The Nonparametric Approach to Applied Welfare Analysis

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Abstract

Changes in total surplus are traditional measures of economic welfare. We propose necessary and sufficient conditions for rationalizing individual and aggregate consumer demand data with individual quasilinear and homothetic utility functions. Under these conditions, consumer surplus is a valid measure of consumer welfare. For nonmarketed goods, we propose necessary and sufficient conditions on input market data for efficient production , i.e. production at minimum cost. Under these conditions we derive a cost function for the nonmarketed good, where producer surplus is the area above the marginal cost curve.

Keywords: Welfare economics, Quasilinear utilities, Homothetic Utilities, Nonmarketed goods, Afriat inequalities

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0.1 Introduction

Theoretical models of consumer demand assume consumer's choice is derived from the individual maximizing a concave utility function subject to a budget constraint. Given a finite set of price and consumption choices, we say that the data set is rationalizable if there exists a concave, continuous and monotonic utility function such that these choices are the maxima of this function over the budget set.

Afriat (1967) provided the first necessary and sufficient conditions for a finite data set to be a result of the consumer maximizing her utility subject to a budget constraint. In a series of lucid papers Varian (1982), (1983) and (1984) increased our understanding and appreciation of Afriat's seminal contributions to demand theory, Afriat (1967), and the theory of production, Afriat (1972a). As a consequence there is now a growing literature on the testable restrictions of strategic and non-strategic behavior of households and firms in market economies —see the survey of Carvajal, et al (2004).

Quasilinear utility functions are used in a wide range of areas in economics, including theoretical mechanism design, public economics, industrial organization and international trade. In applied partial equilibrium models we often measure economic welfare in terms of total surplus, i.e., the sum of consumer and producer surplus, and deadweight loss. As is well known, consumer surplus is a valid measure of consumer welfare only if the consumer's demand derives from maximizing a homothetic or quasilinear utility function subject to her budget constraint —see section 11.5 in Silberberg (1990). Both Afriat (1972b) and Varian (1983) proposed a necessary and sufficient combinatorial condition for rationalizing data sets, consisting of market prices and consumer demands, with homothetic utility functions. This condition is the homothetic axiom of revealed preference or HARP. To our knowledge, there is no comparable result in the literature for quasilinear rationalizations of consumer demand data. In this paper we show that a combinatorial condition introduced in Rockafellar (1970) to characterize the subgradient correspondence for convex real-valued functions on R^n, cyclical monotonicity, is a necessary and sufficient condition for a finite data set to be rationalizable with a quasilinear utility function.[1] We then extend our analysis to aggre-

[1]In the paper by Rochet (1987) "A necessary and sufficient condition for rationalizability in a quasilinear context" published in the *Journal of Mathematical Economics* he defines rationalizability as implementability of an action profile via compensatory transfers. Therefore the results presented in his paper are of a different nature.

gate demand data. That is, given aggregate demands, income distributions and market prices we give necessary and sufficient conditions for the data to be rationalized as the sum of individual demands derived from maximizing quasilinear utility functions subject to a budget constraint. Our analysis differs from Varian's in that it is the form of the Afriat inequalities for the homothetic and quasilinear case that constitute the core of our analysis. We show in the case of aggregate data, where individual data is not observed, that these inequalities reduce to convex and linear inequalities respectively. As a consequence in both cases we can determine in polynomial time if they are solvable and, if they are, compute the solution in polynomial time. This is certainly not true in general.[2]

On the other hand, measuring producer surplus for nonmarketed goods such as health, education or environmental amenities and ascertaining if these goods are produced efficiently, i.e., at minimum cost, are important policy issues. Our contribution to the literature on nonmarketed goods is the observation that Afriat's combinatorial condition, cyclical consistency or CC, and equivalently Varian's generalized axiom of revealed preference or GARP, are necessary and sufficient conditions for rationalizing a finite data set, consisting of factor demands and factor prices, with a concave, monotone and continuous production function. Hence they constitute necessary and sufficient conditions for nonmarketed goods to be produced at minimum cost for some production function. If these conditions hold, then the supply curve for the nonmarketed good is the marginal cost curve of the associated cost function and producer surplus is well defined.

0.2 Rationalizing Individual Demand Data with Quasilinear Utilities

In this section we provide necessary and sufficient conditions for a data set to be the result of the consumer maximizing the widely used qualinear utility function subject to a budget constraint.

Definition 1 Let (p_r, x_r), $r = 1, ..., N$ be given. The data is *quasilinear rationalizable* if for some $y_r > 0$ and $I > 0$, $\forall r$ x_r solves

$$\max_{x \epsilon R^n_{++}} U(x) + y_r$$

[2]See Brown and Kannan for further discussion.

$$\text{s.t. } p_r x + y_r = I$$

for some concave U and numeraire good y_r.

Theorem 1 The following are equivalent:

1. *The data* (p_r, x_r), $r = 1, ..., N$ *is quasilinear rationalizable by a continuous, concave, strictly monotone utility function* U*.*

2. *The data* (p_r, x_r), $r = 1, ..., N$ *satisfies Afriat's inequalities with constant marginal utilities of income, that is, there exists* $G_r > 0$ *and* $\lambda > 0$ *for* $r = 1, \ldots, N$ *such that*

$$G_r \leq G_l + \lambda p_l \cdot (x_r - x_l) \;\; \forall r, l = 1, \ldots, N$$

or equivalently there exist $U_r > 0$ for $r = 1, \ldots, N$

$$U_r \leq U_l + p_l \cdot (x_r - x_l) \;\; \forall r, l = 1, \ldots, N$$

where $U_r = \frac{G_r}{\lambda}$

3. *The data* (p_r, x_r), $r = 1, ..., N$ *is "cyclically monotone", that is, if for any given subset of the data* $\{(p_s, x_s)\}_{s=1}^m$*:*

$$p_1 \cdot (x_2 - x_1) + p_2 \cdot (x_3 - x_2) + \cdots + p_m \cdot (x_1 - x_m) \geq 0$$

Proof:

(1) $\Rightarrow$ (2): First note that if U is concave on R^n, then $\beta \in R^n$ is a subgradient of U at x if for all $y \in R^n : U(y) \leq U(x) + \beta \cdot (y - x)$. Let $\partial U(x)$ be the set of subgradients of U at x. From the FOC of the quasilinear utility maximization problem we know:

$$\exists \beta_r \,\epsilon\, \partial U(x), \text{ s.t. } \beta_r = \lambda_r p_r \text{ where } \lambda_r = 1$$

Also, U being concave implies that $U(x_r) \leq U(x_l) + \beta_l(x_r - x_l)$ for $r, l = 1, 2, ..., N$. Since $\beta_l = p_l \; \forall l = 1, \ldots, N$ we get $U(x_r) \leq U(x_l) + p_l(x_r - x_l) \; \forall r, l = 1, ..., N$

(2) $\Rightarrow$ (3): For any set of pairs $\{(x_s, p_s)\}_{s=1}^m$ we need that: $p_0 \cdot (x_1 - x_0) + p_1(x_2 - x_1) + ... + p_m(x_0 - x_m) \geq 0$.

From the Afriat inequalities with constant marginal utilities of income we know:

$$U_1 - U_0 \leq p_0 \cdot (x_1 - x_0)$$

$$\cdots$$

$$U_0 - U_m \leq p_m(x_0 - x_m)$$

Adding up these inequalities we see that the left hand sides cancel and the resulting condition defines cyclical monotonicity.

(3) $\Rightarrow$ (1): Let $U(x) = \inf\{p_m \cdot (x - x_m) + \ldots + p_1 \cdot (x_2 - x_1)\}$ where the infimum is taken over all finite subsets of data, then $U(x)$ is a concave function on R^n and p_r is the subgradient of U at $x = x_r$ (this construction is due to Rockafellar (1970) in his proof of Theorem 24.8). Hence if $\lambda_r = 1$ for $r = 1, \ldots, N$ then $p_r = \partial U(x_r)$ constitutes a solution to the first order conditions of the quasilinear maximization problem.

If we require strict inequalities in (2) of Theorem 1, then it follows from Lemma 2 in Chiappori and Rochet (1987) that the rationalization can be chosen to be a C^∞ function. It then follows from Roy's identity that $x(p) = -\frac{\partial V(p)}{\partial p}$. Hence for any line integral we see that $\int_{p_1}^{p_2} x(p)dp = -\int_{p_1}^{p_2} \frac{\partial V(p)}{\partial p} dp = V(p_1) - V(p_2)$. That is, consumer welfare is well-defined and the change in consumer surplus induced by a change in market prices is the change in consumer's welfare.

0.3 Rationalizing Aggregate Demand Data with Quasi-linear and Homothetic Utilities

Consumer surplus is the area under the aggregate demand function. This function is estimated from a finite set of observations of market prices and aggreagate demand. Assuming we have the income distribution for each observation, we can derive the Afriat inequalities for each household, where individual consumptions are unknown, but are required to sum up to the observed aggregate demand. Setting marginal utilities of income equal to one, we have a system of linear inequalities in utility levels and individual demands, which can be solved in polynomial time. For the homothetic case, the Afriat inequalities reduce to $U_i \leq U_j p_j \cdot (x_i - x_j) \ \ \forall i, j$. Changing variables where $U_i = e^{z_i}$ we derive a family of smooth convex inequalities of the form $e^{z_i - z_j} - p_j \cdot (x_i - x_j) \leq 0$, which can also be solved in polynomial time using interior point methods.

0.4 Rationalizing the Production of Nonmarketed Goods

Health, education and environmental amenities are all examples of nonmarketed goods. To compute producer surplus for such goods, we must derive the supply curve, given only factor demand and prices, since demand data and prices for these goods are not observed. An important policy issue is whether these goods are produced efficiently given the factor demand and prices. In fact, as we show, there may be no concave, monotone and continuous production function that rationalizes the input data. If one does exist, we can rationalize the data and derive the supply curve for the nonmarketed good.[3]

Definition 3 Let (p_r, x_r), $r = 1, \ldots, N$ be given. A production, F, rationalizes the data if for all $r = 1, \ldots, N$ there exists q_r such that x_r solves:

$$\max_{x \in R^n_{++}} q_r F(x) - p_r \cdot x$$

where F is a concave function.

The rationalization is contained in Theorem 2, where the output price and quantity, represented by q_r and $F(x)$ are unknown.

Theorem 2 The following conditions are equivalent:

1. *There exists a concave, monotone, continuous, non-satiated production function that rationalizes the data.*

2. *The data (p_r, x_r), $r = 1, ..., N$ satisfies Afriat inequalities, that is, there exists $F_r > 0$ and $q_r > 0$ for $r = 1, \ldots, N$ such that*

$$F_r \leq F_l + \frac{1}{q_l} p_l \cdot (x_r - x_l) \;\; \forall r, l = 1, \ldots, N$$

   where q_l is the marginal cost of producing F_l.

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[3] If we write the cost minimization problem of the firm, $\min_{x \in R^n} p \cdot x$ s.t. $F(x) \geq y$, from the F.O.C. we find $p = \mu F'(x)$, where μ is the Lagrange multiplier associated with the constraint, and therefore is equal to the marginal cost of producing one more unit of output at the optimum. Therefore it is easy to see from the FOC of the profit maximization problem that the output price, q_r, is the marginal cost of production. The inequalities in (2) in Theorem 2 are the same as those in condition (3) of Theorem 2 in Varian (1984), where he assumes that the production levels F_r are observable.

$F(x) = min_{1 \leq l \leq r}\{F_l + \frac{1}{q_l} p_l (x - x_l)\}$ is Afriat's utility (production) function derived from a solution to the Afriat inequalities. The associated expenditure (cost) function is $c(y; p) = \min_{x \in R^n} p \cdot x$ s.t. $F(x) \geq y$. In the production setting, the supply curve is the marginal cost curve.

3. *The data* (p_r, x_r), $r = 1, ..., N$ *satisfies "cyclical consistency", that is,*

$$p_r x_r \geq p_r x_s,\ p_s x_s \geq p_s x_t, \cdots, p_q x_q \geq p_q x_r$$

implies

$$p_r x_r = p_r x_s,\ p_s x_s = p_s x_t, \cdots, p_q x_q = p_q x_r$$

Proof: This is Afriat's (1967) result where we let $F = U$ and $\lambda_r = \frac{1}{q_r}$.

References

[1] AFRIAT, S.N.: The Construction of a Utility Function from Expenditure Data, International Economic Review **8**, 67-77 (1967)

[2] ——— : Efficiency Estimates of Production Functions, International Economic Review **13**, 568-598 (1972a)

[3] ——— : The Theory of International Comparison of Real Income and Prices, in Daly, J.D. (ed) International Comparisons of Prices and Output, New York, National Bureau of Economic Research 1972b

[4] ——— : On the Constructability of Consistent Price Indices Between Several Periods Simultaneously, in Deaton (ed.), Essays in Applied Demand Analysis, Cambridge, Cambridge University Press 1981

[5] Brown, D.J., Kannan, O.: Decision Methods for Solving Systems of Walrasian Inequalities, Cowles Foundation Discussion Paper 1508 (2005)

[6] Carvajal, A., Ray, I., Snyder, S.: Equilibrium Behavior in markets and games: testable restrictions and identification, Journal of Mathematical Economics **40**, 1-40 (2004)

[7] CHIAPPORI, P.A., ROCHET, J.C.: Revealed Preferences and Differentiable Demand, Econometrica **55**, 687-691 (1987)

[8] ROCHET, J.C.: A necessary and sufficient condition for rationalizability in a quasilinear context, Journal of Mathematical Economics **16**, 191-200 (1987)

[9] ROCKAFELLAR, R.T.: Convex Analysis, Princeton, NJ: Princeton University Press 1970

[10] SILBERBERG, E.: The Structure of Economics, McGraw Hill, New York 1990

[11] VARIAN, H.: The Nonparamteric Approach to Demand Analysis, Econometrica **50**, 945-973 (1982)

[12] ——— : Non-parametric Tests of Consumer Behavior, Review of Economic Studies **50**, 99-210 (1983)

[13] ——— : The Nonparametric Approach to Production Analysis, Econometrica **52** 579-597 (1984)