Notes on the Overlapping Generations Model

1. Introduction

The Neoclassical Model of Growth is widely used in growth theory, business cycle theory, and quantitative applications in public finance.

The usual formulation of the model assumes that in the economy *markets are complete*, so that even if there is uncertainty, and/or even if agents are heterogeneous, in general it is possible to write an equivalent *representative agent* version of the model economy. This is the so-called *perfect aggregation* result under complete markets.

For this reason, very often the representative agent is assumed to exist beforehand. The implication is that in the usual formulation of the Neoclassical Model there is no heterogeneity.

- The lack of heterogeneity in the model is clearly at odds with the observations from actual economies: agents are heterogeneous at least in gender, age, ability, wealth, preferences...

The Neoclassical model of growth ignores all this heterogeneity, hence it remains silent about its effects, for instance on (optimal) policy.

- In actual economies markets are far from complete: there are borrowing constraints, lack of many insurance markets...

Again, it is hard to believe that market completeness is irrelevant. A basic issue is to try to measure how far is the model from "reality". Aiyagari (1994), Huggett (1993, 1997), Krusell and Smith (1998)..., choose an appropriate metric and asses the differences between complete and incomplete markets ($\clubsuit$).

The Overlapping Generations (OLG) Model introduced by Allais (1947), Samuelson (1958), and Diamond (1965) takes explicitly into account heterogeneity in "age", or more generally, heterogeneity in the position over the life-cycle.

This is a relevant dimension, as "old" and "young" consumer/workers clearly differ in their ability to supply labor, saving decisions (hence consumption), and the taxes they pay. For this reason, the OLG model is widely used in applied economic analysis.

(♣) Rios-Rull (1994) studies the effects of market incompleteness using an OLG economy.

2. Basic Model

- Time is discrete $t = 1, 2, 3, ...$ and goes on forever.

- Consumers live for two periods. In each time period a new generation (of measure one) is born.

- Let (e_t^t, e_{t+1}^t) denote the endowment of the generation born in t in the first and in the second period of their lives, and let (c_t^t, c_{t+1}^t) denote their consumptions. The endowment is the only available consumption good, and it is NOT storable.

- In the first period there is already an initial generation of "olds", endowed with e_1^0. In some applications we will assume additionally that they are endowed with *outside money*, m.

  Outside money is money that *is not a liability* to any agent inside the economy, and thus, it is money that (potentially) represents an asset.

  Inside money *is both, an asset* to some agents *and a liability* to some others (for instance, bank deposits are an asset to the private sector and a liability to the banking sector).

- Preferences are given by

$$u_t(c) = U(c_t^t) + \beta U(c_{t+1}^t),$$

except for the initial old: $u_0(c) = U(c_1^0)$.

U is strictly concave, strictly increasing, and twice continuously differentiable.

Definition 1: An allocation is a sequence c_1^0, $\{c_t^t, c_{t+1}^t\}_{t=1}^{\infty}$.

Definition 2: An allocation is feasible if $c_t^{t-1}, c_t^t \geq 0$ for all $t \geq 1$, and if

$$c_t^{t-1} + c_t^t = e_t^{t-1} + e_t^t, \text{ for all } t \geq 1.$$

Definition 3: An allocation c_1^0, $\{c_t^t, c_{t+1}^t\}_{t=1}^{\infty}$ is Pareto optimal if it is feasible and if there is no other allocation $\widehat{c}_1^0$, $\{\widehat{c}_t^t, \widehat{c}_{t+1}^t\}_{t=1}^{\infty}$ such that

$$u_t(\widehat{c}) \geq u_t(c), \text{ for all } t \geq 0,$$

with strict inequality for at least one $t \geq 0$.

Efficient allocations (Pareto optimal allocations) in the Neoclassical model are easily found by solving the Ramsey (planner's) problem: maximize the utility of the representative agent subject to the resource/feasibility constraint. If there is agent heterogeneity a generalized version of this procedure is to choose a social welfare function, such as a weighted average of the different types of agents...

Who is the representative agent in an OLG economy?

In OLG economies agents care only about their own welfare, which is materialized over a finite number of periods, *even though the economy lasts for ever!* Hence current generations put zero weight on the welfare of future generations.

How do we compare allocations (policies) that affect both current and not-yet-born generations? What if policies have an impact on fertility decisions (Conde-Ruiz, Giménez, and Pérez-Nievas (2010)?

A common approach is to assume that current generations care about the welfare of the society, including its future welfare. Hence the *current generations* may ask the planner to take into account all the future by considering the present discounted value of current and future utilities, using a social discount rate.

$$\begin{aligned} \max_{c} \quad & u_0(c^0) + \textstyle\sum_{t=1}^{\infty} \delta^{t-1} u_t(c^t) \\ \text{s. to} \quad & c_t^{t-1} + c_t^t = e_t^{t-1} + e_t^t, \forall t \geq 1, \end{aligned} \tag{1}$$

where δ is the social discount rate.

3. Equilibrium

3.1 Sequential equilibrium

It is natural to assume trade takes place sequentially over time in spot markets. We normalize the price of the actual consumption good to 1, let s_t^t be the saving of generation t undertaken in period t, and we let r_{t+1} be the interest rate paid in period $t + 1$ (in units of consumption goods of period $t + 1$).

Definition 4: Given m, a Sequential Markets (SM) equilibrium is an allocation c_1^0, $\{c_t^t, c_{t+1}^t\}_{t=1}^{\infty}$, and interest rates $\{r_t\}_{t=1}^{\infty}$ such that:

1. Given $\{r_t\}_{t=1}^{\infty}$, the allocation solves, for each $t \geq 1$,

$$\max_{\{c_t^t, c_{t+1}^t\}} \quad u_t(c)$$

$$\text{s. to} \quad c_t^t + s_t^t \leq e_t^t, \tag{2}$$

$$c_{t+1}^t \leq e_{t+1}^t + (1 + r_{t+1})s_t^t. \tag{3}$$

2. Given r_1, c_1^0 solves,

$$\max_{c_1^0} \quad u_0(c)$$

$$\text{s. to} \quad c_1^0 \leq e_1^0 + (1 + r_1)m. \tag{4}$$

3. For all $t \geq 1$,

$$c_t^{t-1} + c_t^t = e_t^{t-1} + e_t^t, \tag{5}$$

(goods market clearing condition).

Notice that, by Walras' law, the asset market necessarily clears in every period.

Take Eq. (2) for generation $t+1$ and Eq. (3) for generation t, and add them up to get

$$c_{t+1}^t + c_{t+1}^{t+1} + s_{t+1}^{t+1} = e_{t+1}^t + e_{t+1}^{t+1} + (1 + r_{t+1})s_t^t,$$

and combine with Eq. (5) to get

$$s_{t+1}^{t+1} = (1 + r_{t+1})s_t^t,$$

hence $s_1^1 = (1 + r_1)m$, and so, we finally get that

$$s_t^t = \prod_{\tau=1}^{t} (1 + r_\tau)m,$$

i.e., saving (the amount of forgone period-t consumption) equals the value of outside money.

3.2 Arrow-Debreu equilibrium

In an Arrow-Debreu world markets only open once (in a sense, *before* the actual start of the economy). At that moment agents engage in trade of contracts which entitle a certain amount of consumption goods (or convey an obligation of delivery). Once trades are completed, markets close, and agents keep executing their contracts over time.

Let p_t the price of the contract that provides one unit of consumption goods in period t. Then

Definition 5: Given m, an Arrow-Debreu (AD) equilibrium is an allocation c_1^0, $\{c_t^t, c_{t+1}^t\}_{t=1}^{\infty}$, and prices $\{p_t\}_{t=1}^{\infty}$ such that:

1. Given $\{p_t\}_{t=1}^{\infty}$, the allocation solves, for each $t \geq 1$,

$$\begin{aligned} \max_{\{c_t^t, c_{t+1}^t\}} \quad & u_t(c) \\ \text{s. to} \quad & p_t c_t^t + p_{t+1} c_{t+1}^t \leq p_t e_t^t + p_{t+1} e_{t+1}^t. \end{aligned} \tag{6}$$

2. Given p_1, c_1^0 solves,

$$\begin{aligned} \max_{c_1^0} \quad & u_0(c) \\ \text{s. to} \quad & p_1 c_1^0 \leq p_1 e_1^0 + m. \end{aligned} \tag{7}$$

3. For all $t \geq 1$,

$$c_t^{t-1} + c_t^t = e_t^{t-1} + e_t^t, \tag{8}$$

(goods market clearing condition).

Notice that we have normalized the price of money to 1.

Of course, the AD equilibrium and the SM equilibrium are intimately connected: it can be shown that the allocation in AD equilibrium is also the allocation one obtains in SM equilibrium.

4. The Offer curve

The offer curve is a useful tool to analyze graphically the equilibrium in the OLG model (we focus on SM equilibrium).

We take the offer curve as delivering the optimal (c_t^t, c_{t+1}^t) as a function of the return r_{t+1}, $OC(r_{t+1})$, and we represent it in the (c_t^t, c_{t+1}^t) plane.

Notice that:
1. As r_{t+1} increases it is more desirable to save, which would make the ratio c_t/c_{t+1} smaller. Also, as r_{t+1} decreases it is more desirable to borrow, hence the ratio c_t/c_{t+1} would increase.

2. There must be a return such that $(c_t^t, c_{t+1}^t) = (e_t^t, e_{t+1}^t)$, hence the offer curve goes through the endowments.

3. Furthermore, any point in the offer curve provides a utility at least as high as the utility of the initial endowment. This means that the offer curve "dominates" the initial indifference curve.

4. The slope of the ray from initial endowments and any point of the Offer curve is the return that optimally delivers that consumption bundle.

5. Inefficient equilibria

Unlike the equilibrium corresponding the Neoclassical model of growth, the equilibrium in the OLG world is not necessarily optimal: in some cases it is possible to redistribute the available resources and improve some (or *all*) agents' welfare without making worse off any of them. That is, the Fundamental Theorems of Welfare do not necessarily apply to OLG economies.

- Clearly, there is nothing to be gained by implementing a redistribution among agents of the same cohort (among the "young" alone, or among the "old" alone). Hence the only possibility is to redistribute between the "old" AND the "young".

Consider an OLG economy with $m = 0$. The autarky equilibrium corresponding to this case is such that $s_t^t = 0$ for all t, hence:

$$c_t^t = e_t^t, \text{ and } c_{t+1}^t = e_{t+1}^t.$$

It is natural to ask whether a hypothetical planner could increase an "old" 's welfare by reducing the consumption of a currently "young" agent, and then compensating her/him in $t+1$ (when that agent will be "old") by transferring some consumption goods from the "new young" born in $t+1$.

This scheme looks like a tax on young agents to be redistributed among old agents.

We are interested in $du_t(c)/d\tau$ taking into account that $c_t^t = e_t^t - \tau$ and that $c_{t+1}^t = e_{t+1}^t + \tau$, which is given by

$$du_t(c) = -U'(c_t^t)d\tau + \beta U'(c_{t+1}^t)d\tau.$$

Hence

$$\frac{du_t(c)}{d\tau}\left\{\begin{array}{l} > \\ = \\ < \end{array}\right. 0, \text{ as } -U'(c_t^t) + \beta U'(c_{t+1}^t)\left\{\begin{array}{l} > \\ = \\ < \end{array}\right. 0 \qquad (9)$$

1. Starting from $\tau = 0$, it is clear that the welfare of the initial olds increases whenever they receive a transfer of consumption goods.

2. For the young agents in the first period, in equilibrium we have that:

$$U'(c_t^t) = \beta(1 + r_{t+1})U'(c_{t+1}^t)$$

hence it follows from Eq. (9) that

$$\begin{aligned} -U'(c_t^t) + \beta U'(c_{t+1}^t) &= -U'(c_{t+1}^t)\beta[(1 + r_{t+1}) - 1] \\ &= -U'(c_{t+1}^t)\beta r_{t+1}. \end{aligned}$$

Therefore if $r_{t+1} > 0$ (Classical case) then the equilibrium is **efficient**, as no redistribution would increase welfare, and if $r_{t+1} < 0$ (Samuelson case) then the equilibrium is **inefficient**, because there are redistributions that would increase welfare from the first generation onwards.

The previous result suggests that

- Low (negative) interest rates are potentially a source of inefficiency. (Is this related to the current downturn after several years of markedly low interest rates?)

- We take goods from one young and give them to one old. If there is population growth, we can take the same amount of goods but from *two* young agents, reducing their losses, and still compensate the olds. That is, if there is population growth a positive interest rate may also be inefficient.

- If a current young agent is willing to give up some consumption units in exchange of “valueless” *blue paper*, thinking that when old some other agent will do the same, then the *blue paper becomes valuable*.

In OLG economies we have:

- An infinite number of goods (as in the Neoclassical model), and in addition, an infinite number of generations/agents (or otherwise no first generation would ever take a piece of paper). This is the “double infinity” problem (K. Shell) of OLG economies (the infinity of goods in the Neoclassical model is “not a problem”, i.e., it does not create inefficiencies in itself).

- In a monetary equilibrium all agents believe that the next generation will find the blue paper valuable.

- When confidence in money is lost, no one accepts it and it looses its value: prices of goods in terms of money go to infinity, i.e., HYPERINFLATION.

Let's incorporate a more precise notion of equilibrium into the $OC(r_{t+1})$.

Definition 6: A stationary economy is such that $e_t^{t-1} = e^o$ and $e_t^t = e^y$ for all t. An equilibrium is stationary if $c_t^{t-1} = c^o$, $c_t^{t-1} = c^o$ for all t, and if either $r_t = r$ or $p_{t+1}/p_t = \pi$ for all t.

Let the excess demand of a "young" in a stationary economy be given by

$$y(r_{t+1}) = c_t^t(r_{t+1}) - e^y$$

given r_{t+1}, and let the excess demand of and "old" be given by

$$z(r_{t+1}) = c_{t+1}^t(r_{t+1}) - e^o.$$

The two equations above summarize utility maximization for a consumer given the return. The optimal choices, in addition, satisfy the intertemporal budget, which we write as

$$y(r_{t+1}) + \frac{z(r_{t+1})}{1 + r_{t+1}} = 0. \tag{10}$$

It follows that

$$\frac{z(r_{t+1})}{y(r_{t+1})} = -(1 + r_{t+1}), \tag{11}$$

and that

$$z(r_{t+1}) \geq 0 \iff y(r_{t+1}) \leq 0. \tag{12}$$

If y and z were known, then the first equation above could be used to determine r_{t+1}. That equation also implies that $y = 0$ and $z = 0$ is a point of the offer curve.

Finally, the market clearing condition in terms of excess demand functions is simply given by

$$y(r_{t+1}) + z(r_t) = 0. \tag{13}$$

It is possible to construct graphically an equilibrium in the (y, z) plane, as follows:

1. Choose $r_1 > -1$, and notice that $z_1 = m(1 + r_1)$. So z_1 becomes known, and from Eq. (13) we get the implied y_2.

2. With y_2 and z_1 known, use the Offer Curve to get z_2.

3. Once y_2 and z_2 are known, it is straightforward to determine r_2 from the corresponding budget constraint in the second period.

4. Repeat 1 to 3, to get the complete sequence of equilibrium allocations and returns.

The initial choice for r_1 is essentially unrestricted: there is a continuum of equilibria!!!

- All equilibria converge to the autarky equilibrium.

- In all those equilibra the value of money declines toward zero.

- Over time, allocations and prices are different across equilibria, but can be made arbitrarily close to each other. Hence, we loose the local uniqueness property.

Summary

1. Unlike in the standard Neoclassical model, in the OLG model the equilibrium allocation is not always efficient.

2. Unlike in the standard Neoclassical model, in the OLG model outside money may be valuable in equilibrium.

3. Unlike in the standard Neoclassical model, in the OLG model there may be a continuum of equilibria.