3. General Equilibrium Under Uncertainty

- In General Equilibrium Theory, the “allocation” of a given quantity of each commodity implies its final consumption with its corresponding utility score.

- Under uncertainty, the final consumption of a given allocation will depend on the state of nature in such a way that equal quantities of the same commodity could produce different utility scores.

- This raises the issues of:
  - existence of equilibrium in this setting
  - information on the state of the nature (basis for “Information Economics”)
  - possibility of incomplete markets and the need for “artificial markets” (basis for “Financial Economics”)
  - other
3.1 Economies with Contingent Commodities

Everything in the economy (commodities, initial endowments, preferences) will depend on the state of nature. For simplicity, we will not consider economies with production.

States of nature are identified with the possible outcomes of uncertainty. The number of different states of nature is given by $S$. The set $S$ is the set of all the possible states of nature

$$S = \{1, 2, \ldots, S\}$$

We will assume that:

- $S$ is a finite set ($S < \infty$)
- All the elements in $S$ are mutually exclusive
- $S$ is exhaustive
- $S$ is known to everybody
Definition. State Contingent Commodity

The allocation $x_{js}^i \in \mathbb{R}^n$ entitles individual $i \in \mathcal{I}$ to receive (or deliver) $x_{js}^i$ units of commodity $j \in \{1, 2, \ldots, n\}$ if and only if state $s \in S$ occurs.

$\begin{align*}
x_j^i \rightarrow \begin{cases} 
1 & \rightarrow \ x_{j1}^i \\
2 & \rightarrow \ x_{j2}^i \\
\vdots & \vdots \\
S & \rightarrow \ x_{js}^i 
\end{cases}
\end{align*}$

Accordingly,

$x^i = (x_{11}^i, \ldots, x_{1n}^i, x_{12}^i, \ldots, x_{n2}^i, \ldots, x_{1S}^i, \ldots, x_{nS}^i) \in \mathbb{R}^{nS}$

is a state contingent vector, which could be seen as:

$x^i = (x_{11}^i, \ldots, x_{1S}^i, x_{21}^i, \ldots, x_{2S}^i, \ldots, x_{n1}^i, \ldots, x_{nS}^i) \in \mathbb{R}^{nS}$

$x_1$ as R.V. $x_2$ as R.V. $x_n$ as R.V.
• Likewise, initial endowments are of the form

\[ \omega_i = (\omega_{i1}^{\text{State 1}}, \ldots, \omega_{i1n_1}^{\text{State 1}}, \omega_{i2}^{\text{State 2}}, \ldots, \omega_{i2n_2}^{\text{State 2}}, \ldots, \omega_{iS}^{\text{State S}}, \ldots, \omega_{in_S}^{\text{State S}}) \in \mathbb{R}_{+}^{n_S} \]

meaning that the initial endowment of individual \( i \in \mathcal{I} \) if state \( s \in S \) occurs is \( (\omega_{1s}^i, \omega_{2s}^i, \ldots, \omega_{ns}^i) \)

• Preferences will also depend on the state of nature

\[ \succsim^i \text{ are defined on } \mathbb{R}^{n_S} \]

These are “ex ante” preferences as they evaluate random variables before uncertainty is resolved.
Example:

Let \( \pi_s \in [0, 1] \) be the probability of state \( s \in S \) and let \( u^i : \mathbb{R}^n \to \mathbb{R} \) the usual utility function. Then, preferences \( \succeq^i \) on \( \mathbb{R}^{nS} \) could be represented by

\[
x^i \succeq^i \hat{x}^i \\
\sum_{s \in S} \pi_s u^i(x^i_1, \ldots, x^i_n) \geq \sum_{s \in S} \pi_s u^i(\hat{x}^i_1, \ldots, \hat{x}^i_n)
\]

which corresponds to the von Neumann-Morgenstern Expected Utility assumption.
Comments

- Very “rough” model because of the requirements on the set of states of nature $S$

- In a pure exchange economy, trade across states is “physically” impossible

- Some specific “mechanism” (or system) is needed to organize trade in a feasible manner

- All individuals must have the same information, and it must be common knowledge

- Although the model remains static, there are implicit dynamics as decisions are taken before uncertainty is resolved (ex-ante) but trade is implemented after the state of nature is known (ex-post)
3.2 Arrow-Debreu Equilibrium

Corresponds to the “straightforward” generalization of the General Equilibrium Model to accommodate uncertainty.

- There are markets for every contingent commodity
- Markets open before the realization of uncertainty
- Each contingent commodity \( j.s \) has a price \( p_{j.s} \)
- Deliveries are contingent, prices are not
- All individuals must have the same information, and all know that all have the same information, and all know that all know that all have the same information, and ...

A “State Contingent Economy” (or Economy Under Uncertainty) is characterized by

\[
\mathcal{U} = \{ \mathcal{I}, \mathcal{S}, \{\succ_i\}_{i \in I}, \{\omega^i\}_{i \in I} \}
\]
where
\[ \omega^i = \left( \omega^i_{11}, \ldots, \omega^i_{n1}, \omega^i_{12}, \ldots, \omega^i_{n2}, \ldots, \omega^i_{1S}, \ldots, \omega^i_{nS} \right) \in \mathbb{R}^{nS}_+ \]

**Definition. Arrow-Debreu Equilibrium**

An allocation \( x^* \in \mathbb{R}^{nIS} \) together with a price vector \( p^* \in \mathbb{R}^{nS}_+ \) are an Arrow-Debreu Equilibrium if

(i) For all \( i \in I \), \( x^i \) maximizes \( \succeq^i \) on \( B^i \) where

\[ B^i = \{ x \in \mathbb{R}^{nS} \mid p^* \cdot x \leq p^* \cdot \omega^i \} \]

(ii) \( x^* \) is feasible, that is,

\[ \sum_{i \in I} x^{i} \leq \sum_{i \in I} \omega^i \]

- Strict Convexity \( \Rightarrow \) Risk Aversion
• Formally, $\mathcal{U} \leftrightarrow \mathcal{E}^S$, hence the same results as before hold in this case (Existence, Core Equivalence, Welfare Theorems)

**Claim**

$x^*$ is Pareto efficient ex ante $\Rightarrow x^*$ is Pareto efficient ex post

**Proof.** Suppose not, that it, suppose that $s \in \mathcal{S}$ occurs and that there exists $\hat{x}_s \in \mathbb{R}^{nI}$ such that

(i) $\sum_{i \in I} \hat{x}_s^i \leq \sum_{i \in I} \omega_s^i$

(ii) $\hat{x}_s^i \succeq_i^i x_s^i \quad \forall i \in I$ with at least one strict preference

Then,

$$(x_1^i, x_2^i, \ldots, \hat{x}_s^i, \ldots, x_S^i) \succeq^i (x_1^i, x_2^i, \ldots, x_S^i) \quad \forall i \in I$$

with at least one strict preference. Then, $(x_1^i, x_2^i, \ldots, x_S^i)$ would not be Pareto efficient!!

**Interpretation**

Although markets open before uncertainty is revealed, if they were re-opened once the state of nature is known there would not be incentives to further trade
Radner Equilibrium

In the Arrow-Debreu model, all trade takes place simultaneously and before uncertainty is revealed, which is not very realistic. Moreover, although delivery is contingent upon the state of nature that is revealed, payment is not. That is, individuals must pay the contingent commodity in full.
Next, we will assume that trade takes place sequentially (... at least to some extent ...)

Sequence of events in the Arrow-Debreu model

\( t=0 \) All markets exist and are opened. Individuals choose \( x^i \in \mathbb{R}^{n_S} \) and pay prices \( p \in \mathbb{R}_{+}^{n_S} \)
These are markets for delivery at \( t=1 \) (forward markets)

\( t=1 \) Once uncertainty is revealed, and the state of nature \( s \) becomes known, all contracts are executed. Each individual \( i \in I \) receives \( x^i_s \in \mathbb{R}^n \). Notice that if markets were re-opened for trade at \( t=1 \) (spot markets), no further trade would take place.
Sequence of events in the Radner model

$t=0$ Only the market for the first contingent commodity, $x_1 \in \mathbb{R}^S$, exists (forward market). Prices for $x_1$ are $q = (q_1, \ldots, q_S) \in \mathbb{R}_+^S$ (forward prices). This market is for delivery at $t=1$ (forward market).

At the same time, individuals “form expectations” regarding prices of all the other commodities for each possible state at time $t=1$, $p = (p_1, \ldots, p_S) \in \mathbb{R}_+^{nS}$ (spot prices).

Faced with forward prices $q \in \mathbb{R}_+^S$ and expected spot prices $p \in \mathbb{R}_+^{nS}$, individuals formulate and buy forward consumption -or trading- plans for the contingent commodity $z^i \in \mathbb{R}^S$ and decide spot consumption plans for each possible state of nature $x^i = (x^i_1, \ldots, x^i_S) \in \mathbb{R}_+^{nS}$

$t=1$ Once uncertainty is revealed, and the state of nature $s$ becomes known, contracts regarding $z$ are executed. Then, individuals buy their consumption plans $x^i_s \in \mathbb{R}_+^n$ ($i \in \mathcal{I}$) in the spot markets.
Individuals plans *must* satisfy usual budget constraint(s). Thus, each individual *decision problem* is given by

\[
\begin{align*}
\max & \quad U^i(x^i_1, \ldots, x^i_S) \\
\text{s.t.} & \quad \sum_{s \in S} q_s \cdot z^i_s \leq 0 \\
& \quad p_s \cdot x^i_s \leq p_s \cdot \omega^i_s + p_{1s} z^i_s \quad \forall s \in S \\
& \quad (x^i_1, \ldots, x^i_S) \in \mathbb{R}^{nS} \\
& \quad (z^i_1, \ldots, z^i_S) \in \mathbb{R}^S
\end{align*}
\]

(1)

- \( U^i \) represents \( \succeq^i \) on \( \mathbb{R}^{nS}_+ \)
- \( z^i_s < -\omega^i_{1s} \) “selling short”
- All have same expectations on spot prices. Expectations are “self fulfilling”
Definition. Radner Equilibrium

A collection composed of forward prices $q^* \in \mathbb{R}_+^S$, spot prices $p^* \in \mathbb{R}_+^{nS}$, contingent consumption-trading plans at $t=0$, $z^i \in \mathbb{R}^S$, and consumption plans at $t=1$, $x^i \in \mathbb{R}_+^{nS}$, for all individuals $i \in I$ constitute a Radner equilibrium if:

(i) For all $i \in I$, $z^i$ and $x^i$ solve the individuals maximization problem (1), and

(ii) $\sum_{i \in I} z^i \leq 0$ and $\sum_{i \in I} x^i_s \leq \sum_{i \in I} \omega_i^s$ for every $s \in S$
**Proposition 3.2.1 Arrow-Debreu vs. Radner**

(i) If \( x^* \in \mathbb{R}^{nIS} \) and \( p^* \in \mathbb{R}^{nS}_{++} \) are an *Arrow-Debreu equilibrium*, then \( \exists q^* \in \mathbb{R}^{S}_{++} \) and contingent consumption-trading plans \( z^* = (z^*_1, z^*_2, \ldots, z^*_I) \in \mathbb{R}^{IS} \) such that \( q^*, p^*, z^*, x^* \) are a *Radner equilibrium*

(ii) If \( z^* \in \mathbb{R}^{IS}, x^* \in \mathbb{R}^{nIS}_+, q^* \in \mathbb{R}^{S}_{++}, \) and \( p^* \in \mathbb{R}^{nS}_{++} \) are a *Radner equilibrium*, then there are “multipliers” \((\mu_1, \ldots, \mu_S) \in \mathbb{R}^{S}_{++}\) such that the allocation \( x^* \in \mathbb{R}^{nIS}_+ \) and the prices \((\mu_1 p_1, \ldots, \mu_s p_s) \in \mathbb{R}^{nS}_{++}\) are an *Arrow-Debreu equilibrium*

**Proof.**

(i) Let

\[
B_{AD}^i = \{(x_1^i, \ldots, x_S^i) \in \mathbb{R}^{nS}_+ \mid \sum_{s \in S} p_s^* \cdot (x_s^i - \omega_s^i) \leq 0\}
\]

the budget constraint for individual \( i \in \mathcal{I} \) that corresponds to the Arrow-Debreu equilibrium
at prices $p^*$. Analogously, let

$$B^i_R = \{ (x^i_1, \ldots, x^i_S) \in \mathbb{R}^{nS}_+ \mid \exists (z^i_1, \ldots, z^i_S) \in \mathbb{R}^S \text{ such that}$$

$$\sum_{s \in S} q^*_s z^i_s \leq 0 \quad \text{and}$$

$$p^*_s \cdot (x^i_s - \omega^i_s) \leq p^*_{1s} z^i_s \quad \forall s \in S \}$$

the budget constraint that individual $i \in \mathcal{I}$ faces in the Radner equilibrium at prices $p^*$ and $q^*$.

We will show that $B^i_{AD} = B^i_R$.

Indeed, suppose that $x^i = (x^i_1, \ldots, x^i_S) \in B^i_{AD}$ and, for every $s \in S$, define

$$q^*_s = p^*_{1s}$$

$$z^i_s = \frac{1}{p^*_s} p^*_{s} \cdot (x^i_s - \omega^i_s)$$
Then,
\[
\sum_{s \in S} q_s^* z^i_s = \sum_{s \in S} p^*_1 z^i_s = \sum_{s \in S} p^*_s \cdot (x^i_s - \omega^i_s) \leq 0
\]

and
\[
p^*_s \cdot (x^i_s - \omega^i_s) = p^*_1 z^i_s \quad \forall s \in S
\]

Hence, we have that \( x^i \in B^i_{AD} \Rightarrow x^i \in B^i_R \).

Conversely, assume now that \( x^i \in B^i_R \). That is, for some \((z^i_1, \ldots, z^i_S)\) we have that, for every \( s \in S \),
\[
\sum_{s \in S} q_s^* z^i_s \leq 0
\]
\[
p^*_s \cdot (x^i_s - \omega^i_s) \leq p^*_1 z^i_s \quad \forall s \in S
\]

Summing over \( s \)
\[
\sum_{s \in S} p^*_s \cdot (x^i_s - \omega^i_s) \leq \sum_{s \in S} p^*_1 z^i_s = \sum_{s \in S} q_s^* z^i_s \leq 0
\]

Hence, \( x^i \in B^i_R \Rightarrow x^i \in B^i_{AD} \).

Therefore, if \( x^* \in \mathbb{R}^{nIS} \) and \( p^* \in \mathbb{R}^{nS} \) constitute an \textit{Arrow-Debreu equilibrium}, then, \( x^* \) and
$p^*$ together with $q^* = (p_{11}^*, \ldots, p_{1S}^*)$ and $z^* \in \mathbb{R}^{IS}$ defined by

$$z_{s}^i = \frac{1}{p^*_{1s}} p_s \cdot (x_{s}^i - \omega_s^i)$$

are a Radner equilibrium because *market clearing* is satisfied ...

$$\sum_{i \in I} z_{s}^i = \frac{1}{p^*_{1s}} p_s^* \cdot \sum_{i \in I} (x_{s}^i - \omega_s^i) \leq 0$$

(ii) Choose $\mu_s$ so that $\mu_s p_{1s}^* = q_s^*$ and rewrite $B_R^i$ as

$$B_R^i = \{(x_1^i, \ldots, x_S^i) \in \mathbb{R}_+^{nS} \mid \exists (z_1^i, \ldots, z_S^i) \in \mathbb{R}^S \text{ such that}$$

$$\sum_{s \in S} q_s^* z_s^i \leq 0 \text{ and}$$

$$\mu_s p_{1s}^* \cdot (x_s^i - \omega_s^i) \leq q_s^* z_s^i \quad \forall s \in S\}$$
To prove (ii), we can proceed as in (i), to prove that $B^i_R = B^i_{AD}$, where $B^i_{AD}$ in this case is

$$B^i_{AD} = \{(x^i_1, \ldots, x^i_S) \in \mathbb{R}^{n_S^+} \mid \sum_{s \in S} \mu_s p^*_s \cdot (x^i_s - \omega^i_s) \leq 0\}$$
3.3 Asset Markets

In the Radner model we had:

\[(z_1, z_2, \ldots, z_S) \quad S \text{ contingent commodities}\]

that were available at (forward) prices \((q_1, q_2, \ldots, q_S)\)

Hence, at time \(t = 0\) the “shopping basket” was composed of
Contingent Title for Mr. $i \in I$
At time $t = 1$, you are entitled to receive (deliver) $z^i_1$ units of commodity 1 if state 1 occurs
Your bill: $q_1 z^i_1$

Contingent Title for Mr. $i \in I$
At time $t = 1$, you are entitled to receive (deliver) $z^i_2$ units of commodity 1 if state 2 occurs
Your bill: $q_2 z^i_2$

::

Contingent Title for Mr. $i \in I$
At time $t = 1$, you are entitled to receive (deliver) $z^i_S$ units of commodity 1 if state $S$ occurs
Your bill: $q_S z^i_S$

Hence, the total cost of such contingent consumption-trading plan is

$$q_1 z^i_1 + q_2 z^i_2 + \cdots + q_S z^i_S$$
In “real life” we do not observe this kind of markets, but we have assets (securities, insurance policies), which allow for the transfer of wealth across states.

An asset is a title to receive “wealth” at time $t = 1$ in amounts that depend on the state of nature.

For simplicity, we will assume that all assets payback in units of good 1. This is the return of the asset

$$
\text{asset} \quad \begin{cases}
    r_1 & \text{if state 1 occurs} \\
    r_2 & \text{if state 2 occurs} \\
    \vdots & \vdots \\
    r_S & \text{if state S occurs}
\end{cases}
$$

**Definition. Asset**

A unit of an asset is a title to receive an amount $r_s$ of commodity 1 at time $t = 1$ if state $s \in S$ occurs.

Hence, an asset is fully characterized by its return

$$
r = (r_1, r_2, \ldots, r_S) \in \mathbb{R}^S
$$
Examples

(i) \( r = (1, 1, \ldots, 1) \)

Entitles the “owner” to receive 1 unit of commodity 1 in whatever state of nature

- If \( S = 1 \) then it is “riskless”
- If \( S > 1 \) then it is “not riskless”

(ii) \( r = (0, 0, \ldots, 1_s, \ldots, 0) \) is an “Arrow Security”

It is equivalent to the \( z_s = 1 \)

Assumption 3.3.1 Asset Structure

In any economy, there is a given collection of \( A \) different assets available. It is called the asset structure

\[ A = \{1, 2, \ldots, A\} \]
• Each asset \( a \in A \) is characterized by its return
\[
\mathbf{r}_a = (r_{a1}, r_{a2}, \ldots, r_{aS}) \in \mathbb{R}^S
\]

• Each asset \( a \in A \) has a price \( \tilde{q}_a \in \mathbb{R}_+ \).

• The price vector
\[
\tilde{\mathbf{q}} = (\tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_A) \in \mathbb{R}_+^A
\]
describes the prices prevailing at time \( t = 0 \)

• The portfolio of individual \( i \in I \) enumerates how many units of each available asset the individual owns
\[
\tilde{\mathbf{z}}^i = (\tilde{z}_1^i, \tilde{z}_2^i, \ldots, \tilde{z}_A^i) \in \mathbb{R}^A
\]
The cost of the portfolio is
\[
\tilde{q}_1 \tilde{z}_1^i + \cdots + \tilde{q}_A \tilde{z}_A^i
\]
and its return is given by
\[
(\tilde{z}_1^i r_1 + \cdots + \tilde{z}_A^i r_A) \in \mathbb{R}^S
\]
As in the Radner model, individuals' plans must satisfy usual budget constraint(s). In this case, each individual decision problem is given by

\[
\begin{align*}
\max & \quad U^i(x^i_1, \ldots, x^i_S) \\
\text{s.t.} & \quad \sum_{a \in A} \tilde{q}_a \cdot \tilde{z}^i_a \leq 0 \\
& \quad p_s \cdot x^i_s \leq p_s \cdot \omega^i_s + p_{1s} \sum_{a \in A} \tilde{z}^i_a r_{as} \quad \forall s \in S \\
& \quad (x^i_1, \ldots, x^i_S) \in \mathbb{R}^{n_S} \\
& \quad (\tilde{z}^i_1, \ldots, \tilde{z}^i_A) \in \mathbb{R}^A
\end{align*}
\]
Definition. Radner Equilibrium in Asset Markets
A collection composed of forward prices $\tilde{q}^* \in \mathbb{R}^A_+$, spot prices $p^* \in \mathbb{R}^{nS}_+$, asset portfolios at $t=0$, $\tilde{z}^{*i} \in \mathbb{R}^A$, and consumption plans at $t=1$, $x^{*i} \in \mathbb{R}^{nS}_+$, for all individuals $i \in I$ constitute a Radner equilibrium in asset markets if:

(i) For all $i \in I$, $\tilde{z}^{*i}$ and $x^{*i}$ solve the individuals maximization problem (2), and

(ii) $\sum_{i \in I} \tilde{z}^{*i} \leq 0$ and $\sum_{i \in I} x^{*i}_s \leq \sum_{i \in I} \omega^i_s$ for every $s \in S$

(We will normalize $p_{1s} = 1$ for every $s \in S$)

As defined, the Radner equilibrium in asset markets “looks” formally equivalent to that in the previous model. After all, an asset “looks like” a contingent consumption-trading plan in which the $S$ contingent commodities are sold together in a unique bundle.
Can we, therefore, apply Proposition 3.2.1 to asset markets?
Notice !

Buying (in the Radner model) the contingent consumption-trading plan \( z^i = (z^i_1, \ldots, z^i_S) \) at prices \( q = (q_1, \ldots, q_z) \) is equivalent to buying (in the Assets model) one unit of asset \( a \) whose returns are \( r_a = (r_{a1}, \ldots, r_{aS}) = (z^i_1, \ldots, z^i_S) \) at the price \( \tilde{q}_a \), where

\[
\tilde{q}_a = q_1 z^i_1 + \cdots + q_S z^i_S
\]

Question 1 ?

If we buy (in the Assets model) one unit of asset \( a \) whose returns are \( r_a = (r_{a1}, \ldots, r_{aS}) \) at the price \( \tilde{q}_a \), do they exist prices \((q_1, \ldots, q_S)\) such that asset \( a \) is equivalent to buying (in the Radner model) the contingent consumption-trading plan \( z^i = (z^i_1, \ldots, z^i_S) = (r_{a1}, \ldots, r_{aS}) \) at these prices so that

\[
\tilde{q}_a = q_1 r_{a1} + \cdots + q_S r_{aS}
\]

Question 2 ?

Are there enough assets in \( A \) to reproduce any contingent consumption-trading plan in the Radner model ?
For **Question 1** we have the following result

**Proposition 3.3.1**

Assume that for every asset \( a \in A \) we have that \( r_a \geq 0, \) \( (r_a \neq 0) \). Then, for any vector \( \tilde{q} \in \mathbb{R}^A \) of asset prices that might arise in a *Radner equilibrium in asset markets* there are “multipliers” \( \mu = (\mu_1, \ldots, \mu_S) \geq 0 \) such that

\[
\tilde{q}_a = \mu_1 r_{a1} + \cdots + \mu_S r_{aS}
\]

**Proof.** *MGW* pp. 701-703  □

For **Question 2**, consider the following example
Example

Suppose that $S = \{1, 2\}$ and that, according to a Radner equilibrium in the previous model, individual $i \in I$ must implement the following contingent consumption-trading plan

$$z^*i = (2, -3)$$

- Suppose that $A = \{1\}$ with return $r_1 = (1, -1)$. Clearly, in this case $z^*i$ can not be reproduced by any combination of assets in $A$.

- Suppose that $A = \{1, 2\}$ with returns $r_1 = (1, -1)$ and $r_2 = (-3, 2)$. In this case, the portfolio $\tilde{z}^i = (\tilde{z}^i_1, \tilde{z}^i_2) = (5, 1)$ reproduces $z^*i$ since

$$\tilde{z}^i_1 r_1 + \tilde{z}^i_2 r_2 = 5(1, -1) + 1(-3, 2) = (2, -3) = z^*i$$

- Suppose that $A = \{1, 2\}$ with returns $r_1 = (1, -1)$ and $r_2 = (2, -2)$. Can $z^*i$ be reproduced by some portfolio in this case?

Therefore, the return vectors must satisfy some conditions.
**Definition.  Definition Return Matrix**

The return matrix $R$ is a $A \times S$ matrix whose $a^{th}$ row is the return vector of the $a^{th}$ asset.

\[
R = \begin{bmatrix}
    r_{11} & r_{12} & \cdots & r_{1S} \\
    r_{21} & r_{22} & \cdots & r_{2S} \\
    \vdots & \vdots & \ddots & \vdots \\
    r_{A1} & r_{A2} & \cdots & r_{AS}
\end{bmatrix}
\]

**Definition.  Complete structure**

An asset structure $A$ with return matrix $R$ is said to be complete if $\text{rank } R = S$.

- The “rank” of a matrix is the *maximum number of linearly independent rows* the matrix contains.
- Notice, in particular, that according to the definition above, if $A$ is complete then $A \geq S$. 
We can now give an answer to Question 2

**Proposition 3.3.2** Arrow-Debreu vs. Radner in asset markets

If $\mathcal{A}$ is complete, then:

(i) If $x^* \in \mathbb{R}^{nIS}$ and $p^* \in \mathbb{R}^{nS}_+$ are an Arrow-Debreu equilibrium, then $\exists \tilde{q}^* \in \mathbb{R}^A_+$ and portfolios $\tilde{z}^* = (\tilde{z}^1, \tilde{z}^2, \ldots, \tilde{z}^I) \in \mathbb{R}^{IA}$ such that $\tilde{q}^*, p^*, \tilde{z}^*, x^*$ are a Radner equilibrium in the asset markets.

(ii) If $\tilde{z}^* \in \mathbb{R}^{IA}$, $x^* \in \mathbb{R}^{nIS}_+$, $\tilde{q}^* \in \mathbb{R}^A_+$, and $p^* \in \mathbb{R}^{nS}_+$ are a Radner equilibrium in the asset markets, then there are “multipliers” $(\mu_1, \ldots, \mu_S) \in \mathbb{R}^S_+$ such that the allocation $x^* \in \mathbb{R}^{nIS}_+$ and the prices $(\mu_1p_1, \ldots, \mu_sp_s) \in \mathbb{R}^{nS}_+$ are an Arrow-Debreu equilibrium.

**Proof.** MGW p. 705. It is very much like the proof of Proposition 3.2.1
3.4 Incomplete Markets

As seen before, markets are complete if the asset structure $\mathcal{A}$ is such that

$$\text{rank } R = S$$

that is, there are enough assets in $\mathcal{A}$ to reproduce any transfer of wealth across states as if all $nS$ markets were open at time $t = 0$.

For simplicity, let us assume that $A < S$ (there are less assets than \textit{states of nature}) and that $n = 1$ (only one commodity at each state).

Recall the budget constraint in 2

$$p_s \cdot x_s^i \leq p_s \cdot \omega_s^i + p_{1s} \sum_{a \in \mathcal{A}} \tilde{z}_a^i r_{as} \quad \forall s \in S$$
3.- Equilibrium under uncertainty

Under the assumption $n = 1$, such constraint becomes

$$x^i_{1s} = \omega^i_{1s} + \sum_{a \in A} \tilde{z}^i_a r_{as} \quad \forall s \in S$$
Hence, the consumption bundle \( x^i \) is completely determined by the portfolio \( \tilde{z}^i \).
We can, therefore, define the *utility induced by portfolio* \( \tilde{z}^i \) as

\[
\tilde{U}^i(\tilde{z}^i) = U^i(\omega_{11}^i + \sum_{a \in A} \tilde{z}^i_a r_{a1}, \ldots, \omega_{1S}^i + \sum_{a \in A} \tilde{z}^i_a r_{aS})
\]

**Definition.** *Constrained Pareto Efficiency*

The asset allocation \((\tilde{z}^1, \ldots, \tilde{z}^I) \in \mathbb{R}^{SI}\) is constrained Pareto efficient if it is feasible and there is no other feasible asset allocation \((\tilde{z}'^1, \ldots, \tilde{z}'^I)\) such that

\[
\tilde{U}^i(\tilde{z}'^1, \ldots, \tilde{z}'^I) \geq \tilde{U}^i(\tilde{z}^1, \ldots, \tilde{z}^i)
\]

for every \( i \in I \) with at least one strict preference.
Example:

\[ I = \{1, 2\} \quad n = 1 \]

\[ S = \{1, 2\} \quad A = \{1\} \quad r = (-1, 1) \]

Being \( \tilde{z}^i \) the amount of (the only) asset chosen by individual \( i \), each individual decision problem is given by

\[
\max \quad U^i(x^i_1, x^i_2)
\]

s.t. \( \tilde{q}\tilde{z}^i \leq 0 \)

\[ x^i_1 \leq \omega^i_1 - \tilde{z}^i \]

\[ x^i_2 \leq \omega^i_2 + \tilde{z}^i \]

(a) Assume

\[ U^i(x^i_1, x^i_2) = \sqrt{x^i_1 x^i_2} \quad (i = \{1, 2\}) \]

\[ \omega^1 = (2, 1) \quad \omega^2 = (1, 2) \]
This case can be represented as in the Edgeworth box below

Notice that, given the asset structure in which the unique asset available is the one with returns (-1,1), the only “transfers” available between states are those that correspond to an “exchange rate” of one-to-one. Hence, in the Edgeworth box above, the **feasible allocations are only** those in the *brown* 45 degrees line
Then, given symmetry and the cobb-douglas properties, $x^1 = x^2 = (1.5, 1.5)$, $\tilde{z}^1 = -\tilde{z}^2 = 0.5$ is an equilibrium which is both Pareto efficient and Constrained Pareto efficient.
(b) Assume all the same except for individual 2

\[ U^2(x_1^2, x_2^2) = x_1^{\frac{1}{3}} x_2^{\frac{2}{3}} \]

Notice that since now individual 2 assigns twice as much weight to state 2 that to state 1
(according to the coefficients of his utility), he will demand to consume twice as much in state 2 that in state 1 (because prices are 1 in each state). Since individual 1 has not changed, his “optima” will be the same as before.

Now there is no equilibrium since individual 1 still has the same “optima” as before, but now the “optima” for individual 2 is $x^*2 = (1, 2)$, $\tilde{z}^*2 = 0$. Clearly, the demands of the two individuals
are incompatibles with the equilibrium definition.

The only “Constrained Pareto efficient” allocation is $x^{*i} = \omega^i$. Indeed, if we consider only the feasible allocations (those in the brown line), we can not improve the utility of any individual 1 without harming individual 2. Notice, though, that is not Pareto efficient