

3. General Equilibrium Under Uncertainty

- In General Equilibrium Theory, the “allocation” of a given quantity of each commodity implies its final consumption with its corresponding utility score
- Under uncertainty, the final consumption of a given allocation will depend on the state of nature in such a way that equal quantities of the same commodity could produce different utility scores.
- This raises the issues of:
 - existence of equilibrium in this setting
 - information on the state of the nature (basis for “Information Economics”)
 - possibility of incomplete markets and the need for “artificial markets” (basis for “Financial Economics”)
 - other

3.1 Economies with Contingent Commodities

Everything in the economy (commodities, initial endowments, preferences) will depend on the *state of nature*. For simplicity, we will not consider economies with production.

States of nature are identified with the possible outcomes of uncertainty. The number of different *states of nature* is given by S . The set \mathcal{S} is the set of all the possible *states of nature*

$$\mathcal{S} = \{1, 2, \dots, S\}$$

We will assume that:

- \mathcal{S} is a finite set ($S < \infty$)
- All the elements in \mathcal{S} are *mutually exclusive*
- \mathcal{S} is *exhaustive*
- \mathcal{S} is known to everybody

Definition. *State Contingent Commodity*

The allocation $x_{j_s}^i \in \mathbb{R}^n$ entitles individual $i \in \mathcal{I}$ to receive (or deliver) $x_{j_s}^i$ units of commodity $j \in \{1, 2, \dots, n\}$ if and only if state $s \in \mathcal{S}$ occurs.

$$x_j^i \rightarrow \begin{cases} 1 \rightarrow & x_{j1}^i \\ 2 \rightarrow & x_{j2}^i \\ \vdots & \vdots \\ S \rightarrow & x_{jS}^i \end{cases}$$

Accordingly,

$$x^i = (\underbrace{x_{11}^i, \dots, x_{n1}^i}_{\text{State 1}}, \underbrace{x_{12}^i, \dots, x_{n2}^i}_{\text{State 2}}, \dots, \underbrace{x_{1S}^i, \dots, x_{nS}^i}_{\text{State S}}) \in \mathbb{R}^{nS}$$

is a state contingent vector, which could be seen as:

$$x^i = (\underbrace{x_{11}^i, \dots, x_{1S}^i}_{x_1 \text{ as R.V.}}, \underbrace{x_{21}^i, \dots, x_{2S}^i}_{x_2 \text{ as R.V.}}, \dots, \underbrace{x_{n1}^i, \dots, x_{nS}^i}_{x_n \text{ as R.V.}}) \in \mathbb{R}^{nS}$$

- Likewise, initial endowments are of the form

$$\omega^i = (\underbrace{\omega_{11}^i, \dots, \omega_{n1}^i}_{\text{State 1}}, \underbrace{\omega_{12}^i, \dots, \omega_{n2}^i}_{\text{State 2}}, \dots, \underbrace{\omega_{1S}^i, \dots, \omega_{nS}^i}_{\text{State } S}) \in \mathbb{R}_+^{nS}$$

meaning that the initial endowment of individual $i \in \mathcal{I}$ if state $s \in \mathcal{S}$ occurs is $(\omega_{1s}^i, \omega_{2s}^i, \dots, \omega_{ns}^i)$

- Preferences will also depend on the state of nature

$$\succsim^i \text{ are defined on } \mathbb{R}^{nS}$$

These are “ex ante” preferences as they evaluate random variables before uncertainty is resolved

Example:

Let $\pi_s \in [0, 1]$ be the probability of state $s \in \mathcal{S}$ and let $u^i : \mathbb{R}^n \rightarrow \mathbb{R}$ the usual utility function. Then, preferences \succsim^i on $\mathbb{R}^{n\mathcal{S}}$ could be represented by

$$x^i \succsim^i \hat{x}^i$$
$$\Downarrow$$
$$\sum_{s \in \mathcal{S}} \pi_s u^i(x_{1s}^i, \dots, x_{ns}^i) \geq \sum_{s \in \mathcal{S}} \pi_s u^i(\hat{x}_{1s}^i, \dots, \hat{x}_{ns}^i)$$

which corresponds to the von Neumann-Morgenstern Expected Utility assumption

Comments

- Very “rough” model because of the requirements on the set of *states of nature* \mathcal{S}
- In a *pure exchange economy*, trade across states is “physically” impossible
- Some specific “mechanism” (or system) is needed to organize trade in a feasible manner
- All individuals must have the same information, and it must be *common knowledge*
- Although the model remains static, there are *implicit dynamics* as decisions are taken before uncertainty is resolved (ex-ante) but trade is implemented after the *state of nature* is known (ex-post)

3.2 Arrow-Debreu Equilibrium

Corresponds to the “straightforward” generalization of the General Equilibrium Model to accommodate uncertainty.

- There are markets for every contingent commodity
- Markets open before the realization of uncertainty
- Each contingent commodity j_s has a price p_{j_s}
- Deliveries are contingent, prices are not
- All individuals must have the same information, and all know that all have the same information, and all know that all know that all have the same information, and ...

A “State Contingent Economy” (or Economy Under Uncertainty) is characterized by

$$\mathcal{U} = \{\mathcal{I}, \mathcal{S}, \{\succsim^i\}_{i \in \mathcal{I}}, \{\omega^i\}_{i \in \mathcal{I}}\}$$

where

$$\omega^i = \left(\underbrace{\omega_{11}^i, \dots, \omega_{n1}^i}_{\text{State 1}}, \underbrace{\omega_{12}^i, \dots, \omega_{n2}^i}_{\text{State 2}}, \dots, \underbrace{\omega_{1S}^i, \dots, \omega_{nS}^i}_{\text{State } S} \right) \in \mathbb{R}_+^{nS}$$

Definition. *Arrow-Debreu Equilibrium*

An allocation $x^* \in \mathbb{R}^{nIS}$ together with a price vector $p^* \in \mathbb{R}_+^{nS}$ are an Arrow-Debreu Equilibrium if

(i) For all $i \in \mathcal{I}$, x^{*i} maximizes \succsim^i on B^i where

$$B^i = \{x \in \mathbb{R}^{nS} \mid p^* \cdot x \leq p^* \cdot \omega^i\}$$

(ii) x^* is feasible, that is,

$$\sum_{i \in \mathcal{I}} x^{*i} \leq \sum_{i \in \mathcal{I}} \omega^i$$

- Strict Convexity \Rightarrow Risk Aversion

- Formally, $\mathcal{U} \leftrightarrow \mathcal{E}^S$, hence the same results as before hold in this case (Existence, Core Equivalence, Welfare Theorems)

Claim

x^* is Pareto efficient *ex ante* $\Rightarrow x^*$ is Pareto efficient *ex post*

Proof. Suppose not, that is, suppose that $s \in \mathcal{S}$ occurs and that there exists $\hat{x}_s \in \mathbb{R}^{nI}$ such that

$$(i) \quad \sum_{i \in \mathcal{I}} \hat{x}_s^i \leq \sum_{i \in \mathcal{I}} \omega_s^i$$

$$(ii) \quad \hat{x}_s^i \succ^i x_s^{*i} \quad \forall i \in \mathcal{I} \text{ with at least one strict preference}$$

Then,

$$(x_1^{*i}, x_2^{*i}, \dots, \hat{x}_s^i, \dots, x_S^{*i}) \succ^i (x_1^{*i}, x_2^{*i}, \dots, x_S^{*i}) \quad \forall i \in \mathcal{I}$$

with at least one strict preference. Then, $(x_1^{*i}, x_2^{*i}, \dots, x_S^{*i})$ would not be Pareto efficient !! \square

Interpretation

Although markets open before uncertainty is revealed, if they were re-opened once the state of nature is known there would not be incentives to further trade

Radner Equilibrium

In the Arrow-Debreu model, all trade takes place simultaneously and before uncertainty is revealed, which is not very realistic. Moreover, although delivery is contingent upon the state of nature that is revealed, payment is not. That is, individuals *must* pay the contingent commodity in full.

Next, we will assume that trade takes place sequentially (... at least to some extent ...)

Sequence of events in the Arrow-Debreu model

t=0 All markets exist and are opened. Individuals choose $x^i \in \mathbb{R}^{nS}$ and pay prices $p \in \mathbb{R}_+^{nS}$

These are markets for delivery at t=1 (forward markets)

t=1 Once uncertainty is revealed, and the state of nature s becomes known, all contracts are executed. Each individual $i \in \mathcal{I}$ receives $x_s^i \in \mathbb{R}^n$. Notice that if markets were re-opened for trade at $t=1$ (spot markets), no further trade would take place.

Sequence of events in the Radner model

- t=0** Only the market for the **first** contingent commodity, $x_1 \in \mathbb{R}^S$, exists (forward market). Prices for x_1 are $q = (q_1, \dots, q_S) \in \mathbb{R}_+^S$ (forward prices). This market is for delivery at t=1 (forward market).
 At the same time, individuals “form expectations” regarding prices of all the other commodities for each possible state at time $t=1$, $p = (p_1, \dots, p_S) \in \mathbb{R}_+^{nS}$ (spot prices).
 Faced with forward prices $q \in \mathbb{R}_+^S$ and expected spot prices $p \in \mathbb{R}_+^{nS}$, individuals formulate and **buy** forward consumption -or trading- plans for the contingent commodity $z^i \in \mathbb{R}^S$ and **decide** spot consumption plans for each possible state of nature $x^i = (x_1^i, \dots, x_S^i) \in \mathbb{R}_+^{nS}$.
- t=1** Once uncertainty is revealed, and the state of nature s becomes known, contracts regarding z are executed. Then, individuals **buy** their consumption plans $x_s^i \in \mathbb{R}_+^n$ ($i \in \mathcal{I}$) in the spot markets.

Individuals plans *must* satisfy usual budget constraint(s). Thus, each individual *decision problem* is given by

$$\begin{aligned}
 & \max \quad U^i(x_1^i, \dots, x_S^i) \\
 & \text{s.t.} \quad \sum_{s \in \mathcal{S}} q_s \cdot z_s^i \leq 0 \\
 & \quad \quad p_s \cdot x_s^i \leq p_s \cdot \omega_s^i + p_{1s} z_s^i \quad \forall s \in \mathcal{S} \\
 & \quad \quad (x_1^i, \dots, x_S^i) \in \mathbb{R}_+^{nS} \\
 & \quad \quad (z_1^i, \dots, z_S^i) \in \mathbb{R}^S
 \end{aligned} \tag{1}$$

- U^i represents \succsim^i on \mathbb{R}_+^{nS}
- $z_s^i < -\omega_{1s}^i$ “selling short”
- All have same expectations on spot prices. Expectations are “self fulfilling”

Definition. *Radner Equilibrium*

A collection composed of forward prices $q^* \in \mathbb{R}_+^S$, spot prices $p^* \in \mathbb{R}_+^{nS}$, contingent consumption-trading plans at $t=0$, $z^{*i} \in \mathbb{R}^S$, and consumption plans at $t=1$, $x^{*i} \in \mathbb{R}_+^{nS}$, for all individuals $i \in \mathcal{I}$ constitute a Radner equilibrium if:

(i) For all $i \in \mathcal{I}$, z^{*i} and x^{*i} solve the individuals maximization problem (1), and

(ii) $\sum_{i \in \mathcal{I}} z^{*i} \leq 0$ and $\sum_{i \in \mathcal{I}} x_s^{*i} \leq \sum_{i \in \mathcal{I}} \omega_s^i$ for every $s \in \mathcal{S}$

Proposition 3.2.1 Arrow-Debreu vs. Radner

- (i) If $x^* \in \mathbb{R}^{nIS}$ and $p^* \in \mathbb{R}_{++}^{nS}$ are an *Arrow-Debreu equilibrium*, then $\exists q^* \in \mathbb{R}_{++}^S$ and contingent consumption-trading plans $z^* = (z^{*1}, z^{*2}, \dots, z^{*I}) \in \mathbb{R}^{IS}$ such that q^*, p^*, z^*, x^* are a *Radner equilibrium*
- (ii) If $z^* \in \mathbb{R}^{IS}$, $x^* \in \mathbb{R}_+^{nIS}$, $q^* \in \mathbb{R}_{++}^S$, and $p^* \in \mathbb{R}_{++}^{nS}$ are a *Radner equilibrium*, then there are “multipliers” $(\mu_1, \dots, \mu_S) \in \mathbb{R}_{++}^S$ such that the allocation $x^* \in \mathbb{R}_+^{nIS}$ and the prices $(\mu_1 p_1, \dots, \mu_S p_S) \in \mathbb{R}_{++}^{nS}$ are an *Arrow-Debreu equilibrium*

Proof.

(i) Let

$$B_{AD}^i = \left\{ (x_1^i, \dots, x_S^i) \in \mathbb{R}_+^{nS} \mid \sum_{s \in \mathcal{S}} p_s^* \cdot (x_s^i - \omega_s^i) \leq 0 \right\}$$

the budget constraint for individual $i \in \mathcal{I}$ that corresponds to the Arrow-Debreu equilibrium

at prices p^* . Analogously, let

$$B_R^i = \left\{ (x_1^i, \dots, x_S^i) \in \mathbb{R}_+^{nS} \mid \right.$$

$$\left. \exists (z_1^i, \dots, z_S^i) \in \mathbb{R}^S \text{ such that} \right.$$

$$\left. \sum_{s \in \mathcal{S}} q_s^* z_s^i \leq 0 \text{ and} \right.$$

$$\left. p_s^* \cdot (x_s^i - \omega_s^i) \leq p_{1s}^* z_s^i \quad \forall s \in \mathcal{S} \right\}$$

the budget constraint that individual $i \in \mathcal{I}$ faces in the Radner equilibrium at prices p^* and q^* .

We will show that $B_{AD}^i = B_R^i$.

Indeed, suppose that $x^i = (x_1^i, \dots, x_S^i) \in B_{AD}^i$ and, for every $s \in \mathcal{S}$, define

$$q_s^* = p_{1s}^*$$

$$z_s^i = \frac{1}{p_{1s}^*} p_s^* \cdot (x_s^i - \omega_s^i)$$

Then,

$$\sum_{s \in \mathcal{S}} q_s^* z_s^i = \sum_{s \in \mathcal{S}} p_{1s}^* z_s^i = \sum_{s \in \mathcal{S}} p_s^* \cdot (x_s^i - \omega_s^i) \leq 0$$

and

$$p_s^* \cdot (x_s^i - \omega_s^i) = p_{1s}^* z_s^i \quad \forall s \in \mathcal{S}$$

Hence, we have that $x^i \in B_{AD}^i \Rightarrow x^i \in B_R^i$.

Conversely, assume now that $x^i \in B_R^i$. That is, for some (z_1^i, \dots, z_S^i) we have that, for every $s \in \mathcal{S}$,

$$\sum_{s \in \mathcal{S}} q_s^* z_s^i \leq 0$$

$$p_s^* \cdot (x_s^i - \omega_s^i) \leq p_{1s}^* z_s^i \quad \forall s \in \mathcal{S}$$

Summing over s

$$\sum_{s \in \mathcal{S}} p_s^* \cdot (x_s^i - \omega_s^i) \leq \sum_{s \in \mathcal{S}} p_{1s}^* z_s^i = \sum_{s \in \mathcal{S}} q_s^* z_s^i \leq 0$$

Hence, $x^i \in B_R^i \Rightarrow x^i \in B_{AD}^i$

Therefore, if $x^* \in \mathbb{R}^{nIS}$ and $p^* \in \mathbb{R}^{nS}$ constitute an Arrow-Debreu equilibrium, then, x^* and

p^* together with $q^* = (p_{11}^*, \dots, p_{1S}^*)$ and $z^* \in \mathbb{R}^{IS}$ defined by

$$z_s^{*i} = \frac{1}{p_{1s}} p_s \cdot (x_s^{*i} - \omega_s^i)$$

are a Radner equilibrium because *market clearing* is satisfied ...

$$\sum_{i \in \mathcal{I}} z_s^{*i} = \frac{1}{p_{1s}^*} p_s^* \cdot \underbrace{\sum_{i \in \mathcal{I}} (x_i^* - \omega_s^i)}_{\leq 0} \leq 0$$

(ii) Choose μ_s so that $\mu_s p_{1s}^* = q_s^*$ and rewrite B_R^i as

$$B_R^i = \{(x_1^i, \dots, x_S^i) \in \mathbb{R}_+^{nS} \mid$$

$$\exists (z_1^i, \dots, z_S^i) \in \mathbb{R}^S \text{ such that}$$

$$\sum_{s \in \mathcal{S}} q_s^* z_s^i \leq 0 \text{ and}$$

$$\mu_s p_s^* \cdot (x_s^i - \omega_s^i) \leq q_s^* z_s^i \quad \forall s \in \mathcal{S}\}$$

To prove **(ii)**, we can proceed as in **(i)**, to prove that $B_R^i = B_{AD}^i$, where B_{AD}^i in this case is

$$B_{AD}^i = \{(x_1^i, \dots, x_S^i) \in \mathbb{R}_+^{nS} \mid \sum_{s \in \mathcal{S}} \mu_s p_s^* \cdot (x_s^i - \omega_s^i) \leq 0\}$$

□

3.3 Asset Markets

In the Radner model we had:

(z_1, z_2, \dots, z_S) S contingent commodities

that were available at (forward) prices (q_1, q_2, \dots, q_S)

Hence, at time $t = 0$ the “shopping basket” was composed of

Contingent Title for Mr. $i \in \mathcal{I}$

At time $t = 1$, you are entitled to receive (deliver) z_1^i units of commodity 1 if state 1 occurs

Your bill: $q_1 z_1^i$

Contingent Title for Mr. $i \in \mathcal{I}$

At time $t = 1$, you are entitled to receive (deliver) z_2^i units of commodity 1 if state 2 occurs

Your bill: $q_2 z_2^i$

⋮

Contingent Title for Mr. $i \in \mathcal{I}$

At time $t = 1$, you are entitled to receive (deliver) z_S^i units of commodity 1 if state S occurs

Your bill: $q_S z_S^i$

Hence, the total cost of such *contingent consumption-trading plan* is

$$q_1 z_1^i + q_2 z_2^i + \cdots + q_S z_S^i$$

In “real life” we do not observe this kind of markets, but we have *assets* (securities, insurance policies), which allow for the transfer of wealth across states

An *asset* is a title to receive “wealth” at time $t = 1$ in amounts that depend on the *state of nature*.

For simplicity, we will assume that all *assets* payback in units of good 1. This is the return of the asset

$$asset \quad \left\{ \begin{array}{ll} r_1 & \text{if state 1 occurs} \\ r_2 & \text{if state 2 occurs} \\ \vdots & \vdots \\ r_S & \text{if state S occurs} \end{array} \right.$$

Definition. *Asset*

A unit of an asset is a title to receive an amount r_s of commodity 1 at time $t = 1$ if state $s \in S$ occurs

Hence, an *asset* is fully characterized by its *return*

$$r = (r_1, r_2, \dots, r_S) \in \mathbb{R}^S$$

Examples

(i) $r = (1, 1, \dots, 1)$

Entitles the “owner” to receive 1 unit of commodity 1 in whatever state of nature

- If $S = 1$ then it is “riskless”
- If $S > 1$ then it is “not riskless”

(ii) $r = (0, 0, \dots, \underbrace{1}_s, \dots, 0)$ is an “Arrow Security”

It is equivalent to the $z_s = 1$

Assumption 3.3.1 Asset Structure

In any economy, there is a given collection of A different assets available. It is called the *asset structure*

$$\mathcal{A} = \{1, 2, \dots, A\}$$

- Each asset $a \in \mathcal{A}$ is characterized by its *return*

$$r_a = (r_{a1}, r_{a2}, \dots, r_{aS}) \in \mathbb{R}^S$$

- Each asset $a \in \mathcal{A}$ has a *price* $\tilde{q}_a \in \mathbb{R}_+$.

- The *price vector*

$$\tilde{q} = (\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_A) \in \mathbb{R}_+^A$$

describes the prices prevailing at time $t = 0$

- The *portfolio* of individual $i \in \mathcal{I}$ enumerates how many units of each available asset the individual owns

$$\tilde{z}^i = (\tilde{z}_1^i, \tilde{z}_2^i, \dots, \tilde{z}_A^i) \in \mathbb{R}^A$$

The *cost of the portfolio* is

$$\tilde{q}_1 \tilde{z}_1^i + \dots + \tilde{q}_A \tilde{z}_A^i$$

and its return is given by

$$(\tilde{z}_1^i r_1 + \dots + \tilde{z}_A^i r_A) \in \mathbb{R}^S$$

As in the Radner model, individuals' plans *must* satisfy usual budget constraint(s). In this case, each individual *decision problem* is given by

$$\begin{aligned}
 & \max \quad U^i(x_1^i, \dots, x_S^i) \\
 & \text{s.t.} \quad \sum_{a \in \mathcal{A}} \tilde{q}_a \cdot \tilde{z}_a^i \leq 0 \\
 & \quad \quad p_s \cdot x_s^i \leq p_s \cdot \omega_s^i + p_{1s} \sum_{a \in \mathcal{A}} \tilde{z}_a^i r_{as} \quad \forall s \in \mathcal{S} \quad (2) \\
 & \quad \quad (x_1^i, \dots, x_S^i) \in \mathbb{R}_+^{nS} \\
 & \quad \quad (\tilde{z}_1^i, \dots, \tilde{z}_A^i) \in \mathbb{R}^A
 \end{aligned}$$

Definition. *Radner Equilibrium in Asset Markets*

A collection composed of forward prices $\tilde{q}^* \in \mathbb{R}_+^A$, spot prices $p^* \in \mathbb{R}_+^{nS}$, asset portfolios at $t=0$, $\tilde{z}^{*i} \in \mathbb{R}^A$, and consumption plans at $t=1$, $x^{*i} \in \mathbb{R}_+^{nS}$, for all individuals $i \in \mathcal{I}$ constitute a Radner equilibrium in asset markets if:

(i) For all $i \in \mathcal{I}$, \tilde{z}^{*i} and x^{*i} solve the individuals maximization problem (2), and

(ii) $\sum_{i \in \mathcal{I}} \tilde{z}^{*i} \leq 0$ and $\sum_{i \in \mathcal{I}} x_s^{*i} \leq \sum_{i \in \mathcal{I}} \omega_s^i$ for every $s \in \mathcal{S}$

(We will *normalize* $p_{1s} = 1$ for every $s \in \mathcal{S}$)

As defined, the Radner equilibrium in asset markets “looks” formally equivalent to that in the previous model. After all, an asset “looks like” a *contingent consumption-trading plan* in which the S contingent commodities are sold together in a unique bundle.

Can we, therefore, apply **Proposition 3.2.1** to *asset markets*?

Notice !

Buying (in the Radner model) the *contingent consumption-trading plan* $z^i = (z_1^i, \dots, z_S^i)$ at prices $q = (q_1, \dots, q_S)$ is equivalent to buying (in the Assets model) one unit of *asset a* whose returns are $r_a = (r_{a1}, \dots, r_{aS}) = (z_1^i, \dots, z_S^i)$ at the price \tilde{q}_a , where

$$\tilde{q}_a = q_1 z_1^i + \dots + q_S z_S^i$$

Question 1 ?

If we buy (in the Assets model) one unit of *asset a* whose returns are $r_a = (r_{a1}, \dots, r_{aS})$ at the price \tilde{q}_a , **do they exist** prices (q_1, \dots, q_S) such that *asset a* is equivalent to buying (in the Radner model) the *contingent consumption-trading plan* $z^i = (z_1^i, \dots, z_S^i) = (r_{a1}, \dots, r_{aS})$ at these prices ? so that

$$\tilde{q}_a = q_1 r_{a1} + \dots + q_S r_{aS}$$

Question 2 ?

Are there *enough* assets in \mathcal{A} to reproduce any contingent consumption-trading plan in the Radner model ?

For **Question 1** we have the following result

Proposition 3.3.1

Assume that for every asset $a \in \mathcal{A}$ we have that $r_a \geq 0$, ($r_a \neq 0$). Then, for any vector $\tilde{q} \in \mathbb{R}^A$ of asset prices that might arise in a *Radner equilibrium in asset markets* there are “multipliers” $\mu = (\mu_1, \dots, \mu_S) \geq 0$ such that

$$\tilde{q}_a = \mu_1 r_{a1} + \dots + \mu_S r_{aS}$$

Proof. *MGW* pp. 701-703 \square

For **Question 2**, consider the following example

Example

Suppose that $\mathcal{S} = \{1, 2\}$ and that, according to a *Radner equilibrium* in the previous model, individual $i \in \mathcal{I}$ must implement the following *contingent consumption-trading plan*

$$z^{*i} = (2, -3)$$

- Suppose that $\mathcal{A} = \{1\}$ with return $r_1 = (1, -1)$.
Clearly, in this case z^{*i} **can not be reproduced** by any combination of assets in \mathcal{A}
- Suppose that $\mathcal{A} = \{1, 2\}$ with returns $r_1 = (1, -1)$ and $r_2 = (-3, 2)$
In this case, the *portfolio* $\tilde{z}^i = (\tilde{z}_1^i, \tilde{z}_2^i) = (5, 1)$ **reproduces** z^{*i} since

$$\tilde{z}_1^i r_1 + \tilde{z}_2^i r_2 = 5(1, -1) + 1(-3, 2) = (2, -3) = z^{*i}$$

- Suppose that $\mathcal{A} = \{1, 2\}$ with returns $r_1 = (1, -1)$ and $r_2 = (2, -2)$. Can z^{*i} **be reproduced** by some portfolio in this case ?

Therefore, the return vectors must satisfy some conditions

Definition. *Definition Return Matrix*

The return matrix R is a $A \times S$ matrix whose a^{th} row is the return vector of the a^{th} asset

$$R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1S} \\ r_{21} & r_{22} & \cdots & r_{2S} \\ \vdots & \vdots & \ddots & \vdots \\ r_{A1} & r_{A2} & \cdots & r_{AS} \end{bmatrix}$$

Definition. *Complete structure*

An asset structure \mathcal{A} with return matrix R is said to be complete if $\text{rank } R = S$

- The “rank” of a matrix is the *maximum number of linearly independent rows* the matrix contains
- Notice, in particular, that according to the definition above, if \mathcal{A} is complete then $A \geq S$

We can now give an answer to **Question 2**

Proposition 3.3.2 Arrow-Debreu vs. Radner in asset markets

If \mathcal{A} is complete, then:

- (i) If $x^* \in \mathbb{R}^{nIS}$ and $p^* \in \mathbb{R}_{++}^{nS}$ are an *Arrow-Debreu equilibrium*, then $\exists \tilde{q}^* \in \mathbb{R}_{++}^A$ and portfolios $\tilde{z}^* = (\tilde{z}^{*1}, \tilde{z}^{*2}, \dots, \tilde{z}^{*I}) \in \mathbb{R}^{IA}$ such that $\tilde{q}^*, p^*, \tilde{z}^*, x^*$ are a *Radner equilibrium in the asset markets*
- (ii) If $\tilde{z}^* \in \mathbb{R}^{IA}$, $x^* \in \mathbb{R}_+^{nIS}$, $\tilde{q}^* \in \mathbb{R}_{++}^A$, and $p^* \in \mathbb{R}_{++}^{nS}$ are a *Radner equilibrium in the asset markets*, then there are “multipliers” $(\mu_1, \dots, \mu_S) \in \mathbb{R}_{++}^S$ such that the allocation $x^* \in \mathbb{R}_+^{nIS}$ and the prices $(\mu_1 p_1, \dots, \mu_S p_S) \in \mathbb{R}_{++}^{nS}$ are an *Arrow-Debreu equilibrium*

Proof. *MGW* p. 705. It is very much like the proof of **Proposition 3.2.1** \square

3.4 Incomplete Markets

As seen before, markets are complete if the asset structure \mathcal{A} is such that

$$\text{rank } R = S$$

that is, there are enough assets in \mathcal{A} to reproduce any transfer of wealth across states *as if* all nS markets were open at time $t = 0$

For simplicity, let us assume that $A < S$ (there are less assets than *states of nature*) and that $n = 1$ (only one commodity at each state)

Recall the budget constraint in 2

$$p_s \cdot x_s^i \leq p_s \cdot \omega_s^i + p_{1s} \sum_{a \in \mathcal{A}} \tilde{z}_a^i r_{as} \quad \forall s \in \mathcal{S}$$

3.- Equilibrium under uncertainty

Under the assumption $n = 1$, such constraint becomes

$$x_{1s}^i = \omega_{1s}^i + \sum_{a \in \mathcal{A}} \tilde{z}_a^i r_{as} \quad \forall s \in \mathcal{S}$$

Hence, the consumption bundle x^i is completely determined by the portfolio \tilde{z}^i . We can, therefore, define the *utility induced by portfolio* \tilde{z}^i as

$$\tilde{U}^i(\tilde{z}^i) = U^i\left(\underbrace{\omega_{11}^i + \sum_{a \in \mathcal{A}} \tilde{z}_a^i r_{a1}}_{x_{11}^i}, \dots, \underbrace{\omega_{1S}^i + \sum_{a \in \mathcal{A}} \tilde{z}_a^i r_{aS}}_{x_{1S}^i}\right)$$

Definition. *Constrained Pareto Efficiency*

The asset allocation $(\tilde{z}^1, \dots, \tilde{z}^I) \in \mathbb{R}^{SI}$ is constrained Pareto efficient if it is feasible and there is no other feasible asset allocation $(\tilde{z}'^1, \dots, \tilde{z}'^I)$ such that

$$\tilde{U}^i(\tilde{z}'^1, \dots, \tilde{z}'^I) \geq \tilde{U}^i(\tilde{z}^1, \dots, \tilde{z}^i)$$

for every $i \in \mathcal{I}$ with at least one strict preference

Example:

$$\mathcal{I} = \{1, 2\} \quad n = 1$$

$$\mathcal{S} = \{1, 2\} \quad \mathcal{A} = \{1\} \quad r = (-1, 1)$$

Being \tilde{z}^i the amount of (the only) asset chosen by individual i , each individual decision problem is given by

$$\max \quad U^i(x_1^i, x_2^i)$$

$$\text{s.t.} \quad \tilde{q}\tilde{z}^i \leq 0$$

$$x_1^i \leq \omega_1^i - \tilde{z}^i$$

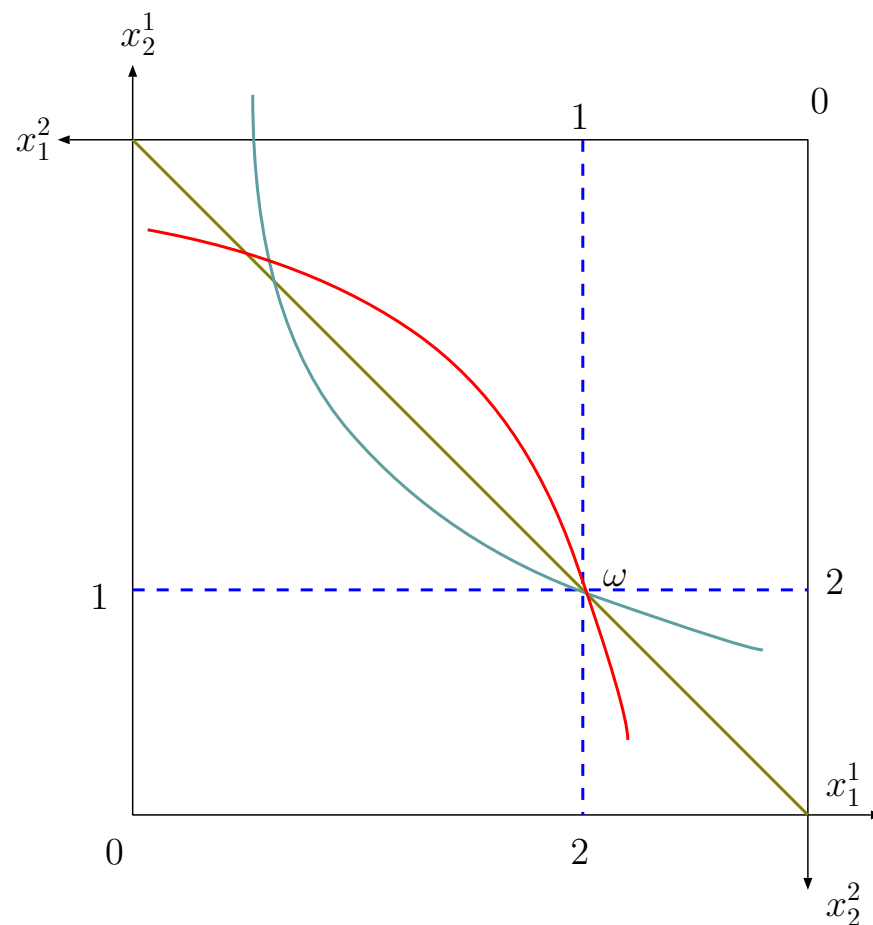
$$x_2^i \leq \omega_2^i + \tilde{z}^i$$

(a) Assume

$$U^i(x_1^i, x_2^i) = \sqrt{x_1^i x_2^i} \quad (i = \{1, 2\})$$

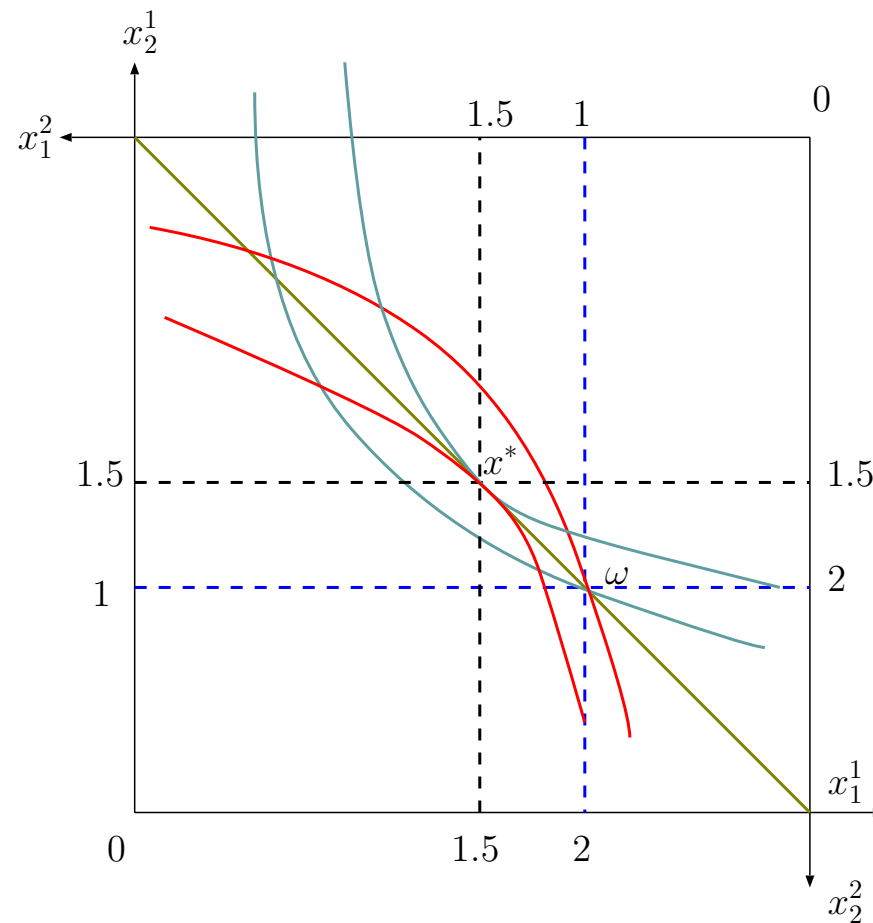
$$\omega^1 = (2, 1) \quad \omega^2 = (1, 2)$$

This case can be represented as in the Edgeworth box below



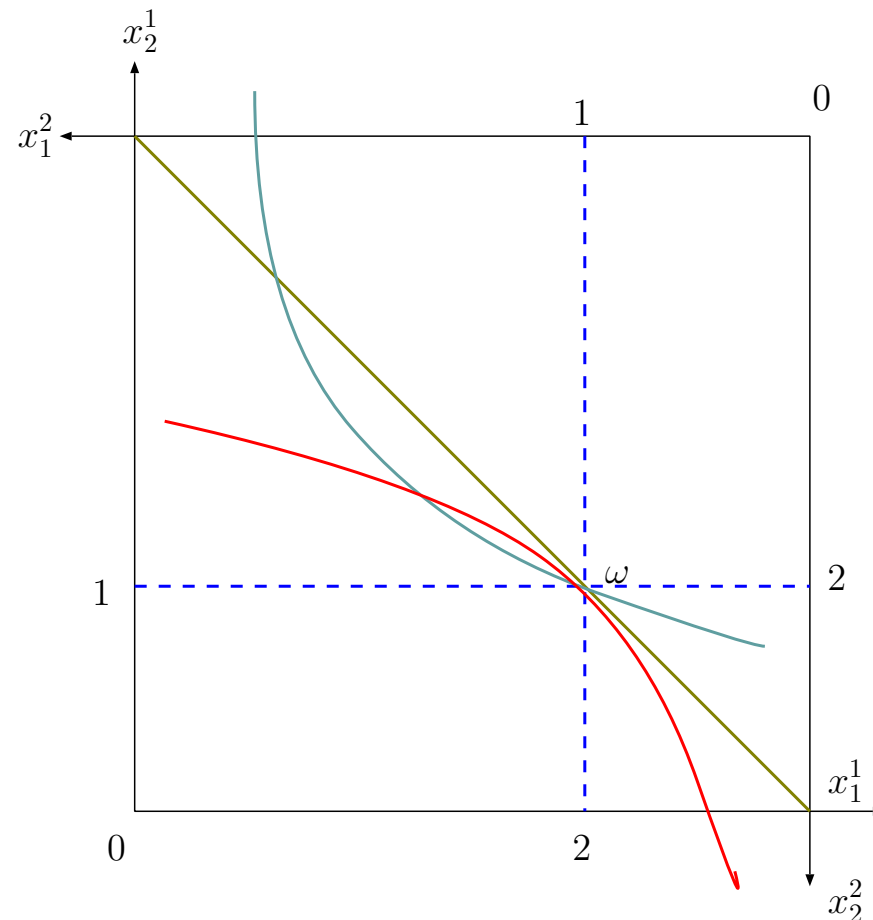
Notice that, given the asset structure in which the unique asset available is the one with returns $(-1,1)$, the only “transfers” available between states are those that correspond to an “exchange rate” of one-to-one. Hence, in the Edgeworth box above, the **feasible allocations are only** those in the *brown* 45 degrees line

Then, given symmetry and the cobb-douglas properties, $x^{*1} = x^{*2} = (1.5, 1.5)$, $\tilde{z}^{*1} = -\tilde{z}^{*2} = 0.5$ is an equilibrium which is both Pareto efficient and Constrained Pareto efficient



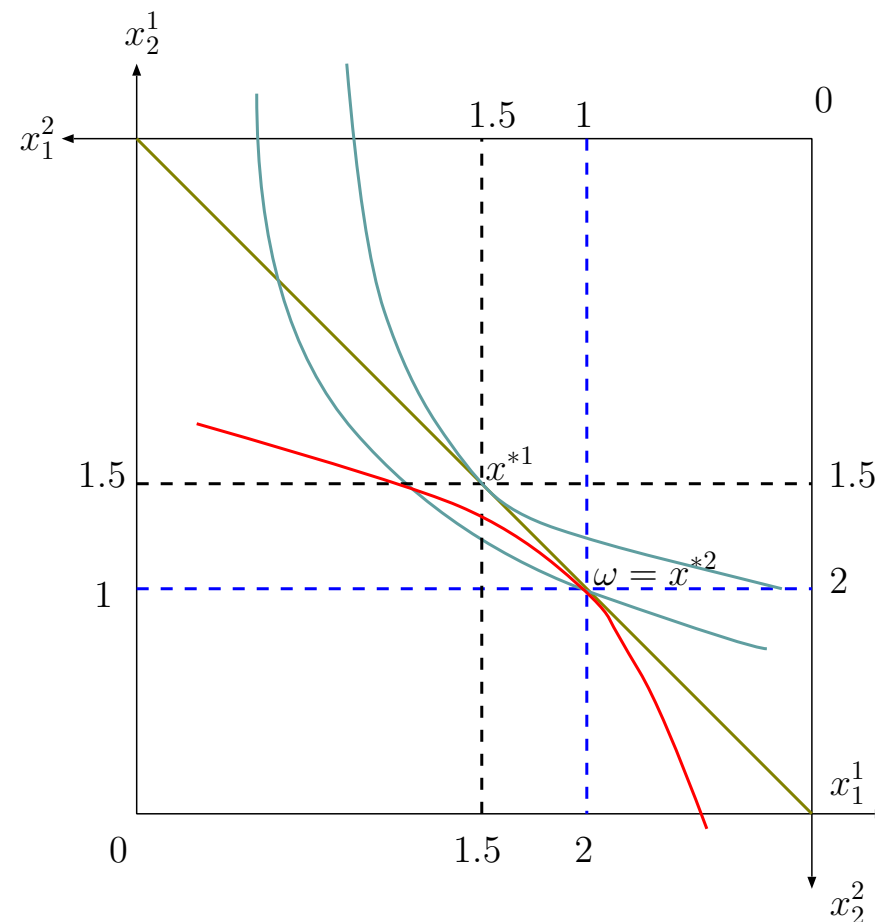
(b) Assume all the same except for individual 2

$$U^2(x_1^2, x_2^2) = x_1^{2\frac{1}{3}} x_2^{2\frac{2}{3}}$$



Notice that since now individual 2 assigns *twice as much weight to state 2 that to state 1*

(according to the coefficients of his utility), he will demand to consume twice as much in state 2 that in state 1 (because prices are 1 in each state). Since individual 1 has not changed, his “optima” will be the same as before.



Now there is no equilibrium since individual 1 still has the same “optima” as before, but now the “optima” for individual 2 is $x^{*2} = (1, 2)$, $\tilde{z}^{*2} = 0$. Clearly, the demands of the two individuals

are incompatibles with the equilibrium definition.

The only “Constrained Pareto efficient” allocation is $x^{*i} = \omega^i$. Indeed, if we consider *only* the feasible allocations (those in the brown line), we can not improve the utility of any individual 1 without harming individual 2. Notice, though, that is not Pareto efficient