1. Expected utility, risk aversion and stochastic dominance

The objective of this part is to examine the choice of under uncertainty. We divide this chapter in 3 sections.

The first part begins by developing a formal apparatus for modeling risk. We then apply this framework to the study of preferences over risky alternatives. Finally, we examine conditions of the preferences that guarantee the existence of a utility function that represents these preferences.

In the second part, we focus on the particular case in which the outcomes are monetary payoffs. Obviously, this case is very interesting in the area of finance. In this part we present the concept of risk aversion and its measures.

In the last part, we are interested in the comparison of two risky assets in the case in which we have a limited knowledge of individuals' preferences. This comparison leads us to the three concepts of stochastic dominance.

1.1 Expected utility

1.1.1 Description of risky alternatives

Let us suppose that an agent faces a choice among a number of risky alternatives. Each risky alternative may result in one of a number of possible outcomes, but which outcome will occur is uncertain at the time that he must make his choice.

Notation:

X = The set of all possible outcomes.

Examples:

X= A set of consumption bundles.

X= A set of monetary payoffs.

To simplify, in this part we make the following assumptions:

1. The number of possible outcomes of X is finite.
2. The probabilities of the different outcomes in a risky alternative are objectively known.

The concept used to represent a risky alternative is the lottery.

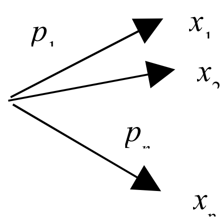
Definition:

A **simple lottery** L is a list $L = (x_1, x_2, ..., x_n; p_1, ..., p_n)$, where

i) $x_i \in X$, $i = 1, \dots, n$

ii) $p_i \geq 0$ and $\sum_{i=1}^{n} p_i = 1$, where p_i is interpreted as the probability of outcome x_i occurring.

Generally to represent a lottery, we use a tree



Notice that in a simple lottery the outcomes are certain. A more general concept, a compound lottery, allows the outcomes of a lottery themselves to be simple lotteries.

Definition:

Given K simple lotteries L_k, $k = 1,...K$, and probabilities $\alpha_k \geq 0$ with $\sum_{k=1}^{K} \alpha_k = 1$, the **compound lottery** $(L_1,...,L_K;\alpha_1,...\alpha_K)$ is the risky alternative that yields the simple lottery L_k with probability α_k, $k = 1,...K$.

For any compound lottery, we can calculate its **reduced lottery** that is a simple lottery that generates the same distribution over the final outcomes.

Exercise: (Illustration of the derivation of a reduced lottery)

1.1. 2 Preferences over lotteries

Having introduced a way to model risky alternatives, we now study the preferences over them corresponding to a fixed agent. In this part we assume that for any risky alternative, only the reduced lottery over final outcomes is of relevance to the agent. This means that given two different compound lotteries with the same reduced lottery, the decision maker is indifferent between them.

Let ℓ denote the set of all simple lotteries over the set of outcomes *X.* According to the previous assumption we assume that the agent's preferences ($\succsim$) are defined on ℓ.

We make the following assumptions

A.1 The decision maker has a preference relation $\succsim$ on ℓ. This means that $\succsim$ satisfy the following properties:

1. $\succsim$ is reflexive ($\forall L \in \ell,\ L \succsim L$)
2. $\succsim$ is complete ($\forall L_1, L_2 \in \ell,$ we either have $L_1 \succsim L_2$ or $L_2 \succsim L_1$)
3. $\succsim$ is transitive ($\forall L_1, L_2, L_3 \in \ell,$ such that $L_1 \succsim L_2$ and $L_2 \succsim L_3$, then $L_1 \succsim L_3$)

A.2 The Archimedian Axiom:

$\forall L_1, L_2, L_3 \in \ell,$ such that $L_1 \succ L_2 \succ L_3$, then there exists $\alpha, \beta \in (0,1)$ such that

$$\alpha L_1 + (1-\alpha) L_3 \succ L_2 \succ \beta L_1 + (1-\beta) L_3$$

Economic intuition:

As $L_1 \succ L_2$, no matter how bad L_3 is, we can combine L_1 and L_3 with α large enough (near to one) such that $\alpha L_1 + (1-\alpha) L_3 \succ L_2$.

As $L_2 \succ L_3$, no matter how good L_1 is, we can combine L_1 and L_3 with β small enough (near to zero) such that $L_2 \succ \beta L_1 + (1-\beta) L_3$.

A.3 The Independence Axiom:

$\forall L_1, L_2, L_3 \in \ell$, and $\alpha \in (0,1]$

$$L_1 \succ L_2 \text{ if and only if } \alpha L_1 + (1-\alpha)L_3 \succ \alpha L_2 + (1-\alpha)L_3$$

Economic intuition:

If we combine two lotteries, L_1 and L_2, with a third lottery in the same way, the preference ordering of the resulting lotteries does not depend on the third lottery.

1.1.3 The expected utility theorem

THE EXPECTED UTILITY THEOREM

Suppose an agent whose preferences ($\succsim$) are defined on ℓ . Then,

1) ($\succsim$) satisfy (A.1)-(A.3)

$$\Updownarrow$$

There exists $u : X \to \Re$ such that

$L_1 \succsim L_2 \Leftrightarrow \sum_{i=1}^{n} p_i^1 u(x_i^1) \geq \sum_{j=1}^{m} p_j^2 u(x_j^2)$, where $L_1 = \left(x_1^1, x_2^1, \dots, x_n^1; p_1^1, \dots, p_n^1\right)$ and $L_2 = \left(x_1^2, x_2^2, \dots, x_m^2; p_1^2, \dots, p_m^2\right)$

2) u and v are two functions that represent these preferences

$$\Updownarrow$$

$v = au + b$, *where* $a, b \in \Re$, *and* $a > 0$

Comments related to this theorem:

1. Consider an individual whose preferences satisfy the previous hypothesis, then he has a utility function that represents her preferences, i.e.,
   There exists $U : \ell \to \Re$ such that $L_1 \succsim L_2 \Leftrightarrow U(L_1) \geq U(L_2)$, for all L_1, L_2.

2. This theorem also states that this utility function $U : \ell \to \Re$ has a form of a expected utility function.

   For any $L = \left(x_1, x_2, \dots, x_n; p_1, \dots, p_n\right)$

$$U(L)=\sum_{i=1}^{n} p_i\, u(x_i\ ),\ \text{where}\ u : X \rightarrow \Re .$$

Notice that any certain outcome has a utility level and the utility of a lottery is measured computing its expected utility level.

3. This utility function is unique, except positive linear transformations.

Notation:

U : the von-Neumann-Morgenstern expected utility function

u: the Bernoulli utility function

1.2 Monetary lotteries and risk aversion

In this section, we focus on risky alternatives whose outcomes are amounts of money.

In Economy, generally we consider money as a continuous variable. Until now we have stated the expected utility theorem assuming a finite number of outcomes. However, this theory can be extended to the case of an infinite domain. Next, we briefly discuss this extension.

1.2.1 Monetary lotteries and the expected utility framework

Notation:

x = amounts of money (continuous variable)

We can describe a monetary lottery by means of a cumulative distribution function, that is,

$$F : \Re \to [0,1]$$

$$F(x) = p(\tilde{x} < x)$$

Therefore, we will take the lottery space to be the set of all distribution functions over nonnegative amounts of money (or more general $[a, \infty)$)

Expected utility theorem:

Consider an agent whose preferences $\succsim$ over ℓ satisfy the assumptions of the theorem, then there exists a utility function U that represents these preferences. Moreover, U has the form of an expected utility function, that is,

$$\exists\ u(.) \ \text{ tq} \ \forall F \ \ U(F) = \int u(x) dF(x) = E(u(\tilde{x})) \ .$$

In addition,

u and v are two functions that represent these preferences

$$\Updownarrow$$

$$v = au + b, \text{ where } a, b \in \Re, \text{ and } a > 0$$

Remark: It is important to distinguish U and u.

- $U()$ is defined on the space of simple lotteries and $u()$ is defined on sure amounts of money.

- U: the von-Neumann-Morgenstern expected utility function
  u: the Bernoulli utility function

Hypothesis: We will assume that u() is strictly increasing and differentiable.

1.2.2 Risk aversion

We begin with a definition of risk aversion very general, in the sense that it does not require the expected utility formulation.

Definition:

An individual is **risk averse** if for any monetary lottery F, the lottery that yields $\int x dF(x)$ with certainty is at least as good as the lottery F.

An individual is **risk neutral** if for any monetary lottery F, the agent is indifferent between the lottery that yields $\int x dF(x)$ with certainty and the monetary lottery F.

An individual is **strictly risk averse** if for any monetary lottery F, the agent strictly prefers the lottery that yields $\int x dF(x)$ with certainty than the lottery F.[1]

[1] In this assumption we assume that the lottery F represents a risky alternative. Otherwise, the individual is indifferent between these two lotteries.

Suppose that the decision maker has preferences that admit an expected utility function representation. Let u(.) be a Bernoulli utility function corresponding to these preferences.

Definition:

An individual is **risk averse** if and only if

$$\int u(x)dF(x) \le u(\int xdF(x)), \text{ for all } F \text{ (\textbf{Jensen's Inequality})}$$

or equivalently ,

$$E(u(\tilde{x})) \le u(E(\tilde{x})) \text{ for all random variable } \tilde{x} .$$

An individual is **risk neutral** if and only if

$$\int u(x)dF(x) = u(\int xdF(x)), \text{ for all } F,$$

or equivalently ,

$$E(u(\tilde{x})) = u(E(\tilde{x})) \text{ for all random variable } \tilde{x} .$$

An individual is **(strictly) risk averse** if and only if

$$\int u(x)dF(x) < u(\int xdF(x)), \text{ for all } F$$

or equivalently ,

$$E(u(\tilde{x})) < u(E(\tilde{x})) \text{ for all random variable } \tilde{x} .$$

Proposition:

Suppose a decision maker with a Bernoulli utility function $u(\cdot)$. Then,

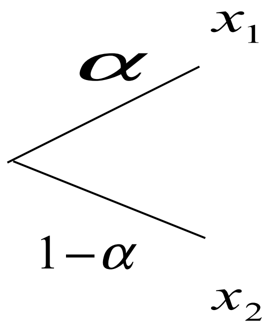
$u(\cdot)$ exhibits (strict) risk aversion $\Leftrightarrow$ $u(\cdot)$ is (strictly) concave

Proof:

$\Rightarrow$ We have to proof that

$$\forall\, x_1, x_2 \text{ and } \forall\, \alpha \in [0,1] \;\; u(\alpha x_1 + (1-\alpha)x_2) \ge \alpha u(x_1) + (1-\alpha)u(x_2) .$$

Consider the following monetary lottery:



Since Jensen’s inequality holds for all the monetary lotteries, in particular for the previous one.

$$E(u(\tilde{x})) \le u(E(\tilde{x}))$$

Now, we develop both sides of this inequality

$$E(u(\tilde{x})) = \alpha u(x_1) + (1-\alpha)u(x_2)$$
$$u(E(\tilde{x})) = u(\alpha x_1 + (1-\alpha)x_2)$$

Therefore, the previous inequality can be written as

$$\alpha u(x_1) + (1-\alpha)u(x_2) \le u(\alpha x_1 + (1-\alpha)x_2) \ .$$

Q.E.D.

$\Leftarrow$ Suppose that u() is concave, then

$$\forall \ x \qquad u(x) \le u(E(\tilde{x})) + u'(E(\tilde{x}))(x - E(\tilde{x}))$$
$$\Downarrow$$
$$u(\tilde{x}) \le u(E(\tilde{x})) + u'(E(\tilde{x}))(\tilde{x} - E(\tilde{x}))$$

$\Downarrow$ **(Taking expected values in both sides)**

$$E(u(\tilde{x})) \le u(E(\tilde{x}))$$

Q.E.D.

Next we introduce two concepts related to risk aversion.

Consider a risk averse agent, with a Bernoulli utility function u(.) and an initial wealth W_0. Let $\tilde{z}$ denote the outcome of a gamble. By risk aversion, we have that the individual prefers $E(\tilde{z})$ to $\tilde{z}$, that is

$$E(u(W_0 + \tilde{z})) \leq u(W_0 + E(\tilde{z})).$$

This inequality tells us that to avoid the risk the individual is willing to pay. The maximum amount of money that the individual is willing to pay is called the **Pratt's risk premium** or **insurance risk premium.**

Definition:

Given a decision maker with a Bernoulli utility function $u()$ and a initial wealth W_0, the **Pratt's risk premium** or **insurance risk premium** of $\tilde{z}$ is a certain amount, denoted by $\Pi(\tilde{z})$, such that

$$E(u(W_0 + \tilde{z})) = u(W_0 + E(\tilde{z}) - \Pi(\tilde{z})).$$

The certain amount $W_0 + E(\tilde{z}) - \Pi(\tilde{z})$ is called **certainty equivalent of** $\tilde{z}$, since it is the amount of money for which the individual is indifferent between $\tilde{z}$ and this certain amount.

Example: Let $\tilde{x} = W_0 + \tilde{z}$. Suppose that it takes two values x_1 and x_2 equally likely.

The following example illustrates the use of the risk aversion concept.

Example: Demand for a risky asset

Consider an individual that wants to invest an initial wealth W_0 in financial assets. This individual has a Bernoulli utility function, u, that holds $u' > 0$ and $u'' < 0$.

Suppose that there exist two assets:

Riskless asset with gross return (constant)

Risky asset with gross return (random variable)

The investor's problem consists in

$$\underset{A,B}{Max} \quad E\left(u\left(R_f B + \tilde{R} A\right)\right)$$

$$s.t. \quad A + B = W_0$$

where

A: quantity invested in the risky asset

B: quantity invested in the riskless asset.

We study this problem assuming that $A \geq 0$. It is important to point out that

1) The fact that we do not allow A<0 means that we are assuming that the investor cannot sell an asset that he does not own ("**short-selling constraints**").

2) The fact that A may be greater than W_0 means that the investor may borrow in the riskless asset.

Since $A + B = W_0$, then $B = W_0 - A$. Substituting this expression, the previous individual's choice problem can be reformulated as:

$$\underset{A}{Max} \quad E\left(u\left(R_f W_0 + \left(\tilde{R} - R_f\right)A\right)\right).$$

Notation: $V(A) = E\left(u\left(R_f W_0 + \left(\tilde{R} - R_f\right)A\right)\right)$.

Property: V is a strictly concave function.

Derivating with respect to A, we have

$$V'(A) = E\left(u'\left(R_f W_0 + \left(\widetilde{R} - R_f\right)A\right)\left(\widetilde{R} - R_f\right)\right)$$

and

$$V''(A) = E\left(u''\left(R_f W_0 + \left(\widetilde{R} - R_f\right)A\right)\left(\widetilde{R} - R_f\right)^2\right) < 0$$

$-$ $+$

The expectation of a negative random variable is negative

We distinguish 3 cases:

1) *V'(A)*>0, $\forall A \geq 0$.

In this case V is strictly increasing $\rightarrow$ A finite solution does not exist.

<u>Example:</u>
If $\widetilde{R} > R_f \rightarrow$ *V'(A)*>0

<u>Economic Intuition:</u>

If $\widetilde{R} > R_f$, then the investor will borrow in the riskless asset and will invest all in the risky asset. Since there is no restriction on the borrowing level, the investor will borrow an infinite quantity to obtain infinite profits.

2) V'(0)$\leq$0

If V'(0)$\leq 0 \rightarrow$ *V'(A)*<0, $\forall A > 0 \rightarrow$ V is strictly decreasing in A $\rightarrow A^* = 0$

Notice that V'(0)= $E\left(u'\left(R_f W_0\right)\left(\widetilde{R} - R_f\right)\right) = u'\left(R_f W_0\right)\left(E\left(\widetilde{R}\right) - R_f\right)$. Therefore
V'(0)$\leq 0 \leftrightarrow E\left(\widetilde{R}\right) \leq R_f$

<u>Intuition:</u>

If the expected return of the risky asset is smaller than the return of the riskless asset, a risk averse agent will not invest in the risky asset.

3) From the previous two cases, we know that $A^* > 0$ and A^* finite implies that $E\left(\widetilde{R}\right) > R_f$ and $\widetilde{R} > R_f$ does not hold.

Conclusion:

A risk averse agent will invest in a risky asset $\leftrightarrow$ $E\left(\widetilde{R}\right) > R_f$

1.2.3 Measures of Risk Aversion

Now we try to measure the extent of risk aversion. We begin with the most used measure of risk aversion which is **the coefficient of absolute risk aversion (also called the Arrow-Pratt's coefficient)**

The Coefficient of Absolute Risk Aversion

Motivation of the coefficient of absolute risk aversion:

Notice that risk aversion is equivalent to the concavity of $u(\ )$, that is, $u'' \leq 0$. Therefore, it seems logical to start considering one possible measure: u''. However, this is not an adequate measure because is not invariant to positive linear transformations. To make it invariant, the simplest modification is to normalize with u'.

Definition:

Given a Bernoulli utility function $u(\ )$, **the coefficient of absolute risk aversion at x is defined as**

$$R_A(x) = -\frac{u''(x)}{u'(x)}.$$

Comments of this expression:

- This measure is invariant to linear transformations.
- The sign minus makes the expression be positive when $u()$ is increasing and concave.

Next, we use this measure in two comparative statics exercises.

1) Comparison of risk attitudes across individuals with different utility functions
2) Comparison of risk attitudes for one individual at different levels of wealth.

Comparison across individuals

Consider two individuals, individual 1 and individual 2, with two Bernoulli utility functions u_1 and u_2, respectively, such that:

1) $u_1' > 0$ and $u_2' > 0$,
2) $u_1'' < 0$ and $u_2'' < 0$.

Next, we want to prove that the coefficient of absolute risk aversion is an effective measure, that is, if individual 1 is more risk averse than individual 2, then it holds $R_A^1(x) \geq R_A^2(x)$, $\forall x$, and vice versa, where

$$R_A^i(x) = -\frac{u_i''(x)}{u_i'(x)}, i = 1,2..$$

Notice that individual 1 is more risk averse than individual 2

$$\Updownarrow$$

∃ G(.), strictly increasing and concave such that $u_1(x) = G(u_2(x))$ ("u_1 is more concave than u_2").

Therefore, we want to show

Lemma 1:

∃ G(.), strictly increasing and concave such that $u_1(x) = G(u_2(x))$ $\Leftrightarrow R_A^1(x) \geq R_A^2(x), \forall x$

$\Rightarrow$ Suppose that $u_1(x) = G(u_2(x))$. Then ,

$$u'_1(x) = G'(u_2(x))u'_2(x) \text{ and}$$

$$u''_1(x) = G''(u_2(x))(u'_2(x))^2 + G'(u_2(x))u''_2(x).$$

Therefore,

$$R_A^1(x) = -\frac{u_1''(x)}{u_1'(x)} = -\frac{G''(u_2(x))(u'_2(x))^2 + G'(u_2(x))u''_2(x)}{G'(u_2(x))u'_2(x)} =$$

$$= -\frac{G''(u_2(x))(u'_2(x))^2}{G'(u_2(x))u'_2(x)} - \frac{G'(u_2(x))u''_2(x)}{G'(u_2(x))u'_2(x)} \geq -\frac{u''_2(x)}{u'_2(x)} = R_A^2(x) \quad (*)$$

$\Leftarrow$ Now we assume that $R_A^1(x) \geq R_A^2(x), \forall x$.

Notice that since u_1 and u_2 are strictly increasing, it is true that exits G(.) such that $u_1(x) = G(u_2(x))$. (Notice that $G = u_1 \circ u_2^{-1}$ and it is differentiable). Therefore, $u'_1(x) = G'(u_2(x))u'_2(x)$, which implies that G'>0.

Moreover, using (*), we know that

$$R_A^1(x) = -\frac{G''(u_2(x))(u'_2(x))^2}{G'(u_2(x))u'_2(x)} + R_A^2(x).$$

Using the fact that $R_A^1(x) \geq R_A^2(x)$, the previous equality tells us that

$-\frac{G''(u_2(x))(u'_2(x))^2}{G'(u_2(x))u'_2(x)} \geq 0$, which implies that $G''(u_2(x)) \leq 0$ because u_2 and G are strictly increasing functions.

Note:

The relation more-risk averse-than relation is a partial ordering of Bernoulli utility functions, since it is not complete. Typically, given two Bernoulli utility functions $R_A^1(x) \geq R_A^2(x)$ at some x, but the contrary inequality holds for other levels.

Lemma 2:

Consider two individuals with strictly increasing and strictly concave Bernoulli utility functions u_1 and u_2, and with identical initial wealth W_0. Then

$$R_A^1(x) \geq R_A^2(x)\ \forall x \Leftrightarrow \Pi_1(\tilde{z}) \geq \Pi_2(\tilde{z})\ \forall \tilde{z}$$

Proof: By Lemma 1, it suffices to show

$\exists$ G(.) strictly increasing and concave such that $u_1(x) = G(u_2(x))$

$\Leftrightarrow \Pi_1(\tilde{z}) \geq \Pi_2(\tilde{z})\ \forall \tilde{z}$

$\Rightarrow$ Using the definition of the Pratt's risk premium for the individual 1, we have

$u_1(W_0 + E(\tilde{z}) - \Pi_1(\tilde{z})) = E(u_1(W_0 + \tilde{z})) = E(G(u_2(W_0 + \tilde{z}))) \leq G(E(u_2(W_0 + \tilde{z}))) =$
$G(u_2(W_0 + E(\tilde{z}) - \Pi_2(\tilde{z}))) = u_1(W_0 + E(\tilde{z}) - \Pi_2(\tilde{z}))$

Therefore,

$$u_1(W_0 + E(\tilde{z}) - \Pi_1(\tilde{z})) \leq u_1(W_0 + E(\tilde{z}) - \Pi_2(\tilde{z})).$$

Using the fact that u_1 is strictly increasing , this inequality implies that

$$W_0 + E(\tilde{z}) - \Pi_1(\tilde{z}) \leq W_0 + E(\tilde{z}) - \Pi_2(\tilde{z}),$$

or equivalently, $\Pi_1(\tilde{z}) \geq \Pi_2(\tilde{z})$.

$\Leftarrow$ Notice that since u_1 and u_2 are strictly increasing, it is true that exits G(.) such that $u_1(x) = G(u_2(x))$ (Notice that $G = u_1 \circ u_2^{-1}$ and it is differentiable**).**

Therefore, $u'_1(x) = G'(u_2(x))u'_2(x)$, which implies that *G'*>0.

To show the concavity of *G*, we will prove Jensen's inequality, that is,

$$E(G(\tilde{x})) \leq G(E(\tilde{x})),\ \forall \tilde{x}.$$

Fix $\tilde{x}$. Then there exists $\tilde{z}$ such that $\tilde{x} = u_2(W_0 + \tilde{z})$. Therefore,

$$E(G(\tilde{x})) = E\left(G\left(u_2(W_0 + \tilde{z})\right)\right) = E\left(u_1\left(W_0 + \tilde{z}\right)\right) = u_1(W_0 + E(\tilde{z}) - \Pi_1(\tilde{z})) \leq$$

$$u_1(W_0 + E(\tilde{z}) - \Pi_2(\tilde{z})) = G\left(u_2(W_0 + E(\tilde{z}) - \Pi_2(\tilde{z}))\right) = G\left(E\left(u_2\left(W_0 + \tilde{z}\right)\right)\right) = G\left(E\left(\tilde{x}\right)\right)$$

Comparison across wealth levels

Typically, richer people are more willing to accept risk than poorer people. Although this might be due to differences in utility functions across people, it is more likely that this is due to differences in the wealth levels. Then, the way to formalize this risk attitude is to assume that $R_A(x)$ is a decreasing function of *x*.

Lemma:

$$\boxed{R_A(x) \text{ is a decreasing function of } x \ \Rightarrow u''' > 0}$$

The Coefficient of Relative Risk Aversion

To understand the concept of relative risk aversion, it is important to point out that the concept of absolute risk aversion is used to compare attitudes toward risky alternatives whose outcomes are absolute gains or absolute losses. But sometimes we consider risky alternatives whose outcomes are percentage gains or losses of current wealth. In this case, we measure the risk aversion by means of the coefficient of relative risk aversion.

Definition:

> Given a Bernoulli utility function $u(\ )$, **the coefficient of relative risk aversion at *x* is defined as**
>
> $$R_R(x) = -\frac{u''(x)}{u'(x)}x = R_A(x)x .$$

Lemma(Relationship between the two coefficients of risk aversion**)**

> Consider an individual with a strictly increasing and strictly concave Bernoulli utility function u. Then
>
> $$\frac{dR_R}{dx} \leq 0 \Rightarrow \frac{dR_A}{dx} < 0 .$$

Proof: Directly follows from $\frac{dR_R}{dx}(x) = \frac{dR_A}{dx}(x) + R_A(x)$.

1.2.4 Risk aversion and portfolio selection

Consider again the portfolio selection problem for an agent with a strictly increasing and strictly concave Bernoulli utility function. Moreover we assume that $E(\tilde{R}) > R_f$. Then

$$\underset{A}{Max} \quad E\left(u\left(R_f W_0 + \left(\tilde{R} - R_f\right)A\right)\right).$$

F.O.C: $$E\left(u'\left(R_f W_0 + \left(\tilde{R} - R_f\right)A\right)\left(\tilde{R} - R_f\right)\right) = 0$$

S.O.C: $$E\left(u''\left(R_f W_0 + \left(\tilde{R} - R_f\right)A\right)\left(\tilde{R} - R_f\right)^2\right) < 0$$

Next, we perform some comparative statics exercises.

1) Comparison of the investment in the risky asset of two agents who differ in their risk attitude.

2) Comparison of the investment in the risky asset of an agent with two distinct initial wealth levels.

1) Consider two individuals with strictly increasing and strictly concave Bernoulli utility functions u_1 and u_2, and with identical initial wealth W_0.

Proposition 1:

Let A_i be the investment in risky asset of agent i, i=1,2. Then,

$$R_A^1(x) \geq R_A^2(x) \ \forall x \Rightarrow A_1 \leq A_2$$

(If individual 1 is more risk averse than individual 2, then the quantity invested in the risky asset of agent 1 is smaller than the one of agent 2)

2) Now, we are interested in studying how vary the investment in the risky asset of an agent when her initial wealth varies.

Proposition 2:

$$\boxed{\begin{array}{l} \dfrac{dR_A}{dx}(x) < 0 \; \forall x \Rightarrow \dfrac{dA}{dW_0} > 0 \\[2ex] \dfrac{dR_A}{dx}(x) = 0 \; \forall x \Rightarrow \dfrac{dA}{dW_0} = 0 \\[2ex] \dfrac{dR_A}{dx}(x) > 0 \; \forall x \Rightarrow \dfrac{dA}{dW_0} < 0 \end{array}}$$

Suppose that an individual has a Bernoulli utility function that exhibits decreasing absolute risk aversion then if the agent becomes richer then he will invest more in the risky asset.

Proposition 3:

Let $a = \frac{A_i}{W_0}$ be the proportion of the initial wealth invested in the risky asset.

Then,

$$\frac{dR_R}{dx}(x) < 0 \, \forall x \Rightarrow \frac{da}{dW_0} > 0$$

$$\frac{dR_R}{dx}(x) = 0 \, \forall x \Rightarrow \frac{da}{dW_0} = 0$$

$$\frac{dR_R}{dx}(x) > 0 \, \forall x \Rightarrow \frac{da}{dW_0} < 0$$

Suppose that an individual has a Bernoulli utility function that exhibits decreasing relative risk aversion then if the agent becomes richer then the proportion of the initial wealth invested in the risky asset increases.

1.3 Stochastic dominance

Suppose that there are two risky assets. The following question is addressed in this part:

Under what conditions can we say that an individual will prefer one asset to another when the only information we have about preferences is that the utility function is increasing or is concave? To answer this question we introduce the concepts of stochastic dominance that are useful to compare random variables.

1.3.1 First degree stochastic dominance

Definition:

$$\boxed{\tilde{x} \underset{\text{FDSD}}{\gtrsim} \tilde{y} \quad \Leftrightarrow \quad \forall\, u(.) \text{ increasing and continuous } \; E(u(\tilde{x})) \geq E(u(\tilde{y}))\,.}$$

Remark:

We ask continuity in order to take expectations. We do not ask differentiability because a function that is continuous and increasing is differentiable almost everywhere.

Property:

$$\boxed{\tilde{x} \underset{\text{FDSD}}{\gtrsim} \tilde{y} \quad \Rightarrow \quad E(\tilde{x}) \geq E(\tilde{y})}$$

Remark:

This property is useful in the sense that if we have $E(\tilde{x}) < E(\tilde{y})$, then $\tilde{x} \underset{\text{FDSD}}{\not\gtrsim} \tilde{y}$

Proof:

Consider u(z)=z.

The opposite implication is not true in general. A counter-example is the following:

Characterizations:

Let $F_x(\cdot)$ and $F_y(\cdot)$ denote the cumulative distribution of $\tilde{x}$ and $\tilde{y}$, respectively.

$$\boxed{\textbf{1. } \tilde{x} \underset{\text{FDSD}}{\gtrsim} \tilde{y} \quad \Leftrightarrow \quad F_x(z) \leq F_y(z), \quad \forall z \in [a,b].}$$

Intuition:

Proof:

$$\boxed{\mathbf{2.}\ \underset{FDSD}{\tilde{x} \succsim \tilde{y}} \quad \Leftrightarrow \tilde{y} \overset{d}{=} \tilde{x} + \tilde{\varepsilon}\text{, with } \tilde{\varepsilon} \le 0\ .}$$

Proof:

$\Rightarrow$ This part is omitted because is very technical.

$\Leftarrow E(u(\tilde{y})) = E(u(\tilde{x} + \tilde{\varepsilon})) \le E(u(\tilde{x}))$

1.3.2 Second degree stochastic dominance

Definition:

$$\boxed{\tilde{x} \succsim \tilde{y} \Leftrightarrow \begin{array}{l}\forall u(.) \text{ concave whose first derivative is continuous } E(u(\tilde{x})) \ge E(u(\tilde{y})) \\ \text{(except on a countable set)}\end{array}.}$$

$\underset{SDSD}{\tilde{x} \succsim \tilde{y}} \equiv \tilde{y}$ is riskier than $\tilde{x}$ in the Rothschild-Stiglitz sense

Properties:

$\underset{SDSD}{>}$

Proof:

$$\tilde{x} \succsim \tilde{y} \Leftrightarrow \begin{array}{l}\forall u(.) \text{ concave whose first derivative is continuous } E(u(\tilde{x})) \ge E(u(\tilde{y})) \\ \text{(except on a countable set)}\end{array}.$$

In particular,

$$\left.\begin{array}{l} u(z) = z \Rightarrow E(\tilde{x}) \ge E(\tilde{y}) \\ u(z) = -z \Rightarrow -E(\tilde{x}) \ge -E(\tilde{y}) \Rightarrow E(\tilde{x}) \le E(\tilde{y}) \end{array}\right\} \Rightarrow E(\tilde{x}) = E(\tilde{y})$$

$$u(z) = -(z - E(\tilde{z}))^2 \Rightarrow -\operatorname{var}(\tilde{x}) \ge -\operatorname{var}(\tilde{y}) \Rightarrow \operatorname{var}(\tilde{x}) \le \operatorname{var}(\tilde{y})$$

The opposite implication is not true. In the Laffont's book there is a numerical counter-example:

Characterizations:

Let $F_x(\cdot)$ and $F_y(\cdot)$ denote the cumulative distribution of $\tilde{x}$ and $\tilde{y}$, respectively.

1. $\tilde{x} \gtrsim \tilde{y}$	$\Leftrightarrow$	i) $\int_a^t \left(F_x(z) - F_y(z)\right)dz \leq 0, \;\; \forall t \in [a,b]$.
SDSD		ii) $\int_a^b \left(F_x(z) - F_y(z)\right)dz = 0$

Intuition:

Proof:

2. $\tilde{x} \underset{\text{SDSD}}{\gtrsim} \tilde{y} \quad \Leftrightarrow \tilde{y} \overset{d}{=} \tilde{x} + \tilde{\varepsilon}$, with $E(\tilde{\varepsilon} \mid \tilde{x}) = 0$.

$\tilde{y}$ is called a mean-preserving spread of $\tilde{x}$ **.**

Intuition:

(If you add noise the new density is more disperse)

Proof:

$\Rightarrow$ This part is omitted because is very technical.

$\Leftarrow$

$$E(u(\tilde{y})) = E(u(\tilde{x}+\tilde{\varepsilon})) = E_{\tilde{x}}\left(E_{\tilde{\varepsilon}}(u(\tilde{x}+\tilde{\varepsilon}) \mid \tilde{x})\right) \leq E_{\tilde{x}}\left(u(E_{\tilde{\varepsilon}}(\tilde{x}+\tilde{\varepsilon}) \mid \tilde{x})\right) = E_{\tilde{x}}(u(\tilde{x})) = E(u(\tilde{x}))$$

1.3.3 Second degree stochastic monotonic dominance

Definition:

$\tilde{x} \underset{\text{SDSMD}}{\gtrsim} \tilde{y} \quad \Leftrightarrow \quad \forall\, u(.)$ increasing and concave $E(u(\tilde{x})) \geq E(u(\tilde{y}))$ **.**

Property:

$\tilde{x} \underset{\text{SDSMD}}{\gtrsim} \tilde{y} \quad \Rightarrow \quad E(\tilde{x}) \geq E(\tilde{y})$

Characterizations:

Let $F_x(\cdot)$ and $F_y(\cdot)$ denote the cumulative distribution of $\tilde{x}$ and $\tilde{y}$, respectively.

1. $\tilde{x} \underset{\text{SDSMD}}{\gtrsim} \tilde{y} \quad \Leftrightarrow \quad$ i) $\int_a^t (F_x(z) - F_y(z))dz \leq 0, \ \forall t \in [a,b]$.

ii) $\int_a^b (F_x(z) - F_y(z))dz \leq 0$

2. $\tilde{x} \underset{\text{SDSMD}}{\gtrsim} \tilde{y} \quad \Leftrightarrow \tilde{y} \overset{d}{=} \tilde{x} + \tilde{\varepsilon}$, with $E(\tilde{\varepsilon} \mid \tilde{x}) \leq 0$.

Application of the concept of stochastic dominance to the simplest portfolio selection problem (Rothschild and Stiglitz (1971))

Consider a risk averse investor with an initial wealth W_0 to be invested in a risky asset, asset 1, with a gross return $\tilde{R}_1$, and a riskless asset with a gross return R_f. Let A_1 denote the optimal amount invested in the risky asset 1. Suppose that A_1 is characterized by the FOC, that is

$$E\left(u'\left(R_f W_0 + \left(\tilde{R}_1 - R_f\right)A_1\right)\left(\tilde{R}_1 - R_f\right)\right) = 0.$$

Imagine now the following situation. There exists another risky asset, asset 2, that is riskier than asset 1 in the Rothschild and Stiglitz sense, $\tilde{R}_1 \underset{\text{SDSD}}{\gtrsim} \tilde{R}_2$.

Question:

If the same individual can invest in the risky asset 2 and the riskless asset, is he going to invest less in the risky asset because now the risky asset is more risky?

Let A_2 denote the optimal amount invested in the risky asset 2. Suppose that A_2 is characterized by the FOC, that is

$$E\left(u'\left(R_f W_0 + \left(\tilde{R}_2 - R_f\right)A_2\right)\left(\tilde{R}_2 - R_f\right)\right) = 0.$$

Notice that a sufficient condition for having $A_2 \leq A_1$ is

$$E\left(u'\left(R_f W_0 + \left(\tilde{R}_2 - R_f\right)A_1\right)\left(\tilde{R}_2 - R_f\right)\right) \leq 0 = E\left(u'\left(R_f W_0 + \left(\tilde{R}_2 - R_f\right)A_2\right)\left(\tilde{R}_2 - R_f\right)\right).$$

Let $g(x) = u'\left(R_f W_0 + \left(x - R_f\right)A_1\right)\left(x - R_f\right)$. If $g(x)$ is concave, then this inequality holds.

Exercise: Prove that $R_R(x) \leq 1, \dfrac{dR_R(x)}{dx} \geq 0$ and $\dfrac{dR_A(x)}{dx} \leq 0$ implies that g is concave.

A utility function that satisfies the tree conditions is

$$u(z) = \frac{z^{\sigma}}{\sigma}, \quad \text{with } 0 < \sigma < 1.$$