Problem Set 1, IDEA-UAB
Solutions

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1. The utility maximization problem of one agent is

$$\max_{c_t^t, c_{t+1}^t} \; u_t(c_t^t, c_{t+1}^t)$$

$$\text{s.t. } \; p_t c_t^t + p_{t+1} c_{t+1}^t \leq p_t w_1 + p_{t+1} w_2$$

The Lagrangian is:

$$L = \frac{((c_t^t)^\gamma (c_{t+1}^t)^{1-\gamma})^{1-\theta} - 1}{1-\theta} + \lambda[p_t w_1 + p_{t+1} w_2 - p_t c_t^t - p_{t+1} c_{t+1}^t]$$

And the FOCs are:

$$[c_t^t]: \; \frac{1}{1-\theta}(1-\theta)((c_t^t)^\gamma (c_{t+1}^t)^{1-\gamma})^{-\theta} \gamma (c_t^t)^{\gamma-1} (c_{t+1}^t)^{1-\gamma} - \lambda p_t = 0$$

$$U_{c_t^t} = \lambda p_t \Leftrightarrow \gamma((c_t^t)^\gamma (c_{t+1}^t)^{1-\gamma})^{-\theta} (c_t^t)^{\gamma-1} (c_{t+1}^t)^{1-\gamma} = \lambda p_t$$

$$[c_{t+1}^t]: \; \frac{1}{1-\theta}(1-\theta)((c_t^t)^\gamma (c_{t+1}^t)^{1-\gamma})^{-\theta} (1-\gamma)(c_t^t)^{\gamma} (c_{t+1}^t)^{-\gamma} - \lambda p_{t+1} = 0$$

$$U_{c_{t+1}^t} = \lambda p_{t+1} \Leftrightarrow (1-\gamma)((c_t^t)^\gamma (c_{t+1}^t)^{1-\gamma})^{-\theta} (c_t^t)^{\gamma} (c_{t+1}^t)^{-\gamma} = \lambda p_{t+1}$$

Divide the two FOCs above to obtain:

$$\frac{U_{c_t^t}}{U_{c_{t+1}^t}} = \frac{p_t}{p_{t+1}} = \frac{\gamma}{1-\gamma} \frac{c_{t+1}^t}{c_t^t} \Leftrightarrow p_{t+1} c_{t+1}^t = \frac{1-\gamma}{\gamma} p_t c_t^t$$

Substituting for the second period of life consumption into the budget constraint yields:

$$p_t c_t + \frac{1-\gamma}{\gamma} p_t c_t^t = p_t w_1 + p_{t+1} w_2 \Leftrightarrow p_t c_t \frac{\gamma + 1 - \gamma}{\gamma} = p_t \left(w_1 + \frac{p_{t+1}}{p_t} w_2 \right)$$

$$\Rightarrow c_t^t = \gamma\left(w_1 + \frac{p_{t+1}}{p_t}w_2\right)$$

The second period of life consumption will then be given by:

$$c_{t+1}^t = \frac{1-\gamma}{\gamma}\frac{p_t}{p_{t+1}}c_t^t \Leftrightarrow c_{t+1}^t = \frac{1-\gamma}{\gamma}\frac{p_t}{p_{t+1}}\gamma\left(w_1 + \frac{p_{t+1}}{p_t}w_2\right)$$

$$c_{t+1}^t = (1-\gamma)\left(w_2 + \frac{p_t}{p_{t+1}}w_1\right)$$

For the initial old we have that:

$$c_1^0 = w_2 + \frac{m}{p_1}$$

2. The excess demand when young is given by:

$$y(p_t, p_{t+1}) = c_t^t - w_1 \Leftrightarrow y(p_t, p_{t+1}) = \gamma\left(w_1 + \frac{p_{t+1}}{p_t}w_2\right) - w_1$$

$$\Rightarrow y(p_t, p_{t+1}) = \gamma\frac{p_{t+1}}{p_t}w_2 - (1-\gamma)w_1$$

The excess demand when old is given by:

$$z(p_t, p_{t+1}) = c_{t+1}^t - w_2 \Leftrightarrow z(p_t, p_{t+1}) = (1-\gamma)\left(w_2 + \frac{p_t}{p_{t+1}}w_1\right) - w_2$$

$$\Rightarrow z(p_t, p_{t+1}) = (1-\gamma)\frac{p_t}{p_{t+1}}w_1 - \gamma w_2$$

For the initial old we have:

$$z_0(p_1, m) = c_1^0 - w_2 \Leftrightarrow z_0(p_1, m) = \frac{m}{p_1}$$

Note that as $\frac{p_{t+1}}{p_t} \in (0, \infty)$ varies, y varies between $(\gamma - 1)w_1$ and ∞ and z varies between $-\gamma w_2$ and ∞.

3. Given m, an ADE equilibrium is an allocation $\hat{c}_1^0$, $\{(\hat{c}_t^t, \hat{c}_{t+1}^t)\}_{t=1}^{\infty}$ and prices $\{p_t\}_{t=1}^{\infty}$ such that:

- Given $\{p_t\}_{t=1}^{\infty}$, $\{(\hat{c}_t^t, \hat{c}_{t+1}^t)\}_{t=1}^{\infty}$ solves

$$\max_{c_t^t, c_{t+1}^t} \ u_t(c_t^t, c_{t+1}^t)$$

$$\text{s.t.} \ \ p_t y(p_t, p_{t+1}) + p_{t+1} z(p_t, p_{t+1}) = 0$$

- Given p_1, c_1^0 solves

$$\max_{c_1^0} \quad u(c_1^0)$$

$$\text{s.t.} \;\; z_0(p_1, m) = \frac{m}{p_1}$$

- Resource balance or goods market clearing

$$y(p_t, p_{t+1}) + z(p_{t-1}, p_t) = 0$$

4. The offer curve is found by solving z as a function of y (by eliminating the price ratio):

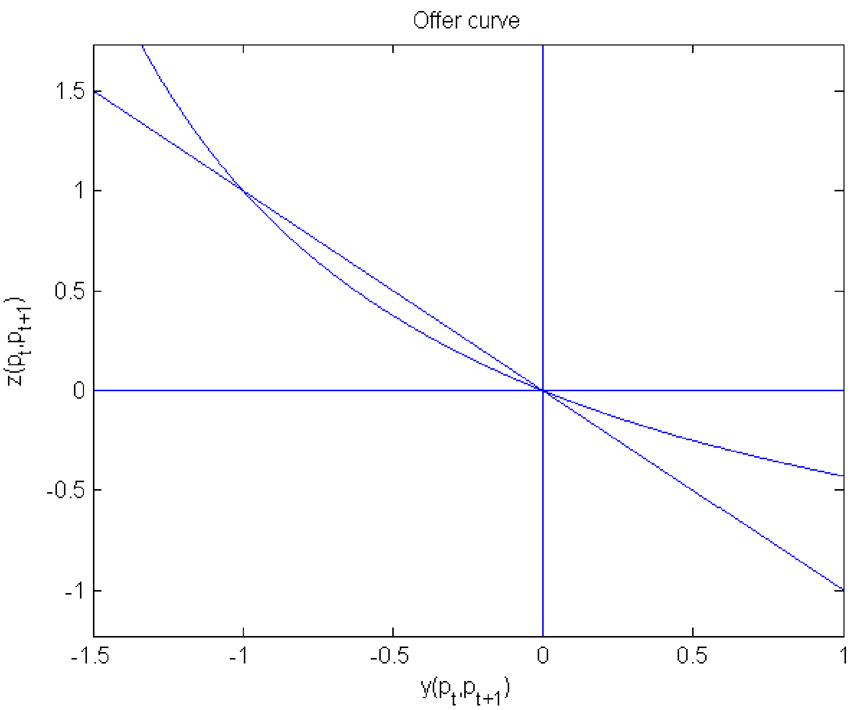
$$y = \gamma \frac{p_{t+1}}{p_t} w_2 - (1-\gamma)w_1 \;\Rightarrow\; y + (1-\gamma)w_1 = \gamma \frac{p_{t+1}}{p_t} w_2$$

$$\Rightarrow\; \frac{p_{t+1}}{p_t} = \frac{y + (1-\gamma)w_1}{\gamma w_2} \;\Rightarrow\; \frac{p_t}{p_{t+1}} = \frac{\gamma w_2}{y + (1-\gamma)w_1}$$

$$z = (1-\gamma)\frac{p_t}{p_{t+1}} w_1 - \gamma w_2 \;\Rightarrow\; z = (1-\gamma)w_1 \frac{\gamma w_2}{y + (1-\gamma)w_1} - \gamma w_2$$

$$z = \gamma w_2 \left[\frac{(1-\gamma)w_1}{y + (1-\gamma)w_1} - 1 \right] \;\Rightarrow\; z = \gamma w_2 \frac{(1-\gamma)w_1 - y - (1-\gamma)w_1}{y + (1-\gamma)w_1}$$

$$\Rightarrow\; z(y) = \frac{-\gamma w_2 y}{y + (1-\gamma)w_1}$$



5. An equilibrium is stationary if $c_{t+1}^t = c_1^0$ (equal consumption when old), $c_t^t = c^y$ (equal consumption when young) and $\frac{p_{t+1}}{p_t} = a$ (a constant), for all t. Given the assumption that each generation has the same endowment structure, a stationary equilibrium necessarily has to satisfy $y(p_t, p_{t+1}) = y$ and $z_0(p_1, m) = z(p_t, p_{t+1}) = z$, for all t. So, they will have to satisfy both $y + z = 0$ and $y + az = 0$. Hence, $a = 1$.

6. When $m = 0$, we have that $z_0(p_1, m) = 0$ for all $p_1 > 0$, which means that the initial old generation has no money and will consume their own endowments, w_2. Then from the goods market clearing condition (resource constraint) it follows that $y(p_1, p_2) = -z_0(p_1, m) = 0$, meaning that the first young generation must consume their endowment when young because they cannot trade with the initial olds. From the offer curve then, we have that

$$z(p_1, p_2) = \frac{-\gamma w_2 y(p_1, p_2)}{y(p_1, p_2) + (1-\gamma)w_1} \Rightarrow z(p_1, p_2) = 0$$

meaning that when old, the first generation will also consume their endowments since they haven't saved anything when young. But then the next young generation is forced to consume their own endowments and so forth. There is no trade whatsoever and every consumer eats its endowment in each period. Formally, we have that:

$$z(p_t, p_{t+1}) = y(p_t, p_{t+1}) = 0 \Rightarrow c_t^t = w_1 \text{ and } c_{t+1}^t = w_2.$$

This is the autarkic equilibrium and it's unique.

7. Define the autarkic interest rate as:

$$1 + \bar{r} = \frac{U_{c_t^t(w_1)}}{U_{c_{t+1}^t}(w_2)}$$

It can be shown that: when $\bar{r} < 0$ (the Samuelson case), the autarkic equilibrium is not Pareto optimal, whereas when $\bar{r} > 0$ (the classical case), the autarkic equilibrium is Pareto optimal. The condition for Pareto optimality will then be given by:

$$\gamma w_2 > (1-\gamma)w_1$$

Consider the alternative allocation given by $c_t^t = w_1 - \tau$ and $c_{t+1}^t = w_2 + \tau$. We want to prove that given the condition above, this allocation does not provide higher utility than the autarkic equilibrium allocation. For that, we compute:

$$du(c_t^t, c_{t+1}^t) = -u_{c_t^t} d\tau + u_{c_{t+1}^t} d\tau$$

Hence,

$$\frac{du(c_t^t, c_{t+1}^t)}{d\tau} = -u_{c_t^t} + u_{c_{t+1}^t} = u_{c_{t+1}^t}\left(1 - \frac{u_{c_t^t}}{u_{c_{t+1}^t}}\right)$$

$$\frac{du(c_t^t, c_{t+1}^t)}{d\tau} = u_{c_{t+1}^t}\left(1 - \frac{\gamma w_2}{(1-\gamma)w_1}\right) \ < 0 \ \text{ if } \ \gamma w_2 > (1-\gamma)w_1$$

Alternatively, we have seen that the first order conditions implies:

$$\frac{U_{c_t^t}}{U_{c_{t+1}^t}} = \frac{p_t}{p_{t+1}} = \frac{\gamma}{1-\gamma}\frac{c_{t+1}^t}{c_t^t}$$

Under autarky, we have

$$\frac{p_t}{p_{t+1}} = \frac{\gamma}{1-\gamma}\frac{w_2}{w_1} \Rightarrow p_{t+1} = \frac{1-\gamma}{\gamma}\frac{w_1}{w_2}p_t$$

For $m = 0$ we can without loss of generality, normalize the price of the first period consumption good $p_1 = 1$ (we can do that because it does not change the real value of the stock of outside money the initial old generation is endowed with). With this normalization, the sequence $\{p_t\}_{t=1}^{\infty}$ can be written as:

$$p_t = \left(\frac{1-\gamma}{\gamma}\frac{w_1}{w_2}\right)^{t-1}$$

If $\sum_{t=1}^{\infty} p_t(w_1 + w_2) < \infty$, then the competitive equilibrium allocation is Pareto efficient. If, however, the value of the aggregate endowment is infinite (at the equilibrium prices), then the competitive equilibrium may not be Pareto optimal. This is sufficient condition. For the value of the aggregate endowment to be finite, $\gamma w_2 > (1-\gamma)w_1$ is required.

8. First, note that from the budget constraint of the initial old we have that:

$$c_1^0 = w_2 + \frac{m}{p_1}$$

It turns out that when $m > 0$ for autarky to be an ADE we need that $p_1 = \infty$, that is the price level is so high in the first period that the stock of money de facto has no value. Since for all other periods we need $\frac{p_t+1}{p_t} = \frac{1-\gamma}{\gamma}\frac{w_1}{w_2}$ to support the autarkic allocation, we have the technical requirement that price levels be infinite with well defined finite price ratios.

When $m > 0$, we can find the stationary equilibria by solving for the intersection of the offer curve and the resource constraint:
Then, it follows that:

$$y + z = 0$$

$$z = \frac{-\gamma w_2 y}{y+(1-\gamma)w_1}$$

$$-y = \frac{-\gamma w_2 y}{y+(1-\gamma)w_1} \quad \Rightarrow \quad y = 0 \quad , \quad z = 0$$

$$y = \gamma w_2 - (1-\gamma)w_1 \quad , \quad z = (1-\gamma)w_1 - \gamma w_2$$

For the second solution to actually be an equilibrium we need that $z > 0$ which is equivalent to $(1-\gamma)w_1 > \gamma w_2$. Then, this is also a stationary equilibrium which is Pareto efficient (it dominates the autarkic equilibrium). Opposite to the condition in the previous exercise.

9. As in Krueger, I construct an equilibrium using the geometric method. All the equilibria with z_0 starting below the z_0 of the monetary stationary equilibrium have the feature that the equilibrium allocations over time converge to the autarkic allocation, with $z_0 > z_1 > z_2 > ... > z_t > 0$ and $lim_{t\to\infty} z_t = 0$ and $0 > y_t > ... > y_1$, with $\lim_{t\to\infty} y_t = 0$. As we approach the autarkic allocation, the slope of the offer curve $\frac{p_t}{p_{t+1}}$, decreases with $\frac{p_t}{p_{t+1}} < \frac{p_{t-1}}{p_t} < \frac{p_{t-2}}{p_{t-1}} < ... < \frac{p_1}{p_2} < 1$. This implies that prices are increasing with $\lim_{t\to\infty} p_t = \infty$. Hence, all non stationary equilibria feature inflation.

10. Given m, a SME is an allocation $\hat{c}_1^0$, $\{\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{s}_t^t\}_{t=1}^{\infty}$ and interest rates $\{r_t\}_{t=1}^{\infty}$ such that

- Given $\{r_t\}_{t=0}^{\infty}$, the allocation $\{\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{s}_t^t\}_{t=1}^{\infty}$ solves

$$\max_{c_t^t, c_{t+1}^t, s_t^t} \quad u_t(c_t^t, c_{t+1}^t)$$

$$\text{s.t. } c_t^t + s_t^t \le w_1$$

$$c_{t+1}^t \le w_2 + (1 + r_{t+1}) s_t^t$$

- Given r_1, c_1^0 solves

$$\max_{c_1^0} \quad u(c_1^0)$$

$$\text{s.t. } c_1^0 \le w_2 + (1 + r_1)m$$

- Resource balance or goods market clearing

$$\hat{c}_t^{t-1} + \hat{c}_t^t = w_1 + w_2$$

The Lagrangian is:

$$L = \frac{((c_t^t)^\gamma (c_{t+1}^t)^{1-\gamma})^{1-\theta} - 1}{1-\theta} + \eta[w_2 + (1+r_{t+1})(w_1 - c_t^t) - c_{t+1}^t]$$

The FOCs are:

$$[c_t^t] : U_{c_t^t} = \eta(1+r_{t+1}) \Longleftrightarrow \gamma((c_t^t)^\gamma (c_{t+1}^t)^{1-\gamma})^{-\theta} (c_t^t)^{\gamma-1} (c_{t+1}^t)^{1-\gamma} = \eta(1+r_{t+1})$$

$$[c_{t+1}^t] : U_{c_{t+1}^t} = \eta \Longleftrightarrow (1-\gamma)((c_t^t)^\gamma (c_{t+1}^t)^{1-\gamma})^{-\theta} (c_t^t)^{\gamma} (c_{t+1}^t)^{-\gamma} = \eta$$

Divide the two FOCs above to obtain:

$$\frac{U_{c_t^t}}{U_{c_{t+1}^t}} = 1 + r_{t+1} = \frac{\gamma}{1-\gamma} \frac{c_{t+1}^t}{c_t^t}$$

From ADE we have that:

$$\frac{U_{c_t^t}}{U_{c_{t+1}^t}} = \frac{p_t}{p_{t+1}} = \frac{\gamma}{1-\gamma} \frac{c_{t+1}^t}{c_t^t}$$

It follows that

$$1 + r_{t+1} = \frac{p_t}{p_{t+1}}$$

Given the equivalence above, it is easy to show that the budget constraints are identical in ADE and SME. So, the SME equilibrium is equivalent to the ADE equilibrium.