# Coercivity Conditions and Variational Inequalities

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**Abstract**

Various coercivity conditions appear in the literature in order to guarantee solutions for the Variational Inequality Problem. We show that these conditions are equivalent to each other and that they are not only sufficient, but also necessary for the set of solutions to be non-empty and bounded.

**Key words:** Variational Inequalities, coercivity, pseudomonotonicity.

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# 1 Introduction

The study of the existence of solutions of Variational Inequalities on unbounded domains usually involves the same sufficient assumptions as for bounded domains, together with a coercivity condition. It is of course desirable to have hypothesis as weak as possible; for this reason various different coercivity conditions have been proposed. Non-coercive problems have also been studied.

In a recent article, Crouzeix [5] studied the variational inequality problem in finite dimensions for multivalued operators which are pseudomonotone in the sense of Karamardian (see [7], [12]). He introduced a new kind of coercivity condition and showed that the latter is not only sufficient, but also necessary for the set of solutions to be non-empty and compact (so in this sense coercivity cannot be relaxed). In this article we extend Crouzeix's results in infinite dimensions: We show that in reflexive Banach spaces if the assumptions used for bounded domains hold, then various coercivity conditions introduced in the literature are equivalent to each other, and also to the fact that the set of solutions is non-empty and bounded. In the finite dimensional case we show in particular that these conditions are also equivalent to the one introduced in [5].

# 2 Solution sets of the Variational Inequality Problem and Coercivity Conditions.

In what follows $K$ will be a non-empty, closed and convex subset of a real Banach space $X$. Let $T : K \to 2^{X^*} \backslash \{\emptyset\}$ be a multivalued operator with non-empty values. We recall that $T$ is called upper hemicontinuous [1], if its restriction to line segments of $K$ is upper semicontinuous, where $X^*$ is equiped with the $w^*$-topology. The operator $T$ is called pseudomonotone (according to Karamardian [7], to be distinguished from the notion defined by Brezis [3]), if for every $x, y \in K$ and $x^* \in T(x)$, $y^* \in T(y)$ the following implication holds:

$$(x^*, y - x) \geq 0 \Rightarrow (y^*, y - x) \geq 0 \tag{1}$$

where $(u^*, u)$ denotes the value of $u^* \in X^*$ at the point $u \in X$.

The Variational Inequality Problem (VIP) for the operator $T$ consists in finding $x \in K$ such that:

$$\forall y \in K, \exists x^* \in T(x) : (x^*, y - x) \geq 0 \tag{2}$$

The set of solutions of the VIP will be denoted by $S$. A solution $x \in S$ will be called a strong solution, if $x^*$ in (2) does not depend on $y$. The set of strong solutions will be denoted by $S_{str}$. It is well known (see for example [10]) that VIP is closely related to the following Dual Variational Inequality Problem (DVIP), which consists in finding $x \in K$ such that:

$$\forall y \in K, \forall y^* \in T(y) : (y^*, y - x) \geq 0 \tag{3}$$

We denote by $S_D$ the set of solutions of the DVIP.

The relations between $S, S_{str}$ and $S_D$ are given in the following well known proposition. We include a proof for the sake of completeness.

**Proposition 1** *(i) If $T$ is pseudomonotone, then $S \subseteq S_D$.*
*(ii) If $T$ is upper hemicontinuous, then $S_D \subseteq S$.*
*(iii) If $T$ has $w^*$-compact and convex values, then $S = S_{str}$.*

**Proof:** (i) is obvious. For (ii), let $x \in S_D$ and suppose to the contrary that for some $y \in K$ and all $x^* \in T(x)$, we have $(x^*, y - x) < 0$. Since in that case the set $\{x^* \in X^* : (x^*, y - x) < 0\}$ is a $w^*$-open neighbourhood of $T(x)$ and $T$ is upper hemicontinuous, then setting $x_t = ty + (1 - t)x$ and taking $t$ close to zero, we obtain the relation $(x_t^*, y - x) < 0$, for all $x_t^* \in T(x_t)$. This in particular implies that $(x_t^*, x_t - x) < 0$, which contradicts the fact that $x \in S_D$. Finally, (iii) is a direct application of the minimax Theorem of Sion [11]. ■

In order to show the existence of a solution for unbounded sets $K$, various coercivity conditions have been used. We single out three of these. Denoting by $\Re(K)$ the set of all weakly compact and convex subsets of $K$, we have:

$$\exists A \in \Re(K), \forall x \in K \backslash A, \forall x^* \in T(x), \exists y \in A : (x^*, x - y) > 0 \tag{C1}$$

$$\exists A \in \Re(K), \forall x \in K \backslash A, \exists y \in A, \forall x^* \in T(x) : (x^*, x - y) > 0 \tag{C2}$$

$$\exists A \in \Re(K), \forall x \in K \backslash A, \exists y \in A, \exists y^* \in T(y) : (y^*, x - y) > 0 \tag{C3}$$

Condition (C1) is standard (see for instance [5]). Condition (C2) is a weaker version of various coercivity conditions (see [4], [6], [12]). Condition (C3) was recently used by Konnov [8] to treat the quasimonotone DVIP.

**Remark 2** *It is obvious that conditions (C1), (C2) and (C3) imply respectively that the (possibly empty) solution sets $S_{str}, S$ and $S_D$ are included in the weakly compact set $A$.*

**Remark 3** *Condition (C2) implies (C1). If $T$ has convex values, then Sion's Minimax Theorem [11] shows that conditions (C1) and (C2) are equivalent. Finally if $T$ is pseudomonotone, then (C3) clearly implies (C2).*

The idea of the proof of the following Theorem is well known. However, we include a proof, since this theorem is usually stated (see [12, Theorem 2.3]) under stronger coercivity assumptions and the additional hypothesis that the values of $T$ are $w^*$-compact.

**Theorem 4** *Let $T$ be an upper hemicontinuous, pseudomonotone operator. Suppose also that (C2) holds. Then $S \neq \emptyset$.*

**Proof:** Let $A \in \Re(K)$ be the one given by (C2). For every finite subset $F$ of $K$, the set $K_F = co(A \cup F)$ is a nonempty, convex and $w$-compact subset of $K$ (where $co(A \cup F)$ denotes as usual the convex hull of the set $A \cup F$). For each $x \in K_F$ we define $G(x) = \{y \in K_F : (x^*, y - x) \leq 0, \forall x^* \in T(x)\}$. The sets $G(x)$ are convex and $w$-compact. If $y = \sum_{i=1}^n \lambda_i x_i$, with $\sum_{i=1}^n \lambda_i = 1$ and $\lambda_i \geq 0$, then for some $i$ we have $y \in G(x_i)$. Indeed, otherwise for all $i$'s there would exist $x_i^* \in T(x_i)$ such that $(x_i^*, y - x_i) > 0$. Since $T$ is pseudomonotone, we would infer that $(y^*, y - x_i) > 0$, for all $y^* \in T(y)$, hence $0 = \sum_{i=1}^n \lambda_i (y^*, y - x_i) > 0$, a contradiction. By Ky Fan's Lemma [9, Lemma 1], $\bigcap_{x \in K_F} G(x) \neq \emptyset$. It is obvious that this intersection coincides with the set $S(F)$ of solutions of DVIP for the operator $T$ in $K_F$. By Proposition 1, $S(F)$ also coincides with the set of the solutions for VIP in $K_F$. Since (C2) holds, we have in particular (see Remark 2) that $S(F) \subseteq A$.

Let now $F_1$ and $F_2$ be two finite subsets of $K$. Since the set $S(F_1 \cup F_2)$ is always contained in the intersection $S(F_1) \cap S(F_2)$, we conclude by induction that the family of all weakly compact sets $S(F)$ (where $F$ is a finite subset of $K$) has the finite intersection property. Hence $\bigcap_{F \text{ finite}} S(F) \neq \emptyset$. It is straighforward to see that the above intersection coincides with the set $S$ of solutions of VIP for the operator $T$ in $K$. ■

¿From now on we assume that the Banach space $X$ is reflexive. In this framework we consider the weakly compact and convex set $K_R = \{x \in K :$

$\|x\| \leq R\}$. It is now easily seen that (C1), (C2) and (C3) can be restated respectively as:

$$\exists R > 0, \forall x \in K \backslash K_R, \forall x^* \in T(x), \exists y \in K_R : (x^*, x - y) > 0$$

$$\exists R > 0, \forall x \in K \backslash K_R, \exists y \in K_R, \forall x^* \in T(x) : (x^*, x - y) > 0$$

$$\exists R > 0, \forall x \in K \backslash K_R, \exists y \in K_R, \exists y^* \in T(y) : (y^*, x - y) > 0$$

We proceed to show that under the usual assumptions on $T$ these coercivity conditions are not only sufficient, but also necessary for the solution set to be weakly compact.

**Theorem 5** *Let $X$ be a reflexive Banach space. Suppose that $T$ is pseudomonotone and $S_D$ is non-empty and bounded. Then (C3) holds.*

**Proof:** Let $x_0 \in S_D$. Set $K_n = \{x \in K : \|x\| \leq n\}$. If (C3) does not hold, then for any $n \in N$ (in particular $n > \|x_0\|$), there exists $x \in K \backslash K_n$, such that for all $y \in K_n$ and all $y^* \in T(y)$, we have

$$(y^*, x - y) \leq 0 \tag{4}$$

Choose $z = \lambda x_0 + (1 - \lambda)x$ with $\lambda \in (0, 1)$ and $n - 1 \leq \|z\| < n$. Since $x_0 \in S_D$, then for any $y \in K_n$ and $y^* \in T(y)$ we have

$$(y^*, y - x_0) \geq 0 \tag{5}$$

which together with (4) implies

$$(y^*, y - z) \geq 0, \forall y^* \in T(y) \tag{6}$$

Hence, $z$ is a solution of DVIP in $K_n$. Now for any $y_1 \in K \backslash K_n$ we can find a $y \in K_n, (y \neq z)$ on the line segment joining $z$ and $y_1$. Then (6) implies $(y^*, y - z) \geq 0, \forall\ y^* \in T(y)$, hence $(y^*, y_1 - y) \geq 0, \forall\ y^* \in T(y)$. Since $T$ is pseudomonotone, we get $(y_1^*, y_1 - y) \geq 0, \forall\ y_1^* \in T(y_1)$, which in particular implies $(y_1^*, y_1 - z) \geq 0, \forall y_1^* \in T(y_1)$, i.e. $z \in S_D$. Given that $\|z\| \geq n - 1$ and $n$ is arbitrarily chosen, this contradicts the assumption that $S_D$ is bounded.■

**Corollary 6** *Let $X$ be a reflexive Banach space and $T$ be an upper hemicontinuous, pseudomonotone operator with $w^*$-compact convex values. Then each of the conditions (C1), (C2) and (C3) is equivalent to the fact that the set $S_{str}$ is non-empty and bounded.*

**Proof:** By Proposition 1, we have $S = S_{str} = S_D$. If $S_{str}$ is non-empty and bounded, then by Theorem 5 condition (C3) holds and so by Remark 3, (C1) and (C2) also hold. Conversely, if any of the coercivity conditions holds, then by Remark 3, conditions (C1) and (C2) hold, so $S = S_{str}$ is bounded (see Remark 2). By Theorem 4, $S$ is also non-empty. ■

# 3 The finite-dimensional case

Let $K$ and $T$ be as before. In this section we limit ourselves to the case $X = R^n$. In [5], Crouzeix considered the following coercivity assumption:

$$K_\infty \bigcap T(K)^o = \{0\} \tag{CR}$$

where $T(K)$ is the image of $K$ under $T$ and $T(K)^o = \{d : (x^*, d) \leq 0, \forall x^* \in T(K)\}$ is the polar cone of $T(K)$. Further, $K_\infty$ is the recession cone of $K$, which in the case of a closed, convex set is defined as follows (see for example [2]):

$$\begin{aligned} K_\infty &= \{d \in R^n : d = \lim_n \frac{x_n}{t_n}, x_n \in K, t_n \to +\infty\} \\ &= \{d \in R^n : \alpha + td \in K, \forall t \geq 0\} \end{aligned} \tag{7}$$

where $\alpha$ is arbitrarily chosen in $K$.

In [5] the following theorem is proved:

**Theorem 7** *Let $T$ be an upper semicontinuous, pseudomonotone operator with compact, convex values. Then (CR) holds if and only if $S_{str}$ is nonempty and compact.*

We intend to show that even if we replace the upper semicontinuity by upper hemicontinuity, (CR) is equivalent to (C1), (C2), (C3), so Theorem 7 remains true.

**Theorem 8** *(i) (CR) implies (C3).*

*(ii) If $S_D \neq \emptyset$, then (C3) implies (CR).*

*(iii) If $T$ is upper hemicontinuous and pseudomonotone, then (C2), (C3) and (CR) are equivalent.*

*(iv) If $T$ is upper hemicontinuous and pseudomonotone with convex values, then (C1), (C2), (C3) and (CR) are equivalent.*

**Proof:** (i) Suppose that (C3) does not hold. Then we can find a sequence $(x_n)_{n\in N} \subset K$, with $\|x_n\| > n$ , such that for every $n \in N$ and all $y \in K, \|y\| \leq n$, we have

$$(y^*, x_n - y) \leq 0, \forall y^* \in T(y) \tag{8}$$

We may assume with no loss of generality that $\frac{x_n}{\|x_n\|} \to d \neq 0$. If to the contrary (CR) holds, then for some $y \in K, y^* \in T(y)$ we should have $(y^*, d) > 0$. The latter implies that for sufficiently large $n$, we get $(y^*, \frac{x_n}{\|x_n\|}) > \frac{1}{2}(y^*, d)$, so in particular $(y^*, x_n) \to +\infty$, which clearly violates (8).

(ii) Let $x \in S_D$, and suppose that (CR) does not hold, i.e. there exists a $d \neq 0, d \in K_\infty$ such that for all $y \in K$, $y^* \in T(y)$, we have $(y^*, d) \leq 0$. Let $y \in K$ be arbitrarily chosen. Then for any $t > 0$, we obviously have

$$(y^*, y - (x + td)) = (y^*, y - x) - t(y^*, d) \geq 0, \forall y^* \in T(y) \tag{9}$$

which implies that $x + td \in S_D$. In particular $S_D$ cannot be bounded, hence (C3) does not hold (see Remark 2).

(iii) According to (i), (CR) implies (C3); by Remark 3, (C3) implies (C2). Thus we have only to show that (C2) implies (CR). Suppose that (C2) holds. Then Theorem 4 implies that $S \neq \emptyset$. By Remark 2, the set $S$ is bounded. By Proposition 1 we have $S = S_D$. Hence, Theorem 5 implies that (C3) holds. Using (ii), we conclude that (CR) holds.

(iv) This is an immediate consequence of (iii) and Remark 3. ■

Combining Corollary 6, Theorem 8(iv) and the fact that $S_{str} = S_D$ is closed, we get the following stronger version of Theorem 7.

**Corollary 9** *Let $T$ be an upper hemicontinuous, pseudomonotone operator with compact, convex values. Then $S_{str}$ is nonempty and compact if and only if any of the conditions (C1), (C2), (C3) or (CR) holds.*

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