Rationalizing and Curve-Fitting Demand Data with Quasilinear Utilities

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In the empirical and theoretical literature a consumer's utility function is often assumed to be quasilinear. In this paper we provide necessary and sufficient conditions for testing if the consumer acts as if she is maximizing a quasilinear utility function over her budget set. If the consumer's choices are inconsistent with maximizing a quasilinear utility function over her budget set, then we compute the "best" quasilinear rationalization of her choices.

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1 Introduction

Theoretical models of consumer demand assume consumer's choice is derived from the individual maximizing a concave utility function subject to a budget constraint. Given a finite set of price and consumption choices, if there exist a concave, continuous and monotonic utility function such that these choices are the maxima of this function over the budget set, then we say that the data set is rationalizable.

Afriat (1967) provided the first necessary and sufficient conditions for a finite data set to be rationalizable, that is for it to be a result of the consumer maximizing her utility subject to a budget constraint. Afriat (1981) and Varian (1983) subsequently derived necessary and sufficient conditions for a finite data set to be rationalizable by homothetic or by separable utility functions.

Quasilinear rationalizations are used in a wide range of areas in economics, including theoretical mechanism design, public economics, industrial organization and international trade, one reason being that in this case changes in consumer surplus are equivalent to changes in consumer welfare.

Quah has shown that homothetic utility functions generate monotone demand functions, i.e. $x(p, I)$ is said to be monotone if $x(p_1, I) - x(p_2, I) \cdot (p_1 - p_2) \leq 0$. This is a multivariable version of the law of demand, i.e. an increase in prices results in a decrease in demand. Demand functions generated from quasilinear utility maximization subject to a budget constraint enjoy an additional property not possessed by monotone demand functions: they are cyclically monotone. Such demand functions are said to satisfy the strong law of demand [3]. Cyclically monotone demand functions not only have downwards sloping demand curves, in the sense that they are monotone functions, but also their line integrals are path-independent and therefore measure the change in consumer welfare for a given multidimensional change in prices.

In the first part of this paper we show that a finite data set consisting of pairs of price vectors and consumption vectors can be rationalized by a quasilinear utility maximization subject to a budget constraint if and only if the data set is cyclically monotone. Moreover we derive a linear program, from the associated Afriat inequalities, that is solvable if and only if the data is cyclically monotone.

Suppose the data is not cyclically monotone, then by our theorem there is no quasilinear rationalization of the data. In this case we ask what is the "best" quasilinear approximation to the data. We address this question in the second section of the paper in two steps. First we show that our nonparametric characterization of quasilinear rationalization extends to maximization of a random quasilinear utility function of the form $u(x) + \epsilon \cdot x + x_0$, subject to a budget constraint. Random quasilinear utility functions have also been discussed by Brown and Wegkamp (2003). Their specification, $v(x, e) = u(x) + \epsilon \cdot x + x_0$ is a special case of the random utility model suggested originally by Brown and Matzkin (1998). If the shocks are assumed to be bounded, as in Brown and Wegkamp and Brown and Matzkin, then we show that the random model, as in the deterministic case, is testable.

If the shocks are not bounded then any data set can be rationalized by a random quasilinear utility function. This suggests another interpretation of ϵ, as slack variables in fitting a quasilinear utility function to demand data. In this case we choose the quasilinear approximation with least absolute deviation as the "best" quasilinear rationalization of her choices.

[3] see Brown and Calsamiglia (2003) for further discussion

2 The Strong Law of Demand and Quasilinear Rationalizability

Afriat (1967) provides the first non-parametric test for consumer behavior. He provides a necessary and sufficient condition on finite data for it to be rationalizable by a neoclassical utility function.

Definition 1 Let (p_r, x_r), $r = 1, ..., N$ be given. A utility function u rationalizes the data if for all $r = 1, ..., N$ x_r solves:

$$\max_{x \epsilon R^n_{++}} u(x)$$

$$s.t. p_r x \leq I = p_r x_r$$

Theorem 1(Afriat 1967) The following conditions are equivalent:

1. *There exists a concave, monotonic, continuous, non-satiated utility function that rationalizes the data.*

2. *The data* (p_r, x_r)*,* $r = 1, ..., N$ *satisfies Afriat inequalities, that is, there exists* $U_r > 0$ *and* $\lambda_r > 0$ *for* $r = 1, \ldots, N$ *such that*

$$U_r \leq U_l + \lambda_l p_l \cdot (x_r - x_l) \;\; \forall r, l = 1, \ldots, N$$

3. *The data* (p_r, x_r)*,* $r = 1, ..., N$ *satisfies "cyclical consistency", that is,*

$$p_r x_r \geq p_r x_s, \; p_s x_s \geq p_s x_t, \cdots, p_q x_q \geq p_q x_r$$

   implies

$$p_r x_r = p_r x_s, \; p_s x_s = p_s x_t, \cdots, p_q x_q = p_q x_r$$

Definition 2 Let (p_r, x_r), $r = 1, ..., N$ be given.The data is *quasilinear rationalizable* if for some $y_r > 0$ and $I > 0$, $\forall r$ x_r solves

$$\max_{x \epsilon R^n_{++}} U(x) + y_r$$

$$s.t. p_r x + y_r = I$$

where U is a concave function.

Definition 3 A demand function $x(p, I) : R^{n+1}_{++} \rightarrow R^n_{++}$ is *cyclically monotone* if for any given I and finite set $\{p_1, ..., p_m\}$ (m arbitrary):

$$x_1 \cdot (p_2 - p_1) + x_2 \cdot (p_3 - p_2) + \cdots + x_m \cdot (p_1 - p_m) \geq 0$$

This is equivalent to $p(x, I)$ being cyclically monotone, that is that for any given I and finite set $\{x_1, ..., x_m\}$ (m arbitrary)

$$p_1 \cdot (x_2 - x_1) + p_2 \cdot (x_3 - x_2) + \cdots + p_m \cdot (x_1 - x_m) \geq 0$$

Definition 4 A demand function $x(p, I) : R^{n+1}_{++} \rightarrow R^n_{++}$ satisfies the *strong law of demand* if it is cyclically monotone.

Definition 5 If u is concave on R^n, then $\beta \in R^n$ is a subgradient of u at x if for all $y \in R^n : u(y) \leq u(x) + \beta \cdot (y - x)$.

Definition 6 If u is a concave function on R^n, then $\partial u(x)$ is the set of subgradients of u at x.

Theorem 2 (Rockafellar 1970) *Let ρ be a multivalued mapping from R^n to R^n. In order that there exists a closed proper concave function f on R^n such that $\rho(x) \subset Df(x)$ for every x, it is necessary and sufficient that ρ be cyclically monotone.*

In his proof he constructs the following function f on R^n:

$$f(x) = \inf\{x^*_m \cdot (x - x_m) + \ldots + x^*_0 \cdot (x_1 - x_0)\}$$

where the infimum is taken over all finite sets of pairs $(x^*_r, x_r), r = 1, \ldots, m$ (m arbitrary) in the graph of $\rho(x)$. Note that if the graph of ρ has only a finite number of elements then the domain of f is all of R^n. Also note that for this f, x^*_i is the subgradient at x_i.

Theorem 3 The following are equivalent:

1. *The data* (p_r, x_r)*,* $r = 1, ..., N$ *is quasilinear rationalizable by a continuous, concave, strictly monotonic utility function* U*.*

2. *The data* (p_r, x_r)*,* $r = 1, ..., N$ *satisfies Afriat's inequalities with constant marginal utilities of income, that is, there exists* $G_r > 0$ *and* $\lambda > 0$ *for* $r = 1, \ldots, N$ *such that*

$$G_r \leq G_l + \lambda p_l \cdot (x_r - x_l) \;\; \forall r, l = 1, \ldots, N$$

   or equivalently there exist $U_r > 0$ for $r = 1, \ldots, N$

$$G_r \leq G_l + p_l \cdot (x_r - x_l) \;\; \forall r, l = 1, \ldots, N$$

   where $G_r = \frac{U_r}{\lambda}$

3. *The data* (p_r, x_r)*,* $r = 1, ..., N$ *satisfies the strong law of demand.*

Proof: (1) $\Rightarrow$ (2): From the FOC of the quasilinear utility maximization problem we know:

$$\exists \beta_r \, \epsilon \, \partial U(x), \text{ s.t. } \beta_r = \lambda_r p_r \text{ where } \lambda_r = 1$$

Also, U being concave implies that $U(x_r) \leq U(x_l) + \beta_l(x_r - x_l)$ for $r, l = 1, 2, ..., N$. Since $\beta_l = p_l \; \forall l = 1, \ldots, N$ we get $U(x_r) \leq U(x_l) + p_l(x_r - x_l) \; \forall r, l = 1, ..., N$

(2) $\Rightarrow$ (3): For any set of pairs $\{(x_s, p_s)\}$, $s = 1, ..., m$ we need that: $p_0 \cdot (x_1 - x_0) + p_1(x_2 - x_1) + ... + p_m(x_0 - x_m) \geq 0$.

From the Afriat inequalities with constant marginal utilities of income we know:

$$U_1 - U_0 \leq p_0 \cdot (x_1 - x_0)$$

$$U_2 - U_1 \leq p_1 \cdot (x_2 - x_1)$$

$$\cdots$$

$$U_0 - U_m \leq p_m(x_0 - x_m)$$

Adding up these inequalities we get that the left hand sides cancel and we get the condition that defines cyclical monotonicity.

(3) $\Rightarrow$ (1): By Theorem 2 we know there exists a utility function U such that p_r is the subgradient of U at x_r, that is:

$$p_r = \partial U(x_r)$$

which together with $\lambda_r = 1$ constitutes a solution to the first order conditions of the quasilinear maximization problem.

If we require strict inequality in (2) of Theorem 3, then it follows from Lemma 2 in Chiappori and Rochet (1987) that the rationalization can be chosen to be a C^∞ function. It then follows from Roy's identity that $x(p) = -\frac{\partial V(p)}{\partial p}$. Hence for any line integral we see that $\int_{p_1}^{p_2} x(p)dp = -\int_{p_1}^{p_2} = \frac{\partial V(p)}{\partial p}dp = h(p_1) - h(p_2)$. That is, consumer welfare is well-defined and the change in consumer surplus induced by a change in market prices is the change in consumer's welfare.

3 Random Quasilinear Rationalizations

If consumers have random quasilinear utility functions can anything happen? No, not if each individual's distribution of utility shocks to her marginal utilities are bounded, where agents have random utility functions of the form $V(x, e) = U(x) + \epsilon \cdot x + x_0$. Assuming $U(x)$ is strictly concave, smooth and monotonic, each realization of ϵ gives rise to a quasilinear utility function having all of the properties previously derived, i.e. for a fixed ϵ, the random demand function $x(p, \epsilon)$ is cyclically monotone. Of course a finite family of observations of such demand functions need not be cyclically monotone, since each observation can in principle be drawn from a "different" cyclically monotone demand function. It is therefore surprising that the hypothesis of random quasilinear rationalization of a data set is refutable, if the shocks are bounded and the bounds are known a priori. As a consequence, this hypothesis is testable in the sense of Brown and Matzkin (1995), i.e. there exists a finite family of polynomial inequalities involving only observations on market data that are solvable if and only if the data can be rationalized with a random quasilinear utility function. These inequalities will involve the known bounds on the shocks.

Definition 7 $U(x) + \epsilon \cdot x + x_0$ is a random quasilinear rationalization of the data (p^r, x^r), $r = 1, ..., N$ if $\exists \epsilon_1, \ldots, \epsilon_r$, x_{0r} and $I > 0$ such that x_r is the solution to

$$\max U(x) + \epsilon_r \cdot x + x_{0r}$$
$$\text{s.t. } p_r \cdot x + x_0 = I$$

Definition 8 The data set (p_r, x_r), $r = 1, ..., N$, satisfies the Random Afriat Inequalities, *RAI*, if it satisfies the Afriat inequalities for a random quasilinear utility function of the form $V(x, \epsilon) = U(x) + \epsilon \cdot x + x_0$. That is, if they satisfy the following:

$$U_r \leq U_l + (p_l - \epsilon_l) \cdot (x_r - x_l) \text{ for } r, l = 1, \ldots, N$$

$$p_r - \epsilon_r >> 0 \text{ for } r = 1, \ldots, N$$
$$U_r > 0 \text{ for } r = 1, \ldots, N$$
$$\epsilon_r >> 0 \forall r = 1, \ldots, N$$

These are linear inequalities in the unknown U_r and ϵ_r. Hence they can be solved in polynomial time using interior point linear programming algorithms.

The utility shock ϵ has compact support if there exists ϵ_{min} and ϵ_{max} such that $\epsilon_{min} \leq \epsilon \leq \epsilon_{max}$, where the quasilinear model is a special case of the random quasilinear model if $\epsilon_{max} = 0$.

Figure 1 shows that for two observations, all possible pairs of budget lines defined by the gradients of $U(x)$ at x_1 and x_2, given consumption x_1 and x_2, violate WARP. Hence rationalizatios with random quasilinear utilities of the form $U(x) + \epsilon \cdot x + x_0$, where ϵ has compact support is refutable.

Figure 2 shows that for two observations, without bounds on the shocks, the model is not refutable. With unbounded shocks we can generate the interior of the positive orthant of the price vector and therefore WARP will never be violated.

Theorem 4 *Let the data set (p_r, x_r), $r = 1, ..., N$ be given. The data can be rationalized by random quasilinear preferences of the form $U(x) + \epsilon \cdot x + x_0$ if and only if it satisfies RAI. Moreover, the model is refutable if and only if the shocks are bounded and the bounds are known a priori.*

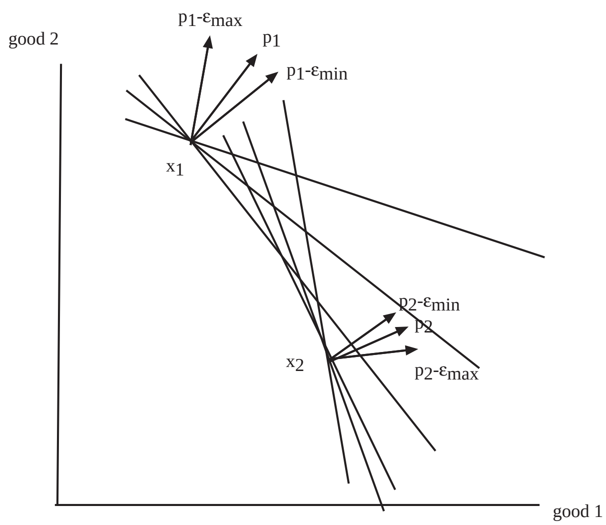

Figure 1: Two goods two observation example with bounded shocks

This result can be viewed as a relaxed version of Theorem 3. If one empirically tests for quasilinear utilities it would be extremely surprising to find that the Afriat inequalities for $\lambda_r = 1\ \forall r$ were satisfied, i.e. that a quasilinear function ratinalizes the data. As in any empirical test we would like to allow for some error in the individual's choice.[4] One way of doing this is to say that individuals make small mistakes in evaluating their marginal utilities when making a consumption choice. This suggests the following family of Afriat inequalities with q's and p's:

$$U_r \leq U_l + q_l \cdot (x_r - x_l)\ \ \forall r, l = 1, \ldots, N$$

$$q_r \leq p_r\ \ \forall r = 1, \ldots, N$$

Note that if we define $\epsilon_r = p_r - q_r$, this reduces to the family of inequalities for random quasilinear utilities. We now compute the solutions to the set of inequalities with unknowns U_r, q_r for $r = 1, \ldots, N$ and pick the solution with least absolute deviation (LAD). This solution is our "best" approximation.

[4] Afriat (1972) and Varian (1990) provide the Critical Cost Efficiency Index as a measure of goodness-of-fit that is measured by the minimal distortion in wealths required to rationalize the data. This is the measure used by Andreoni and Miller (2002) to rationalize altruism with experimental data.

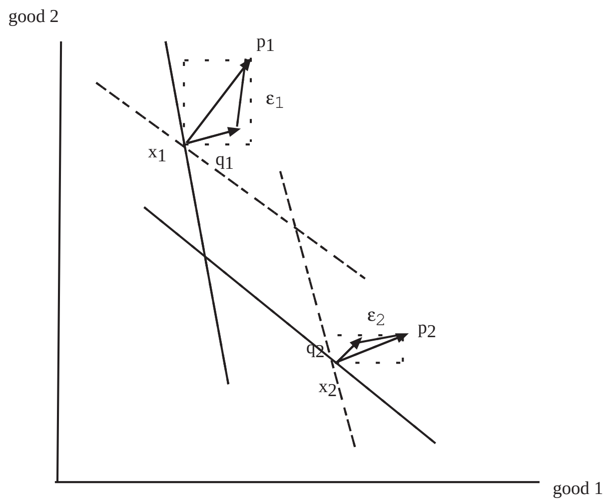

Figure 2: Two goods two observation example with unbounded shocks, $\epsilon_r \leq p_r$ for $r = 1, 2$, and a pair of budget sets where consumption choices do not violate WARP.

Our notion of curve-fitting demand data with quasilinear utilities extends naturally to the wide class of economic models formulated as solutions to a finite family of polynomial inequalities consisting of the Afriat inequalities for consumers or producers and possibly other polynomial inequalities such as budget constraints or feasibility constraints, e.g. market clearing. We simply relax the Afriat inequalities for agents by introducing slack variables for the gradients of the utility and production or cost functions. In particualr, if we apply our relaxation method to the Brown-Matzkin inequalities characterizing a Walrasian equilibrium, we obtain the "best" Walrasian approximation to the observed market data with respect to minimizing least absolute deviations. It is in this sense that economists may use competitive markets as approximations to actual market structures.

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