Notes on the Overlapping Generations Model

1. Social security

In the previous environment we have seen that under some conditions young agents may be willing to give up some of their consumption in exchange of an increase in their consumption when old.

Money is an instrument able to implement such exchanges. The introduction of a social security system is potentially another way to improve welfare.

Assume an OLG economy in which the population grows at rate $n > 0$, so that for each "old" there are 1+n "young" agents. The feasibility constraint in this case reads:

$$c_t^{t-1} + (1+n)c_t^t = e_t^{t-1} + (1+n)e_t^t,$$

which we rewrite in terms of excess demands as:

$$z(r_t) + (1+n)y(r_{t+1}) = 0. \tag{1}$$

All previous definitions and the graphical analysis go through without variation.

We introduce in this economy a pay-as-you-go social security system, such that young agents pay a tax τ, the proceeds of which are distributed as benefit b among the old agents around. We assume that the social security system runs a balanced budget, and since there are $(1+n)$ young agents for each old agent, we have that:

$$b = \tau(1+n).$$

In the absence of money the only possible equilibrium is the autarky equilibrium, which in a stationary economy entails:

$$c_t^t = e^y - \tau, \text{ and } c_{t+1}^t = e^o + \tau(1+n),$$

which conveys a utility level

$$V(\tau) = U(e^y - \tau) + \beta U(e^o + \tau(1+n)).$$

Proceeding as before we can ask, What is the effect of an increase in τ (starting from $\tau = 0$)?

$$V'(0) = -U'(e^y) + \beta(1+n)U'(e^o).$$

Using again the FOC corresponding to the optimal decisions of the agents: $-U'(e^y) + \beta(1+r)U'(e^o) = 0$, we have that

$$V'(0) > 0 \iff n > \frac{U'(e^y)}{\beta U'(e^o)} - 1 = r.$$

2. Fiscal Policy (I): Ricardian Equivalence revisited

In the Neoclassical model of growth with infinitely lived agents in a frictionless environment we know that the timing of taxation (lump-sum) to finance a given sequence of public expenditure has no real effects. That is, in a frictionless economy all that matters is present value calculations: present value of incomes, of consumptions and of taxes. The reason is that the present value of taxes **must exactly match** the present value of public expenditure, no matter when are they levied, and so, lower taxes in the beginning simply imply higher taxes in the future (and vice versa).

The Ricardian Equivalence does not hold in the Neoclassical model when

- there are binding borrowing constraints,
- there is distortionary taxation,
- markets are incomplete,

because in these cases "present value" calculations do not work:

- binding borrowing constraints imply that FOC do not hold with equality,

- distortionary taxation affect relative prices, hence the timing of taxation matters,

- with incomplete markets consumption (and every other decision) becomes state-dependent, hence the timing of taxation matters again.

Is it possible to transfer taxes over generations of an OLG economy, say from "current old" to "future young", without affecting, in some sense, their choices?

In general the answer is *NO* because taxes redistribute wealth over "ages": for instance, there is no way we can compensate an old agent from her/his loss after a tax increase.

Barro (1974) proposes a way to recover a Ricardian proposition kind of result: all that is needed is a way to connect the generations, so that in the end, they behave *"as if"* they were infinitely lived agents.

Consider a stationary OLG exchange economy without money, and assume that $n = 0$.

- There is a government with $g_t = 0$ for all $t \geq 1$, but with $B_1 > 0$. These bonds are an endowment for the initial old generation.

- The government keeps B constant over time, and runs a balanced budget:

$$B + T = B(1 + r)$$

- Let a_t^t be the savings of a young agent for the next period.

- Let a_{t+1}^t be the saving of an old agent in $t+1$ (which was born in t), for the next generation. That is,

$a_{t+1}^t > 0$ is a bequest from old to young, reflecting that an old agent cares about her/his next descendant (i.e., there is *altruism*).

The budget constraint of the typical young reads:

$$c_t^t + a_t^t = e^y - T,$$

and the budget constraint of the typical old reads:

$$c_{t+1}^t + a_{t+1}^t = (a_t^t + a_t^{t-1})(1+r),$$

(the budget constraint of the initial old is $c_1^0 + a_1^0 = B(1+r)$). Notice that the bequest accrues to the recipient in her old age. The inter temporal budget constraint is given by

$$c_t^t + \frac{c_{t+1}^t}{1+r} + \frac{a_{t+1}^t}{1+r} = e^y - T + a_t^{t-1}.$$

As usual, the inter temporal budget constraint describes all feasible possibilities of consumption/bequests, which are constrained by the present value of lifetime wealth $\omega = e^y - T + a_t^{t-1}$. This means that lifetime utility is also a function of ω.

Market clearing in the goods and asset market is given respectively by

$$c_t^{t-1} + c_t^t = e^y, \forall t \geq 1,$$

(remember that here we are assuming that $e^o = 0$) and

$$a_t^{t-1} + a_t^t = B, \forall t \geq 1.$$

We write the preferences of the "representative" generation as

$$u_t(c_t^t, c_{t+1}^t, a_{t+1}^t) = U(c_t^t) + \beta U(c_{t+1}^t) + \alpha V_{t+1}(\omega_{t+1}),$$

with $\alpha \in (0,1)$ and where $V_{t+1}(e_{t+1})$ is the maximal utility of a young agent born in $t+1$ and starting with a lifetime wealth ω_{t+1}, which is given by

$$\omega_{t+1} = e^y - T + a_{t+1}^t.$$

(The utility of the initial old is $\beta U(c_1^0) + \alpha V_1(\omega_1)$).

Notice that

$$\begin{aligned} V_t(\omega_t) = & \max\{U(c_t^t) + \beta U(c_{t+1}^t)\} + \alpha V_{t+1}(\omega_{t+1}) \\ = & \max\{U(c_t^t) + \beta U(c_{t+1}^t)\} \\ + & \alpha \Big(\max\{U(c_{t+1}^{t+1}) + \beta U(c_{t+2}^{t+1})\} + \alpha V_{t+2}(\omega_{t+2}) \Big) \end{aligned}$$

Continuing in this way, and considering the first (old) generation, we write the following utility maximization problem

$$\max_{\{c_t^{t-1}, c_t^t, a_t^{t-1}\}_{t=1}^{\infty}} \left\{ \beta U(c_1^0) + \sum_{t=1}^{\infty} \alpha^t \left(U(c_t^t) + \beta U(c_{t+1}^t) \right) \right\}$$

subject to

$$c_1^0 + a_1^0 = B(1+r),$$

$$c_t^t + \frac{c_{t+1}^t}{1+r} + \frac{a_{t+1}^t}{1+r} = e^y - T + a_t^{t-1}, \forall t \geq 1.$$

The only difference between the problem above and the problem corresponding to the usual infinitely lived agent is that here each period is divided into two sub-periods. The agent consumes in each sub-period (c_t^t, c_{t+1}^t), and the relative price between these consumption goods is $1 + r$.

Hence, it seems that all that is actually needed is to introduce preferences for the wellbeing of our descendants.

The previous finding needs to be clarified in important dimensions:

1. Barro assumes that "old" agents derive utility V from their "children" through the bequests they leave to them, a_{t+1}^{t}. Strictly speaking, then, transfers go from old to young and V only measures the utility the old get from the young.

2. This means, in particular, that in equilibrium we need to have $a_{t+1}^{t} \geq 0$, i.e., we need to make sure that bequest motive is *operative*. In the present context, if bequest were negative it would represent a transfer from young to old: This could be seen as "young" giving gifts to "old", or as if parents were literally stealing the endowments of their children.

3. The critical question, then, is to know under what conditions $a_{t+1}^t > 0$. Notice that under such conditions increases in debt imply both an increase in the resources available to the old, and an increase in the taxes the young will pay. Hence the old "undo" the effect of increased taxation to their descendants by passing to them the increased bonds they hold, in the form of a larger bequest.

Consider the following version of the OLG economy: Agents live for two periods, population grows at rate $n \geq 0$, and the endowments in each period are e^y, e^o. The problem one of the agents needs to solve is

$$\max u = u(c_t^t) + \beta u(c_{t+1}^t)$$

subject to

$$\begin{aligned} c_t^t + s_t^t &\leq e^y, \\ c_{t+1}^t &\leq e^o + (1 + r_{t+1}) s_t^t, \\ c_t^t, c_{t+1}^t, s_t^t &\geq 0. \end{aligned}$$

The solution of the problem above entails the FOC:

$$-u'(e^y - s_t^t) + \beta(1 + r_{t+1}) u'(e^o + (1 + r_{t+1}) s_t^t). \quad (2)$$

Since no money/bonds are assumed to exist, then the only possible equilibrium is the autarky equilibrium, in which $s_t^t = 0$. We define $\bar{r}$ as:

$$1 + \bar{r} = \frac{u'(e^y)}{\beta u'(e^o)}.$$

We simply use the above economy to obtain the equilibrium return in the autarky equilibrium.

We now extend the previous economy to incorporate bequests a^t_{t+1}. The problem of one of the agents is now

$$\max u = u(c^t_t) + \beta u(c^t_{t+1}) + \alpha v^*$$

subject to

$$\begin{aligned} c^t_t &\leq e^y + a^t_t, \\ c^t_{t+1} + (1+n)a^t_{t+1} &\leq e^o, \\ c^t_t, c^t_{t+1}, a^t_{t+1} &\geq 0. \end{aligned}$$

v^* is the maximal utility the next generation will obtain, which is a function of the current bequest, plus the bequest in the following period. For future reference, remember that $\alpha \in (0, 1)$.

Notice that in this economy young agents receive the bequest in the first period. Notice also that money/bonds are not present.

R1. The strict concavity of u is enough to prove that bequests are operative if and only if

$$\alpha > \frac{1+n}{1+\bar{r}}.$$

The interpretation is that bequests are positive only if agents care sufficiently about the welfare of their descendants. The second key result is that

R2. If the economy is inefficient ($\bar{r} < n$), then bequests cannot be operative in the economy with a bequest motive.

Hence, not all OLG economies can be represented by an infinitely lived agent: only those in which the bequest motive is operational,i.e., only those that are efficient.

Interesting references along these lines include:

Weil, P. (JME, 1987) generalizes these ideas to production economies and also looks at an environment with uncertainty.

Kimball, M. (JME, 1987) studies two-sided altruism. This is not enough to rule out dynamic inefficiency.

Abel, A. (REStat., 1988, with M. Warshawsky) looks at the "joy of giving" and altruism. Abel (AER, 1987) also studies operative bequests motives.

3. Production

Consider now including production in an OLG economy similar to the ones we have studied before.

- N_t^t is the number of young agents in period t, and N_t^{t-1} is the number of old agents in period t (they were born in period $t-1$). We assume $N_0^0 = 1$. We also assume $N_t^t = N_{t+1}^t$, hence agents only die at the end of the second period.

- $N_t^t = (1+n)N_t^{t-1} = (1+n)^t N_0^0$.

- Endowments: Young agents are endowed with a unit of time, which they inelastically supply as labor. Old agents are unable to work.

- The initially old agents are endowed with $k_1 > 0$ units of capital. Capital depreciates a the constant rate $\delta \in (0, 1)$ in every period.

- Technology: Constant returns to scale in capital and labor: $Y_t = F(K_t, L_t)$. We have:

$$y_t = Y_t/L_t = \frac{F(K_t, L_t)}{L_t} = F\left(\frac{K_t}{L_t}, 1\right) = f(k_t)$$

- Preferences are as usual: $u_t(c) = U(c_t^t) + \beta U(c_{t+1}^t)$.

In every period t, production requires capital and labor (we assume firms are perfectly competitive in factor and product markets). Labor is provided by the currently young agents. Capital is just the savings from the currently old generation.

Once production has been materialized, young agents earn their wage and choose how much to consume and save for their old (retired) period.

Also, old agents get back their saving plus return, and consume everything (there is no altruism/bequests).

Definition 1: Given k_1, a competitive equilibrium with sequential markets is a list $x^h = \{c_1^0, \{c_t^t, c_{t+1}^t, s_t^t\}_{t=1}^{\infty}\}$ for households, a list $x^f = \{K_t, L_t\}_{t=1}^{\infty}$ for the firm, and a sequence of prices $x^p = \{r_t, w_t\}_{t=1}^{\infty}$ such that:

1. Given k_1 and x^p, x^h solves

$$\max_{\{c_t^t, c_{t+1}^t, s_t\}} U(c_t^t) + \beta U(c_{t+1}^t)$$

subject to

$$\begin{aligned} c_t^t + s_t^t &\leq w_t, \\ c_{t+1}^t &\leq (1 + r_{t+1} - \delta) s_t^t, \\ c_t^t, c_{t+1}^t &\geq 0, \end{aligned}$$

for all $t \geq 1$, and c_1^0 solves

$$\max_{c_0^1 \geq 0} U(c_0^1)$$

subject to

$$c_0^1 \leq (1 + r_1 - \delta)k_1.$$

2. Given x^p, x^f solves

$$\max_{K_t, L_t} F(K_t, L_t) - r_t K_t - w_t L_t.$$

3. Market clearing conditions:

3.1 Goods market:

$$N_t^t c_t^t + N_t^{t-1} c_t^{t-1} + K_{t+1} = F(K_t, L_t) + (1-\delta)K_t.$$

3.2 Asset market:

$$K_{t+1} = N_t^t s_t^t.$$

3.3 Labor market:

$$L_t = N_t^t.$$

Definition 2: A stationary competitive equilibrium with sequential markets is a competitive equilibrium in which $c_t^t = c_1, c_t^{t-1} = c_2, K_t/L_t = k, L_t = N_t, r_t = r, w_t = w$, and in which $s_t^t = s = k_1$ for all $t \geq 1$.

The key equation of the equilibrium is the relationship between savings from the young and capital from the old:

$$K_{t+1} = s_t^t N_t,$$

which we rewrite as

$$\frac{K_{t+1}}{N_t} = s_t^t, \text{ hence } \frac{K_{t+1} N_{t+1}}{N_{t+1} N_t} = s_t^t,$$

and thus

$$k_{t+1} = \frac{s_t^t}{1+n}. \tag{3}$$

This equation delivers a difference equation in k.

To see this, notice that in equilibrium

$$w_t = f(k_t) - f'(k_t)k_t, \text{ and that } r_{t+1} = f'(k_{t+1}),$$

hence the $(c_t^t, c_{t+1}^t, s_t^t)$ that solve the household problem are a functions of (k_t, k_{t+1}).

In particular, $s_t^t = s(f(k_t) - f'(k_t)k_t, f'(k_{t+1}))$, where the precise shape of s depends on the underlying U.

Therefore, we have that Eq. (3) can be written as

$$k_{t+1} = \frac{s(f(k_t) - f'(k_t)k_t, f'(k_{t+1}))}{1+n}. \tag{4}$$

Equilibria and equilibrium dynamics are determined by Eq. (4). For instance, stationary equilibria are given by all k^s such that $k^s = s(f(k^k) - f'(k^s)k^s, f'(k^s))/(1+n)$. Without further assumptions no general result about uniqueness and stability is available.

It can be shown that if k^s represents a stationary level of capital, then it suffices to have

$$0 < \frac{dk_{t+1}}{dk_t}|_{k^s} < 1,$$

in order to guarantee that the steady state is locally stable. In fact, some analytical progress is possible by exploiting the FOC of the household problem, which we write as

$$U'(w_t - s(w_t, r_{t+1})) = \beta(1 + r_{t+1} - \delta)U'((1 + r_{t+1} - \delta)s(w_t, r_{t+1})) \quad (5)$$

Applying the implicit function theorem to the previous equation we get that:

$$s_{w_t}(w_t, r_{t+1}) = \frac{U''(c_t^t)}{U''(c_t^t) + \beta(1 + r_{t+1} - \delta)^2 U''(c_{t+1}^t)},$$

where

$$c_t^t = w_t - s(w_t, r_{t+1}), \text{ and } c_{t+1}^t = (1 + r_{t+1} - \delta) s(w_t, r_{t+1}).$$

It follows that

$$s_{w_t}(w_t, r_{t+1}) \in (0, 1).$$

We also have that

$$s_{r_{t+1}}(w_t, r_{t+1}) = \frac{-\beta U'(c_{t+1}^t) - \beta U''(c_{t+1}^t)(1 + r_{t+1} - \delta)s(w_t, r_{t+1})}{U''(c_t^t) + \beta(1 + r_{t+1} - \delta)^2 U''(c_{t+1}^t)},$$

and so,

$$s_{r_{t+1}}(w_t, r_{t+1}) \begin{array}{c} \geq \\ \leq \end{array} 0.$$

The interpretation is that under standard assumptions saving increases with wage, and yet, saving may increase or decrease with the interest rate. The "nice" case of $s_{r_{t+1}}(w_t, r_{t+1}) > 0$ is obtained under the assumption that wealth effects are weaker than substitution effects (i.e., that consumption in the two consecutive dates are gross substitutes).

The two expressions above are useful because they allow us to write

$$\frac{dk_{t+1}}{k_t} = \frac{-s_{w_t} f''(k_t) k_t + s_{r_{t+1}} f''(k_{t+1})(dk_{t+1}/dk_t)}{1+n},$$

which reduces to

$$\frac{dk_{t+1}}{k_t} = \frac{-s_{w_t} f''(k_t) k_t}{1+n-s_{r_{t+1}} f''(k_{t+1})}.$$

Hence, if $s_{r_{t+1}} \geq 0$ then the condition

$$0 < \frac{dk_{t+1}}{dk_t}|k^s < 1,$$

guarantees *local* uniqueness and monotonic dynamics toward the steady state.

4. Dynamic inefficiency

Competitive equilibria in OLG economies with production may be inefficient in the same way as in exchange economies.

The issue of efficiency/inefficiency is in general cumbersome: showing that an allocation is inefficient is simpler than showing that it is efficient. The reason is that to prove inefficiency it is enough to find a single alternative allocation that improves with respect to the current one. To prove for efficiency, however, one needs to show that there is no other feasible allocation that improves with respect to the initial one. In the case of the OLG economy this amount to having to check many different possibilities of transfers, both inter temporally (as in infinitely lived agent models), but also intra temporally, because agents from different generations coexist in every period.

A simple alternative that is always available from any steady state with positive production (i.e., positive capital) is to implement a jump to a new steady state with a smaller level of capital. This requires to save less today, and keep constant the new level of capital from the following period onwards. If such a change promotes an increase in welfare of at least some generation without decreasing the welfare of any other generation, then clearly the initial steady state is inefficient.

In what follows we will try to check this simple condition.

The feasibility constraint of the economy reads

$$N_t^t c_t^t + N_t^{t-1} c_t^{t-1} + K_{t+1} = F(K_t, L_t) + (1-\delta)K_t$$

so if we rewrite the condition in per capita terms of currently young agents (dividing by N_t) we obtain:

$$c_t^t + \frac{c_t^{t-1}}{1+n} + (1+n)k_{t+1} = f(k_t) + (1-\delta)k_t,$$

where $c_t^{t-1}/(1+n)$ is the amount of consumption that each young agent could get out of the consumption of each old agent. In a steady state the previous condition reads

$$c_1 + \frac{c_2}{1+n} + (1+n)k^s = f(k^s) + (1-\delta)k^s.$$

If we take consumption per capita of a young agent as $c = c_1 + c_2/(1+n)$, then we have that

$$c = f(k^s) - (n+\delta)k^s.$$

Notice that

$$\frac{dc}{dk^s} = f'(k^s) - (n+\delta).$$

This is interesting because if $f'(k^s) - (n+\delta) < 0$, then the above calculation suggests that consumption (i.e., welfare) could increase by reducing k^s. Hence, we need to check what is the effect on welfare of a reduction in the capital stock.

To this end, suppose that starting at the steady state, the stock of capital is reduced by $-dk^s$, and that $(k^s)' =$

$k^s - dk^s$ will be kept constant in all future periods (that is, we implement a one period transition to a new steady state, which of course, is feasible). There are two effects to consider:

1) On the consumption of the current generation:

$$\Delta c_t = (1+n)dk^s > 0.$$

2) On the consumption of all future generations (for $\tau \geq 1$):

$$\Delta c_{t+\tau} = -[f'(k^s) - (n+\delta)]dk^s,$$

We conclude that reducing saving is desirable from the perspective of the current generation, because their consumption increases. For the future generations that will live with a smaller amount of capital, we have that if $f'(k^s) - (n + \delta) < 0$, then their consumption (hence welfare) will increase.

Therefore, if $f'(k^s) - (n + \delta) < 0$ then we are able to implement another allocation in which all agents are better of, and so, if $f'(k^s) - (n + \delta) < 0$ then the competitive steady state is inefficient: at that steady state saving is too large. In this case the economy is said to be *dynamically inefficient*.

We conclude this section with a final result.

Let $r_t = f'(k_t)$ and keep the same assumptions as before, and let n_t be the growth rate of the population in

t. Then a feasible allocation is optimal if and only if

$$\sum_{t=1}^{\infty} \prod_{\tau=1}^{t} \frac{1 + r_{t+1} - \delta}{1 + n_{\tau+1}} = +\infty.$$

(This result is due to D. Cass, 1972). It is clear from the previous result that the steady state is efficient if and only if $f'(k^*) - \delta > n$.

The intuition for the case of constant n is as follows. One can decrease the consumption of young by d_1, and increase their consumption when old by d_2, and *leave them indifferent* by choosing d_2 such that:

$$d_1 u'(c_t^t) = \beta d_2 u'(c_{t+1}^t)$$

holds. Since the Euler equation requires that $u'(c_t^t) = \beta(1 + r_{t+1} - \delta)u'(c_{t+1}^t)$, combining the two expressions we have

$$d_1(1 + r_{t+1} - \delta) = d_2,$$

and in general, that

$$d_t = d_0 \prod_{\tau=1}^{t} (1 + r_{\tau+1} - \delta).$$

Hence, if d_t remains bounded (when interest rates do not grow too fast), one could implement transfers from young to old without affecting their wellbeing, except for the first generation, that would be better off. Hence, if the interest rates are low, one can implement a Pareto improvement, which renders inefficient the initial allocation.

5. Fiscal Policy (II): About the effects of government debt

Diamond (1965) studied the effects of having outstanding debt, and he distinguished between *internal debt* (hold by domestic residents), and *external debt* (in the hands of foreign residents).

We assume that the government collects lump sum taxes from current young in order to satisfy its pay the service of the debt in period t:

$$B_t(1 + r_t - \delta) = B_{t+1} + N_t\tau,$$

where $1 + r_t - \delta$ is the same return of capital, B_t is the amount of debt issued in the previous period (the one that is returned in t), $N_t\tau$ is total revenue from taxation to young agents, and B_{t+1} is the amount of outstanding debt that will be returned in $t+1$. Dividing by N_t we get

$$\tau = (r_t - \delta - n)b,$$

where b is a constant debt-labor ratio.

5.1 External debt

All debt is in the hands of foreign residents, so the only effect of the outstanding debt is that young agents pay its service:

$$c_t^t + s_t^t = w_t - \tau = w_t - (r_t - \delta - n)b.$$

The equilibrium condition in the asset markets is:

$$k_{t+1} = \frac{1}{1+n} s(w_t - (r_t - \delta - n)b, r_{t+1}).$$

Proceeding in the usual way we get:

$$\frac{dk_{t+1}}{dk_t} = \frac{-s_w(k_t + b)f''(k_t)}{1 + n - s_r f''(k_{t+1})},$$

and we continue assuming that $\frac{dk_{t+1}}{dk_t} \in (0,1)$ around the steady state. We then get that:

$$\frac{dk_{t+1}}{db} = \frac{-s_w(f'(k_t) - (n+\delta))}{1 + n - s_r f''(k_{t+1})}.$$

Suppose we are at a steady state k^* with some b^*, and that the government increases marginally this b^*. Then:

- If $f'(k_t) - (n+\delta) < 0$ (i.e., the economy is dynamically inefficient), then the stock of capital tends to increase with b. This means that in the new steady state:
  - the interest rate is smaller,
  - the wage rate is higher,

– taxes are higher.

The effects of these changes at the new steady state utility are indeterminate.

- If $f'(k_t) - (n + \delta) > 0$, then the stock of capital tends to decrease, and at the new steady state the utility level is smaller than initially.

- This calculation ignores the transition: after the increase in b, it takes some time to reduce k^*, and taking properly into account the transition may easily reverse the previous conclusion.

5.2 Internal debt

In this case we need to modify the market clearing condition of the assets market:

$$K_{t+1} + B_{t+1} = N_t s(w_t - (r_t - \delta - n)b, r_{t+1}),$$

as bonds now are in the hands of the domestic residents. Notice that with internal debt not all saving is converted into capital: part of the forgone consumption is devoted to buy bonds.

The above condition can be rewritten as

$$k_{t+1} = \frac{1}{1+n} s(w_t - (r_t - \delta - n)b, r_{t+1}) - b.$$

In this case we have that

$$\frac{dk_{t+1}}{db} = \frac{-s_w(f'(k_t) - (n+\delta))}{1+n-s_r f''(k_{t+1})} - (1+n) < 0,$$

where we have used again the stability condition and the fact that $s_w \in (0,1)$. This means that starting from a steady state k^*, we have that

- increasing the bonds to labor ratio reduces the steady state level of capital, and so,

- If the initial steady state was dynamically inefficient, welfare is larger in the second steady state. Conversely, welfare decreases after an increase in the debt to labor ratio when the initial steady state was efficient.

6. A preview of Monetary Policy

An important issue in monetary policy is whether the quantity of "green" (or "blue") paper can actually have real effects. In frictionless environments such as the NMG the answer is *no*, simply because "money" plays no role.

In an OLG economy, however, money may play a role: it may promote a welfare improvement by helping to shift consumption goods from young to old through trade. It is natural therefore to ask whether these trades, hence welfare, increase or decrease with the amount of money.

Consider the SM arrangement of an exchange economy in which population is constant: $N_t = N$ for all t. In addition, there are M units of green paper.

We write the utility maximization problem of an agent born in $t \geq 1$ as

$$\max_{\{c_t^t, c_{t+1}^t, s_t^t\}} u(c_t^t) + \beta u(c_{t+1}^t)$$

subject to

$$\begin{aligned} p_t c_t^t + s_t^t &= \ p_t w^y, \\ p_{t+1} c_{t+1}^t &= \ p_{t+1} w^o + s_t^t, \end{aligned}$$

where s_t^t is the amount of nominal money (which we also use as the numeraire). The optimality condition corresponding to the problem is

$$u'(c_t^t) = \beta \frac{p_{t+1}}{p_t} u'(c_{t+1}^t)$$

Combining the two budget constraints we get

$$c_t^t + \frac{p_{t+1}}{p_t} c_{t+1}^t = w^y + \frac{p_{t+1}}{p_t} w^o.$$

The notion of SM equilibrium entails the following market clearing condition for consumption goods: $N_t c_t^t + N_{t-1} c_{t+1}^t = N_t w^y + N_{t-1} w^o$, which we rewrite as

$$c_t^t + c_{t+1}^t = w^y + w^o,$$

(remember that population remains constant over time). Combining the budget constraint and the feasibility constraint corresponding to the stationary equilibrium (i.e., $c_t^t = c^y$ and $c_{t+1}^t = c^0$) we obtain:

$$c_{t+1}^t \left(\frac{p_{t+1}}{p_t} - 1 \right) = w^o \left(\frac{p_{t+1}}{p_t} - 1 \right),$$

and thus:

$$(c_{t+1}^t - w^o) \left(\frac{p_{t+1}}{p_t} - 1 \right) = 0$$

Hence, in any SM stationary equilibrium we have that

- Either $c^o = w^o$, in which case $c^y = w^y$ (the non monetary equilibrium), and so no one holds money (agents do not expect to be able to exchange money for goods in the future, so money has no value),

- or $c^o \neq w^o$, in which case $c^y \neq w^y$ (the monetary equilibrium), and $s_t^t/p_t = w^y - c^y$ remains constant over time, and $p_t = p_{t+1}$ for all t.

Since in the monetary equilibrium market clearing in the money market requires $s_t N_t = M$, then if two economies are identical in every respect except in the amount of green paper (say M and $\widehat{M}$), we have that the corresponding monetary equilibrium are such that $s_t^t/p_t = \widehat{s}_t^t/\widehat{p}_t$, and so, the quantity of money has no real effects (*money is neutral*).

To see this, combine the optimality condition from the consumers problem in each economy corresponding to the stationary monetary equilibrium to get

$$\frac{u'(w^y - s/p)}{u'(w^o + s/p)} = \frac{u'(w^y - \widehat{s}/\widehat{p})}{u'(w^o + \widehat{s}/\widehat{p})}.$$

Since u is strictly increasing and strictly concave, then $u'(w^y - s/p)/u'(w^o + s/p)$ is monotone, which implies that $s/p = \widehat{s}/\widehat{p}$. The budget constraint of the agents in each environment then implies that consumption when young and when old is the same in both economies. That is, differences in the quantity of nominal money are eliminated by a convenient price adjustment, and there are no differences in consumption streams and real balances (money in real terms).

A related issue is whether the speed at which money is introduced can have any real effects: Is money super-neutral?

Suppose now that there is a government and that it increases money supply over time according to:

$$M_{t+1} = (1+\sigma)M_t,$$

with $\sigma \geq 0$. New money is introduced in the economy by means of a nominal transfer X_{t+1} to each old agent in $t+1$, so that the budget constraint of the government reads:

$$M_{t+1} - M_t = N_0 X_{t+1},$$

which can be rewritten as

$$M_{t+1} - M_t = N_0 p_{t+1} x,$$

where x is the constant value of the transfer in real terms.

We specialize preferences to the log case:

$$\max_{\{c_t^t, c_{t+1}^t, s_t^t\}} \log c_t^t + \beta \log c_{t+1}^t,$$

subject to the budget constraints:

$$p_t c_t^t + s_t^t = p_t w^y, \text{ and } p_{t+1} c_{t+1}^t = p_{t+1} w^o + s_t^t + x p_{t+1},$$

which we rewrite as

$$c_t^t + m_t^t = w^y, \text{ and } c_{t+1}^t = w^o + \frac{m_t^t}{(1 + \pi_{t+1})} + x,$$

where $m_t^t = s_t^t / p_t$ stands for real balances, and where $(1 + \pi_{t+1}) = p_{t+1} / p_t$.

The FOC of the problem reads

$$\frac{1}{w^y - m_t^t} = \frac{\beta(1+\pi_{t+1})}{(w^o + m_t^t/(1+\pi_{t+1}) + x)(1+\pi_{t+1})},$$

so that we get

$$m_t^t = \frac{\beta w^y - (1+\pi_{t+1})(w^o + x)}{1+\beta}. \qquad (6)$$

We are interested in stationary equilibria:

- $m_t^t = m$, $\forall t$,

- $\pi_t = \pi$, $\forall t$.

We also rewrite the budget constraint of the government as

$$m_{t+1} - \frac{m_t}{(1+\pi_{t+1})} = N_0 x,$$

and after imposing stationarity we obtain:

$$m = \frac{N_0 x(1+\pi)}{\pi}. \tag{7}$$

Combining Eq. (6) and Eq. (7) we obtain:

$$\frac{N_0 x(1+\pi)}{\pi} = \frac{\beta w^y - (1+\pi)(w^o + x)}{1+\beta},$$

which delivers a polynomial of degree 2 in π. Hence, there are two different levels of π that are compatible with constant real balances in equilibrium.

Since in equilibrium m is a function of σ (through π, see Eq. (7)), then differences in σ do have real effects, and so, money *is not super-neutral*.

- In the high-σ, high-π, welfare is smaller than in the low-σ, low-π.

- It can be shown that the bad equilibrium is stable (non stationary equilibria converge to the bad equilibrium).

- Bruno and Fischer (1990) and Marcet and Sargent (1989) study equilibrium dynamics under several learning mechanisms.