralized market. Thus, these agents will not need money in this market. With probability n an agent will only consume in the second market and, given assumptions that I discuss later, needs money to buy the consumption good. It is assumed, without loss of generality, that $n \geq \frac{1}{2}$.

Credit Market: At any period of time a credit market opens in which agents can reallocate their money balances before going into trade. Financial intermediation is done by perfectly competitive banks that accept deposits from

producers and extend loan contracts to consumers. The net borrowing of a certain agent is denoted by l_t . Given their respective situations, producers may become net lenders, deposit their money in the bank, and earn the interest at the nominal rate i_t . Consumers may become net borrowers and finance higher consumption in the next market, at the same rate i_t . Agents are anonymous and there is lack of enforcement and certain lack of record keeping, which implies that banks cannot punish agents who default on their loan contracts.

As it was mentioned before, agents at the end of every period invest in a certain type of asset, a , that happens to yield an idiosyncratic stochastic return, δ , at the end of next period. One way to interpret this heterogeneity is having many people investing in the same type of asset. Then, for reasons specific to each agent, the final return of the asset is different across individuals. For instance, investors with different capacity to access a financial market may enter the same project in different degrees of participation and, therefore, obtain different returns from it. Another interpretation is that in many cases agents in different countries investing in the same assets face different taxation on asset returns. While modeling the reasons as to why agents are heterogeneous is out of the scope of this paper, there would be many ways to do it without altering the main results.

We assume in this paper that at the beginning of every period each agent only observes its own return on investment, δ , and does not know anyone else's trading histories. Only banks have the technology to verify the asset returns of the economy in the credit market. Therefore, agents cannot trade this asset in

any of the decentralized markets. Banks, however, can sign optimal debt contracts based on the value of the collateral asset, δa , that will allow each agent to borrow, at most, up to her particular return on the stored asset. In case of default, banks can seize the collateral asset and transfer its monetary value back to lenders. Lacker (2001) discusses how the use of collateral is part of an optimal debt contract. The main conditions that support this type of contract are that the collateral asset has more value than the avoided debt repayments, and that borrowers value the collateral asset more highly than the lender. The borrowing constraints that I introduce would be consistent with this notion of optimal credit contracts. As will be shown later, for unconstrained borrowers the value of the asset will be strictly higher than the debt value, whereas for constrained borrowers it will be exactly the same as the value of repayments.³ Borrowers also assign more value to the stored asset from its liquidity premium.

Each agent will then collect an idiosyncratic value, δa , at the end of every period. I assume that these shocks follow a certain distribution, $F(\delta)$, continuous and such that $\delta \in [0, \bar{\delta}]$. Thus, at worst there may be some agent whose investment is worth nothing, and $\bar{\delta}$ is the highest that the collateral of any given agent can be valued. This upper bound will be defined later.⁴ In order to be consistent with the assumptions and with equilibrium I impose that $\beta E(\delta) < 1$, where 1 is the normalized price of consumption in the centralized market. Notice that when $\beta E(\delta) > 1$ agents would want to store an infinite amount of the good.⁵ The total supply of the asset is A .⁶

Decentralized Market: When the credit market closes, trade takes place in a

decentralized market where agents meet in a bilateral and anonymous manner. At the point of entering the second market there are two types of agents: those who want to consume but cannot produce, *buyers* (b), and those who can produce but do not wish to consume, *sellers* (s). The terms of trade are determined in a competitive pricing procedure. This bargaining protocol makes it easier to grasp the intuition of the results.⁷

Centralized Market: Finally, in the third market agents work, consume, and readjust their portfolios for the next period. In order to simplify the analysis and get a clear intuition of the functioning of the model I assume, without loss of generality, that loan and deposit contracts cannot be rolled over. As a consequence all financial contracts will have a duration of only one period.⁸ Thus, in the third market agents repay their loans or redeem their deposits, and decide the amount of money and collateral to be taken into next period.

Consumers in market 2 get utility $u(q)$ from consuming an amount $q > 0$ of the good, where $u'(q) > 0$, $u''(q) < 0$, $u'(0) = +\infty$, and $u'(\infty) = 0$. Producers, on the other hand, incur in a cost $c(q)$ for producing q units of output in the second market, where $c'(q) > 0$, $c''(q) \geq 0$, $c'(0) = 0$. For the sake of simplicity I assume that $c(q) = q$. Agents in the centralized market get utility $U(x)$ from consumption with $U'(x) > 0$, $U''(x) \leq 0$, $U'(0) = \infty$, and $U'(+\infty) = 0$.⁹ Also, a technology is available in this market that allows agents to produce one unit of output with one unit of labor generating one unit of disutility.

2.1 Money

There is an intrinsically worthless object available to agents in the economy that is called (fiat) money. Such object is perfectly divisible and can be stored in any amount $m_t \geq 0$. Papers that model search frictions with money as a medium of exchange, from the early work of Kiyotaki and Wright (1989, 1993) to more modern models, provide a good description of those frictions that make money essential. The same basic assumptions are also established here. In particular, since trade in the decentralized market occurs in bilateral and anonymous meetings where trading histories are private information, and the liquidity needs of each agent are different, we generate the need to use fiat money as a medium of exchange (Kocherlakota (1998), Wallace (2001)).

The per capita money stock in market 3 at any given period in time t is denoted by M_t . The gross rate of growth of money supply is given by $\gamma = 1 + \tau = M_t/M_{t-1}$, where $\gamma > 0$. Changes in the money supply take place through deterministic lump-sum injections or withdrawals of money, τM_t , at the beginning of every period. Therefore, the net change in the aggregate money stock would be $\tau M_{t-1} = (\gamma - 1)M_{t-1}$. I consider a monetary authority that takes the money growth rate, τ , as its policy variable. In particular, the money growth rate, τ , will be set so that it is consistent with the interest rate, i , that is determined by market clearing in each situation.¹⁰ Henceforth, in order to keep notation simple, variables corresponding to the next period will be indexed by $+1$, and those corresponding to the previous one will be indexed by -1 .

2.2 First-best allocation

For any given stationary allocation (q, a) , where q is consumption in the decentralized market and a is asset holdings, the expected lifetime utility of a representative agent at the beginning of a period is

$$(1 - \beta)\mathcal{W} = \tag{1}$$

$$U(x) - x + \int_0^{\delta_c} [u(q_c) - q_c] d\delta + \int_{\delta_c}^{\bar{\delta}} [u(q_u) - q_u] d\delta + [\beta E(\delta) - 1] a.$$

Notice that I differentiate both utility and production cost for constrained, $(u(q_c), q_c)$, and unconstrained borrowers, $(u(q_u), q_u)$. These two types are identified by a critical value, δ_c , which will be defined below. The first two terms represent the utility from consuming and producing in the centralized market. The third and fourth terms are the expected utility from consuming and producing the good in market 2. Finally, the last term reports the utility from producing collateral and receiving its return in the following period.

The first-best allocation for this economy, as would be chosen by a social planner, can be expressed by a vector (x^*, q_j^*, a^*) , $j = c, u$, satisfying $U'(x^*) = 1$, and $q_j = q^*$, $j = c, u$, where q^* solves $u'(q) = 1$. Furthermore, $a^* = 0$ if $\beta E(\delta) < 1$, and $a^* = A$ if $\beta E(\delta) \geq 1$. In other words, a social planner is never going to choose a positive amount of collateral when it is costly, unless there is an indirect return to it that makes it valuable, as will be shown below. Lagos and Wright (2005) and some others show that this allocation is achieved by running $\tau = \beta - 1 = \tau_{FR}$, which implements the Friedman rule. With such

policy, however, a credit market generates no real value at all within this type of framework. In this paper I want to consider economies where credit plays a role. Therefore, I will focus on monetary policies in a neighborhood of τ_{FR} .

3 Equilibrium

Let P denote the price of the good in the centralized market. Then, $\phi = 1/P$ denotes the value of money, the real price of money in that market. I focus on symmetric stationary monetary equilibria where all agents of a given type behave in the same way, and real allocations (including money balances) are constant through time. In particular,

$$\phi M = \phi_{-1} M_{-1} \equiv z. \quad (2)$$

In these equilibria $\phi_{-1}/\phi = P/P_{-1} = M/M_{-1} = 1 + \tau$, which means that I am restricting my attention to equilibria where the growth rate of money supply τ is constant. In what follows, I will solve and study the model backwards to better track the decisions of the agents.

3.1 Third Market

At this stage agents get the return from having stored the real asset in the last period, δa , and decide how much to work, h , how much to consume, x , and how to readjust their portfolios for next period. That is, how much money, m_{+1} , and

how much of the real asset, a_{+1} , to carry into next period. Finally, in order to settle financial claims, agents who were consumers in market 2 and borrowed l units of money have to pay back $(1+i)l$ from their loan contracts. This amount is transferred to agents who were producers and thus they redeem their deposits.

Given a distribution $F(\delta)$ and a policy arrangement τ , let $V_3(m, l, a)$ be the expected lifetime utility of entering the third market with m units of money, l net borrowing, and a units of the stored asset. The problem of a representative agent in this market is

$$\begin{aligned} V_3(m, l, a) &= \max_{x, h, m_{+1}, a_{+1}} [U(x) - h + \beta V_1(m_{+1}, a_{+1})] \\ \text{s.t.} \quad &x + \phi m_{+1} + a_{+1} = h + \phi m - \phi(1+i)l + \delta a. \end{aligned} \quad (3)$$

Substituting the budget constraint into (3) gives us

$$\begin{aligned} V_3(m, l, a) &= \phi[m - (1+i)l] + \delta a \\ &+ \max_{x, m_{+1}, a_{+1}} [U(x) - x - \phi m_{+1} - a_{+1} + \beta V_1(m_{+1}, a_{+1})]. \end{aligned}$$

The first order conditions are

$$\begin{aligned} U'(x^*) &= 1 \\ \phi &\geq \beta V_1^m(m_{+1}, a_{+1}) \quad (= 0, \text{ if } m_{+1} > 0) \end{aligned} \quad (4)$$

$$1 \geq \beta V_1^a(m_{+1}, a_{+1}) \quad (= 0, \text{ if } a_{+1} > 0) \quad (5)$$

The superscript indicates the variable with respect to which the function is being differentiated. Thus, V_1^m is the marginal value of taking an additional unit of money into the first market of next period, and equation (5) is the asset pricing equation. The envelope conditions are

$$V_3^m = \phi; V_3^l = -\phi(1+i); V_3^a = \delta. \quad (6)$$

As can be observed, the value function is linear. This, together with the assumptions on preferences and production costs listed in the description of the model, ensures that there are no wealth effects. Thus, a degenerate distribution of money and the asset is obtained such that all agents enter the following period with same amount of money and collateral.¹¹

3.2 Second Market

An agent that walks into the second market with m units of money, net borrowing l , and a units of collateral has lifetime utility, $V_{2j}(m_j, l_j, a)$, $j = b, s$, as described below

$$\begin{aligned} V_{2s}(m_s, l_s, a) &= \max_{q_s} -c(q_s) + V_3(m_s + pq_s, l_s, a), \\ V_{2b}(m_b, l_b, a) &= \max_{q_b} u(q_b) + V_3(m_b - pq_b, l_b, a). \end{aligned}$$

where $m_j = m + l_j$, $j = b, s$. I analyze here a competitive pricing solution to the problem of each agent. Thus, p will be the competitive price paid in exchange

for the quantity of the good traded. A buyer who consumes q_b units of the good in market 2 enters market 3 with $m_b - pq_b$ units of money. Similarly, an seller who has produced and sold q_s units of the good in market 2 enters market 3 with $m_s + pq_s$ units of money. This market carries a generic market clearing condition $nq_b = (1 - n)q_s$. From now on I will adopt the following notation, $q = q_b = \frac{1-n}{n}q_s$.

The problem to be solved by the sellers then would be

$$\max_{q_s} -q_s + V_3(m_s + pq_s, l_s, a) \quad (7)$$

The first order condition is $-1 + pV_3^m = 0$, which using (6) implies $p = 1/\phi$. On the other side, the buyer faces the maximization

$$\begin{aligned} \max_q & u(q) + V_3(m_b - pq, l_b, a) \\ \text{s.t.} & \quad pq \leq m_b \end{aligned} \quad (8)$$

whose first order condition, again using (6), can be expressed as

$$\lambda_q = \phi [u'(q) - 1], \quad (9)$$

where λ_q is the multiplier associated to the constraint. Recall that the optimal amount of consumption, q^* , is defined by $u'(q^*) = 1$. The equilibrium characterization in this market ultimately depends on the buyer's real money balances, z_b . If $z_b \geq z^*$ such that $u'(q(z^*)) = 1$, then it is optimal to spend $z_b = z^*$ in pur-

chasing q^* , and $\lambda_q = 0$. However, agents will only choose to carry $z_b \geq z^*$ with them as long as $\tau = \beta - 1$.¹² Since we are focusing on policies with $\tau > \beta - 1$ equilibrium in this market features $z_b < z^*$. Given the maintained assumptions this implies that $q(z_b) < q^*$. Therefore $u'(q) > 1$, which means $\lambda_q > 0$, implying $q = z_b$.

All types of consumers spend all their money in acquiring the consumption good in this market and trades are not efficient. Also, since the budget constraint binds for at any equilibrium, the interest rate basically acts as a tax on consumption. Finally, using the envelope theorem and equations (6) and (9) we obtain

$$\begin{aligned} V_{2j}^l &= -\phi(1+i), \quad V_{2j}^a = \delta, \quad j = b, s. \\ V_{2s}^m &= V_3^m = \phi \\ V_{2b}^m &= V_3^m + \lambda_q = \phi u'(q). \end{aligned} \tag{10}$$

3.3 First Market

An agent who enters the first market with m units of money and a units of the real asset has expected lifetime utility

$$V_1(m, a) = \int [nV_{1b}(m, a) + (1-n)V_{1s}(m, a)] dF(\delta) \tag{11}$$

where $V_{1j} = \max_{l_j} V_{2j}(m_j, l_j, a)$ and $m_j = m + \tau M + l_j$, $j = b, s$. Agents face a borrowing constraint that depends on the value of collateral at any point in time.

As we explained before, this constraint captures the essentials of an optimal debt contract in an environment with limited enforcement: the real amount of loan plus interest must not be greater than the real value of available collateral, $\phi(1+i)l \leq \delta a$. The key notion here is that each agent will face a different constraint according to the particular value of her collateral. Once trading types have realized, an agent of type $j = b, s$ solves the following problem:

$$\max_{l_j} V_{2j}(m_j, l_j, a) \quad (12)$$

$$\text{s.t. } 0 \leq m + \tau M + l_j,$$

$$\phi(1+i)l_j \leq \delta a. \quad (13)$$

The first order condition for this problem is

$$V_{2j}^m + \lambda_j + V_{2j}^l - \lambda_j^l = 0$$

where λ_j is the multiplier corresponding to the first constraint and λ_j^l is the multiplier for the borrowing constraint. It is easy to prove that sellers will become net lenders in equilibrium, $\lambda_s > 0$.¹³ Since they will earn a positive interest, it is always profitable for them to lend the money that they will not need. Obviously, for these agents it is also the case that $\lambda_j^l = 0$, since their net borrowing is negative. Thus, using (10) the previous first order condition for sellers becomes

$$\lambda_s = \phi i. \quad (14)$$

As long as $i > 0$ (implied by $\tau > \beta - 1$), then $\lambda_s > 0$ and sellers deposit all their money. In a similar fashion, one can easily see that agents of type b will always become net borrowers in any equilibrium since money is required to buy consumption. Assumptions on preferences, like the usual Inada conditions, guarantee that $m + \tau M + l_b > 0$, that is, $\lambda_b = 0$. Berentsen et al. (2007) prove that financial intermediation improves welfare (consumption) away from the Friedman rule. In other words, $l_b = 0$ is optimal only when $\tau = \beta - 1$. Therefore, using (10) and $\lambda_b = 0$, the necessary and sufficient conditions for an optimal solution for buyers can be written

$$\begin{aligned} \frac{u'(q)}{1+i} - 1 &= \lambda_b^l, \\ \lambda_b^l [\phi(1+i)l_b - \delta a] &= 0. \end{aligned} \tag{15}$$

Any optimal solution will then have $\lambda_b = 0$, $\lambda_b^l \geq 0$, and $l_b > 0$. Henceforth I will use l instead of l_b . Notice that for the economies that we focus on, $\lambda_b^l \geq 0$ in fact implies $\lambda_q > 0$, and therefore $q = z_b$. Most models of borrowing with collateral consider borrowing constraints that are either always binding or not.¹⁴ However, I consider at this point two types consumers: those who are borrowing constrained and those who are not. The constraint will be binding for the first ($\lambda_b^l > 0$), but not so for the second ($\lambda_b^l = 0$). In other words, for the constrained agents $u'(q_c) > 1 + i$, whereas for the unconstrained $u'(q_u) = 1 + i$, where q_c and q_u are the amounts of good in the decentralized market consumed by constrained and unconstrained consumers, respectively. Furthermore, the real loan demand

schedule can be expressed as

$$\phi l_b = \begin{cases} \phi l(i) = u'^{-1}(1+i) - z & \text{if } u'(q) = 1+i, \\ \phi l(i) = \frac{\delta a}{1+i} & \text{if } u'(q) > 1+i. \end{cases} \quad (16)$$

Recall that $q = z + \phi l$. The unconstrained loan demand is well defined since the first derivative of the utility function is itself continuously differentiable and strictly decreasing. These expressions tell us that unconstrained buyers take loans until the marginal benefit of doing so is equal to its marginal cost. Trades are efficient in this case. Constrained agents are not able to reach this efficiency and will borrow as much as they are allowed to. In equilibrium the marginal utility of constrained borrowers will be higher than that of unconstrained agents. Notice, however, that the fact that unconstrained borrowers get efficient trades in this market does not mean that they actually get the efficient consumption $q_u(z, a) = q^*$. On the contrary, for positive interest rates the equation $u'(q_u) = 1+i$ actually means that $u'(q_u) > 1$. Therefore, $q_u(z, a) < q^*$. Moreover, since $q = z + \phi l_b$ and $u'(q)$ is decreasing in q , consumption for constrained borrowers is such that $q_c < q_u < q^*$, which obviously implies that $(\phi l)_c < (\phi l)_u$ as well.

The following lemma defines the critical value that separates constrained from unconstrained borrowers.

Lemma 1. *Given the maintained assumptions on utility and the distribution of the value of collateral, there exists a critical value, δ_c , that lies in the interior*

of the support of the distribution $F(\delta)$, and is defined by

$$u'^{-1}(1+i) - z = \frac{\delta_c a}{1+i}. \quad (17)$$

Proof. See Appendix.

Q.E.D.

The credit market clearing condition, that I state below, will establish a unique equilibrium interest rate. And given real balances, z , and asset investment, a , from previous period, equation (17) uniquely determines a collateral value such that all agents whose asset is worth less than that will borrow all the way up to their asset value, and below the efficient borrowing level. We can actually describe the real loan demand schedule in terms of the return to collateral:

$$\phi l_b = \begin{cases} \phi l(i) = u'^{-1}(1+i) - z & \text{if } \delta \geq \delta_c, \\ \phi l(i) = \frac{\delta a}{1+i} & \text{if } \delta < \delta_c. \end{cases} \quad (18)$$

For $\delta < \delta_c$, the loan demand is increasing in δ and for $\delta \geq \delta_c$ is constant, given that unconstrained borrowers take loans only until the marginal benefit of doing so is equal to its marginal cost. Also, both loan demands are naturally decreasing in the interest rate. Market clearing in the loan market requires that the total amount of loans demanded by constrained and unconstrained borrowers does

not exceed the total amount of money deposited by lenders, $(1 - n)z$,

$$(1 - n)z = \int_0^{\delta_c(z,a,i)} \frac{\delta a}{1 + i} d\delta + \int_{\delta_c(z,a,i)}^{\bar{\delta}} [u'^{-1}(1 + i) - z] d\delta. \quad (19)$$

3.4 Steady State

Differentiating (11) and using the envelope theorem and conditions (14) and (15), the marginal values of money and collateral can be expressed as ¹⁵

$$V_1^m = \int \phi [nu'(q) + (1 - n)(1 + i)] dF(\delta), \quad (20)$$

and

$$V_1^a = \int \left[n\delta \left(\frac{u'(q)}{1 + i} - 1 \right) + \delta \right] dF(\delta). \quad (21)$$

The marginal value of money has the following interpretation. If the agent is a buyer an additional unit of money yields $u'(q)$ units of marginal utility in the second market. If the agent is a seller she can deposit the additional unit of money which will yield the nominal return, $1 + i$. The credit market increases the marginal value of money because agents can earn interest on idle cash.

Remember that equation (5) is the pricing equation for the asset that is used as collateral. The price of this asset is always that of the good in the centralized market. However, the pricing equation will also reflect both the expected return and the liquidity properties of the collateral asset: every additional unit of

collateral allows the borrower to take loans that will give them net additional utility $u'(q) - (1 + i)$. Using (5) and (21), and rearranging them, we can derive the expression for the liquidity premium of collateral

$$1 - \beta E(\delta) = \beta \int \left[n \left(\delta \frac{u'(q)}{1+i} - \delta \right) \right] dF(\delta). \quad (22)$$

The cost of storing an additional unit of collateral is 1, and the expected return of doing so is $\beta \int \delta dF(\delta) = \beta E(\delta)$. Thus, the right-hand side of (22) is the collateral's liquidity premium. However, we imposed that $\beta \bar{\delta} < 1$, which means that this premium is positive. Agents need an incentive to hold collateral other than its investment return. The liquidity properties of collateral provide this incentive.

Plugging (20) and (21) into (4) and (5) respectively, and using equation (17) along with the fact that $u'(q) = 1 + i$ for unconstrained borrowers, we can rearrange to obtain the equilibrium conditions:

$$\frac{1 + \tau - \beta(1 + i)}{\beta(1 + i)} = \int_0^{\delta_c(z, a, i)} n \left(\frac{u' \left(z + \frac{\delta a}{1+i} \right)}{1+i} - 1 \right) dF(\delta), \quad (23)$$

$$\frac{1 - \beta E(\delta)}{\beta} \geq \int_0^{\delta_c(z, a, i)} n \delta \left(\frac{u' \left(z + \frac{\delta a}{1+i} \right)}{1+i} - 1 \right) dF(\delta). \quad (24)$$

The terms in parenthesis on the right-hand sides of these equations reflect the excess of marginal utility with respect to the marginal cost of borrowing in the credit market. Thus, the right-hand side of equation (23) can be interpreted

as the average excess of marginal utility over the marginal cost of borrowing among constrained borrowers.

Definition 1. *Given a distribution function $F(\delta)$, with $\delta \in [0, \bar{\delta}]$, a symmetric stationary monetary equilibrium with credit is a set of value functions V_{ij} , $i = 1, 2$, $j = b, s$, and V_3 , that satisfy the Bellman equations, a bounded choice of real balances and collateral holdings, (z, a) , a price that clears the credit market, i , a distribution of borrowers, δ_c , and a policy τ that satisfy (24), (23), (19), and (17) at every period, with $m_t = M_t$ and $a_t = A$, all t .*

We need the following Lemma before we proceed to prove that an equilibrium exists.

Lemma 2. *In an economy with credit, there exists an upper bound to the money growth rate, $\bar{\tau}$, implicitly defined by*

$$u'^{-1}(1 + \bar{i}) - z = 0. \quad (25)$$

Proof. See Appendix.

Q.E.D.

Proposition 1. *For any $\tau \in (\beta - 1, \bar{\tau})$, there exists a unique symmetric stationary monetary equilibrium with collateralized credit.*

Proof. The idea behind this proposition is that for this type of economies the first-best allocation can always be implemented with $\tau = \beta - 1$. Stationary

real money balances are then z^* such that $u'(q(z^*)) = 1$. The pricing equations (23) and (24) boil down to $i = 0$ and $1 - \beta E(\delta) > 0$. Therefore, the optimal choice would be $a = 0$ and there would be no need for credit. End of the story. However, we are interested in the behavior of the economy in a neighborhood of τ_{FR} .¹⁶ For these economies it will now be optimal to choose $a > 0$, collateralized credit will achieve better allocations, and equation (24) always holds with equality, i.e., agents optimally choose to store $a_t = A$ every period.¹⁷

Once we know i , q_u is pinned down by $u'(q_u) = 1 + i$. Similarly, given, i , z , and a , all the equations of the form $q_c^k = z + \frac{\delta_k a}{1+i}$ will determine consumption for each constrained agent, q_c^k , for each $\delta_k \in [0, \delta_c)$, and (17) will determine δ_c . In short, we are down to 4 equilibrium equations, (17), (19), (23), (24), in 4 equilibrium variables, z , δ_c , i , τ . There is a way to solve the system recursively, but the algebra is considerably tedious and makes the intuition and mechanisms in the model a lot less clear. I discuss this alternative procedure in some detail in the Appendix and focus here on a more clear and intuitive approach.

Since τ is linearly and uniquely determined by (23), we can use (17) to form a system of two implicit equations, (19) and (24), in two unknowns, z and i . First, let us compute the integrals in the market clearing condition, which will look like

$$(1 - n)z = \frac{\delta_c^2 a}{2(1 + i)} + [u'^{-1}(1 + i) - z](\bar{\delta} - \delta_c),$$

and we can use (17) to rewrite it as

$$T_1(i, z, \bar{\delta}) \equiv (1 - n)z - [u'^{-1}(1 + i) - z]\bar{\delta} + [u'^{-1}(1 + i) - z]^2 \frac{1 + i}{2a} = 0, \quad (26)$$

Notice that this equation could be differently rearranged to obtain an expression for the interest rate

$$1 + i = u' \left[z \left(1 + \frac{1 - n}{\bar{\delta} - \delta_c/2} \right) \right], \quad (27)$$

and obviously unconstrained consumption

$$q_u = z \left(1 + \frac{1 - n}{\bar{\delta} - \delta_c/2} \right), \quad (28)$$

both as a function of the aggregate value of collateral, $\bar{\delta}$, and the fraction of constrained borrowers, δ_c . Finally, (24) can also be rewritten as

$$T_2(i, z, \bar{\delta}) \equiv \beta \int_0^{\delta_c(i, z)} n \delta \left(\frac{u' \left(z + \frac{\delta a}{1 + i} \right)}{1 + i} - 1 \right) dF(\delta) + \beta E(\delta) - 1 = 0. \quad (29)$$

Recall that given z , i , and $\bar{\delta}$, then all q_c and q_u , τ , and δ_c are automatically pinned down. Thus, I can deal with our initial system of equations as an implicit system $T : \mathbf{R}^3 \rightarrow \mathbf{R}^2$, where $T = (T_1(i, z, \bar{\delta}), T_2(i, z, \bar{\delta}))$, treating i and z as the endogenous variables and $\bar{\delta}$ as the exogenous one. Berentsen et al. (2007), and Berentsen and Monnet (2008), among others, show existence and uniqueness of equilibrium for this type of monetary economies with credit. The same

standard assumptions are made in this paper about concavity of the value function and properties of the constraint set that guarantee existence and uniqueness of equilibrium with collateralized credit. Thus, the only thing left to show is that the equilibrium mappings are all well-behaved. The details on this last part can be found in the Appendix. *Q.E.D.*

The equilibrium system cannot be solved analytically. However, we are not as interested in the exact or numerical solution as we are on the qualitative properties of the model. In particular, it will be significantly interesting to analyze the behavior of the equilibrium when the aggregate value of collateral changes.

3.4.1 Comparative Statics

The approach here is to approximate the system $T(i, z, \bar{\delta})$ by differentiating it, considering that i and z may be implicit functions of $\bar{\delta}$. We do not know how the exact stationary equilibrium looks like, but we don't need it. All we want to know is the direction of the changes in i and z when $\bar{\delta}$ varies, for any equilibrium. By total differentiating T we should be able to obtain an expression for $\frac{\partial i}{\partial \bar{\delta}}$ and $\frac{\partial z}{\partial \bar{\delta}}$ that would be given by

$$\begin{pmatrix} di \\ dz \end{pmatrix} = - \begin{pmatrix} \frac{\partial T_1}{\partial i} & \frac{\partial T_1}{\partial z} \\ \frac{\partial T_2}{\partial i} & \frac{\partial T_2}{\partial z} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial T_1}{\partial \bar{\delta}} \\ \frac{\partial T_2}{\partial \bar{\delta}} \end{pmatrix} d\bar{\delta}. \quad (30)$$

It is shown in the Appendix that $\det(D_{(i,z)}T) < 0$. Therefore, we know that there exist continuously differentiable relationships $i = \mathbf{i}(\bar{\delta})$ and $z = \mathbf{z}(\bar{\delta})$. Furthermore, details are also provided that show that these relationships around the equilibrium are such that $\frac{\partial i}{\partial \bar{\delta}} < 0$ and $\frac{\partial z}{\partial \bar{\delta}} > 0$. As a direct consequence of this, $\frac{\partial \delta_c}{\partial \bar{\delta}} < 0$. Also, since $\frac{\partial T_2}{\partial \bar{\delta}} = \beta \frac{\partial E(\bar{\delta})}{\partial \bar{\delta}} > 0$, the right hand side of (24) is decreasing in the aggregate value of collateral. This implies that the right hand side of (23) is also decreasing in $\bar{\delta}$. We can use this to differentiate (23) to obtain that $\frac{\partial \tau}{\partial \bar{\delta}} < 0$. Finally, it is straightforward to differentiate the condition $u'(q_u) = 1 + i$ to obtain $u''(q_u) \frac{\partial q_u}{\partial \bar{\delta}} = \frac{\partial i}{\partial \bar{\delta}}$, and conclude that $\frac{\partial q_u}{\partial \bar{\delta}} > 0$. Similarly, all the conditions $q_c^k = z + \frac{\delta_k a}{1+i}$, $\delta_k \in [0, \delta_c)$, can be used to obtain that $\frac{\partial q_c^k}{\partial \bar{\delta}} > 0$.

In terms of welfare, consumption of constrained and unconstrained types strictly increases with the value of collateral. In the case of constrained agents both real balances are higher and it is cheaper to borrow (lower interest rates). Consumption of unconstrained agents, however, may be seen as depending only on interest rates, although indeed also depends on $\bar{\delta}$ and δ_c . Thus, whenever the aggregate value of collateral is higher interest rates will be lower and their consumption increases. However, in their case welfare is higher not because borrowing is cheaper, but mainly because real balances are higher. Notice that the real amount of loans for unconstrained consumers, $u'^{-1}(1+i) - z$, is not strictly increasing in the value of collateral. The term $u'^{-1}(1+i)$ increases with collateral, but does so in a non-linear manner, whereas the term $-z$ decreases linearly with collateral. Thus, borrowing by unconstrained agents is small for

very low and very high values of collateral while it is highest for moderate values of collateral. That their borrowing is low for high values of the asset should not be interpreted as their consumption being low, since it is for those values of collateral that money balances are highest.

In the opposite case there is a surprising result for both types: for low values of collateral money balances are low as well. One would expect that real balances would be higher in order to compensate for the reduced borrowing capacity. However, there is another mechanism at work. When the asset is worth little, it is very costly for society to store it and the monetary authority tries to discourage its investment with high interest rates induced by high money growth rates, which worsen money balances. Therefore, even investment in low-yield assets that can be used as collateral has a greater effect on real allocations in this economy. In a world with homogeneous borrowers whenever the return to collateral was low the monetary authority could always reduce money supply, thereby increasing money balances and reducing the need to borrow and the impact of real assets on the final allocation. This could not be achieved in a heterogeneous economy because there would always be a fraction of constrained agents.

The most interesting results relate to interest rates. From equation (27) equilibrium interest rates depend on both the value of collateral and the number of borrowing constrained agents. This is an interesting feature that is not present in other money search models. In homogeneous borrowers models either interest rates would not depend on the return to collateral (all unconstrained) or

they would not depend on the fraction of constrained agents (all constrained). Probably, one of the main results is that interest rates increase when the aggregate value of collateral falls. This result may seem at first a bit counterintuitive: it would be reasonable to expect lower interest rates that helped relax borrowing constraints. Our result is somehow in line with the literature on general equilibrium with incomplete markets (aka GEIM), of which Geanakoplos and Zame (2007) is one example. In their models, when the value of assets increases collateral requirements are lowered and consumer can get bigger loans at lower prices. We find the same result but the mechanisms at work in our model are different. First, when the asset used as collateral loses value, the monetary authority induces higher interest rates to discourage investment in such costly asset. Second, the fraction of constrained borrowers is higher, so there are more people borrowing up to their limits, but always less than $u'^{-1}(1+i) - z$. Therefore, higher money growth rate that worsens real balances would prevent an excess loan supply.

Finally, there is yet another interesting effect in this model. Since $\frac{\partial T_2}{\partial i} > 0$ we know that the liquidity premium of the asset is increasing in the interest rate. This means that when collateral loses value it becomes even more costly, but higher interest rates can make it regain value through a larger liquidity premium. Thus, in our model the asset's liquidity premium really plays an important role. It is not just what standard asset pricing theory would tell us: the difference between the price of the asset and its expected return. It also reflects the indirect liquidity that consumers obtain in the decentralized market, which

depends crucially on the degree of borrowing heterogeneity, reflected by δ_c . All of these are mechanisms not present in the general GEIM literature, but in any case these effects give support to the GEIM models' claim that central banks might find it useful to follow the movements in the value of the assets that are used as collateral when designing monetary policy. This is in a sense in contrast with the conclusion in Bernanke and Gertler (1999, 2001) that monetary policy should not respond to changes in asset prices, unless they help to forecast inflationary pressures.

As a final remark, it would be interesting to compare the results in this economy with those of homogenous economies. It would not be an easy task because we have heterogeneity only on one side and a continuum of idiosyncratic shocks. However, we can approximate those economies by considering the extreme cases where $\bar{\delta}$ is high so that $\delta_c \rightarrow 0$, and where $\bar{\delta}$ is low so that $\delta_c \rightarrow \bar{\delta}$. In the former interest rates are close to zero and real balances close to z^* , while in the latter interest rates are high and real balances close to zero. In a word, in an all-unconstrained economy interest rates are lower and consumption is high, whereas in an all-constrained economy the interest rate is high, real balances and consumption are low.¹⁸

4 Conclusion

In this paper I have built a model of money and credit where agents can invest in an asset that may be collateralized in contingencies, even though the oppor-

tunity cost of holding it is positive. A series of interesting results is obtained for a large set of economies with moderate inflation. First, I abstract from the effect of the price of collateral on the equilibrium dynamics in order to isolate the role of endogenous constraints and heterogeneous borrowing capacity. This modeling approach allows me to endogenously determine the monetary policy instrument in response to changes in the value of collateral. In other models, for a predetermined money growth rate the amount borrowed would depend on the interest rate, being highest near the Friedman rule, therefore making constraints more likely to bind. However, in this paper the money growth rate will adjust to changes in the value of collateral, so the amount borrowed will depend both on changes in the interest rate and money balances. Equilibrium interest rates, consumption, and liquidity premia depend on the fraction of constrained agents and on the aggregate value of collateral.

Another key innovative element in the model is that the return of the asset is idiosyncratic and random to each agent. There will always be a fraction of agents that will be borrowing constrained, this fraction being endogenously determined in equilibrium. When the aggregate value of collateral falls more people are constrained, which raises the liquidity premium of the asset, and interestingly interest rates increase. To further surprise, the latter is achieved by implementing a money growth rate that lowers real money balances, which can also be explained by collateral and money being complements in this model. The behavior of interest rates uncovers interesting mechanisms that are not all present in other models. First, monetary authority discourages inefficient in-

vestment. Second, the fraction of people borrowing constrained is higher and monetary policy worsens money balances to prevent an excessive loan supply. Finally, when the aggregate return to collateral falls higher interest rates can make it recover value through its liquidity premium.

A Appendix

Welfare Function. When we discussed the first-best allocation I stated the welfare function that any social planner would have to maximize in order to achieve its objective. I now show how to derive the welfare function, equation (1). Recall that $M_{+1} = M + \tau M$, and that the portfolio of a buyer entering the third market is $(m, l, a) = (0, l_b, a)$, and that of a seller is $(m, l, a) = (m + \tau M + l_b, l_s, a)$. First, we need hours worked:

$$\begin{aligned} h_b &= x - (E(\delta) - 1)a + \phi(1 + i)l_b \\ h_s &= x - (E(\delta) - 1)a - \phi(1 + i)l_s - \phi(m + \tau M + l_b) \end{aligned}$$

Total hours worked $h = nh_b + (1 - n)h_s$. Using the market clearing conditions, this expression reduces to $h = x - (E(\delta) - 1)a$. Thus, welfare can be computed as

$$\begin{aligned} \mathcal{W} &= U(x) - x - a + \int_0^{\delta_c} [u(q_c) - q_c] d\delta + \int_{\delta_c}^{\bar{\delta}} [u(q_u) - q_u] d\delta + \\ &\sum_{t=1}^{\infty} \beta^t \left\{ \int_0^{\delta_c} [u(q_c) - q_c] d\delta + \int_{\delta_c}^{\bar{\delta}} [u(q_u) - q_u] d\delta + U(x) - x + (E(\delta) - 1)a \right\} \end{aligned}$$

$$= \frac{U(x) - x + \int_0^{\delta_c} [u(q_c) - q_c] d\delta + \int_{\delta_c}^{\bar{\delta}} [u(q_u) - q_u] d\delta + (\beta E(\delta) - 1)a}{1 - \beta}.$$

Q.E.D.

Proof of Lemma 1. We want to show that as long as the aggregate value of collateral is not too low ($\bar{\delta} > 0$), there will always be a fraction of constrained and unconstrained borrowers. A sufficient condition on this value will also be discussed. Let us define the following function:

$$b(\delta) \equiv (1 + i)\phi l(i) - \delta a,$$

where $\phi l(i)$ is the real amount borrowed. It will be useful to notice two things. First, since $b(\delta)$ is a continuous and monotonically decreasing function of δ on a compact set $[0, \bar{\delta}]$, we know that there exists a unique $\delta_c \in (0, \bar{\delta})$ such that $b(\delta_c) = 0 \in (b(\bar{\delta}), b(0))$, as long as $b(\bar{\delta}) < 0$. Second, for any possible $F(\delta)$ and feasible i , there are two real loan demands: $\phi l(i) = u'^{-1}(1 + i) - z$, and $\phi l(i) = \frac{\delta a}{1+i}$, both decreasing and non-linear in i . Also, the maximum amount of money that can be borrowed is $\frac{1-n}{n}z$ and the minimum is zero. The logic behind many of the arguments that follow is that whatever i the credit market determines some agents could be better off by borrowing the unconstrained amount at that i and then default. In other words, if all agents borrowed the unconstrained amount $\phi l(i) = u'^{-1}(1 + i) - z$ at a given interest rate, it can be possible that for some δ , $V_{2b}(m_b, l_b, a; \lambda_b^l = 0) \leq V_{2b}(m_b, l_b, a; \lambda_b^l > 0)$, i.e., $u(u'^{-1}(1 + i)) - (1 + i)(u'^{-1}(1 + i) - z) + \delta a \leq u(u'^{-1}(1 + i))$, which means

$\delta a \leq (1+i)(u'^{-1}(1+i) - z)$. Therefore, it boils down to studying how the term $(1+i)\phi l(i)$ behaves and whether there can be situations where the borrowing constraint binds, or not, for all agents.

Let us first consider the case in which agents can borrow a very high amount. In this case constraints are more likely to bind for all agents. If it binds for the consumer with the highest value of collateral, then it will also bind for the rest. Well, an agent who borrows the highest amount enjoys consumption $q = z + \frac{1-n}{n}z = \frac{z}{n}$. If there is just one such agent then interest rates will be $1+i = u'(\frac{z}{n})$, and his total debt $(1+i)\phi l(i) = u'(\frac{z}{n})\frac{1-n}{n}z$. Notice now that at the end of the discussion of an alternative procedure to solve for the steady state, the existence of a unique solution required, among other things, that $\bar{\delta} > (1+i)2(1-n)z/a$. This implies that there are at least some agents whose asset value is near $\bar{\delta}$ for whom $b(\delta) = (1+i)\phi l(i) - \delta a < 0$. And since for $\delta = 0$, $b(0) > 0$, we know that there exists a unique $\delta_c \in (0, \bar{\delta})$ such that $b(\delta_c) = 0$.

Let us now turn to the case in which interest rates and money growth rate are such that agent's borrowing demand is so low that makes it more likely that all agents face a non-binding constraint. This can happen in two situations, namely $i \rightarrow 0$ or $i \rightarrow \bar{i}$, where \bar{i} is defined by $u'^{-1}(1+\bar{i}) - z(\bar{\tau}) = 0$. The former case refers to the case in which real balances are so high that little is needed to achieve high consumption. The latter means that interest rates and money growth rate are high enough so that it is too expensive to borrow. For any $i \in (0, \bar{i})$, we know that $(1+i)\phi l(i) > 0$, therefore $b(i, 0) > 0$. Also, for any interest rate near 0 or \bar{i} , $b(i, \bar{\delta}) < 0$. Therefore, we can conclude that for any

$i \in (0, \bar{i})$ and any $\tau \in (\beta - 1, \bar{\tau})$, there always exist a $\delta_c \in (0, \bar{\delta})$ such that (17) holds. *Q.E.D.*

Proof of Lemma 2. The real amount borrowed by both constrained and unconstrained agents is continuously decreasing in the interest rate. Then, for an interest rate high enough the loan demanded would be zero. Since the amount borrowed by a constrained agent is always lower than that of an unconstrained one, if the above is true for the unconstrained it will also be true for the constrained. So let us consider the loan demand function for the unconstrained, $u'^{-1}(1+i) - z$. For $i \rightarrow 0$, $u'^{-1}(1+i) \rightarrow z^* > z$, and the loan function would be positive. On the contrary, for $i \rightarrow \infty$, $u'^{-1}(1+i) \rightarrow 0 < z$, and the loan function would be negative. By the Intermediate Value Theorem we know there is an interest rate \bar{i} such that $u'^{-1}(1+\bar{i}) - z = 0$. Associated to each interest rate there is a money growth rate. This equation implicitly defines such $\bar{\tau}$, which could be computed by multiplying both sides of the equation by $\phi_{-1}M > 0$,

$$\frac{u'^{-1}(1+\bar{i})}{\phi_{-1}M} - \frac{\phi M}{\phi_{-1}M} = 0 \Rightarrow 1 + \bar{\tau} = \frac{M/P_{-1}}{u'^{-1}(1+\bar{i})}.$$

Q.E.D.

Proof of Proposition 1. We have to analyze the behavior of the implicit system $T(i, z, \bar{\delta})$. In particular, we have to study its monotonicity and behavior at the extreme points of its domain in order to apply the Intermediate Value Theorem. Since the study of monotonicity will also be needed for the analysis

of changes in the value of collateral, we will start there. We have to differentiate the system with respect to both i and z ,¹⁹

$$\begin{aligned}
\frac{\partial T_1}{\partial i} &= -\frac{\bar{\delta} - \delta_c}{u''(q_u)} + \frac{\delta_c^2 a}{2(1+i)^2} > 0, \\
\frac{\partial T_1}{\partial z} &= \bar{\delta} - \delta_c + 1 - n > 0, \\
\frac{\partial T_2}{\partial i} &= \beta \left[\int_0^{\delta_c(i,z)} n\delta \frac{(1+i)u''(q_c) \left(\frac{\delta a}{(1+i)^2}\right) - u'(q_c)}{(1+i)^2} dF(\delta) \right. \\
&\quad \left. + n\delta_c \underbrace{\left(\frac{u' \left(z + \frac{\delta_c a}{1+i} \right)}{1+i} - 1 \right)}_{=0} \frac{\partial \delta_c}{\partial i} \right] > 0, \\
\frac{\partial T_2}{\partial z} &= \beta \int_0^{\delta_c(i,z)} n\delta \frac{u''(q_c)}{1+i} dF(\delta) + n\delta_c \underbrace{\left(\frac{u' \left(z + \frac{\delta_c a}{1+i} \right)}{1+i} - 1 \right)}_{=0} \frac{\partial \delta_c}{\partial z} < 0.
\end{aligned}$$

Several things are worth noting. I write a generic q_c for $z + \frac{\delta a}{1+i}$ inside the integrals. The first and second derivatives are clearly positive, while the fourth is negative. The third derivative requires a bit more explanation. Since the denominator is just positive, the sign of this derivative will be determined by the sign of the numerator inside the integral, which we can write as $-u''(q_c) \frac{\delta a}{1+i} - u'(q_c) = -u''(q_c)(q_c - z) - u'(q_c) = -[u'(q_c) + u''(q_c)q_c] + u''(q_c)z$. We could also write this term in a more recognizable form by adding and subtracting $u'(z)$ from it. We would obtain $[u'(z) - u'(q_c) + u''(q_c)(z - q_c)] - u'(z)$. Now, $q_c > z$ and $u''(\cdot)$ is a negative-valued convex function. Then we know that the term in brackets is positive and probably greater than the last $u'(z)$. But to be sure we

just need that $zu''(q_c) > u'(q_c) + u''(q_c)q_c$. If we now divide both sides of this last expression by $u''(q_c)$ we get that $z < q_c \left(1 - \frac{u'(q_c)}{u''(q_c)q_c}\right)$. That is, we just need that

$$z < q_c \left(1 - \frac{1}{RRA}\right), \quad (\text{a.1})$$

where RRA stands for the coefficient of relative risk aversion. In words, as long as inflation is not too low (z not too high), and the coefficient of relative risk aversion is greater than 1, then $\partial T_2/\partial i > 0$.²⁰ Even though these two equilibrium equations cannot be further reduced keeping things manageable, since the magnitude of the derivatives of each equation is clearly different it would suffice to show that they intersect just once within the feasible domain. This domain is $(0, \bar{i}) \times (0, z^*)$, with $\lim_{(i,z) \rightarrow (0^-, z^{*+})} \delta_c = u'(\frac{z}{n})^{\frac{1-n}{n}} \frac{z}{a} \approx \bar{\delta}$ and $\lim_{(i,z) \rightarrow (\bar{i}^+, 0^-)} \delta_c = (1+\bar{i})[u'^{-1}(1+\bar{i}) - z]^{\frac{1}{a}} \approx 0$. Remember that we consider i and z arbitrarily near their limits, but without reaching them, otherwise borrowing would be zero and our exercise would be pointless. Thus,

$$T_1(0, z^*) = -\frac{\bar{\delta}}{2} \frac{1-n}{n} z + (1-n)z > 0,$$

$$T_2(0, z^*) = \beta E(\delta) - 1 < 0.$$

On the other hand,

$$T_1(\bar{i}, 0) = u'^{-1}(1+\bar{i})(\delta_c - \bar{\delta}) < 0,$$

$$T_2(\bar{i}, 0) = \beta \int_0^{\delta_c} n\delta \left(\frac{u'(0)}{1+i} - 1 \right) dF(\delta) + \beta E(\delta) - 1 > 0.$$

Q.E.D.

Comparative Statics. In the previous proof we have computed the derivatives of the equilibrium equations with respect to i and z . In order to be able to apply the Implicit Function Theorem on the system T we have to make sure that $D_{(i,z)}T$ is invertible, i.e., $\det(D_{(i,z)}T) \neq 0$:

$$\begin{aligned} \det(D_{(i,z)}T) &= \frac{\partial T_1}{\partial i} \frac{\partial T_2}{\partial z} - \frac{\partial T_2}{\partial i} \frac{\partial T_1}{\partial z} = \left(\frac{\bar{\delta} - \delta_c}{-u''(q_u)} + \frac{\delta_c^2 a}{2(1+i)^2} \right) \beta \int_0^{\delta_c} n\delta \frac{u''(q_c)}{1+i} dF(\delta) \\ &\quad - (\bar{\delta} - \delta_c + 1 - n) \beta \int_0^{\delta_c} n\delta \frac{-u''(q_c)(q_c - z) - u'(q_c)}{(1+i)^2} dF(\delta) < 0. \end{aligned}$$

Once the inverse of $D_{(i,z)}T$ is taken, we can write the equation (30) as follows

$$\begin{pmatrix} \frac{\partial i}{\partial \bar{\delta}} \\ \frac{\partial z}{\partial \bar{\delta}} \end{pmatrix} = -\frac{1}{\det(D_{(i,z)}T)} \begin{pmatrix} \frac{\partial T_2}{\partial z} & -\frac{\partial T_1}{\partial z} \\ -\frac{\partial T_2}{\partial i} & \frac{\partial T_1}{\partial i} \end{pmatrix} \begin{pmatrix} \frac{\partial T_1}{\partial \bar{\delta}} \\ \frac{\partial T_2}{\partial \bar{\delta}} \end{pmatrix}.$$

Then,

$$\frac{\partial z}{\partial \bar{\delta}} = -\frac{1}{\det(D_{(i,z)}T)} \left(-\frac{\partial T_2}{\partial i} \frac{\partial T_1}{\partial \bar{\delta}} + \frac{\partial T_1}{\partial i} \frac{\partial T_2}{\partial \bar{\delta}} \right),$$

which is clearly positive. In the case of the interest rate we have

$$\frac{\partial i}{\partial \bar{\delta}} = -\frac{1}{\det(D_{(i,z)}T)} \left(\frac{\partial T_2}{\partial z} \frac{\partial T_1}{\partial \bar{\delta}} - \frac{\partial T_1}{\partial z} \frac{\partial T_2}{\partial \bar{\delta}} \right),$$

which is negative. This relationship may not be so obvious at first sight, but one just has to realize that $|\frac{\partial T_2}{\partial \bar{\delta}}| > |\frac{\partial T_2}{\partial z}|$ and $|\frac{\partial T_1}{\partial z}| > |\frac{\partial T_1}{\partial \bar{\delta}}|$.²¹ As for the distribution of borrowers,

$$\begin{aligned} \frac{\partial \delta_c}{\partial \bar{\delta}} &= \frac{1}{a} \left[\left(u'^{-1}(1+i) - z \right) \frac{\partial i}{\partial \bar{\delta}} + \frac{1+i}{u''(q_u)} \frac{\partial i}{\partial \bar{\delta}} - (1+i) \frac{\partial z}{\partial \bar{\delta}} \right] \\ &= \frac{1}{a} \left[\left((q_u - z) + \frac{u'(q_u)}{u''(q_u)} \right) \frac{\partial i}{\partial \bar{\delta}} - u'(q_u) \frac{\partial z}{\partial \bar{\delta}} \right] \\ &= \frac{1}{a} \left[\underbrace{\frac{u''(q_u)(q_u - z) + u'(q_u)}{u''(q_u)}}_{>0} \underbrace{\frac{\partial i}{\partial \bar{\delta}}}_{<0} - u'(q_u) \frac{\partial z}{\partial \bar{\delta}} \right] < 0. \end{aligned}$$

Finally, if we differentiate (23) with respect to $\bar{\delta}$ (recall that the derivative of the right hand side is negative) we have that:

$$\frac{1}{\beta(1+i)} \frac{\partial \tau}{\partial \bar{\delta}} - \frac{1+\tau}{\beta(1+i)^2} \frac{\partial i}{\partial \bar{\delta}} < 0,$$

implying that

$$\frac{\partial \tau}{\partial \bar{\delta}} < \frac{1+\tau}{1+i} \frac{\partial i}{\partial \bar{\delta}} < 0.$$

Q.E.D.

Alternative solution approach. Computing the integrals in (19) and using (17) one can develop the expression a little to obtain a quadratic equation in z ,

$$z^2 + z \left[\frac{(\bar{\delta} + 1 - n)2a}{1+i} - 2u'^{-1}(1+i) \right] - \frac{\bar{\delta}2a}{1+i} u'^{-1}(1+i) + \left(u'^{-1}(1+i) \right)^2 = 0.$$

After some more algebra, the exact solution would give us an expression for z as a function only of parameters and the interest rate i :

$$z = u'^{-1}(1+i) - \frac{(\bar{\delta} + 1 - n)a}{1+i} \pm \sqrt{\left(\frac{(\bar{\delta} + 1 - n)a}{1+i}\right)^2 - \frac{2(1-n)a}{1+i}u'^{-1}(1+i)}.$$

Indeed there is only one solution to this equation: the one that carries the 'plus' sign before the square root. Notice that since (17) must hold in any steady state, we can define real balances as

$$z = u'^{-1}(1+i) - \frac{\delta_c a}{1+i}.$$

But in the solution to the quadratic equation, before the square root we are subtracting from $u'^{-1}(1+i)$ more than $\frac{\delta_c a}{1+i}$. Therefore, the only possible solution is that which adds the square root to the previous term. Now, we have z as a function of i , which means that we can also express δ_c as a function of i . All we have to do now is substitute $z = \mathbf{z}(i)$ and $\delta_c = \mathbf{d}(i)$ in (24) and (23) to obtain i and τ as a function of parameters and the aggregate value of collateral. With the maintained assumptions on properties of the utility function and distribution of returns, the equations that determine these variables are well behaved, but the derivations above give us a taste of how unmanageable the expressions become. Notice, finally, that for the unique solution to be a real number we need the term inside the square root to be positive, which implies that $\bar{\delta} > (1+i)2(1-n)z/a$. This result will be useful for the proof of lemma 1. *Q.E.D.*

Notes

¹ Monacelli (2006) more specifically reports on the considerable degree of co-movement between total private consumption and household mortgage debt in the last fifteen years, as well as the significant increase in secured debt as a percentage of household debt.

² According to Ferraris and Watanabe (2008) the value of all commercial and industrial loans intermediated by US banks and secured by collateral amount to 46.9% of the total value of loans in 2006. In the case of commercial loans, in 2004 up to 47.9% of US households were holding debt secured by their houses. In this sense, Iacoviello (2005) also reports on a large proportion of consumption borrowing secured by real estate.

³ Constrained agents will therefore be indifferent between defaulting and paying back.

⁴ Berentsen et al. (2007) and Ferraris and Watanabe (2008) both model deterministic returns to the asset that is used as collateral.

⁵ If $\beta E(\delta) = 1$ agents would be indifferent between storing assets and not storing at all. Furthermore, as will be discussed later, in this case the only monetary policy consistent with equilibrium is $\tau = \beta - 1$. In this situation my economy basically reduces to the one in Lagos and Wright.

⁶ I discuss other arrangements regarding the supply of the asset in the section on equilibrium.

⁷ This pricing protocol is equivalent to a take-it-or-leave-it offers by the buyer protocol but a lot simpler and cleaner. Rocheteau and Wright (2005) provide a comprehensive study of several relevant market structures and their different results. They show that some results may vary according to different pricing procedures. The quantity traded under bargaining may be lower than under competitive pricing due to the classic holdup problem also pointed out in Lagos and Wright (2005). I consider the price taking approach.

⁸ With quasi-linear utility agents are happy to pay off all debt at the end of every period.

While one-period debt contracts are very convenient, in equilibrium they turn out to be sufficient to cover the needs of any trading situation.

⁹ Lagos and Wright (2005) explain how these assumptions, along with quasi-linearity of preferences, lead to a degenerate distribution of money and asset holdings at the beginning of every period. In particular, they show how the assumptions regarding the utility functions allow to derive technical conditions to ensure that in equilibrium all agents produce and consume in the centralized market.

¹⁰ This approach to monetary policy is similar to the way Central Banks inject/withdraw money through open market operations in order to induce an equilibrium interest rate.

¹¹ This result also depends on the strict monotonicity of the value function in the first market and its partial derivatives, as it is discussed in Lagos and Wright.

¹² See Rocheteau and Wright (2005) and Lagos and Wright (2005).

¹³ See Berentsen, Camera and Waller (2007) or Berentsen and Waller (2006).

¹⁴ Monacelli (2006) makes technical assumptions to ensure that around the steady state both the budget and borrowing constraints of all the agents are satisfied with equality. As he argues in his paper, this was needed in order to be able to log-linearize the model. Iacoviello (2005) also builds a model such that borrowing constraints could be non-binding only in some uncertainty cases. However, he assumes that uncertainty is sufficiently small so that they will always be binding for all agents.

¹⁵ It is straightforward to check that the value function is concave in both m and a .

¹⁶ Berentsen and Waller (2006) provide a discussion on the feasibility of the Friedman rule in environments with limited enforcement similar to the one in this paper.

¹⁷ Alternatively, we could take the approach that the real asset is not provided in fixed supply. Then, we could show that there exists $\theta \in \mathbf{R}_+$ such that $u'^{-1}(1+i) - z = \theta \frac{\bar{\delta} a}{1+i}$, and use this equation to determine a as a function of z and i . However, this approach would imply nothing more than the fraction of constrained borrowers being a unique function of $\bar{\delta}$,

i.e. $\delta_c = \theta\bar{\delta}$. But this is exactly the same qualitative implications that we will obtain with the current modeling approach.

¹⁸ In a homogeneous economy with the price of the asset being endogenously determined consumption does not need to be lower in this case.

¹⁹ Using the inverse function theorem, the derivative of $u'^{-1}(1+i)$ with respect to i would be $\frac{1}{u''(q_u)} < 0$.

²⁰ Choi and Menezes (1992) point that considerably many comparative static results depend on whether relative risk aversion is equal to, greater, or lower than one, noting that this coefficient being greater than one may indeed be the case in a wide variety of models.

²¹ Since both the left and right hand side of (24) are decreasing in $\bar{\delta}$, it is clear that the magnitude of the change in T_2 with respect to z is less than with respect to $\bar{\delta}$.

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