On the Subdifferentials of Quasiconvex and Pseudoconvex Functions and Cyclic Monotonicity[1]

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ABSTRACT. The notions of cyclic quasimonotonicity and cyclic pseudomonotonicity are introduced. A classical result of convex analysis concerning the cyclic monotonicity of the (Fenchel-Moreau) subdifferential of a convex function is extended to corresponding results for the Clarke-Rockafellar subdifferential of quasiconvex and pseudoconvex functions.
The notion of proper quasimonotonicity is also introduced. It is shown that this new notion retains the characteristic property of quasimonotonicity (i.e. a lower semicontinuous function is quasiconvex if and only if its Clarke-Rockafellar subdifferential is properly quasimonotone), while it is also related to the KKM property of multivalued maps; this makes it more useful in applications to variational inequalities.

1. Introduction

Let X be a Banach space and $f : X \to R\bigcup\{+\infty\}$ a lower semicontinuous (lsc) function. According to a relatively recent result of Correa, Joffre and Thibault (see [7] for reflexive and [8] for arbitrary Banach spaces), the function f is convex if and only if its Clarke-Rockafellar subdifferential ∂f is monotone. In the same line of research, much work has been done to characterize the generalized convexity of lsc functions by a corresponding generalized monotonicity of the subdifferential. Thus Luc [15] and, independently, Aussel, Corvellec and Lassonde [2], showed that f is quasiconvex if and only if ∂f is quasimonotone. Similarly, Penot and Quang [16] showed that if the function f is also radially continuous, then f is pseudoconvex if and only if ∂f is pseudomonotone (in the sense of Karamardian and Schaible [14], as generalized for multivalued operators by Yao [20]). In section

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2, we review these results, together with some notation and definitions, and show that in most cases the radial continuity assumption is not necessary.

However, since the Clarke-Rockafellar subdifferential of a convex function coincides with the classical Fenchel-Moreau subdifferential [19], it is not only monotone, but also cyclically monotone [17]. In section 3 of this work, we define analogous notions of cyclic quasimonotonicity and cyclic pseudomonotonicity and show that the subdifferential of quasimonotone and pseudomonotone functions have these properties respectively. Cyclic generalized monotonicity is not just a stronger property than the corresponding generalized monotonicity, but it expresses a behavior of a specific kind; In particular, an operator can even be strongly monotone without being cyclically quasimonotone.

Cyclic (generalized) monotonicity describes the behavior of an operator around a "cycle" consisting of a finite number of points. In section 4 we consider instead the convex hull of such a cycle. We show that the definitions of monotone and pseudomonotone operators can be equivalently stated in terms of this convex hull. This is not so for quasimonotone operators; this leads to the introduction of the new notion of a properly quasimonotone operator. We show that this new notion, while retaining the important characteristics of quasimonotonicity (in particular, f is quasiconvex if and only if ∂f is properly quasimonotone) is often easier to handle; in particular, it is closely related to the KKM property of multivalued maps. We show this by an application to Variational Inequalities. In addition, quasimonotonicity and proper quasimonotonicity are identical on one dimensional spaces, which is probably the reason why the latter escaped attention.

2. Relations between generalized convexity and generalized monotonicity

We denote by X^* the dual of X and by (x^*, x) the value of $x^* \in X^*$ at $x \in X$. For $x, y \in X$ we set $[x, y] = \{tx + (1-t)y : 0 \leq t \leq 1\}$ and define $(x, y]$, $[x, y)$ and (x, y) analogously. Given a lsc function $f : X \to R\bigcup\{+\infty\}$ with domain $dom(f) := \{x \in X : f(x) < +\infty\} \neq \emptyset$, the Clarke-Rockafellar generalized derivative of f at $x_0 \in dom(f)$ in the direction $d \in X$ is given by (see [19]):

$$f^{\uparrow}(x_0, d) = \sup_{\varepsilon > 0} \limsup_{\substack{x \to_f x_0 \\ t \searrow 0}} \inf_{d' \in B_\varepsilon(d)} \frac{f(x + td') - f(x)}{t} \tag{2.1}$$

where $B_\varepsilon(d) = \{d' \in X : \|d' - d\| < \varepsilon\}$, $t \searrow 0$ indicates the fact that $t > 0$ and $t \to 0$, and $x \to_f x_o$ means that both $x \to x_o$ and $f(x) \to f(x_o)$.

The Clarke-Rockafellar subdifferential of f at x_0 is defined by

$$\partial f(x_0) = \{x^* \in X : (x^*, d) \le f^\uparrow(x_0, d), \forall d \in X\}. \tag{2.2}$$

We recall that a function f is called quasiconvex, if for any $x, y \in X$ and $z \in [x, y]$ we have

$$f(z) \le \max\{f(x), f(y)\} \tag{2.3}$$

A lsc function f is called pseudoconvex [16], if for every $x, y \in X$, the following implication holds:

$$\exists x^* \in \partial f(x) : (x^*, y - x) \ge 0 \Longrightarrow f(x) \le f(y) \tag{2.4}$$

It is known [16] that a lsc pseudoconvex function which is also radially continuous (i.e. its restriction to line segments is continuous), is quasiconvex. Both quasiconvexity and pseudoconvexity of functions are often used in Optimization and other areas of applied mathematics when a convexity assumption would be too restrictive [5].

Let $T : X \to 2^{X^*}$ be a multivalued operator with domain $D(T) = \{x \in X : T(x) \neq \emptyset\}$. The operator T is called

(i) cyclically monotone, if for every $x_1, x_2, ..., x_n \in X$ and every $x_1^* \in T(x_1), x_2^* \in T(x_2), ..., x_n^* \in T(x_n)$ we have

$$\sum_{i=1}^{n} (x_i^*, x_{i+1} - x_i) \le 0 \tag{2.5}$$

(where $x_{n+1} := x_1$).

(ii) monotone, if for any $x, y \in X$, $x^* \in T(x)$ and $y^* \in T(y)$ we have

$$(y^* - x^*, y - x) \ge 0 \tag{2.6}$$

(iii) pseudomonotone, if for any $x, y \in X$, $x^* \in T(x)$ and $y^* \in T(y)$ the following implication holds:

$$(x^*, y - x) \ge 0 \Longrightarrow (y^*, y - x) \ge 0 \tag{2.7}$$

or equivalently,

$$(x^*, y - x) > 0 \Longrightarrow (y^*, y - x) > 0 \tag{2.8}$$

(iv) quasimonotone, if for any $x, y \in X$, $x^* \in T(x)$ and $y^* \in T(y)$ the following implication holds:

$$(x^*, y - x) > 0 \Longrightarrow (y^*, y - x) \geq 0 \tag{2.9}$$

The above properties were listed from the strongest to the weakest. We recall the hitherto known results connecting generalized convexity with generalized monotonicity:

Theorem 2.1. *Let $f : X \to R\bigcup\{+\infty\}$ be a lower semicontinuous function. Then*
(i) f is convex if and only if ∂f is monotone [8]. In this case ∂f is also cyclically monotone (see for instance [17]).
(ii) f is quasiconvex if and only if ∂f is quasimonotone (see [2] or [15]).
(iii) Let f be also radially continuous. Then f is pseudoconvex if and only if ∂f is pseudomonotone (see [4] or [16]).

We now show that pseudoconvexity of a function f implies quasiconvexity of f and pseudomonotonicity of ∂f, even without the radial continuity assumption:

Proposition 2.2. *Let $f : X \to R\bigcup\{\infty\}$ be a lsc, pseudoconvex function with convex domain. Then:*
(i) f is quasiconvex
(ii) ∂f is pseudomonotone.

Proof: (i) Suppose that for some $x_1, x_2 \in dom(f)$ and some $y \in (x_1, x_2)$ we have $f(y) > \max\{f(x_1), f(x_2)\}$. Set $m = \max\{f(x_1), f(x_2)\}$. Since f is lower semicontinuous, there exists some $\varepsilon > 0$ such that $f(y') > m$, for all $y' \in B_\varepsilon(y)$. From (2.4) it follows (see also [4]) that the sets of local and global minimizers of the function f coincide; hence the point y cannot be a local minimizer, so there exists $w \in B_\varepsilon(y)$ such that $f(w) < f(y)$. Applying Zagrodny's Mean Value Theorem [21, Theorem 4.3] to the segment $[w, y]$, we obtain $u \in [w, y)$, a sequence $u_n \to u$ and $u_n^* \in \partial f(u_n)$, such that $(u_n^*, y - u_n) > 0$. Since $y \in co\{x_1, x_2\}$ it follows that $(u_n^*, x_i - u_n) > 0$, for some $i \in \{1, 2\}$. Using relation (2.4) we get

$m \geq f(x_i) \geq f(u_n)$ and, since f is lower semicontinuous, $m \geq f(u)$. This clearly contradicts the fact that $u \in B_\varepsilon(y)$.

(ii) Let $x^* \in \partial f(x)$ be such that $(x^*, y-x) > 0$. By part (i), f is quasiconvex, so applying Theorem 2.1(ii) we conclude that ∂f is quasimonotone. Hence $(y^*, y-x) \geq 0$, for all $y^* \in \partial f(y)$. Suppose to the contrary that for some $y^* \in \partial f(y)$ we have $(y^*, y-x) = 0$. From relation (2.4) we obtain $f(x) \geq f(y)$.

On the other hand, since $f^{\uparrow}(x; y-x) > 0$, there exists $\varepsilon_1 > 0$, such that for some $x_n \to x$, $t_n \searrow 0$ and for all $y' \in B_{\varepsilon_1}(y)$, we have $f(x_n + t_n(y' - x_n)) > f(x_n)$. Quasiconvexity of f implies $f(y') > f(x_n)$, for every $y' \in B_{\varepsilon_1}(y)$. In particular $f(y') \geq f(x)$ (since f is lsc), hence $f(y') \geq f(y)$. The latter shows that y is a local minimizer, hence a global one. This is a contradiction, since we have at least $f(y) > f(x_n)$.■

It is still an open question whether pseudomonotonicity of ∂f implies pseudoconvexity of f, without the radial continuity assumption. As a partial result, we have the following proposition, which will be of use in the next section.

Proposition 2.3. *Let f be a lsc function such that ∂f is pseudomonotone. Then f has the following properties:*

(i) If $0 \in \partial f(x)$, then x is a global minimizer

(ii) $\exists x^* \in \partial f(x) : (x^*, y-x) > 0 \Longrightarrow f(y) > f(x)$

Proof: (i) Suppose that $f(y) < f(x)$. Then using again Zagrodny's Mean Value Theorem, we can find a sequence $z_n \to z \in [y, x)$ and $z_n^* \in \partial f(z_n)$, such that $(z_n^*, x - z_n) > 0$. By pseudomonotonicity, $(x^*, x - z_n) > 0$ for all $x^* \in \partial f(x)$, i.e. $0 \notin \partial f(x)$.

(ii) Let us assume that for some $x^* \in \partial f(x)$ we have $(x^*, y-x) > 0$. We may choose $\varepsilon > 0$ such that $(x^*, y' - x) > 0$, for all $y' \in B_\varepsilon(y)$. Since ∂f is obviously quasimonotone, from Theorem 2.1(ii) we conclude that f is quasiconvex; it then follows that $f(y) \geq f(x)$ (see for instance Theorem 2.1 in [4]). Suppose to the contrary that $f(x) = f(y)$. Then $f(y') \geq f(x) = f(y)$, so f has a local minimum at y. It follows that $0 \in \partial f(y)$ (see for instance [21, Th.2.2(c)]). However ∂f is pseudomonotone, hence we should have (see relation (2.8)) $(y^*, y-x) > 0$, for all $y^* \in \partial f(y)$, a contradiction.■

3. Generalized cyclic monotonicity

We first introduce cyclic quasimonotonicity.

Definition 3.1. *An operator $T : X \to 2^{X^*}$ is called cyclically quasimonotone, if for every $x_1, x_2, ..., x_n \in X$, there exists an $i \in \{1, 2, ..., n\}$ such that*

$$(x_i^*, x_{i+1} - x_i) \le 0, \forall x_i^* \in T(x_i) \tag{3.1}$$

(where $x_{n+1} := x_1$).

It is easy to see that a cyclically monotone operator is cyclically quasimonotone, while a cyclically quasimonotone operator is quasimonotone. Cyclic quasimonotonicity is considerably more restrictive than quasimonotonicity (see Example 3.5 below). However, this property characterizes subdifferentials of quasiconvex functions, as shown by the next theorem.

Theorem 3.2. *Let $f : X \to R \bigcup \{+\infty\}$ be a lower semicontinuous function. Then f is quasiconvex if and only if ∂f is cyclically quasimonotone.*

Proof: In view of Theorem 2.1(ii), we have only to prove that if f is quasiconvex then ∂f is cyclically quasimonotone.

Assume to the contrary that there exist $x_1, x_2, ..., x_n \in D(\partial f)$ and $x_i^* \in \partial f(x_i)$ such that $(x_i^*, x_{i+1} - x_i) > 0$, for $i = 1, 2, ..., n$ (where as usual $x_{n+1} = x_1$). It follows that $f^{\uparrow}(x_i, x_{i+1} - x_i) > 0$. In particular, for every i there exists $\varepsilon_i > 0$, $\delta_i > 0$ such that

$$\limsup_{\substack{x_i' \to_f x_i \\ t \searrow 0}} \inf_{d \in B_{\varepsilon_i}(x_{i+1} - x_i)} \frac{f(x_i' + td) - f(x_i')}{t} > \delta_i > 0. \tag{3.2}$$

We set $\varepsilon = \min\limits_{i=1,2..n} \varepsilon_i$ and $\delta = \min\limits_{i=1,2..n} \delta_i$. For any $y \in B_{\frac{\varepsilon}{2}}(x_i)$ and $x_{i+1}' \in B_{\frac{\varepsilon}{2}}(x_{i+1})$ we have $y - x_{i+1}' \in B_\varepsilon(x_{i+1} - x_i)$; hence we can choose $\bar{x}_i \in B_{\frac{\varepsilon}{2}}(x_i)$ and $t_i \in (0, 1)$ such that

$$\inf_{x_{i+1}' \in B_{\frac{\varepsilon}{2}}(x_{i+1})} \frac{f(\bar{x}_i + t_i(x_{i+1}' - \bar{x}_i)) - f(\bar{x}_i)}{t_i} > \delta > 0 \tag{3.3}$$

or equivalently

$$f(\bar{x}_i + t_i(x_{i+1}' - \bar{x}_i)) > f(\bar{x}_i) + t_i\delta, \ \forall x_{i+1}' \in B_{\frac{\varepsilon}{2}}(x_{i+1}) \tag{3.4}$$

for $i = 1, 2, .., n$.

Now for every i we choose $x'_{i+1} = \bar{x}_{i+1}$, hence (3.4) becomes

$$f(\bar{x}_i + t_i(\bar{x}_{i+1} - \bar{x_i})) > f(\bar{x_i}) + t_i\delta \tag{3.5}$$

for $i = 1, 2, ..., n$.

Since f is quasiconvex, (3.5) implies that

$$f(\bar{x}_{i+1}) \geq f(\bar{x}_i + t_i(\bar{x}_{i+1} - \bar{x_i})) \tag{3.6}$$

for $i = 1, 2, ..., n$. Combining with (3.5) and adding for $i = 1, 2, ..., n$, we get $0 > \delta(\sum_{i=1}^{n} t_i)$, a contradiction.■

In [18] it was proved that the subdifferential of a convex function is a maximal monotone and maximal cyclically monotone operator. An analogous property does not hold for quasiconvex functions, since for the quasiconvex function $f(x) = sgn(x)\sqrt{\mid x \mid}$, $x \in R$, it is known (see [15]) that ∂f is not maximal quasimonotone. The following proposition shows that it is neither maximal cyclically quasimonotone:

Proposition 3.3. *Every quasimonotone operator $T : R \to 2^R$ is cyclically quasimonotone.*

Proof: We assume to the contrary that the operator T is quasimonotone and there exist $x_1, x_2, ..., x_n \in R$, $x_i^* \in T(x_i)$, such that

$$(x_i^*, x_{i+1} - x_i) > 0 \tag{3.7}$$

for $i = 1, 2, ..., n$ (where $x_{n+1} = x_1$). Set $x_M = \max\limits_{i=1,2,...,n} x_i$. Then relation (3.7) implies that $x_M^* < 0$. On the other hand, since $x_{M-1} < x_M$, we conclude from (3.7) that $x_{M-1}^* > 0$. Thus $(x_{M-1}^*, x_M - x_{M-1}) > 0$, while $(x_M^*, x_M - x_{M-1}) < 0$, which contradicts the definition of quasimonotonicity.■

We now introduce cyclic pseudomonotonicity:

Definition 3.4. *An operator $T : X \to 2^{X^*}$ is called cyclically pseudomonotone, if for every $x_1, x_2, ..., x_n \in X$, the following implication holds:*

$$\begin{gathered} \exists i \in \{1, 2, ..., n\}, \exists x_i^* \in T(x_i) : (x_i^*, x_{i+1} - x_i) > 0 \Longrightarrow \\ \exists j \in \{1, 2, ..., n\}, \forall x_j^* \in T(x_j) : (x_j^*, x_{j+1} - x_j) < 0 \end{gathered} \tag{3.8}$$

(where $x_{n+1} := x_1$).

One can easily check that every cyclically monotone operator is cyclically pseudomonotone, while every cyclically pseudomonotone operator is pseudomonotone and cyclically quasimonotone. On the other hand, the following example shows that cyclic generalized monotonicity differs essentially from generalized monotonicity:

Example 3.5. *Let $T : R^2 \to R^2$ be defined by $T(a,b) = (\frac{a}{2} - b, a + \frac{b}{2})$. Then the operator T is monotone (and even strongly monotone, i.e. satisfies $(T(x) - T(y), x-y) \geq k\|x-y\|^2$ for all $x, y \in R^2$ where k is a constant). In particular, T is pseudomonotone and quasimonotone. However, it is not cyclically quasimonotone, as one sees by considering the points $x_1 = (1,0), x_2 = (0,1), x_3 = (-1,0)$ and $x_4 = (0,-1)$.*

We now show the following strengthening of Theorem 2.1(iii).

Theorem 3.6. *Let $f : X \to R\bigcup\{+\infty\}$ be a lsc function. If f is pseudoconvex, then ∂f is cyclically pseudomonotone. Conversely, if ∂f is pseudomonotone and f is radially continuous, then f is pseudoconvex.*

Proof: Again we have only to show that if f is pseudoconvex then ∂f is cyclically pseudomonotone. Assume to the contrary that there exist $x_1, x_2, ..., x_n \in D(\partial f)$ and $x_i^* \in \partial f(x_i)$ such that $(x_i^*, x_{i+1} - x_i) \geq 0$, for $i = 1, 2, ..., n$ (where $x_{n+1} = x_1$), while for some i_o and some $x_{i_o}^* \in \partial f(x_{i_o})$ we have

$$(x_{i_o}^*, x_{i_o+1} - x_{i_o}) > 0 \tag{3.9}$$

By the definition of pseudoconvexity (relation (2.4)) we have $f(x_{i+1}) \geq f(x_i)$, for $i = 1, 2, ..., n$, hence all $f(x_i)$ are equal. In particular, $f(x_{i_o+1}) = f(x_{i_o})$, which contradicts (3.9) in view of Proposition 2.3.■

4. Proper Quasimonotonicity

The definitions of monotonicity and pseudomonotonicity have an equivalent formulation, which involves a finite cycle of points and its convex hull:

Proposition 4.1. *(i) An operator T is monotone, if and only if for any $x_1, x_2, ..., x_n \in X$ and every $y = \sum_{i=1}^n \lambda_i x_i$, with $\sum_{i=1}^n \lambda_i = 1$ and $\lambda_i > 0$, one has*

$$\sum_{i=1}^n \lambda_i \sup_{x_i^* \in T(x_i)} (x_i^*, y - x_i) \leq 0. \tag{4.1}$$

(ii) An operator T with convex domain $D(T)$ is pseudomonotone, if and only if for any $x_1, x_2, ..., x_n \in X$ and every $y = \sum_{i=1}^n \lambda_i x_i$, with $\sum_{i=1}^n \lambda_i = 1$ and $\lambda_i > 0$, the following implication holds:

$$\begin{aligned} &\exists i \in \{1, 2, ...n\}, \exists x_i^* \in T(x_i) : (x_i^*, y - x_i) > 0 \Longrightarrow \\ &\exists j \in \{1, 2, ...n\}, \forall x_j^* \in T(x_j) : (x_j^*, y - x_j) < 0. \end{aligned} \tag{4.2}$$

Proof: If the operator T satisfies condition (4.1) (respectively (4.2)), then by choosing $y = \frac{x_1+x_2}{2}$, we conclude that it is monotone (respectively pseudomonotone). Hence it remains to show the two opposite directions.

Let us first suppose that T is monotone. Then for any $x_1, x_2, ..., x_n \in X$, any $x_i^* \in T(x_i)$ (for $i = 1, 2, ...n$) and any $y = \sum_{j=1}^n \lambda_j x_j$, with $\sum_{j=1}^n \lambda_j = 1$ and $\lambda_j > 0$, we have:

$$\sum_{i=1}^n \lambda_i (x_i^*, y - x_i) = \sum_{i=1}^n \sum_{j=1}^n \lambda_i \lambda_j (x_i^*, x_j - x_i) =$$

$$\sum_{i>j} \lambda_i \lambda_j [(x_i^*, x_j - x_i) + (x_j^*, x_i - x_j)] = \sum_{i>j} \lambda_i \lambda_j (x_i^* - x_j^*, x_j - x_i) \leq 0$$

where the last inequality is a consequence of the monotonicity of T. Hence T satisfies relation (4.1).

We now suppose that the operator T is pseudomonotone. If relation (4.2) does not hold, then there exist $x_1, x_2, ..., x_n \in X$, $x_i^* \in T(x_i)$ for $i = 1, 2, ...n$, and some $y = \sum_{j=1}^n \lambda_j x_j$ with $\sum_{j=1}^n \lambda_j = 1$ and $\lambda_j > 0$, such that

$$(x_i^*, y - x_i) \geq 0 \tag{4.3}$$

while for at least one i (say $i = 1$),

$$(x_1^*, y - x_1) > 0. \tag{4.4}$$

In particular we have $x_1, x_2, ..., x_n \in D(T)$, hence $T(y) \neq \emptyset$. Choose any $y^* \in T(y)$. Relations (2.7) and (4.3) show that

$$(y^*, y - x_i) \geq 0 \tag{4.5}$$

for all $y^* \in T(y)$ and all i's. Since $\sum_i \lambda_i (y^*, y - x_i) = 0$, relations (4.5) show that $(y^*, y - x_i) = 0$ for all i's. On the other hand, relation (4.4) together with (2.8) imply that $(y^*, y - x_1) > 0$, a contradiction. ■

In view of the above Proposition, one could seek an equivalent formulation for the definition of quasimonotonicity, which would involve again the convex hull of a finite cycle. However, in contrast to monotone and pseudomonotone operators, this leads to a different, more restrictive definition:

Definition 4.2. *An operator $T : X \to 2^{X^*}$ is called properly quasimonotone, if for every $x_1, x_2, ..., x_n \in X$ and every $y = \sum_{i=1}^n \lambda_i x_i$, with $\sum_{i=1}^n \lambda_i = 1$ and $\lambda_i > 0$, there exists i such that*

$$\forall x_i^* \in T(x_i) : (x_i^*, y - x_i) \leq 0. \tag{4.6}$$

Choosing $y = \frac{x_1+x_2}{2}$, we see that a properly quasimonotone operator is quasimonotone. As in Proposition 3.3, it is easy to show that the converse is true whenever $X = R$; however, it is not true in general, as the following example shows.

Example 4.3. *Let $X = R^2$, $x_1 = (0,1)$, $x_2 = (0,0)$, $x_3 = (1,0)$. We define $T : R^2 \to R^2$ by $T(x_1) = (-1,-1)$, $T(x_2) = (1,0)$, $T(x_3) = (0,1)$ and $T(x) = 0$ otherwise. It is easy to check that T is quasimonotone but not properly quasimonotone (it suffices to consider $y = \frac{x_1+x_2+x_3}{3}$).*

The class of properly quasimonotone operators, though strictly smaller than the class of quasimonotone operators, is in a sense not much smaller. This is shown in the next proposition.

Proposition 4.4. *(i) Every pseudomonotone operator with convex domain is properly quasimonotone.*
(ii) Every cyclically quasimonotone operator is properly quasimonotone

Proof: (i) This is an obvious consequence of Proposition 4.1(ii).
(ii) Suppose that the operator T is not properly quasimonotone. Then there would exist $x_1, x_2, ..., x_n \in D(T)$, $x_i^* \in T(x_i)$ and $y = \sum_{i=1}^n \lambda_i x_i$ with $\lambda_i > 0$, such that

$$(x_i^*, y - x_i) > 0 \tag{4.7}$$

for $i = 1, 2, ..., n$. Set $x_{i(1)} = x_1$. Relation (4.7) implies that $\sum_j \lambda_j (x_{i(1)}^*, x_j - x_{i(1)}) > 0$. It follows that for some $x_j \neq x_1$ we have $(x_{i(1)}^*, x_j - x_{i(1)}) > 0$. We set $x_{i(2)} = x_j$ and apply relation (4.7) again. Continuing in this way, we define a sequence $x_{i(1)}, x_{i(2)}, ...$ such that

$$(x^*_{i(k)}, x_{i(k+1)} - x_{i(k)}) > 0 \tag{4.8}$$

for all $k \in N$.

Since the set $\{x_1, x_2, ..., x_n\}$ is finite, there exist $m, k \in N$, $m < k$ such that $x_{i(k+1)} = x_{i(m)}$. Thus, for the finite sequence of points $x_{i(m)}, x_{i(m+1)}, ..., x_{i(k)}$ relation (4.8) holds. This means that T is not cyclically quasimonotone. ■

Combining Proposition 4.4(ii) and Theorem 3.2, we get the following corollary.

Corollary 4.5. *A lower semicontinuous function f is quasiconvex if and only if ∂f is properly quasimonotone.*

The converse of Proposition 4.4 does not hold. For instance, the operator T defined in Example 3.5 is properly quasimonotone (since it is monotone, hence pseudomonotone), but not cyclically quasimonotone. On the other hand, any subdifferential of a continuous quasiconvex function f is properly quasimonotone, but not pseudomonotone unless f is also pseudoconvex. Thus, between the various generalized monotonicity properties we considered, the following strict implications hold, and none other:

$$\begin{array}{ccc}
\text{cyclically monotone} & \longrightarrow & \text{monotone} \\
\downarrow & & \downarrow \\
\text{cyclically pseudomonotone} & \longrightarrow & \text{pseudomonotone} \\
\downarrow & & \downarrow \\
\text{cyclically quasimonotone} & \longrightarrow & \text{properly quasimonotone} \\
 & & \downarrow \\
 & & \text{quasimonotone}
\end{array}$$

Note that the implication (pseudomonotone → properly quasimonotone) holds under the assumption that the domain of the operator is convex.

We recall that a multivalued mapping $G : X \to 2^{X^*}$ is called KKM [11], if for any $x_1, x_2, ..., x_n \in X$ and any $y \in co\{x_1, x_2, ..., x_n\}$ one has $y \in \bigcup_i G(x_i)$. It is easy to see that an operator $T : X \to 2^{X^*}$ is properly quasimonotone if and only if the multivalued mapping $G : X \to 2^{X^*}$ defined by

$$G(x) = \{y \in K : \sup_{x^* \in T(x)} (x^*, y - x) \leq 0\} \tag{4.9}$$

is KKM. This suggests an obvious application to Variational Inequalities. All known theorems of existence of solutions for quasimonotone Variational Inequality Problems require extra assumptions on the domain of the operator (see [12]) and, in case of a multivalued operator, on its values (see [9]). As the following theorem shows, existence of solutions for properly quasimonotone operators requires very weak assumptions. We first recall from [1] the following definition.

Definition 4.6. *The operator $T : X \to 2^{X^*}$ is called upper hemicontinuous, if its restriction to line segments of its domain is upper semicontinuous, when X^* is equipped to the weak-$*$ topology.*

We now have:

Theorem 4.7. *Let K be a nonempty, convex and w-compact subset of X. If T is a properly quasimonotone, upper hemicontinuous operator with $K \subseteq D(T)$, then there exists an $x_0 \in K$, such that for every $x \in K$, there exists $x^* \in T(x_0)$ such that:*

$$(x^*, x - x_0) \geq 0 \tag{4.10}$$

Proof: Since the multivalued map G defined by (4.9) is KKM, and the sets $G(x)$ are obviously weakly closed, by Ky Fan's Lemma [10] one has $\bigcap_{x \in K} G(x) \neq \emptyset$. Take any $x_0 \in \bigcap_{x \in K} G(x)$. We shall show that x_0 is actually a solution of (4.10).

We assume to the contrary, that for some $x \in K$ and all $x^* \in T(x_0)$ we have $(x^*, x - x_0) < 0$. The set $V = \{x^* \in X^* : (x^*, x - x_0) < 0\}$ is a w*- neighborhood of $T(x_0)$; hence, if we set $x_t = tx + (1-t)x_0$, by the upper hemicontinuity assumption, we have $T(x_t) \in V$ for all t sufficiently small. Since $x_t - x_0 = t(x - x_0)$, this means that $(x^*, x_t - x_0) < 0$ for all $x^* \in T(x_t)$, i.e. $x_0 \notin G(x_t)$. This contradicts the definition of x_0.■

We conclude with a final remark. The notion of a quasimonotone operator was introduced to describe a property that characterizes the subdifferential of a lsc quasiconvex function. Since proper quasimonotonicity does exactly the same thing and is directly related to the KKM property, it is possibly a good candidate to replace quasimonotonicity in most theoretical and practical applications.

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