CONNECTEDNESS OF THE EFFICIENT SET FOR
THREE-OBJECTIVE QUASICONCAVE MAXIMIZATION PROBLEMS

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ABSTRACT. For three-objective maximization problems involving continuous, semistrictly quasiconcave functions over a compact, convex set it is shown that the set of efficient solutions is connected. With that, an open problem stated by Choo, Schaible and Chew in 1985 is solved.

KEY WORDS. Multi-objective maximization, semistrictly quasiconcave functions, efficient solution set, connectedness.

1. Introduction

Consider the multi-objective maximization problem

$$\max_{x \in S} F(x) = (f_1(x), ..., f_n(x)) \tag{1}$$

where $S \subseteq R^k$ is nonempty, compact and convex and $f_i : S \to R$ are continuous. It is well known that the set of efficient (Pareto-optimal) solutions E is connected if all functions f_i are concave (Ref.1). This topological property of E is algorithmicly important.

Several authors have tried to relax the concavity assumption without giving up connectedness of the efficient set, usually working with particular kinds of quasiconcave functions: e.g. Refs.2-7.

If $n = 1$, then E coincides with the set of optimal solutions of (1) which is convex, hence connected, as long as f_1 is quasiconcave. However for $n = 2$, quasiconcavity of f_1 and f_2 does not guarantee connectedness of E in general (Ref.3). It tuns out that semistrict quasiconcavity (previously called strict quasiconcavity (Ref.8)) is sufficient to ensure connectedness of E (Ref.3).

The case $n = 3$ is treated in Ref.4. With proof techniques different from those in Ref.3. it is shown that the closure of the efficient frontier $F(E)$ is connected if f_1, f_2 and f_3 are semistrictly quasiconcave. However, connectedness of E could not be established under these assumptions nor a counterexample be given in Ref.4. It has remained an open problem in the literature until now (Ref.7). Only partial answers for particular kinds of generalized concave functions could be proved, using various approaches; e.g. Ref.2, Refs.5-7 and the references therein. These preliminary results seem to point towards a positive answer to the conjecture that E is connected if $f_1, ..., f_n$ are semistrictly quasiconcave, for any number of functions.

In this paper we will show that the conjecture is true for $n = 3$, i.e.

E is connected if f_1, f_2 and f_3 are semistrictly quasiconcave. Though the approach differs from the one in Ref.4, the paper borrows heavily from the work in Ref.4 in addition to that in Ref.3.

2. Notation and Relevant Results

Let $x = (x_1, x_2, ..., x_k)$ and $y = (y_1, y_2, ..., y_k)$ be two vectors in R^k. We denote by $[x, y]$ the line segment $\{tx + (1-t)y : t \in [0,1]\}$ and we define the open line segment (x, y) analogously. We write $x \leq y$, if $x_i \leq y_i$ for all $i = 1, 2, ..., k$. We also write $x \lneqq y$, if $x \leq y$ and $x \neq y$.

Let $S \subset R^k$ be convex. A function $f : S \longrightarrow R$ is called semistrictly quasiconcave, if $x, y \in S$ and $f(x) < f(y)$ imply $f(x) < f(z)$ for all $z \in (x, y)$. An upper semicontinuous, semistrictly quasiconcave function is quasiconcave (Ref.8).

Given a vector-valued function $F = (f_1, f_2, ..., f_n)$ on S, a point $\overline{x} \in S$ is called *efficient* (with respect to F), if $x \in S$ and $F(\overline{x}) \leq F(x)$ imply $F(\overline{x}) = F(x)$. The set of all efficient points is denoted by E. Its image $F(E)$ is called the *efficient frontier.* The connectedness of E is related to the connectedness of $F(E)$ through the following theorem(Ref.4, Theorem 6).

Theorem 2.1 *If S is compact and convex and $f_1, f_2, ..., f_n$ are continuous and quasiconcave, then E is connected if and only if $F(E)$ is connected.*

Throughout the paper we shall assume that S is nonempty, compact and convex and f_i are continuous and semistrictly quasiconcave. In the special case of $n = 2$, the efficient frontier is the graph of a continuous, strictly decreasing function. To see this, we define

$\overline{f}_i = max\{f_i(x) : x \in S\}$, $i = 1, 2$
$\underline{f}_1 = max\{f_1(x) : f_2(x) = \overline{f}_2\}$
$\underline{f}_2 = max\{f_2(x) : f_1(x) = \overline{f}_1\}$.

The following lemma is a combination of Lemmas 1 and 2 in Ref.3.

Lemma 2.1 *Let $n = 2$. If $x \in E$, then $\underline{f}_i \leq f_i(x) \leq \overline{f}_i$, $i = 1, 2$. Conversely, if for $i = 1$ or $i = 2$ we have $\underline{f}_i \leq t \leq \overline{f}_i$, then there exists $x \in E$ such that $f_i(x) = t$.*

We now have:

Theorem 2.2 *Let $n = 2$. Then there exists a continuous, strictly decreasing function g from $[\underline{f}_1, \overline{f}_1]$ onto $[\underline{f}_2, \overline{f}_2]$ such that*

$$F(E) = \{(t, g(t)) : t \in [\underline{f}_1, \overline{f}_1]\}. \tag{2}$$

Proof. By the previous lemma, for any $(t_1, t_2) \in F(E)$ we have $t_1 \in [\underline{f}_1, \overline{f}_1]$. Conversely, for any $t_1 \in [\underline{f}_1, \overline{f}_1]$, we may find $t_2 \in [\underline{f}_2, \overline{f}_2]$ such that $(t_1, t_2) \in F(E)$. This t_2 is uniquely determined by t_1; indeed if we had $(t_1, t_2') \in F(E)$ with, say $t_2 < t_2'$, then we would have $(t_1, t_2) \lneqq (t_1, t_2')$ which contradicts $(t_1, t_2) \in F(E)$. Setting $t_2 = g(t_1)$, we see that (2) holds. Using again the above argument, we infer that g is strictly decreasing. Applying again Lemma 2.1, this time to the second coordinate, we infer that g is a map onto $[\underline{f}_2, \overline{f}_2]$. Hence, g is continuous. ■

Theorem 2.2 shows that in the case of $n = 2$, $F(E)$ is pathwise connected and closed. Since E is the inverse image of $F(E)$, E is also closed, and by Theorem 2.1 it is also connected (see also Ref.3, Theorems 2 and 3).

In what follows, we shall restrict ourselves to the case $n = 3$. Besides the set E of the efficient points of S with respect to the vector-valued function $F = (f_1, f_2, f_3)$, we also define the vector-valued functions F^i by $F^1 = (f_2, f_3)$, $F^2 = (f_1, f_3)$ and $F^3 = (f_1, f_2)$ and the corresponding sets of efficient points E^i.

Furthermore, for any $t \in R$, we define the sets $S_t^i = \{x \in S : f_i(x) \geq t\}$, $i = 1, 2, 3$. Since f_i are continuous and semistrictly quasiconcave, S_t^i are closed and convex. We denote by E_t^i the set of the efficient points of S_t^i with respect to F^i. Obviously, E^i and E_t^i are closed and connected subsets of S.

The proof of the following theorem is contained in the proof of Lemma 3 in Ref. 4. We refer the reader to Ref.4 for the definition of lower semicontinuity of multifunctions.

Theorem 2.3 *The multifunctions $t \to F(E_t^i)$, $i = 1, 2, 3$ are lower semicontinuous.*

Finally, we need the following:

Lemma 2.2 *Suppose that $x \in E_t^i$ and $x \notin E^i$. Then $x \in E$ and $f_i(x) = t$.*

Proof. The fact that $x \in E$ is the content of Theorem 7 in Ref. 4.

To prove that $f_i(x) = t$, we may suppose that $i = 1$. Since $x \in E_t^1$, we have $f_1(x) \geq t$. Let us suppose that $f_1(x) > t$. Since $x \notin E^1$, there exists $y \in S$ such that $F^1(x) \lneqq F^1(y)$. By semistrict quasiconcavity (and quasiconcavity) of f_2 and f_3, we have: $F^1(x) \lneqq F^1(z)$, for any $z \in (x, y)$. Since $f_1(x) > t$, taking $z \in (x, y)$ close enough to x we would have $f_1(z) > t$, i.e. $z \in S_t^1$, which contradicts the fact that $x \in E_t^1$. ■

3. The Main Result

The sets E^i, $i = 1, 2, 3$ are closed subsets of S, hence compact. We define

$$t_i = max\{f_i(x) : x \in E^i\} \tag{3}$$

The outline of the proof is as follows. We first show that the sets $F(E \cap S_{t_i}^i)$, $i = 1, 2, 3$, are connected. Then we prove that any two of them have a nonempty intersection, hence their union is connected. Finally, we show that any other point in $F(E)$ can be joined to a point in these sets, by a continuous curve lying inside $F(E)$.

We begin with a useful lemma:

Lemma 3.1 *For all $i = 1, 2, 3$ we have $E \cap S_{t_i}^i = \bigcup_{t \geq t_i} E_t^i$.*

Proof. We may suppose that $i = 1$.

Let $x \in E \cap S^1_{t_1}$. Then $f_1(x) \geq t_1$. Obviously, since $x \in E$, x is an efficient point of $S^1_{f_1(x)}$ with respect to F^1. Hence $x \in \bigcup_{t \geq t_1} E^1_t$.

Conversely, let $x \in \bigcup_{t \geq t_1} E^1_t$, i.e. $x \in E^1_t$ for some $t \geq t_1$. If $x \notin E^1$, then $x \in E$, as it follows directly from Lemma 2.2. If on the other hand $x \in E^1$, then for any $y \in S$ with $F(x) \leq F(y)$, we have in particular $F^1(x) \leq F^1(y)$, hence $F^1(x) = F^1(y)$. It follows that $y \in E^1$, hence $f_1(y) \leq t_1 \leq f_1(x)$ which shows that $F(x) = F(y)$, i.e. $x \in E$. ■

The next lemma is an easy application of previous results:

Lemma 3.2 *For all $i = 1, 2, 3$, the set $F(E \cap S^i_{t_i})$ is connected.*

Proof. By the previous lemma, we have $F(E \cap S^i_{t_i}) = F(\bigcup_{t \geq t_i} E^i_t)$. Since the sets E^i_t are connected, so are their images $F(E^i_t)$ under the continuous function F. By Theorem 2.3, the multifunctions $t \to F(E^i_t)$ are lower semi-continuous, so applying Corollary 2.1 of Ref.4, we conclude that $\bigcup_{t \geq t_i} F(E^i_t) = F(\bigcup_{t \geq t_i} E^i_t) = F(E \cap S^i_{t_i})$ is connected for all $i = 1, 2, 3$. ■

We proceed by showing the following.

Lemma 3.3 *For $i \neq j$, we have $E \cap S^i_{t_i} \cap S^j_{t_j} \neq \varnothing$.*

Proof. We may suppose that $i = 1$ and $j = 2$.

By the definition of t_1, there exists $x \in E^1$ such that $f_1(x) = t_1$. Obviously, $x \in E \cap S^1_{t_1}$. Likewise, there exists $y \in E^2$ such that $f_2(y) = t_2$ and $y \in E \cap S^2_{t_2}$.

If $x \in S^2_{t_2}$, then $E \cap S^1_{t_1} \cap S^2_{t_2} \neq \varnothing$. Otherwise, $f_2(x) < t_2 = f_2(y)$. Since $x \in E^1$, we have $f_3(x) > f_3(y)$. This, together with the fact that $y \in E^2$, implies that $f_1(x) < f_1(y)$. Since $f_1(x) = t_1$, we conclude that $y \in S^1_{t_1}$, and so again $E \cap S^1_{t_1} \cap S^2_{t_2} \neq \varnothing$. ■

We finally state our last crucial lemma.

Lemma 3.4 *Let $x \in E$ be such that $f_i(x) < t_i$, $i = 1, 2, 3$. Then there exists* *a point $y \in E \cap S^3_{t_3}$ and a continuous curve with values in $F(E)$, joining $F(x)$ and $F(y)$.*

Proof. Let $F(x) = (a, b, c)$. We set further

$\widetilde{f_1} = max\{f_1(x') : x' \in S^3_{t_3}\}$

$\widetilde{f_2} = max\{f_2(x') : x' \in S^3_{t_3}\}$

$\underset{\sim}{f_1} = max\{f_1(x') : x' \in S^3_{t_3} \ \&\ f_2(x') = \widetilde{f_2}\}$

By Lemma 3.3, we have $S^3_{t_3} \cap S^1_{t_1} \neq \varnothing$. For any $x' \in S^3_{t_3} \cap S^1_{t_1}$, we obviously get $t_1 \leq f_1(x') \leq \widetilde{f_1}$. Since $a = f_1(x) < t_1$, we deduce $a < \widetilde{f_1}$. Similarly, from the fact that $S^3_{t_3} \cap S^2_{t_2} \neq \varnothing$, we deduce $b < \widetilde{f_2}$.

Let $z \in S^3_{t_3}$ be such that $f_2(z) = \widetilde{f_2}$ and $f_1(z) = \underset{\sim}{f_1}$. We then have $f_2(x) < f_2(z)$ and $f_3(x) < t_3 \leq f_3(z)$. Since $x \in E$, we get $f_1(x) > f_1(z) = \underset{\sim}{f_1}$. Hence,

$$\underset{\sim}{f_1} < a < \widetilde{f_1} .$$

Applying now Lemma 2.1 to the set $S^3_{t_3}$ and the function F^3, we infer that there exists $y \in E^3_{t_3}$, such that $f_1(y) = a$. So, we have $f_1(y) = f_1(x)$ and $f_3(y) \geq t_3 > f_3(x)$; recalling that $x \in E$, we get $f_2(y) < f_2(x)$. Consequently, $F^3(y) \lneqq F^3(x)$, so $y \notin E^3$. Then Lemma 2.2 implies that $y \in E$ and $f_3(y) = t_3$.

We now show that $F(x)$ and $F(y)$ can be joined by a continuous curve lying in $F(E)$. We note that x, $y \in E \cap S^1_a$, hence in particular x, $y \in E^1_a$. Applying Theorem 2.2 to the set S^1_a and the vector-valued function F^1, we infer that $F^1(x)$, $F^1(y)$ belong to the graph of a continuous strictly decreasing function. Since $f_2(y) < f_2(x)$, this means that there exists a continuous strictly decreasing function g, defined on $[b', b]$ where $b' = f_2(y)$, such that $(t, g(t)) \in F^1(E^1_a)$, for every $t \in [b', b]$, with $F^1(y) = (b', g(b'))$ and $F^1(x) = (b, g(b))$. For each $b' < t < b$, let $x_t \in E^1_a$ be such that $F^1(x_t) = (t, g(t))$. Then $f_2(x_t) = t < b = f_2(x) \leq \widetilde{f_2} = f_2(z)$. Similarly, we have $f_3(x_t) = g(t) < g(b') = f_3(y) = t_3 \leq f_3(z)$. Hence $F^1(x_t) < F^1(z)$, i.e. $x_t \notin E^1$. Invoking now Lemma 2.2, we get $x_t \in E$ and $f_1(x_t) = a$.

Consequently, the curve $[b', b] \ni t \to F(x_t) = (a, t, g(t))$ is continuous, lies in $F(E)$ and joins $F(x)$ to an element of $F(E \cap S^3_{t_3})$, as asserted. ■

Theorem 3.1 *The set E is connected.*

Proof. By Lemmas 3.2 and 3.3, the set $F(E \cap (S^1_{t_1} \cup S^2_{t_2} \cup S^3_{t_3}))$ is connected. Furthermore, by Lemma 3.4, for any $x \in E \setminus (S^1_{t_1} \cup S^2_{t_2} \cup S^3_{t_3})$, there exists a continuous curve, lying in $F(E)$ and joining $F(x)$ to a point in $F(E \cap (S^1_{t_1} \cup S^2_{t_2} \cup S^3_{t_3}))$. Hence, $F(E)$ is connected. Finally, recalling Theorem 2.1, we infer that E is connected. ■

4. Conclusion

The paper solves an open problem in the literature of generalized concave multi-objective maximization (Ref.4). It is shown that in the three-objective case semistrict quasiconcavity of all three functions guarantees connectedness of the efficient set E. In Ref.4 it could only be shown that the closure of the efficient frontier $F(E)$ is connected.

Since in case of two functions semistrict quasiconcavity guarantees connectedness of E as well, it is conjectured that for *any* number of functions semistrict quasiconcavity ensures connectedness of E. Unfortunately, some of the lemmas in this paper cannot be extended to more than three functions. A new approach seems to be needed to deal with the general case.

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