Microeconomics II
- IDEA -

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Year 2006-2007
(2nd Term)

Syllabus

Introduction

1. General Equilibrium Theory (*4 weeks*)

   (a) Equilibrium in Exchange Economies
   (b) Equilibrium in Perfectly Competitive Economies
   (c) Equilibrium in Production
   (d) Core and Equilibria

2. Social Choice and Welfare Economics (*2 weeks*)

   (a) Arrow's Impossibility Theorem
   (b) Restricted Domains
   (c) Social Choice Functions

3. Partial Equilibrium under Imperfect Competition (*2 weeks*)

   (a) Cournot Competition
   (b) Bertrand Competition
   (c) Stackelberg Competition
   (d) Monopolistic Competition

4. Externalities and Public Goods (3 weeks)

   (a) Bilateral externalities: The inefficiency of equilibrium
   (b) Classical solutions, property rights, and missing markets
   (c) Public Goods: The inefficiency of private provision
   (d) Lindahl Equilibrium

5. Information Economics (*3 weeks*)

   (a) Adverse Selection
   (b) Moral Hazard

References

✔ JEHLE, G.A., RENY, P.J., Advanced Microeconomic Theory, Addison Wesley (2nd edition), 2001.

✔ MAS-COLELL, A., WHINSTON, M.D., GREEN, J., Microeconomic Theory, Oxford University Press, 1995.

✔ TAKAYAMA, A., Mathematical Economics, Cambridge University Press (2nd edition), 1996.

✔ DEBREU, G., Theory of Value, Yale University Press, 1959.

✔ VARIAN, H., Microeconomic Analysis, Norton (3rd. edition), 1992.

Introduction

Microeconomics approaches the study of the economy as a complex system were the actions of self-interested agents lead to order (equilibrium) and not chaos.

Alan Kirman (*paraphrasing*)

"Every individual...generally, indeed, neither intends to promote the public interest, nor knows how much he is promoting it. By preferring the support of domestic to that of foreign industry he intends only his own security; and by directing that industry in such a manner as its produce may be of the greatest value, he intends only his own gain, and he is in this, as in many other cases, led by an invisible hand to promote an end which was no part of his intention."

Adam Smith
The Wealth of Nations, Book IV Chapter II

"Order"

ORDER $\Leftrightarrow$ EQUILIBRIUM

Issues

✔ Existence

✔ Uniqueness

✔ Foundations

✔ Properties

✔ Robustness

✔ ...

1.- General Equilibrium Theory

So far ... (Microeconomics I) ...

One Market

$$\underbrace{\begin{array}{cc}
\text{Consumer} & \text{Producer} \\
\downarrow & \downarrow \\
\text{Preferences} & \text{Technology} \\
\downarrow & \downarrow \\
\text{Demand} & \text{Supply} \\
\downarrow & \downarrow \\
\sum \text{Demands} & \sum \text{Supplies}
\end{array}}_{\textit{Market Equilibrium}}$$

In General Equilibrium ...

$$\left.\begin{array}{c}\text{Market } 1\\ \text{Market } 2\\ \vdots\\ \text{Market } n\end{array}\right\} Eq. \Leftrightarrow \left\{\begin{array}{c}\text{Eq. in Market } 1\\ \text{Eq. in Market } 2\\ \vdots\\ \text{Eq. in Market } n\end{array}\right.$$

Plan

1.1. Exchange Economies

1.2. Perfect Competition *without* production

1.3. Perfect Competition *with* production

1.4. Core and Perfectly Competitive Equilibria

1.1. Equilibrium in (pure) Exchange Economies

- Very simple economy - No Production
- No institutions (No Money, No markets, No prices,...)
- What kind of *equilibrium* (order) might emerge by means of a (iterated) process of *voluntary exchange ?*
- Interesting to understand the basics
- “Benchmark” for other systems

Basic Assumptions of the Economy

✔ n perfectly divisible goods (indexed 1 through n)

✔ I "rational" individuals. Let

$$\mathcal{I} = \{1, 2, \ldots, I\}$$

denote the *Set of Individuals*

✔ Initial Endowments

$$\forall i \in \mathcal{I}, \quad \omega^i = (\omega_1^i, \omega_2^i, \ldots, \omega_n^i) \in \mathbb{R}_+^n$$

✔ *Complete, Reflexive, and Transitive* Individual Preferences

$$\forall i \in \mathcal{I}, \quad \succcurlyeq^i \text{ preferences on } \mathbb{R}_+^n$$

✔ Individuals are self-interested *utility maximizers*

✔ Private ownership economy — Non-coercive trade

Simple Case: <u>2 goods — 2 individuals</u>

- Initial endowment of $i \in \mathcal{I} = \{1, 2\}$

$$\omega^i = (\omega_1^i, \omega_2^i)$$

- Total endowment

$$\begin{aligned} \omega &= (\omega_1, \omega_2) \\ \omega_j &= \omega_j^1 + \omega_j^2, \quad j \in \{1, 2\} \end{aligned}$$

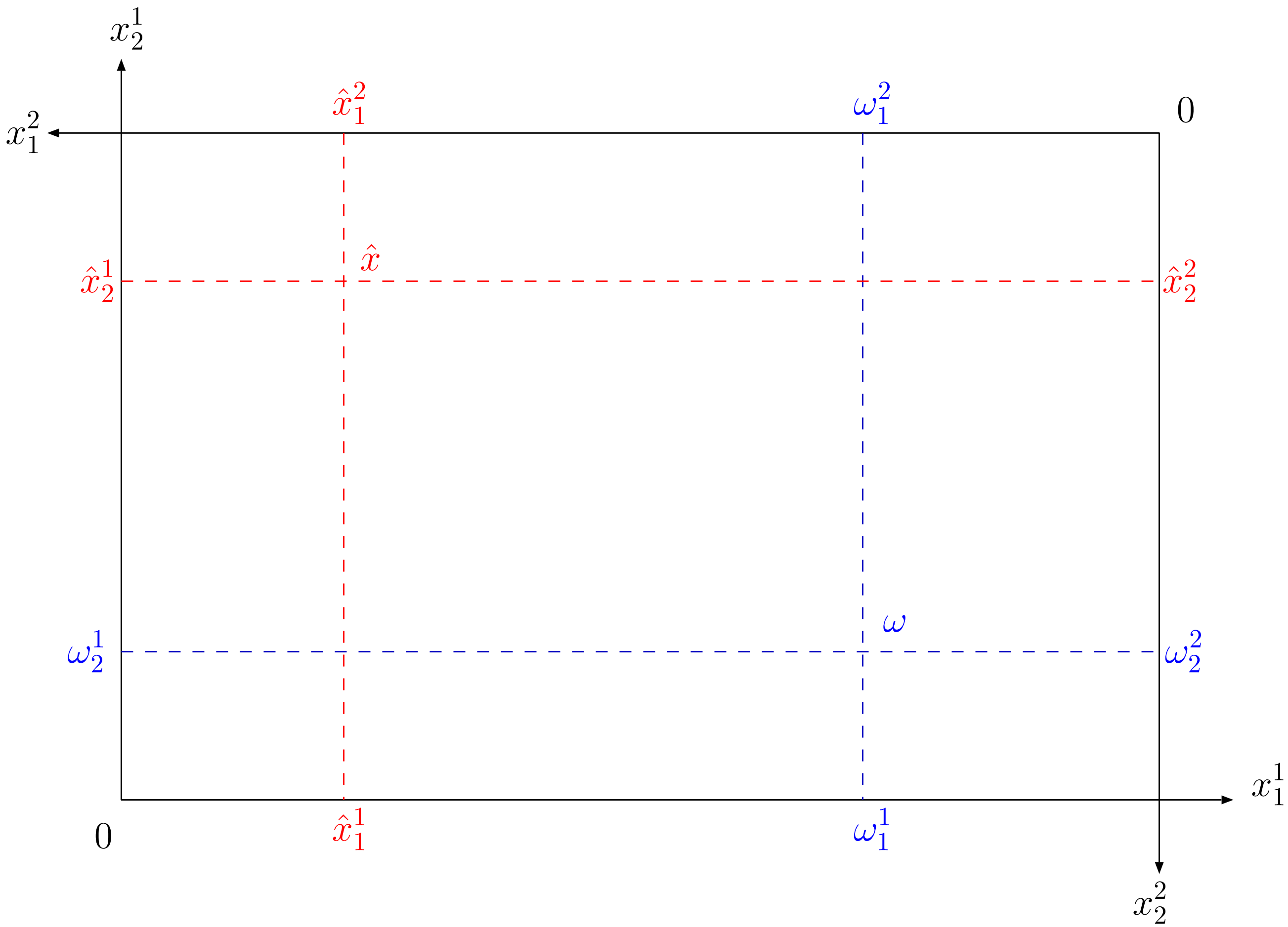
- Allocation

$$x = (x^1, x^2) = (x_1^1, x_2^1, x_1^2, x_2^2)$$

- Feasible allocation

$$\begin{aligned} x_1^1 + x_1^2 &\leq \omega_1 \\ x_2^1 + x_2^2 &\leq \omega_2 \end{aligned}$$

Edgeworth Box Representation (i)



$$\omega = (\omega_1^1, \omega_2^1, \omega_1^2, \omega_2^2)$$

Initial endowments

$$\hat{x} = (\hat{x}_1^1, \hat{x}_2^1, \hat{x}_1^2, \hat{x}_2^2)$$

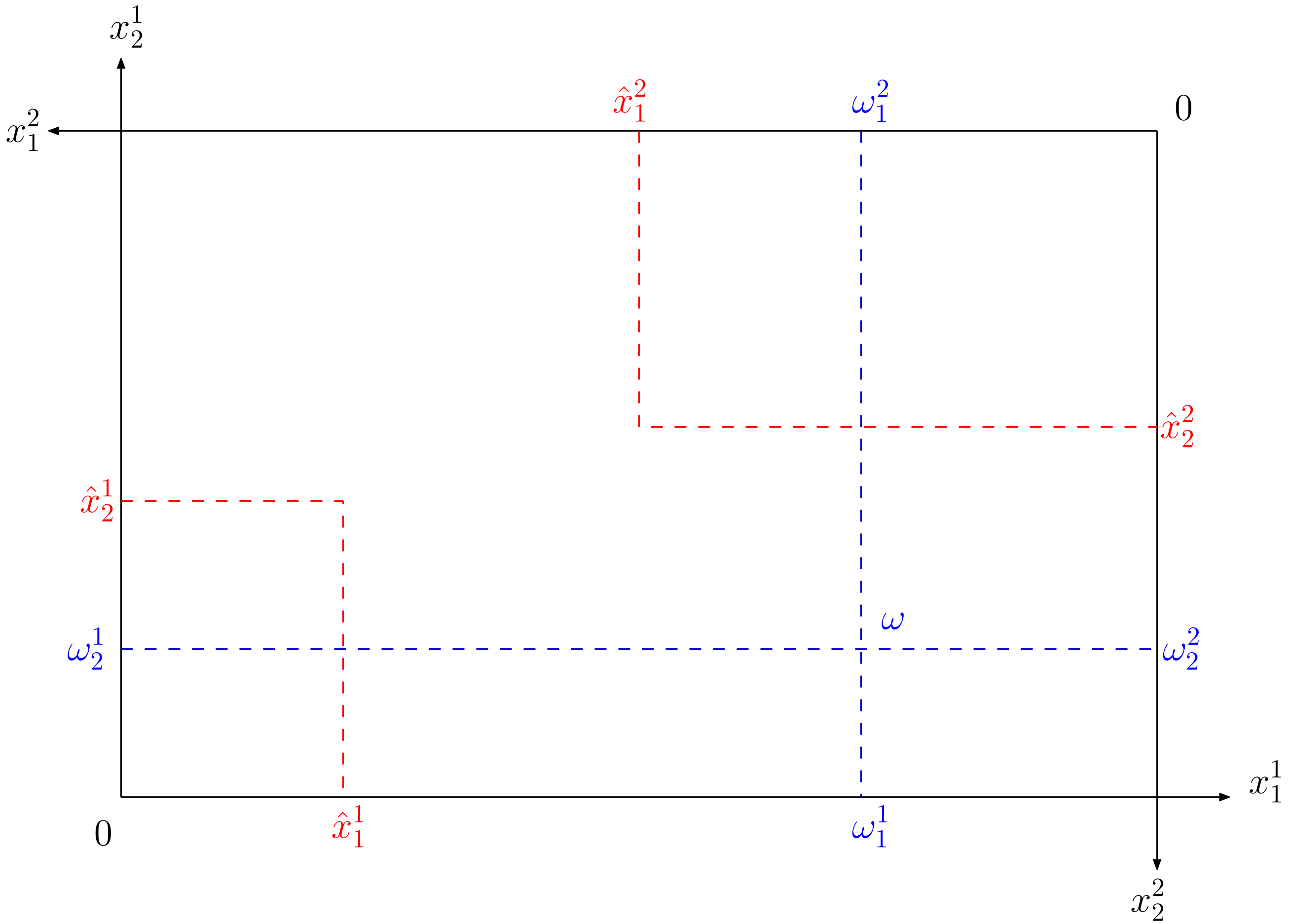
a feasible allocation

Edgeworth Box Representation (ii)

Notice:

Any point in the box $\Rightarrow$ Feasible allocation

Any feasible allocation $\nRightarrow$ Point in the box



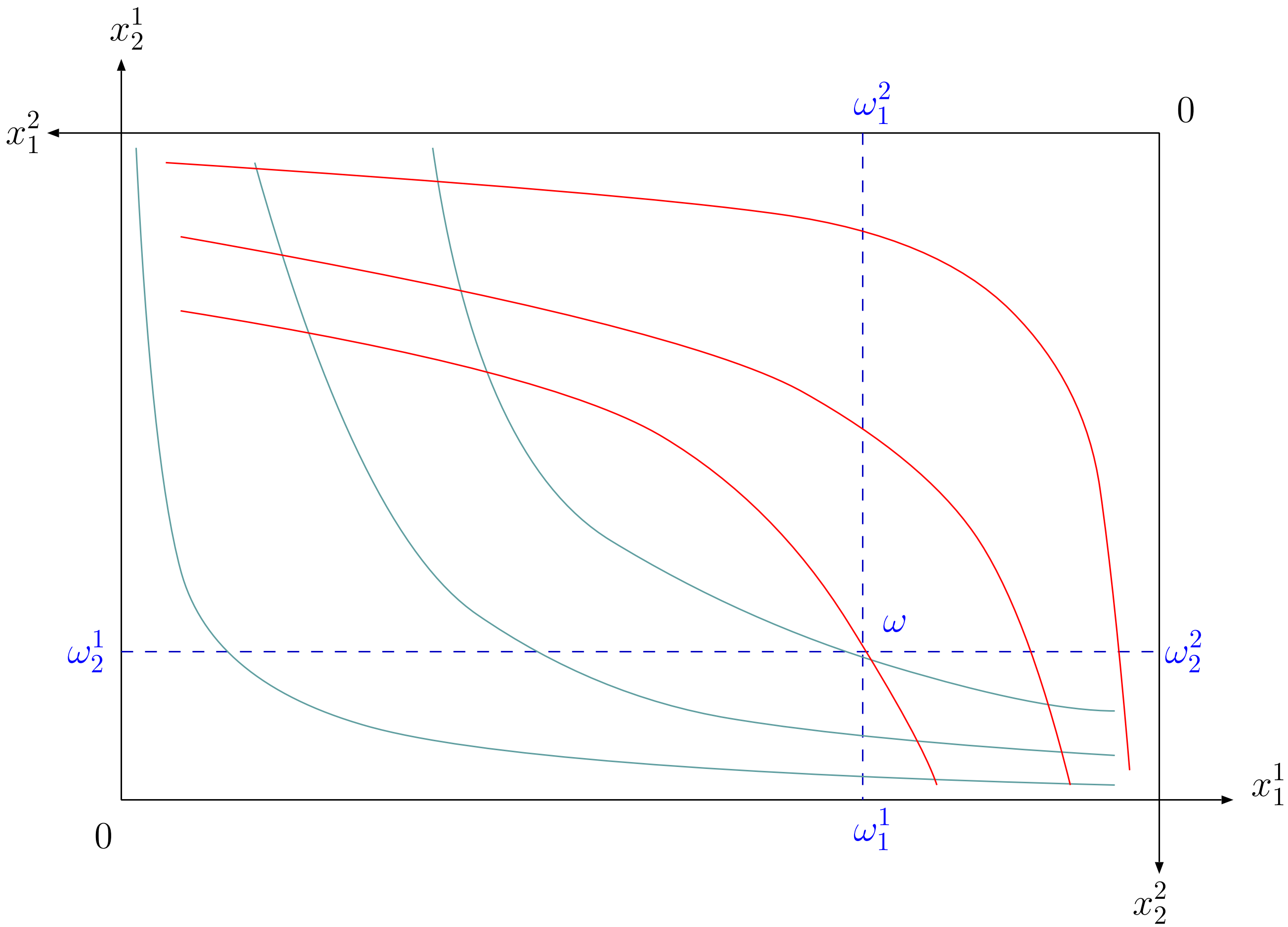
$\omega = (\omega_1^1, \omega_2^1, \omega_1^2, \omega_2^2)$

Initial endowments

$\hat{x} = (\hat{x}_1^1, \hat{x}_2^1, \hat{x}_1^2, \hat{x}_2^2)$

a feasible allocation with *free disposal*

Edgeworth Box Representation (iii)



Adding preferences $\succcurlyeq$

(i) $Complete$
(ii) $Reflexive$
(iii) $Transitive$
(iv) $Monotone$
(v) $Convex$

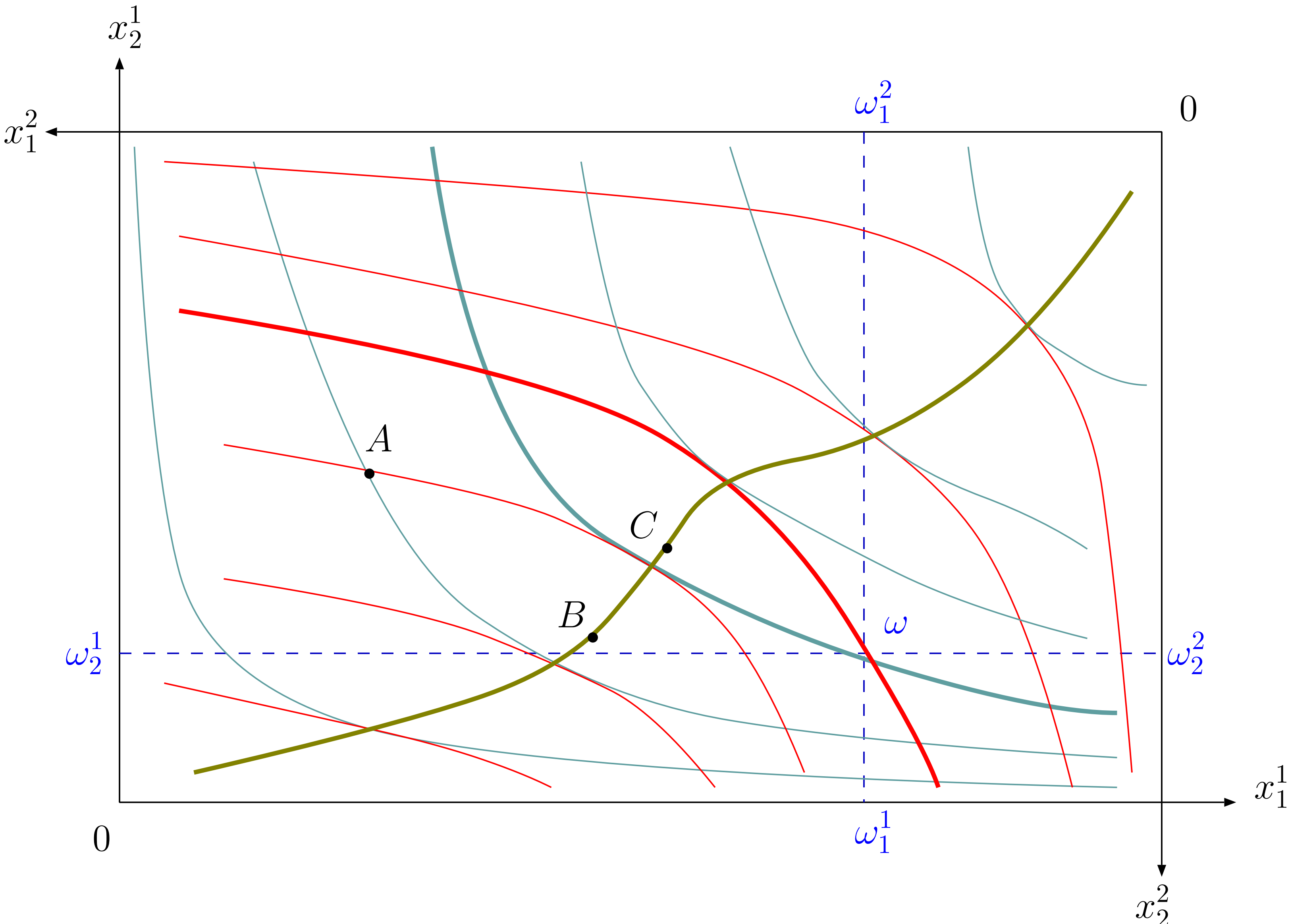
Equilibrium (... stability, order ...)

What conditions should an allocation satisfy to be considered an equilibrium ?

1. *Feasibility*: It has to be feasible

2. *Efficiency*: Minimal requirement
   It should be not possible to improve one individual's utility without harming the other $\rightsquigarrow$ Pareto efficiency

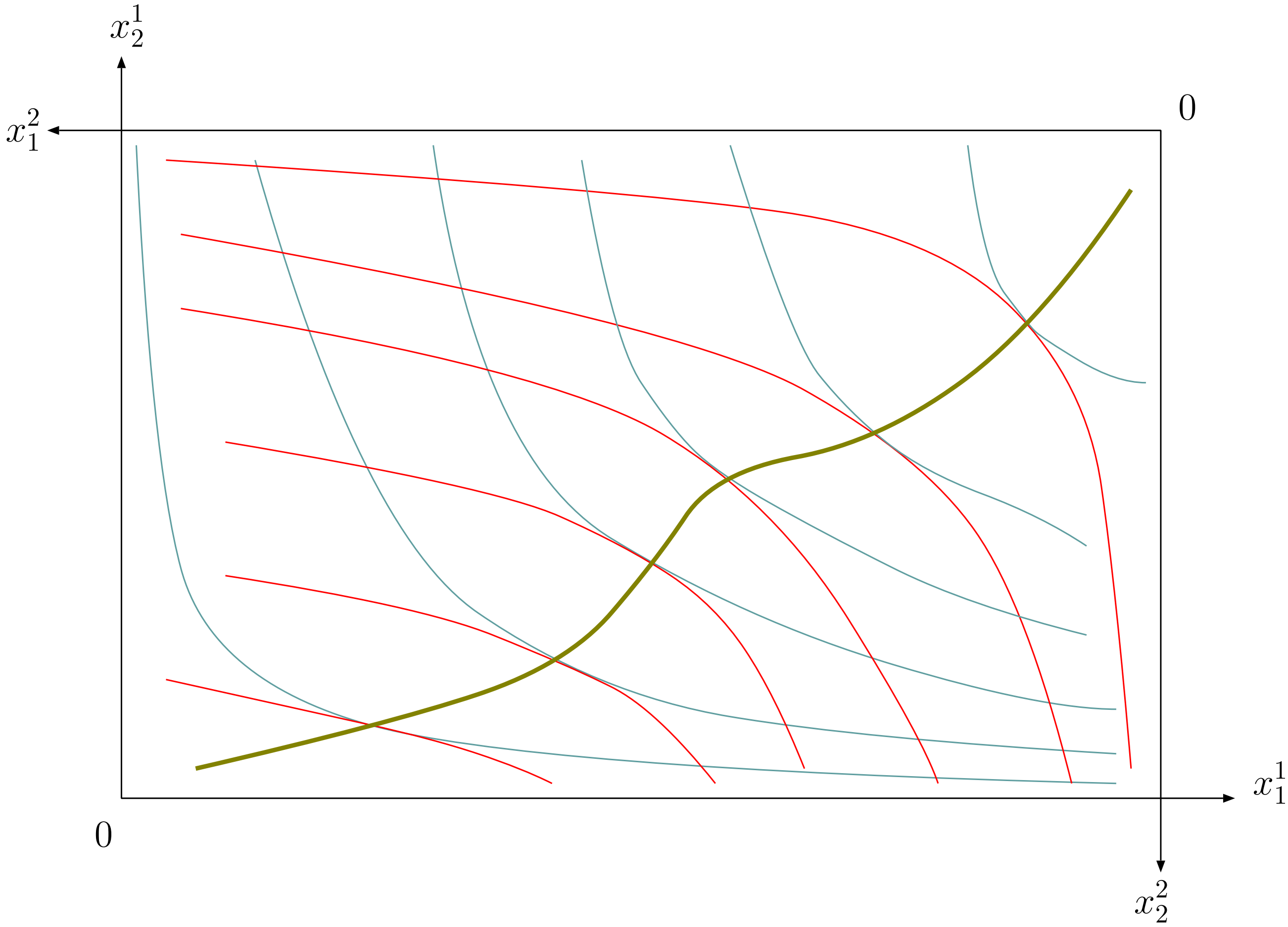
3. *Stability*: Rationality requirement
   Both individuals should be *at least as well as* with their initial endowments $\rightsquigarrow$ Individual Rationality



A is not equilibrium because of B — B is not equilibrium because of C

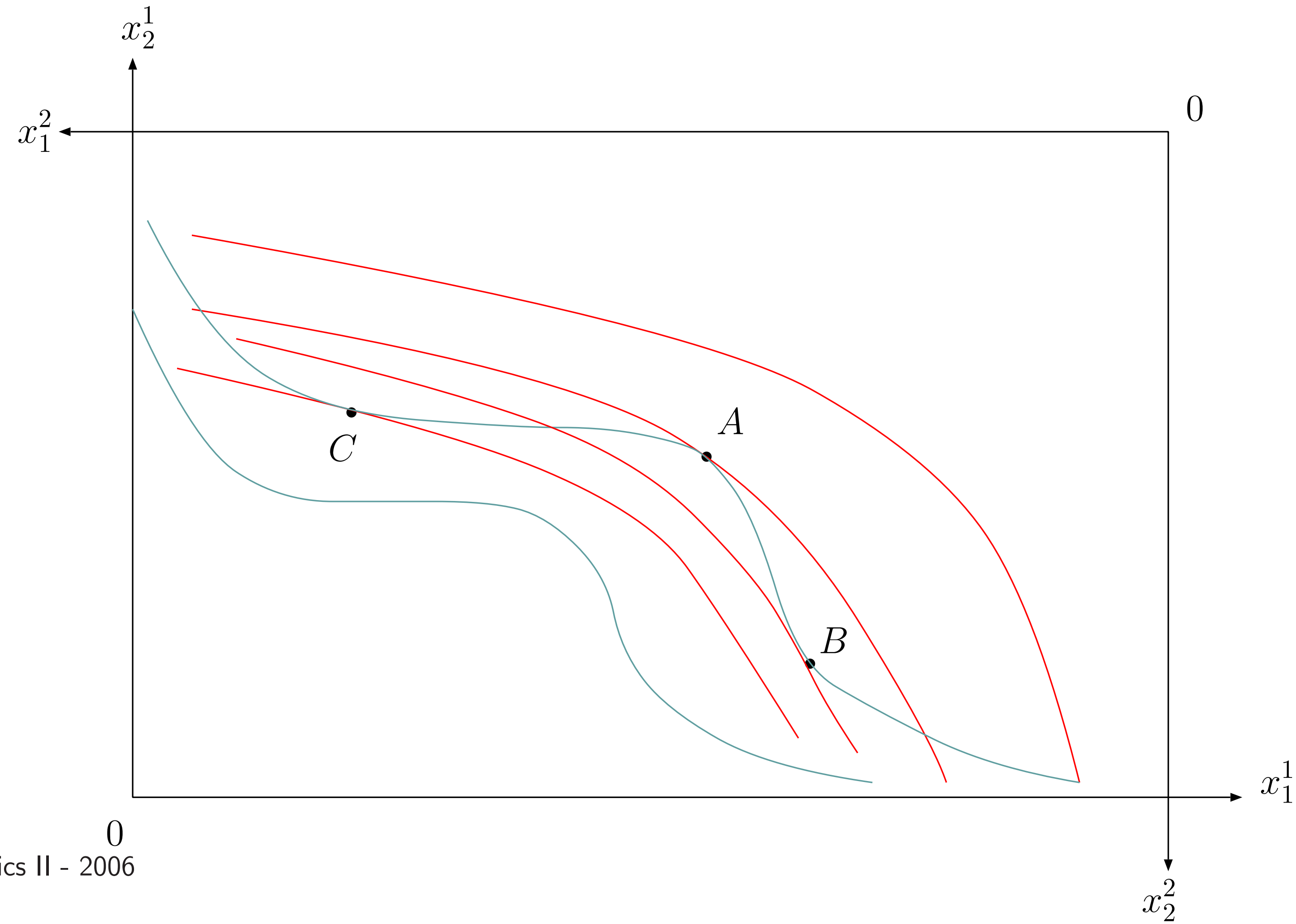
Contract Curve

The set of *efficient* points is called the contract curve. Under the "usual" assumptions, it corresponds to the tangency points between the indifference curves

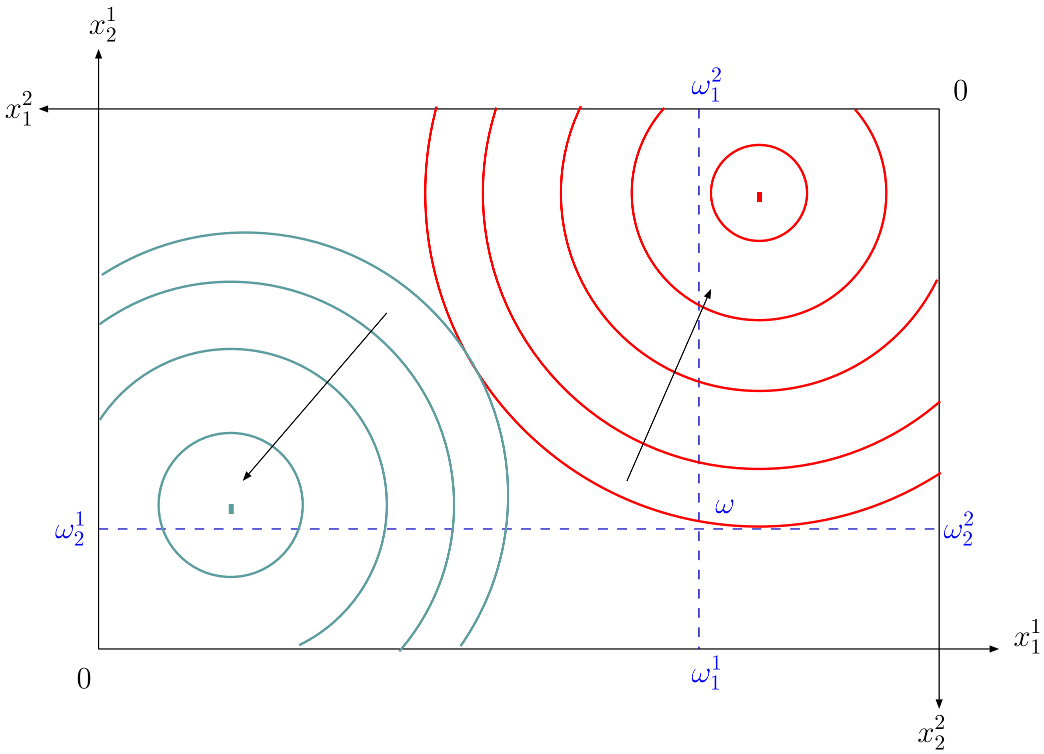


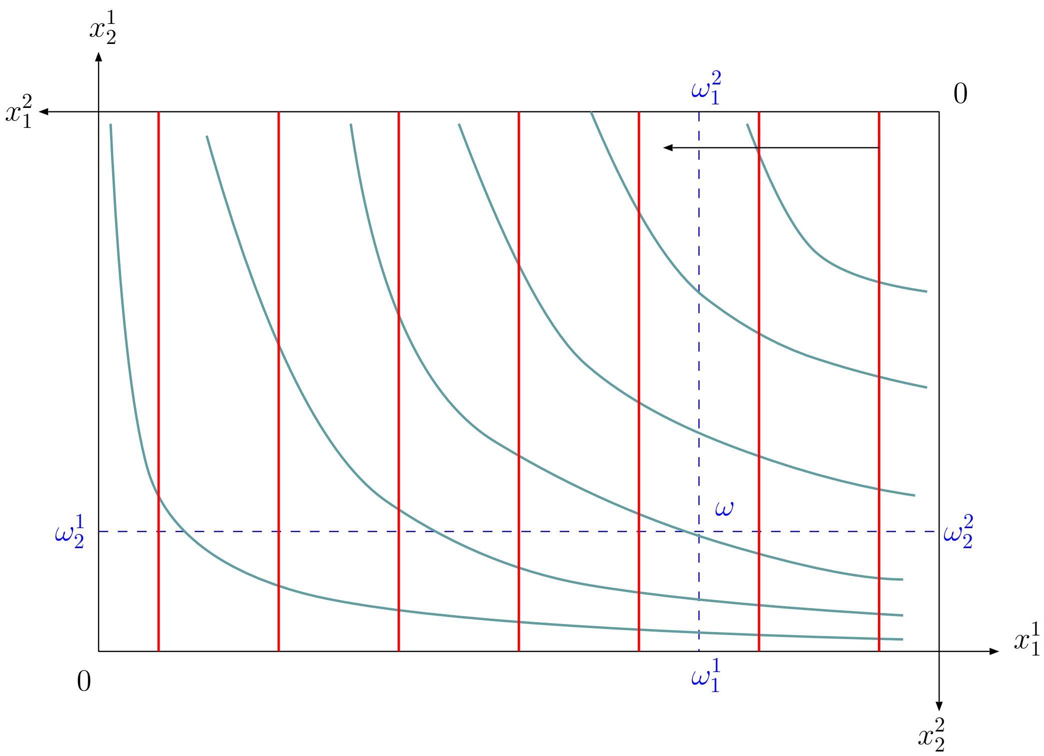
$$\left.\begin{array}{c} \frac{u_2^1(x_1^1,x_2^1)}{u_1^1(x_1^1,x_2^1)} = \frac{u_2^2(x_1^2,x_2^2)}{u_1^2(x_1^2,x_2^2)} \\ x_1^1 + x_1^2 = \omega_1 \\ x_2^1 + x_2^2 = \omega_2 \end{array}\right\} \begin{array}{c} 3\,equations \\ 4\,unknowns \\ x_2^1 = f(x_1^1) \end{array}$$

Without Convexity



Without Monotonicity





General Case: n goods — I individuals

- Initial endowment of $i \in \mathcal{I} = \{1, 2, \ldots, I\}$

$$\omega^i = (\omega_1^i, \omega_2^i, \ldots, \omega_n^1) \in \mathbb{R}_+^n$$

- Preferences of $i \in \mathcal{I} = \{1, 2, \ldots, I\}$

$$\succcurlyeq^i \text{ are } complete,\ reflexive,\ and\, transitive$$

- Total endowment

$$\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in \mathbb{R}_+^n$$
$$\omega_j = \omega_j^1 + \omega_j^2 + \cdots + \omega_j^I, \quad j \in \{1, 2, \ldots, n\}$$

- Allocation

$$x = (x^1, x^2, \ldots, x^I) =$$
$$= (x_1^1, x_2^1, \ldots, x_n^1, \ldots, x_1^I, x_2^I, \ldots, x_n^I) \in \mathbb{R}_+^{nI}$$

Definition. *Exchange economy*
An exchange economy $\mathcal{E}$ is fully characterized by:

$$\mathcal{E} = (\mathcal{I}, \{\succsim^i\}_{i \in \mathcal{I}}, \{\omega^i\}_{i \in \mathcal{I}})$$

Definition. *Feasible allocation*
An allocation $x = (x^1, \ldots, x^I) \in \mathbb{R}_+^{nI}$ is feasible if $\forall j \in \{1, 2, \ldots, n\}$

$$\sum_{i \in \mathcal{I}} x_j^i \leq \sum_{i \in \mathcal{I}} \omega_j^i$$

Definition. *Exhaustive allocation*
An allocation $x = (x^1, \ldots, x^I) \in \mathbb{R}_+^{nI}$ is exhaustive (non-wasteful) if $\forall j \in \{1, 2, \ldots, n\}$

$$\sum_{i \in \mathcal{I}} x_j^i = \sum_{i \in \mathcal{I}} \omega_j^i$$

Definition. *Feasible Set*
Given the initial endowments ω, the set

$$\mathcal{F}(\omega) = \{x \in \mathbb{R}_+^{nI} \mid x \, \text{is feasible}\}$$

is called the Feasible Set

Definition. *Pareto Efficiency*
An allocation $x \in \mathcal{F}(\omega)$ is Pareto efficient if there is no other allocation $y \in \mathcal{F}(\omega)$ such that

$$y^i \succcurlyeq^i x^i \ \ \forall i \in \mathcal{I}$$

with at least one strict preference

Definition. *Blocking Coalition*
Let $S \subset \mathcal{I}$ be a coalition of individuals. We say that S blocks $x \in \mathcal{F}(\omega)$ if $\exists y$ such that

$$(i) \sum_{i \in S} y^i \leq \sum_{i \in S} \omega^i$$
$$(ii)\, y^i \succsim^i x^i \;\forall i \in S$$
$$\text{with at least one strict preference}$$

Definition. *Blocked allocation*
An allocation $x \in \mathcal{F}(\omega)$ is blocked if $\exists S \subset \mathcal{I}$ that blocks x

Definition. *Unblocked allocation*
An allocation $x \in \mathcal{F}(\omega)$ is unblocked if $\nexists S \subset \mathcal{I}$ that blocks x

Definition. *Core*
Given $\mathcal{E} = (\mathcal{I}, \{\succcurlyeq^i\}_{i\in\mathcal{I}}, \{\omega^i\}_{i\in\mathcal{I}})$, *the core of* $\mathcal{E}$ ($\mathcal{C}(\mathcal{E})$) *is the set of all unblocked allocations*

Definition. *Equilibrium*
An allocation x *is an equilibrium of* $\mathcal{E}$ *if* $x \in \mathcal{C}(\mathcal{E})$

Notice ...

$$x \in \mathcal{C}(\omega) \quad \Rightarrow \quad x \text{ is Pareto efficient}$$
$$(\not\Leftarrow)$$

$$x \in \mathcal{C}(\omega) \quad \Rightarrow \quad x \text{ is individually rational}$$
$$(\not\Leftarrow)$$

Example

$\mathcal{I} = \{1, 2, 3\}$, $n = 2$

$u^1(x_1^1, x_2^1) = x_1^1 x_2^1 \quad \omega^1 = (3, 3) \quad u^1(\omega^1) = 9$

$u^2(x_1^2, x_2^2) = x_1^2 x_2^2 \quad \omega^2 = (4, 0) \quad u^2(\omega^2) = 0$

$u^3(x_1^3, x_2^3) = x_1^3 x_2^3 \quad \omega^3 = (0, 4) \quad u^3(\omega^3) = 0$

Consider

$$x^1 = (5, 5),\ x^2 = (1, 1),\ x^3 = (1, 1)$$
$$u^1(x^1) = 25,\ u^2(x^2) = 1,\ u^3(x^3) = 1$$

Clearly,

$$(i)\, x^i \succ^i \omega^i \quad \forall i \in \mathcal{I}$$
$$(ii)\, x \,\mathrm{is\, Pareto\, efficient}$$

Consider the coalition $S = \{2, 3\}$ and the alternative allocation $y^1 = (3, 3),\ y^2 = (2, 2),\ y^3 = (2, 2)$

Now,

$$u^2(y^2) = 4 > u^2(x^2)$$
$$u^3(y^3) = 4 > u^3(x^3)$$
$$y^2,\ y^3 \, \text{is feasible for} \, S$$

Hence, $S = \{2, 3\}$ <u>blocks</u> the allocation x

The Core

The *Core* concept is very important because

✔ It's very intuitive as a minimal requirement for stability

✔ It requires no institutions, no mechanisms, no devices, ...

✔ Extends the idea of <u>individual rationality</u> to groups

✔ It's a "check point" for other equilibrium concepts

✔ Game theoretical concept

... but it has some drawbacks ...

✘ Information requirements

✘ Cost of coordination in coalitions

✘ Implementation

1.2 Equilibrium in Perfectly Competitive Economies

Economic Institution $\rightsquigarrow$ $\boxed{\text{Perfectly Competitive Market System}}$

✔ Self-interested agents (utility/profit maximizers)

✔ "Insignificant" agents (no power to affect prices)

✔ Market Equilibrium $\rightsquigarrow$ Buyers' and Sellers' plans are compatible

✔ General Equilibrium $\rightsquigarrow$ All markets are in equilibrium

✔ (again, for the time being ... <u>no production</u>)

Preliminaries

✔ The *economy* is (still) described by

$$\mathcal{E} = (\mathcal{I}, \{\succcurlyeq^i\}_{i\in\mathcal{I}}, \{\omega^i\}_{i\in\mathcal{I}})$$

✔ *Prices* are attached to each good and are taken "as given" by the agents (perfect competition assumption). The *price vector* is given by:

$$p = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n_{++}$$
$$(p_1, p_2, \ldots, p_n) \gg 0$$

✔ Budget constraint

$$p \cdot x^i \leq p \cdot \omega^i \quad \forall i \in \mathcal{I}$$

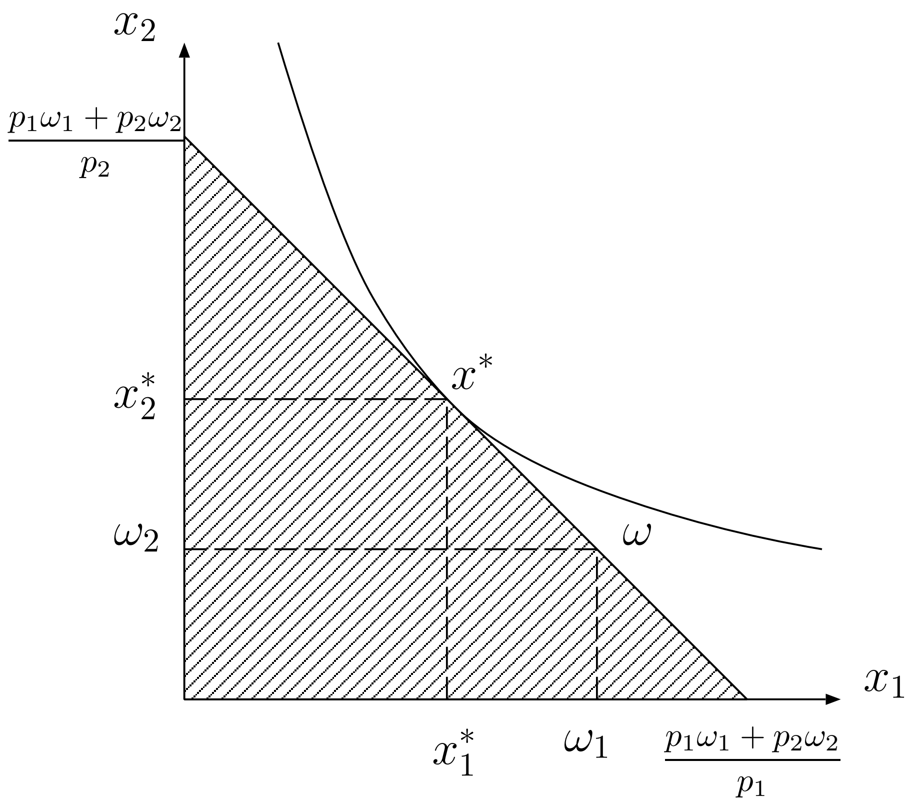
Assumption 1.2.1 The preferences of the individuals ($\succcurlyeq^i$) are represented by a *utility function*

$$u^i : \mathbb{R}^n_+ \to \mathbb{R}$$

that is *continuous, strongly increasing, and strictly quasiconcave* on $\mathbb{R}^n_+$

✔ i's individual problem ($\forall i \in \mathcal{I}$)

$$\max_{x^i \in \mathbb{R}^n_+} u^i(x^i) \quad \text{s.t.} \quad p \cdot x^i \quad \leq \quad p \cdot \omega^i \tag{1}$$



Definition. *Demand function*
The solution to the maximization problem in (1),

$$x^i(p, \omega^i) : \mathbb{R}^n_{++} \times \mathbb{R}^n_+ \rightarrow \mathbb{R}^n_+$$

is called the *<u>demand function</u>* *of individual i*

Proposition 1.2.1 If u^i satisfies Assumption 1.2.1 then,

$$
\begin{array}{ll}
(i) & x^i(p,\omega^i)\, is\, a\, function \\
 & (the\, solution\, to\, (1)\, is\, unique\, for\, each\, p \gg 0) \\
 & \\
(ii) & x^i(p,\omega^i)\,\text{is continuous in}\,p\,\text{on}\,\mathbb{R}^n_{++}
\end{array}
$$

Proof. (... sketch ...)

Existence: Because $p \gg 0 \Rightarrow$ compact budget set

Uniqueness: Because of the strict quasiconcavity of u^i

Continuity: Because of the *Theorem of the Maximum* ($p \gg 0$ is required) $\square$

Definition. *For every $j \in (1, 2, \ldots, n)$,*

$$\sum_{i\in\mathcal{I}} x^i_j(p,\omega^i) \qquad \text{is called the } \underline{\textit{aggregate demand}}$$

$$\sum_{i\in\mathcal{I}} \omega^i_j \qquad \text{is called the } \underline{\textit{aggregate supply}}$$

Definition. *Excess Demand*
The excess demand function in market $j \in \{1, \ldots, n\}$ is given by

$$z_j(p) = \sum_{i \in \mathcal{I}} x_j^i(p, \omega^i) - \sum_{i \in \mathcal{I}} \omega_j^i$$

Aggregate excess demand is the vector

$$z(p) = (z_1(p), z_2(p), \ldots, z_n(p))$$

Theorem 1.2.1 Properties of Excess Demand Functions

If u^i satisfies assumption 1.2.1 then, for all $p \gg 0$

(i) $\quad z(\cdot)\,\text{is}\,continuous\,\text{in}\,p$ $\quad (continuity)$

(ii) $\quad z(\lambda p) = z(p) \quad \forall \lambda > 0$ $\quad (homogeneity)$

(iii) $\quad p \cdot z(p) = 0$ $\quad (Walras'\,law)$

Proof.

(i) Continuity follows from the continuity of the *demand function*

(ii) Homogeneity of degree zero

$$z_j(\lambda p) = \sum_{i\in\mathcal{I}} x_j^i(\lambda p, \omega^i) - \sum_{i\in\mathcal{I}} \omega^i$$

Now ...

$$\begin{array}{ll} x^i(\lambda p, \omega^i) & = \arg\max\ u^i(x^i) \quad \text{s. t. } \lambda p \cdot x^i \leq \lambda p \cdot \omega^i = \\ (if\ \lambda > 0) & = \arg\max\ u^i(x^i) \quad \text{s. t. } p \cdot x^i \leq p \cdot \omega^i = \\ & = x^i(p, \omega^i) \end{array}$$

Hence,

$$z_j(\lambda p) = \sum_{i\in\mathcal{I}} x_j^i(p, \omega^i) - \sum_{i\in\mathcal{I}} \omega^i = z_j(p)$$

(iii) Walras' law. It relies upon u^i being strongly increasing (the budget constrain is *binding*)

$$p\cdot x^i(p,\omega^i) = p\cdot\omega^i \Rightarrow p\cdot(x^i(p,\omega^i) - \omega^i) = 0 \quad \forall i \in \mathcal{I}$$

that is,

$$\sum_{j=1}^{n} p_j (x_j^i(p, \omega^i) - \omega_j^i) = 0 \quad \forall i \in \mathcal{I}$$

Hence,

$$\sum_{i=1}^{I} \sum_{j=1}^{n} p_j (x_j^i(p, \omega^i) - \omega_j^i) = 0$$

$$\sum_{j=1}^{n} \sum_{i=1}^{I} p_j (x_j^i(p, \omega^i) - \omega_j^i) = 0$$

$$\sum_{j=1}^{n} p_j \sum_{i=1}^{I} (x_j^i(p, \omega^i) - \omega_j^i) = 0$$

$$\sum_{j=1}^{n} p_j z_j(p) = 0$$

$$p \cdot z(p) = 0$$

□

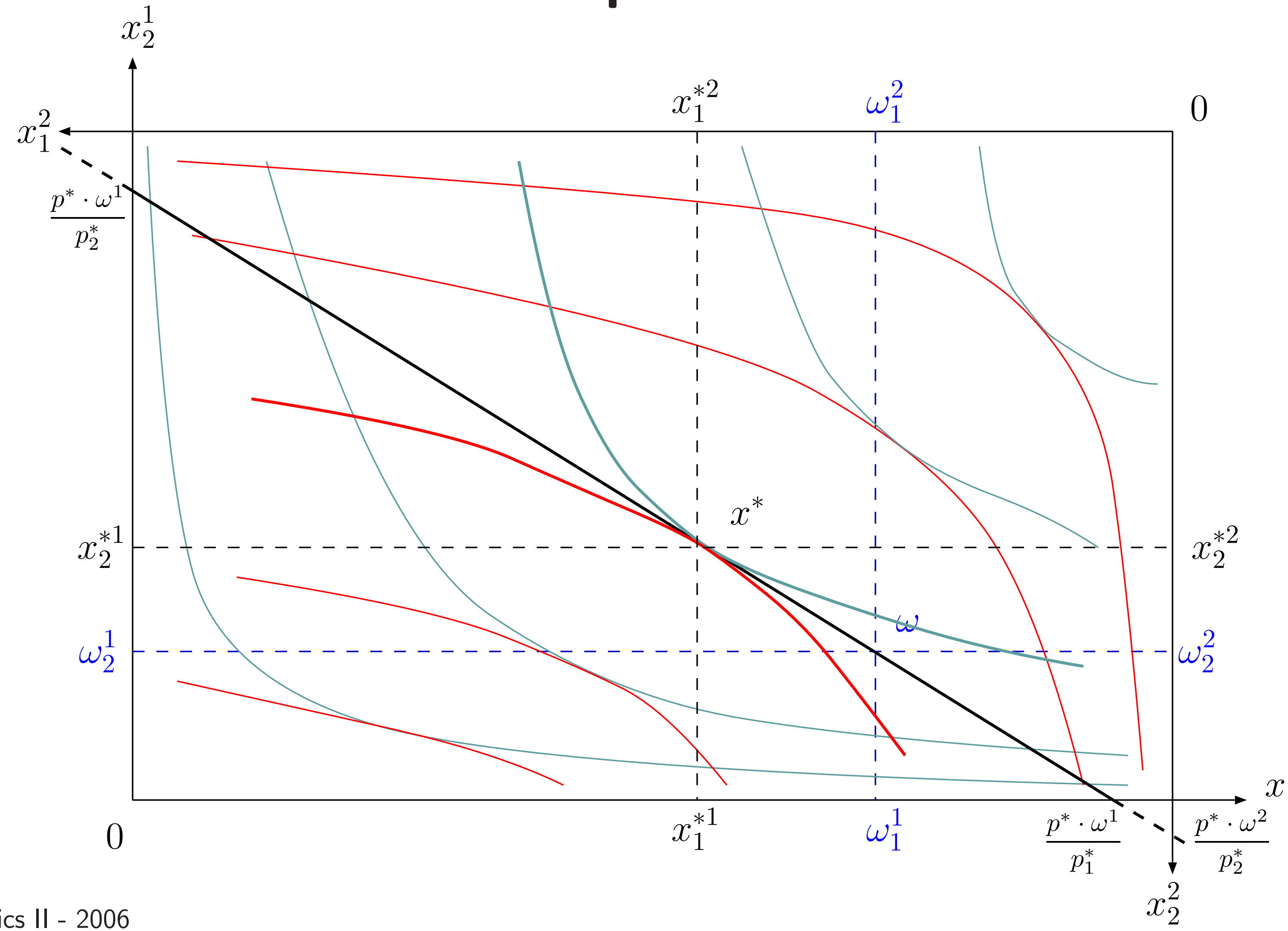
Definition. *Walrasian Equilibrium*
A vector of prices $p^* \in \mathbb{R}^n_{++}$ *is called a* *Walrasian Equilibrium* *if*

$$z(p^*) = 0$$

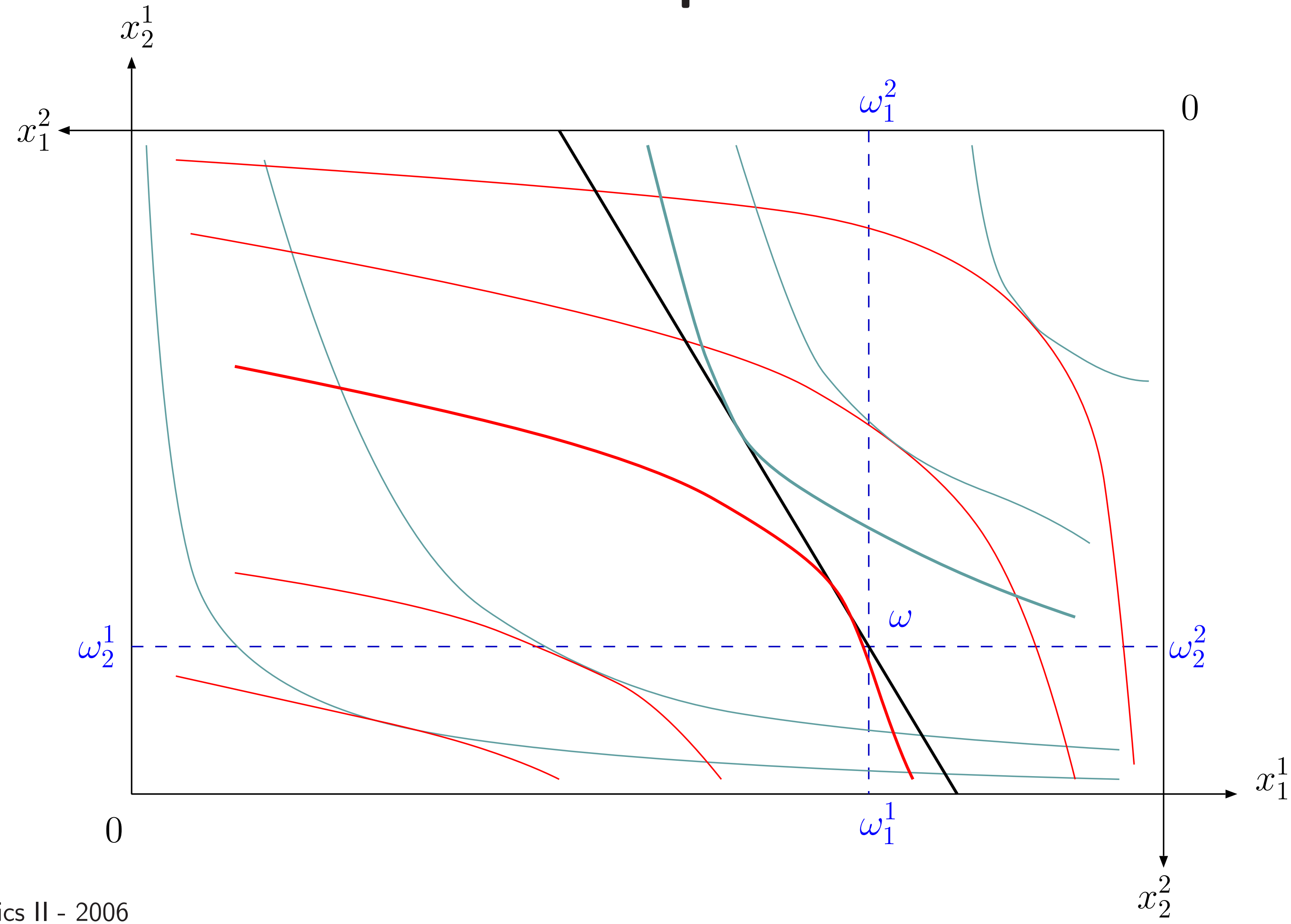
Basic Question: Existence

- Studied since the XIXth century (Walras, Pareto, Edgeworth, Fischer,...)
- *Formal* mathematical proof: McKenzie (1954), Arrow-Debreu (1954), Debreu (1959)

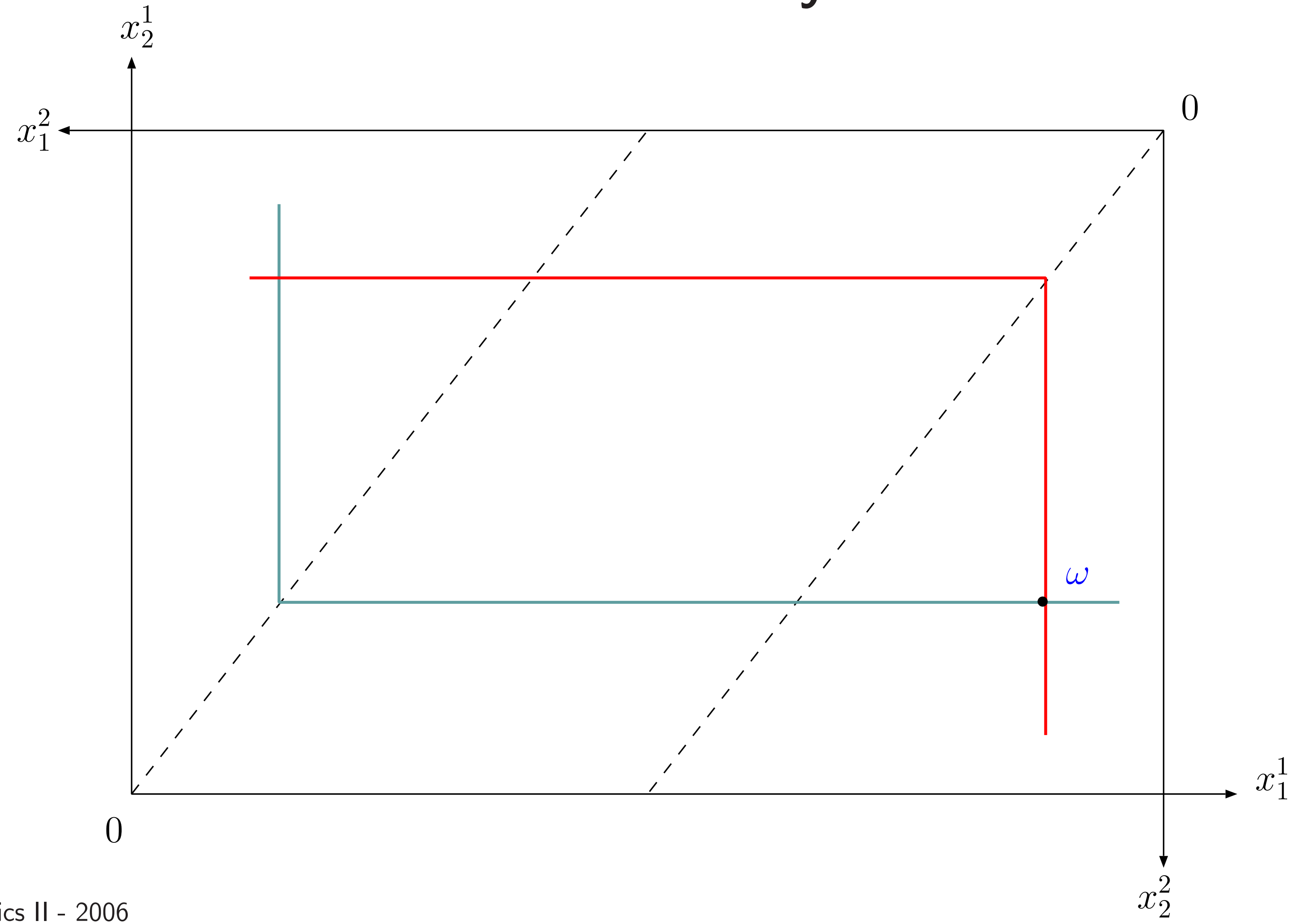
Walrasian Equilibrium



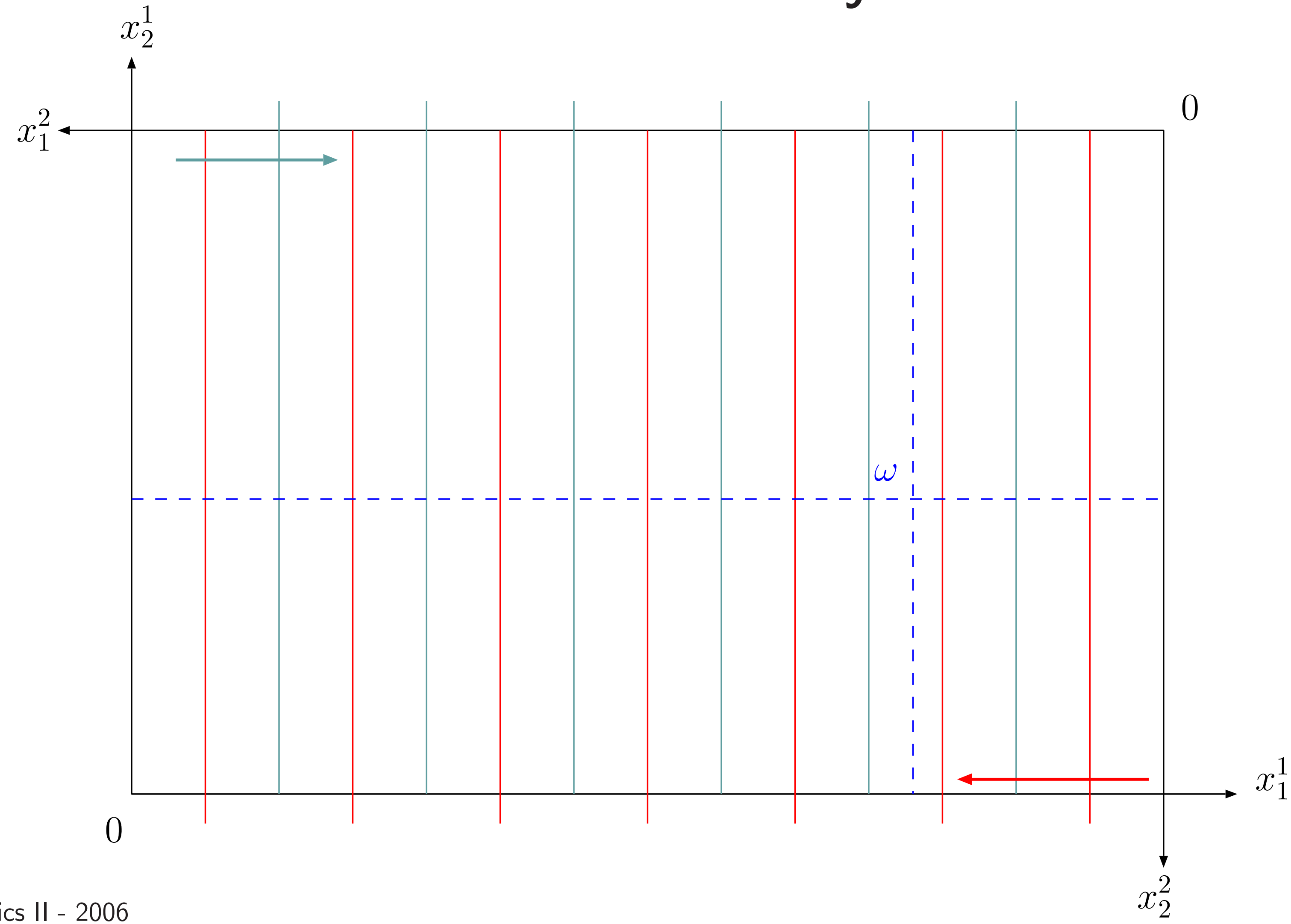
NON Walrasian Equilibrium



Non Strict Convexity



Non Strict Monotonicity



Existence of Walrasian Equilibrium

$$\boxed{\exists p^* \gg 0 \quad \text{s.t.} \quad z(p^*) = 0 \quad ?}$$

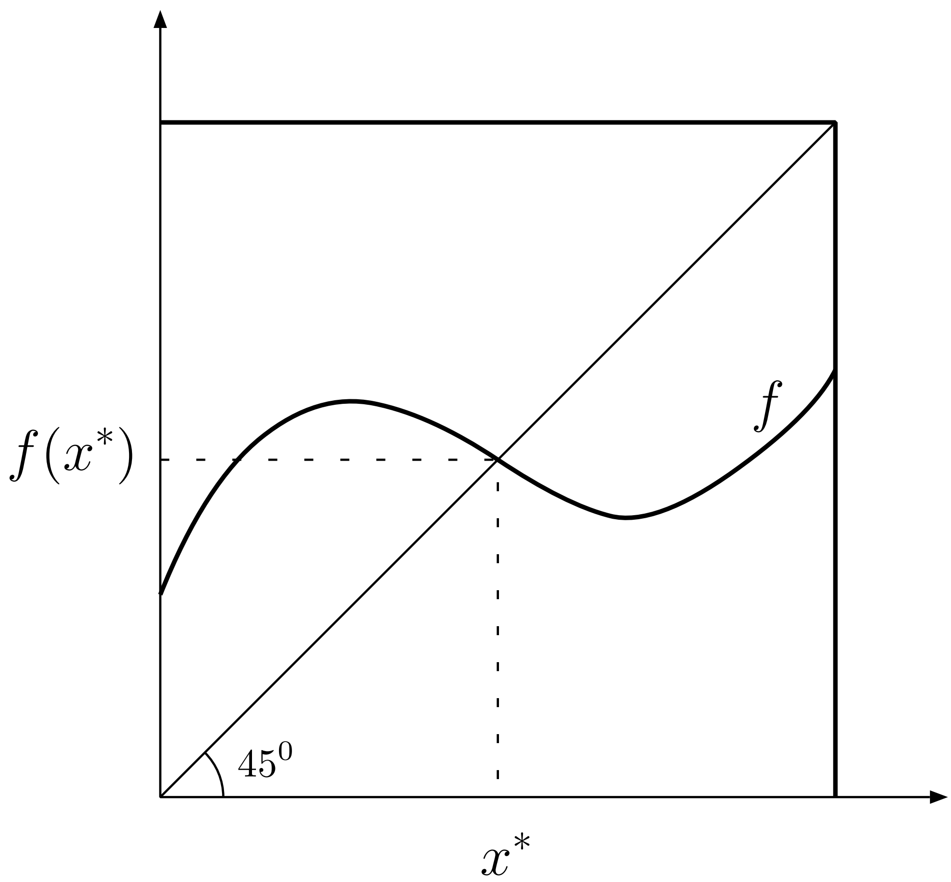
Strategy of the proof

- Brower's Fixed Point Theorem
- Debreu's lemma
- Existence Theorem

Theorem 1.2.2 Brower's Fixed Point Theorem

Let $\mathcal{S}$ be a compact and convex subset of $\mathbb{R}^n$ and $f : \mathcal{S} \to \mathcal{S}$ a continuous function. Then, there exists a point $x^* \in \mathcal{S}$ such that

$$f(x^*) = x^*$$



Definition. *Price Simplex*
The Price Simplex is given by

$$\mathcal{S} = \{p \in \mathbb{R}^n_+ | \sum_{j=1}^{n} p_j = 1\}$$

Lemma. *Debreu's Lemma*
If $z : \mathcal{S} \to \mathbb{R}^n$ is continuous and satisfies the "weak" Walras' law $(p \cdot z(p) \leq 0)$, then there exists $p^* \in \mathcal{S}$ such that $z(p^*) \leq 0$

Notice

- Prices can be *zero*
- "Weak" version of Walras' law
- $z(p^*) \leq 0$ is not an equilibrium since markets are not cleared

Proof. Define $g(p) : \mathcal{S} \rightarrow \mathcal{S}$ by

$$g_j(p) = \frac{p_j + \max(0, z_j(p))}{1 + \sum_{j=1}^{n} \max(0, z_j(p))}$$

Notice that $\sum_{j=1}^{n} g_j(p) = 1$, hence g goes indeed from the simplex $\mathcal{S}$ to the simplex $\mathcal{S}$

Since z is continuous (by assumption), g is also <u>continuous</u>. Also, $\mathcal{S}$ is <u>compact</u> and <u>convex</u>. Hence, we can apply Brower's Fixed Point Theorem to conclude that

$$\exists p^* \in \mathcal{S} \quad \text{s.t.} \quad g(p^*) = p^*$$

Hence, $\forall j \in \{1, 2, \ldots, n\}$

$$p_j^* + \max(0, z_j(p^*)) = p_j^* + p_j^* \sum_{j=1}^{n} \max(0, z_j(p^*))$$

$$\max(0, z_j(p^*)) = p_j^* \sum_{j=1}^{n} \max(0, z_j(p^*)) \qquad (2)$$

Notice that it cannot be that $z_j(p^*) > 0 \, \forall j$ because in such case we would have that $p^* z(p^*) > 0$, which is a CONTRADICTION with the weak Walras' law

Hence, $\exists k \quad \text{s.t.} \quad z_k(p^*) \leq 0$

Consider all "k"s like such. Then, because of 2,

$$0 = p_k^* \sum_{j=1}^{n} \max(0, z_j(p^*)) \quad \forall k \text{ s.t. } z_k(p^*) \leq 0 \quad (3)$$

From 3, two things might happen

1. If $\sum_{j=1}^{n} \max(0, z_j(p^*)) = 0$ then $z_j(p^*) \leq 0 \ \forall j$. DONE !!!

2. If $p_k^* = 0 \ \forall k \text{ s.t. } z_k(p^*) \leq 0$

   Let $J_1 = \{j | z_j(p^*) \leq 0\}$ $(p_j^* = 0 \ \forall j \in J_1)$
   Let $J_2 = \{j | z_j(p^*) > 0\}$

Clearly, $\sum_{j \in J_1} p_j^* z_j(p^*) = 0$.

Also, $\sum_{j \in J_2} p_j^* z_j(p^*) > 0$. Hence

$$p^* \cdot z(p^*) = \sum_{j=1}^{n} p_j^* z_j(p^*) = \sum_{j \in J_1} p_j^* z_j(p^*) + \sum_{j \in J_2} p_j^* z_j(p^*) > 0$$

CONTRADICTION with the weak Walras' law !! $\square$

Theorem 1.2.3 Existence of Walrasian Equilibrium
Let $\mathcal{E} = (\mathcal{I}, \{\succcurlyeq^i\}_{i\in\mathcal{I}}, \{\omega^i\}_{i\in\mathcal{I}})$ be an *economy* such that

(i) $\succcurlyeq^i$ are continuous $\forall i \in \mathcal{I}$ (u^i is continuous)

(ii) $\succcurlyeq^i$ are strictly monotone $\forall i \in \mathcal{I}$ (u^i is strictly increasing)

(iii) $\succcurlyeq^i$ are strictly convex $\forall i \in \mathcal{I}$ (u^i is strictly quasi-concave)

(iv) $\omega^i \gg 0$

Then, there exists $p^* \in \mathbb{R}^n_+$ such that $z(p^*) = 0$

Notice:

1. We are not assuming continuity of the demand (or excess demand) function

2. Prices are still allowed to be *zero !!*

Proof. Choose M to be "big enough", such as

$$M > \max_j \sum_{i=1}^{I} \omega_j^i$$

Define

$$\hat{x}^i(p) = \{x \in \mathbb{R}_+^n |\, x \, \text{maximizes} \succsim^i \text{ on } \hat{B}^i(p)\}$$

where

$$\hat{B}^i(p) = \{x \in \mathbb{R}_+^n |\, p \cdot x \leq p \cdot \omega^i \, \text{and} \, x_j \leq M \, \forall j\}$$

Properties of $\hat{x}^i$:

(i) It is a <u>function</u> (maximization of a quasiconcave function on a convex set)

(ii) It is <u>continuous</u> everywhere (since $\omega^i \gg 0$)

(iii) $p_j = 0 \Rightarrow \hat{x}_j^i = M \quad \forall i \in \mathcal{I}$ (because of strict monotonicity)

(iv) $p \cdot \hat{x}^i(p) \leq p \cdot \omega^i \Rightarrow p \cdot \hat{z}^i(p) \leq 0$

(v) $p_j = 0 \Rightarrow \hat{z}_j(p) > 0$ (desirability)

Hence, we have that $\hat{z}(p) = \hat{x}(p) - \omega$ is

(i) Continuous

(ii) Satisfies $p \cdot \hat{z}(p) \leq 0$ (weak Walras' law)

Therefore, by Debreu's lemma,

$$\exists p^* \in \mathcal{S} \quad \text{s.t.} \quad \hat{z}(p^*) \leq 0$$

But ... is $\hat{z}(p^*) = z(p^*)$? (i.e. is $\hat{x}(p^*) = x(p^*)$?)

Notice first that $\hat{x}^i_j(p^*) < M \quad \forall i, \forall j$

Suppose now that for some $i \in \mathcal{I}$, $x^i(p^*) \neq \hat{x}^i(p^*)$. That is, $\exists \tilde{x}$ such that

(a) $p^* \cdot \tilde{x} \leq p^* \cdot \omega^i$ (feasible)

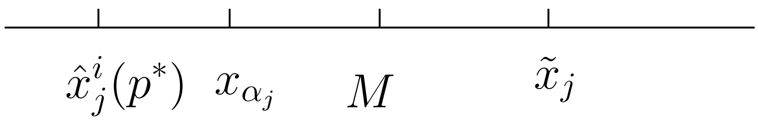
(b) $\tilde{x} \succ^i \hat{x}^i(p^*)$

Then, it must be the case that $\tilde{x}_j > M$ for some j (for otherwise would contradict $\hat{x}^i(p^*)$ being the optimal choice under $\hat{B}^i(p^*)$)

Then, by strict convexity

$$x_\alpha = \alpha\tilde{x} + (1-\alpha)\hat{x}^i(p^*) \succ^i \hat{x}^i(p^*)$$

Hence, for α small enough (close to zero), we have something like:



Therefore, we have that, for α small enough

$$(i) x_\alpha \in \hat{B}^i(p^*)$$
$$(ii) x_\alpha \succ^i \hat{x}^i(p^*)$$

which contradicts $\hat{x}^i(p^*)$ being the optimal choice under $\hat{B}^i(p^*)$

Thus, it must be the case that $\forall i \in \mathcal{I}$, $\hat{x}^i(p^*) = x^i(p^*)$, which implies that $\forall i \in \mathcal{I}$, $\hat{z}^i(p^*) = z^i(p^*)$

We have therefore proved that:

$$\exists p^* \quad \text{s.t.} \quad z(p^*) \leq 0$$

We will prove that, in fact, $z(p^*) = 0$

First, by strict monotonicity, we know that the Budget constraints will be binding for all individuals. That is, the Walras' law will be satisfied in its "strong" version

$$p \cdot z(p) = 0$$

Second, recall that by "desirability" $p_j^* > 0 \; \forall j$

Suppose now that $\exists k \quad \text{s.t.} \quad z_k(p^*) < 0$

In such case, we would have the following:

$$\begin{array}{cccccccccc}
p^* \cdot z(p^*) = & p_1^* & z_1(p^*) & \cdots\cdots & p_k^* & z_k(p^*) & \cdots\cdots & p_n^* & z_n(p^*) & < 0\,!! \\
 & > & \leq & & > & < & & > & \leq & \\
 & \multicolumn{2}{c}{\leq} & & \multicolumn{2}{c}{<} & & \multicolumn{2}{c}{\leq} &
\end{array}$$

We therefore have that

$$\left.\begin{array}{l} \exists p^* \quad \text{s.t.} \quad z(p^*) \leq 0 \\ \text{Strong Monotonicity} \end{array}\right\} \Rightarrow \exists p^* \quad \text{s.t} \quad z(p^*) = 0$$

$\square$

Example

$$u^1(x_1^1, x_2^1) = (x_1^1 x_2^1)^2 \qquad \omega^1 = (18, 4)$$

$$u^2(x_1^2, x_2^2) = \ln(x_1^2) + 2\ln(x_2^2) \quad \omega^2 = (3, 6)$$

These are transformation of *standard* Cobb-Douglas utility functions. Hence, we now that:

$$p_1 x_1^1(p) = \tfrac{1}{2} p \cdot \omega^1 \quad p_1 x_1^2(p) = \tfrac{1}{3} p \cdot \omega^2$$

$$p_2 x_2^1(p) = \tfrac{1}{2} p \cdot \omega^1 \quad p_2 x_2^2(p) = \tfrac{2}{3} p \cdot \omega^2$$

Therefore

$$x_1^1(p) = \tfrac{18p_1 + 4p_2}{2p_1} \quad x_1^2(p) = \tfrac{3p_1 + 6p_2}{3p_1}$$

$$x_2^1(p) = \tfrac{18p_1 + 4p_2}{2p_2} \quad x_2^2(p) = \tfrac{2(3p_1 + 6p_2)}{3p_2}$$

which simplifies to

$$x_1^1(p) = 9 + 2\tfrac{p_2}{p_1} \quad x_1^2(p) = 1 + 2\tfrac{p_2}{p_1}$$

$$x_2^1(p) = 9\tfrac{p_1}{p_2} + 2 \quad x_2^2(p) = 2\tfrac{p_1}{p_2} + 4$$

Hence, the aggregate demands are

$$x_1(p) = \sum_{i=1}^{2} x_1^i(p) = (9 + 2\frac{p_2}{p_1}) + (1 + 2\frac{p_2}{p_1}) = 10 + 4\frac{p_2}{p_1}$$

$$x_2(p) = \sum_{i=1}^{2} x_2^i(p) = (9\frac{p_1}{p_2} + 2) + (2\frac{p_1}{p_2} + 4) = 11\frac{p_1}{p_2} + 6$$

and the corresponding excess demand functions

$$z_1(p) = 10 + 4\frac{p_2}{p_1} - (18 + 3) = 4\frac{p_2}{p_1} - 11$$

$$z_2(p) = 11\frac{p_1}{p_2} + 6 - (4 + 6) = 11\frac{p_1}{p_2} - 4$$

(is Walras' law satisfied ??)

To look for the equilibrium prices we must set $z(p^*) = 0$. Therefore

$z_1(p^*) = 0 \Rightarrow 4\frac{p_2^*}{p_1^*} - 11 = 0 \Rightarrow \frac{p_2^*}{p_1^*} = \frac{11}{4}$

So, all prices satisfying

$$\frac{p_2}{p_1} = \frac{11}{4}$$

constitute a Walrasian Equilibrium

(we should check also that $z_2(p^*) = 0$)

Definition. *Walrasian Equilibrium Allocations*
Let $\mathcal{E} = (\mathcal{I}, \{\succsim^i\}_{i\in\mathcal{I}}, \{\omega^i\}_{i\in\mathcal{I}})$ be an *economy* and let p^*be a *Walrasian Equilibrium (WE)*. Then,

$$x(p^*) \equiv (x^1(p^*), x^2(p^*), \ldots, x^I(p^*))$$

is called a Walrasian Equilibrium Allocation (WEA), and $\mathcal{W}(\mathcal{E})$ denotes the set of all WEA's of $\mathcal{E}$

Given an economy $\mathcal{E}$, does it exist any relation between $\mathcal{W}(\mathcal{E})$ and $\mathcal{C}(\mathcal{E})$?

Theorem 1.2.4 Core and Equilibria

Let $\mathcal{E} = (\mathcal{I}, \{\succsim^i\}_{i\in\mathcal{I}}, \{\omega^i\}_{i\in\mathcal{I}})$ be an *economy* such that $\succsim^i$ satisfy local non-satiation $\forall i \in \mathcal{I}$. Then,

$$\mathcal{W}(\mathcal{E}) \subset \mathcal{C}(\mathcal{E})$$

Proof. Suppose, on the contrary, that $x(p^*) \in \mathcal{W}(\mathcal{E})$ and $x(p^*) \notin \mathcal{C}(\mathcal{E})$

Then, $\exists S \subset \mathcal{I}$ and an allocation y such that

$$\begin{array}{ll} (i) & \sum_{i \in S} y^i = \sum_{i \in S} \omega^i \\ & \\ (ii) & u^i(y^i) \geq u^i(x^i(p^*)) \quad \forall i \in S \\ & \text{with at least one strict } > \end{array}$$

Clearly, (i) implies that

$$p^* \cdot \sum_{i \in S} y^i = p^* \cdot \sum_{i \in S} \omega^i \tag{4}$$

From (ii) we have

1. $\forall i \in S \, \text{s.t.} \, u^i(y^i) > u^i(x^i(p^*))$, it must be the case that $p^* \cdot y^i > p^* \cdot \omega^i = p^* \cdot x^i(p^*))$ for otherwise $x(p^*)$ would not be a WEA

2. $\forall i \in S \, \text{s.t.} \, u^i(y^i) = u^i(x^i(p^*))$, if $p^* \cdot y^i < p^* \omega^i$ then for ε small enough ,

$$p^* \cdot (y^i + \varepsilon \vec{e}) \leq p^* \cdot \omega^i$$

where $\vec{e}$ is the vector corresponding to the direction in which preferences increase (local non-satiation), so that

$$u^i(y^i + \varepsilon\vec{e}) > u^i(y^i) = u^i(x^i(p^*))$$

which contradicts $x(p^*)$ being a WEA.
Hence, $\forall i \in S \,\text{s.t.}\, u^i(y^i) = u^i(x^i(p^*))$ it must be the case that $p^* \cdot y^i \geq p^* \cdot \omega^i$

Items 1. and 2. together imply that $\forall i \in S \quad p^* \cdot y^i \geq p^* \cdot \omega^i$ with at least one strict $>$. Adding up for all $i \in S$

$$p^* \cdot \sum_{i \in S} y^i > p^* \cdot \sum_{i \in S} \omega^i$$

which contradicts (4) $\square$

Hence:

- ✔ The "equilibrium" of a economy with a market system is also an "equilibrium" of the same economy without any institution
- ✔ All the information costs, coalition coordination and implementation problems behind the concept of the Core disappear in a market system
- ✔ Since individuals do not need to meet with each other to do exchange (as in the "pure" exchange case), prices act as "regulators" so that the plans of all individuals are compatible (demand=supply)
- ✘ Nothing guarantees, though, that $\mathcal{C}(\mathcal{E}) \subset \mathcal{W}(\mathcal{E})$
- ✘ Equilibrium prices "exist", but the model does not explain how they are formed and/or computed

Welfare Theorems

Theorem 1.2.5 First Welfare Theorem

Let $\mathcal{E} = (\mathcal{I}, \{\succsim^i\}_{i\in\mathcal{I}}, \{\omega^i\}_{i\in\mathcal{I}})$ be an *economy* such that $\succsim^i$ satisfy local non-satiation $\forall i \in \mathcal{I}$. Then, every Walrasian equilibrium allocation is Pareto efficient

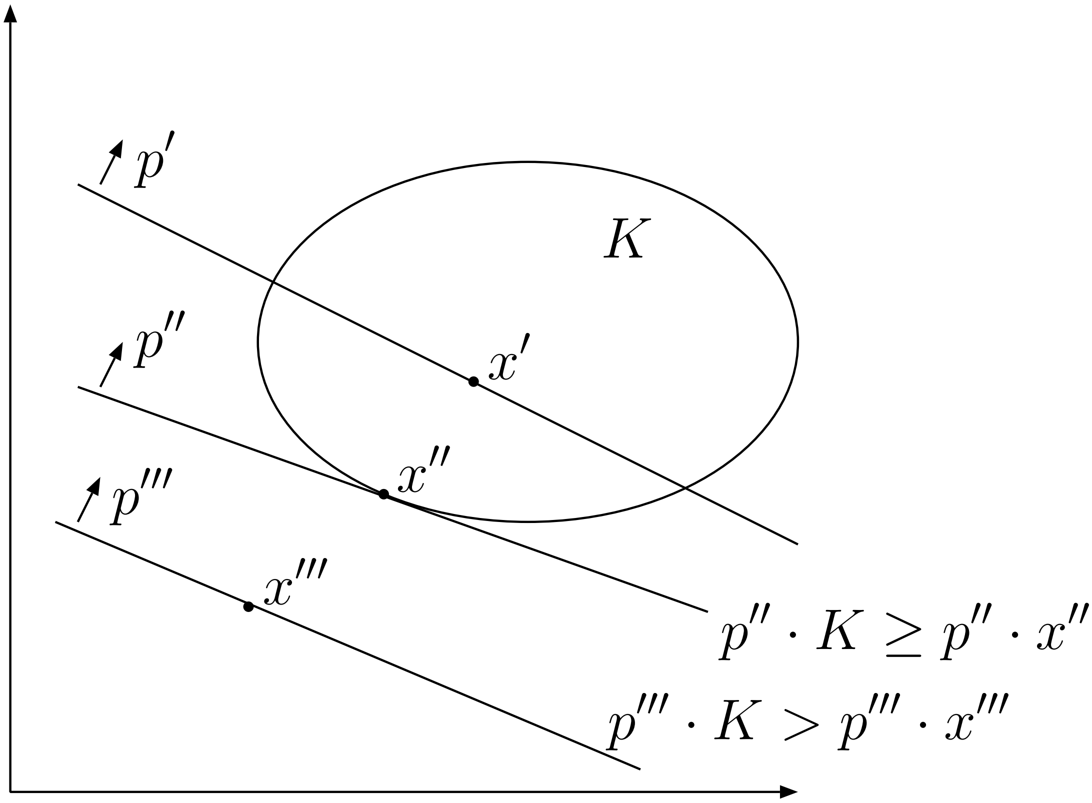
Proof. This is a Corollary of Theorem 1.2.4 $\square$

Theorem 1.2.6 Minkowsky Separation Theorem

Let $K \subset \mathbb{R}^n$ be a convex subset and take $x \in \mathbb{R}^n$. Then, $\exists p \neq 0\,(p \in \mathbb{R}^n)$ such that

$$p \cdot K \geq p \cdot x \Leftrightarrow x \,\text{is not in the interior of}\, K$$

$(p \cdot K \geq p \cdot x \equiv p \cdot k \geq p \cdot x \quad \forall k \in K)$



- The hyperplane generated by $p \cdot x$ is called a "separating hyperplane"

- It "separates" x from the set K

- This would not be possible if the set K were not <u>convex</u>

- This would not be possible if x were in the interior of K

Theorem 1.2.7 Second Welfare Theorem

Let $\mathcal{E} = (\mathcal{I}, \{\succsim^i\}_{i\in\mathcal{I}}, \{\omega^i\}_{i\in\mathcal{I}})$ be an *economy* such that $\succsim^i$ are convex, monotone, and continuous. Let $x^* = (x^{*1}, x^{*2}, \ldots, x^{*I})$ be a Pareto efficient allocation of $\mathcal{E}$. Then, $\exists p^* \neq 0$ such that

(i) $x^i \succsim^i x^{*i} \Rightarrow p^* x^i \geq p^* x^{*i} \quad \forall i \in \mathcal{I}$
(i.e., x^{*i} is *expenditure minimizing* at p^*)

(ii) If $x^{*i} \gg 0 \,\forall i \in \mathcal{I}$ then $x^* \in \mathcal{W}(\mathcal{E}')$ (at prices p^*), where $\mathcal{E}' = (\mathcal{I}, \{\succsim^i\}_{i\in\mathcal{I}}, \{x^{*i}\}_{i\in\mathcal{I}})$

Proof. For each individual $i \in \mathcal{I}$, define

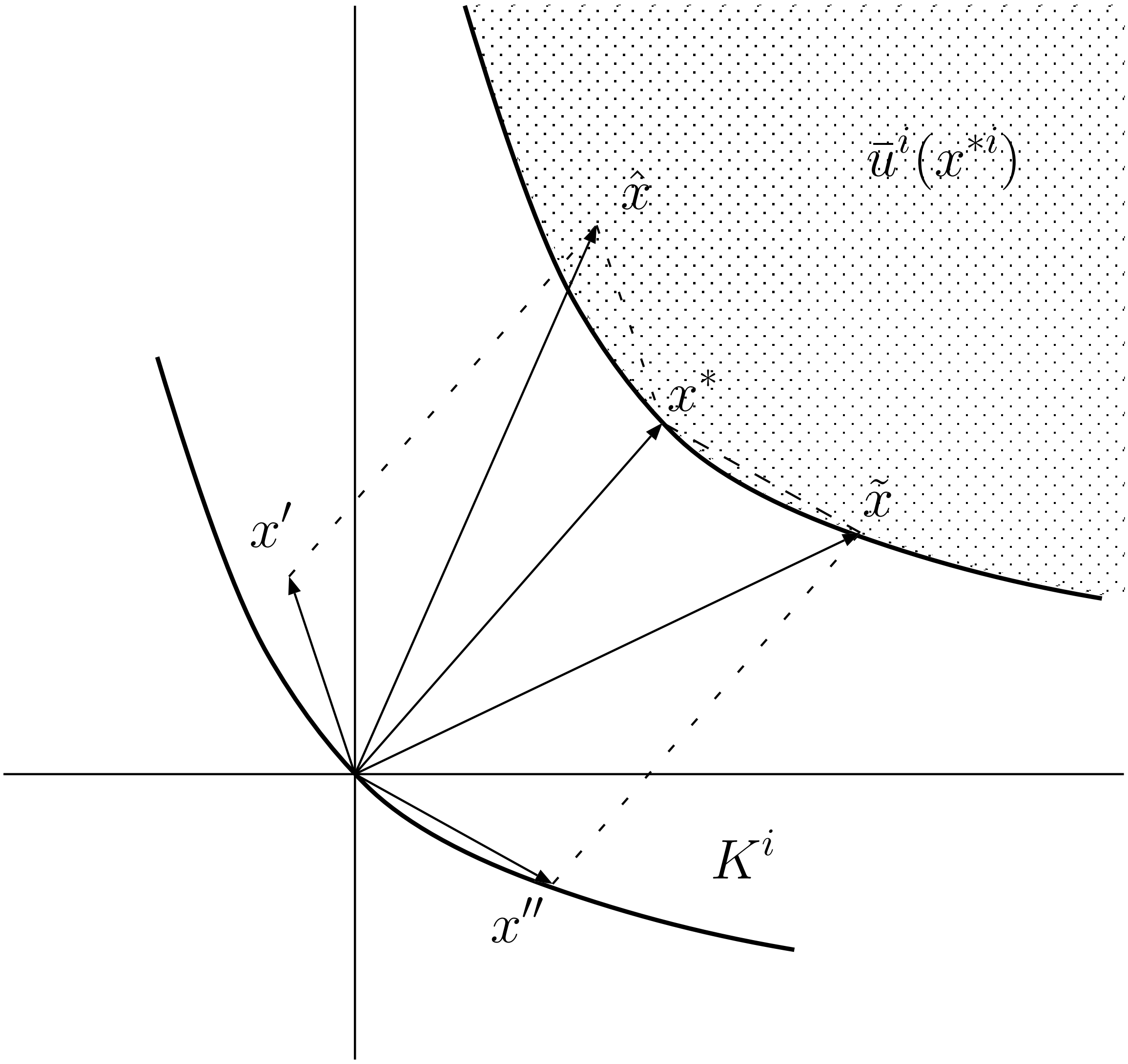
$$K^i = \{x - x^{*i} |\, x \succsim^i x^{*i}\}$$

That is, if

$$\bar{u}^i(x^{*i}) = \{x | x \succsim^i x^{*i}\}$$

(upper contour set), then

$$K^i = \bar{u}^i(x^{*i}) - \{x^{*i}\}$$

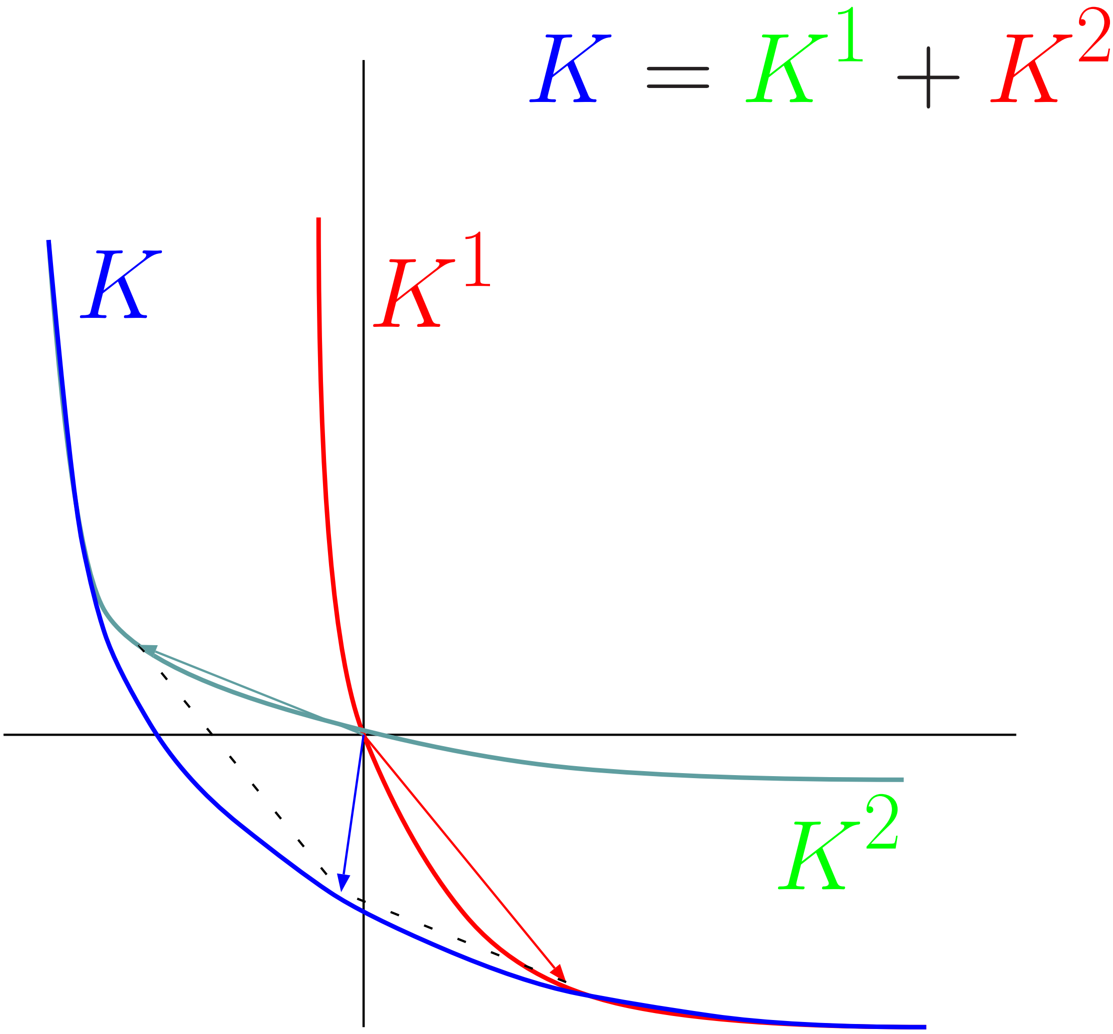


Notice

$$
\begin{array}{ll}
(i) & x' + x^* = \hat{x} \Rightarrow x' = \hat{x} - x^* \Rightarrow x' \in K \\
(ii) & x'' + x^* = \tilde{x} \Rightarrow x'' = \tilde{x} - x^* \Rightarrow x'' \in K \\
(iii) & 0 \in K^i \text{ because } x^{*i} \succsim^i x^{*i} \\
(iv) & \bar{u}^i(x^{*i}) \text{ convex} \Rightarrow K^i \text{ convex}
\end{array}
$$

Let

$$K = \sum_{i \in \mathcal{I}} K^i$$



It is clear the K is convex. Furthermore...

Claim 1 $\mathbb{R}^n_+ \subset K^i \quad \forall i \in \mathcal{I}$

Proof. Let $k \geq 0$ and consider $x = k + x^{*i}$. Clearly

$$x \geq x^{*i} \Rightarrow x \succcurlyeq^i x^{*i} \quad \text{(by monotonicity)}$$

hence,

$$x - x^{*i} \in K^i \Rightarrow k \in K^i$$

$\square$

Claim 2 0 is not *interior* to K

Proof. Suppose the contrary is true: 0 is interior to K. Then,

$$(-\varepsilon, -\varepsilon, \ldots, -\varepsilon) \in K \quad \text{for } \varepsilon \text{ small enough}$$

That is, if $\vec{e} = (1, 1, \ldots, 1)$, we have that, $-\varepsilon \cdot \vec{e} \in K$

In other words, by definition of K,

$$\exists k^1, k^2, \ldots k^n \,\text{such that}\, k^i \in K^i \,\text{and}\, \sum_{i \in \mathcal{I}} k^i = -\varepsilon \cdot \vec{e}$$

Therefore, $\exists x^i \, (i \in \mathcal{I})$ such that

$$x^i \succcurlyeq^i x^{*i} \quad \text{and} \quad k^i = x^i - x^{*i}$$

Hence,

$$\sum_{i \in \mathcal{I}} (x^i - x^{*i}) = \sum_{i \in \mathcal{I}} k^i = -\varepsilon \cdot \vec{e}$$

We therefore have that $\exists x^i \, \text{such that} \, x^i \succcurlyeq^i x^{*i}$ and

$$\sum_{i \in \mathcal{I}} x^i = \sum_{i \in \mathcal{I}} x^{*i} - \varepsilon \cdot \vec{e} \leq \sum_{i \in \mathcal{I}} \omega^i - \varepsilon \cdot \vec{e}$$

Let $\hat{x}^1 = x^1 + \varepsilon \cdot \vec{e}$, $\hat{x}^2 = x^2, \ldots, \hat{x}^I = x^I$. Clearly,

$$\hat{x}^1 \succ^1 x^1 \succcurlyeq^1 x^{*1}, \, \hat{x}^2 \succcurlyeq^2 x^2, \ldots, \hat{x}^I \succcurlyeq^I x^I$$

and

$$\sum_{i\in\mathcal{I}} \hat{x}^i = \hat{x}^1 + \sum_{i=2}^{I} \hat{x}^i = x^1 + \varepsilon \cdot \vec{e} + \sum_{i=2}^{I} x^i =$$

$$= x^1 + \varepsilon \cdot \vec{e} + (\sum_{i\in\mathcal{I}} x^i - x^1) \leq$$

$$\leq x^1 + \varepsilon \cdot \vec{e} + (\sum_{i\in\mathcal{I}} \omega^i - \varepsilon \cdot \vec{e} - x^1) = \sum_{i\in\mathcal{I}} \omega^i$$

Hence,

$$\begin{array}{ll} (i) & \hat{x}^1 \succ^1 x^{*1},\ \hat{x}^2 \succcurlyeq^2 x^2, \ldots, \hat{x}^I \succcurlyeq^I x^I \\ (ii) & \sum_{i\in\mathcal{I}} \hat{x}^i \leq \sum_{i\in\mathcal{I}} \omega^i \quad \text{feasible!} \end{array}$$

which CONTRADICTS x^*being Pareto efficient

Therefore, by Minkowski's ...

$$\exists p^* \neq 0 \quad \text{s.t.} \quad p^* \cdot 0 \leq p^* \cdot K$$

that is,

$$\exists p^* \quad \text{such that} \quad p^* \cdot K \geq 0$$

To prove (i) in the Theorem's statement, consider x^i such that $x^i \succcurlyeq^i x^{*i}$

$$\left.\begin{array}{cc} \text{Then} & x^i - x^{*i} \in K^i \\ \text{also} & 0 \in K^i \, \forall i \in \mathcal{I} \end{array}\right\} \Rightarrow x^i - x^{*i} \in K \, \forall i \in \mathcal{I}$$

Hence, $p^* \cdot (x^i - x^{*i}) \geq 0$. That is,

$$p^* \cdot x^i \geq p^* x^{*i} \quad \text{whenever} \, x^i \succcurlyeq^i x^{*i}$$

To prove (ii), note that since $\mathbb{R}^n_+ \subset K^i$ and $p^* \cdot k \geq 0 \quad \forall k \in K^i$, it must be the case that $p^* \geq 0$

Thus, since $x^{*i} \gg 0$ (by assumption), we have that

$$p^* x^{*i} > 0 \quad \forall i \in \mathcal{I}$$

We want to show that

$$x^{*i} \in \arg\max\{\succcurlyeq^i \quad \text{s.t.} \quad p^* \cdot x \leq p^* \cdot x^{*i}\}$$

Suppose not, that is,

$$\left.\begin{array}{ccc} \exists x \succ^i x^{*i} & \text{s.t} & p^* \cdot x \leq p^* x^{*i} \\ \text{From (i)} & & p^* \cdot x \geq p^* x^{*i} \end{array}\right\} \Rightarrow p^* \cdot x = p^* \cdot x^{*i}$$

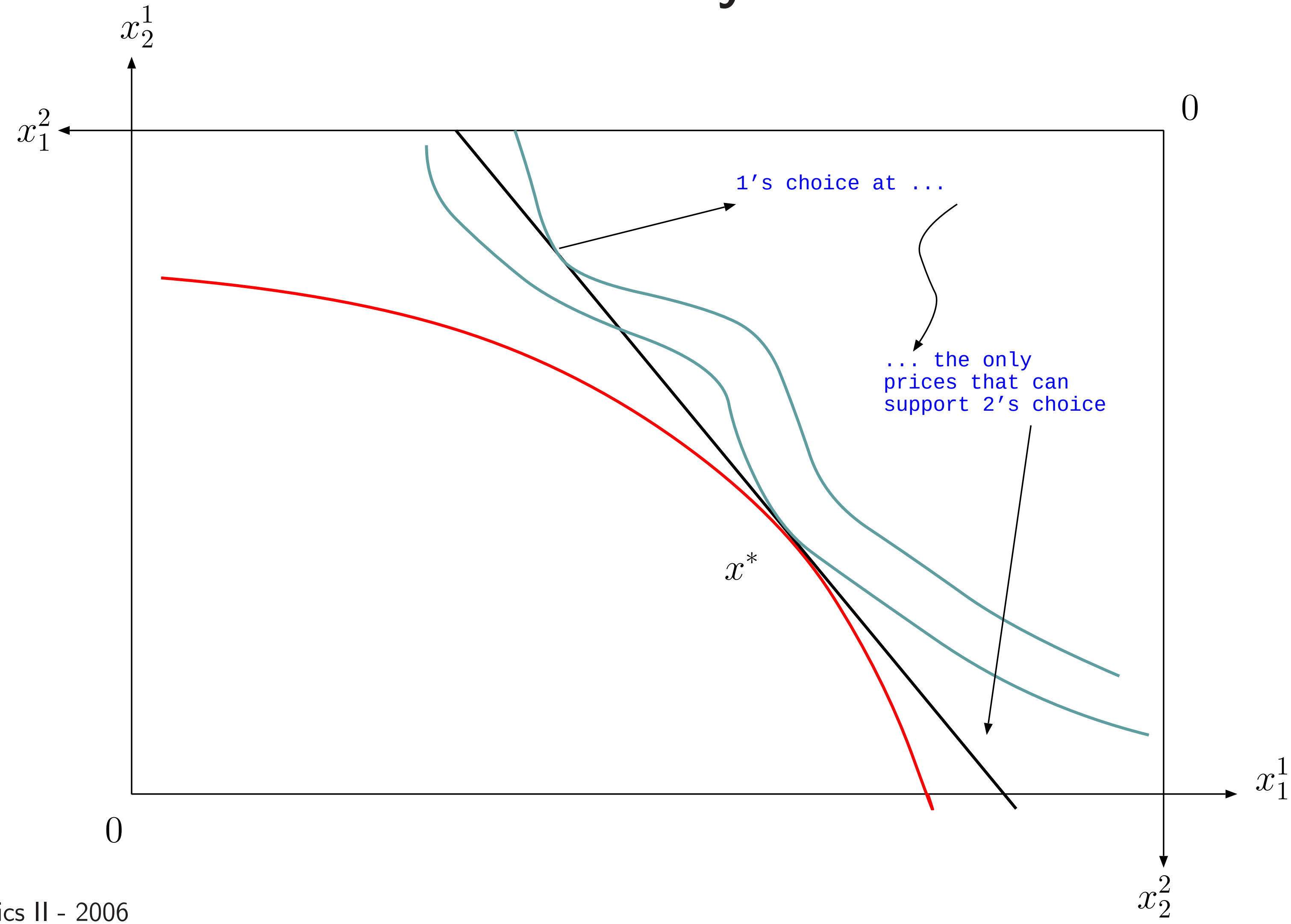
Let $\tilde{x} = \alpha x$, where $\alpha < 1$ is chosen so that $\tilde{x} \succ^i x^{*i}$ (by continuity, such α does exist)

Then, notice that

$$p^* \cdot \tilde{x} = p^* \cdot \alpha x = \alpha p^* \cdot x = \alpha p^* \cdot x^{*i} < p^* x^{*i}$$

which CONTRADICTS (i) $\square$

Non Convexity



Comments

- ✔ Under general assumptions, Walrasian Equilibrium does exist.
- ✔ Under more general assumptions, if a Walrasian Equilibrium exists it is in the *Core* (and hence, *efficient*)
- ✔ Under general assumptions, any *efficient* allocations can be supported as a Walrasian Equilibrium (subject to a redistribution of initial endowments)

1.3 Equilibrium in Perfectly Competitive Economies with Production

Economic Institution $\rightsquigarrow$ $\boxed{\text{Perfectly Competitive Market System}}$

✔ Self-interested agents (utility/profit maximizers)

✔ "Insignificant" agents (no power to affect prices)

✔ General Equilibrium $\rightsquigarrow$ Buyers' and Sellers' plans are compatible in every market

✔ Besides the initial endowments, production (transformation) is possible

Issues:

✘ Resizing Box ?

✘ Profits distribution

✘ Inputs-Outputs "overlapping"

Producers

Individuals, with their initial endowments, can: consume, exchange, and transform (produce) in any feasible combination

For convenience, those individuals involved in a particular production activity will be generically called "firm". The number of firms is denoted by F, and $\mathcal{F} = \{1, 2, \ldots, F\}$ is the set of firms. A generic firm will be denoted by a superindex f.

Definition. *Production Set*

The set $Y^f \subset \mathbb{R}^n$ is called the production set and contains all the production possibilities for firm $f \in \mathcal{F}$

Convention.

$$\forall y \in Y^f \begin{cases} y_j > 0 & \text{Good } j \text{ is an } \textit{output} \\ y_j < 0 & \text{Good } j \text{ is an } \textit{input} \\ y_j = 0 & \text{Good } j \text{ is not used} \end{cases}$$

Example. Neoclassical Production Function

$$y_1 = f(y_2) \quad (y_1 \geq 0)$$
$$Y = \{(y_1, y_2) \in \mathbb{R}^2 \,|\, y_1 = f(-y_2),\, y_2 \leq 0\}$$

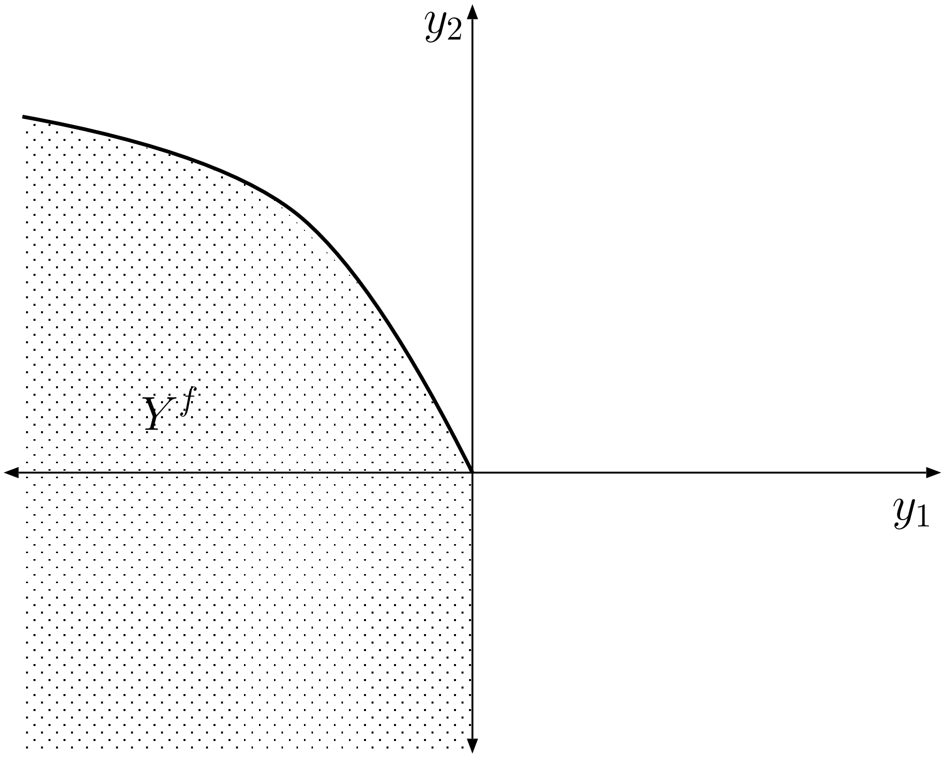
Assumption 1.3.1 For any firm $f \in \mathcal{F}$, the *production set* Y^f satisfies:

(i) $0 \in Y^f$ (possibility of inactivity)

(ii) $\mathbb{R}^n_- \subset Y^f$ (free-disposal)

(iii) $\mathbb{R}^n_+ \cap Y^f = \{0\}$ (no free production)

(iv) Y^f is compact and strictly convex (decreasing returns to scale)



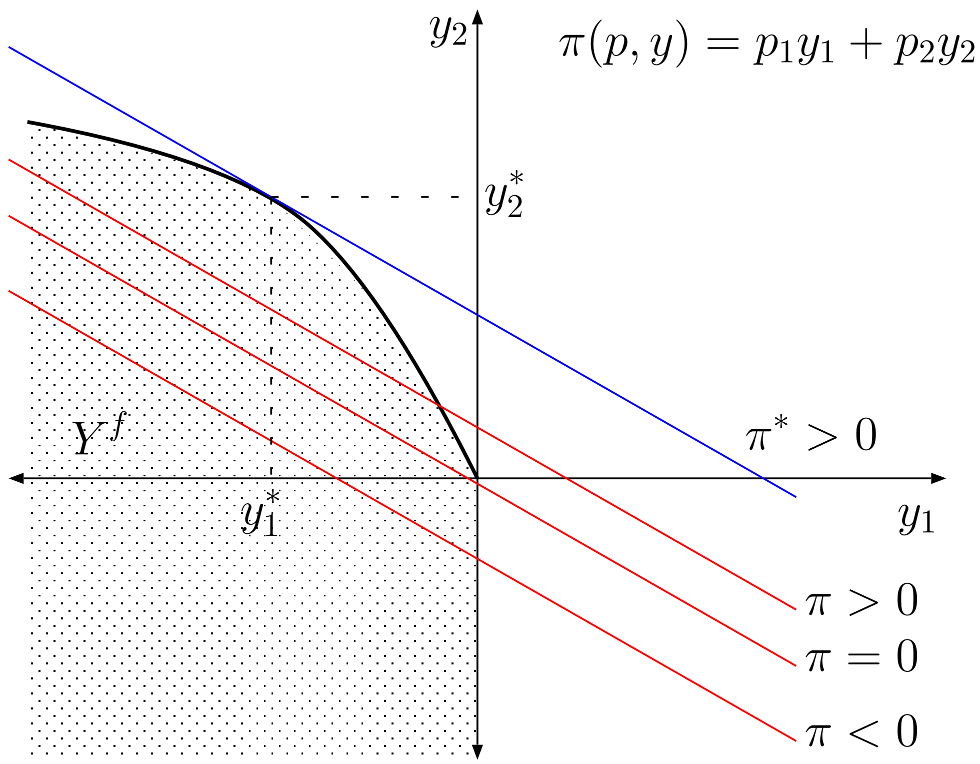
Definition. *Profits*

Given $y \in Y^f$ *and prices* $p \in \mathbb{R}^n_+$, *the* *profits* *of firm* $f \in \mathcal{F}$ *are the net value of* y

$$\pi^f(p, y) = p \cdot y$$

The problem of firm f ($\forall f \in \mathcal{F}$) is

$$\max_{y^f \in \mathbb{R}^n} \; p \cdot y^f \quad \text{s.t.} \quad y^f \in Y^f \tag{5}$$



Definition. *Net Supply Function*
The solution to the maximization problem in (5),

$$y^f(p) : \mathbb{R}^n_+ \rightarrow Y^f \subset \mathbb{R}^n$$

is called the net supply function of firm f

Proposition 1.3.1 If Y^f satisfies Assumption 1.3.1 then,

$$\begin{array}{rl} (i) & y^f(p) \text{ exists and is unique} \\ (ii) & y^f(p) \text{ is homogeneous of degree zero in } p \\ (iii) & y^f(p) \text{ is continuous on } \mathbb{R}^n_{++} \end{array}$$

Proof. (i) and (ii) left as exercises

For continuity (iii), we have to check whether

$$\left.\begin{array}{c} y(t) \xrightarrow[t\to\infty]{} y^* \\ \\ p(t) \xrightarrow[t\to\infty]{} p^* \\ \\ y(t) = y^f(p(t)) \end{array}\right\} \Rightarrow y^* = y^f(p^*)$$

Suppose not, that is, suppose $\exists \hat{y} \in Y^f$ such that

$$p^* \cdot \hat{y} > p^* \cdot y^*$$

Then, since $p(t) \to p^*$ we have that $p(t) \cdot \hat{y} \to p^* \cdot \hat{y}$. Thus, for t large enough,

$$p(t) \cdot \hat{y} > p^* \cdot y^* \tag{6}$$

Also, since $y(t) \rightarrow y^*$ and $p(t) \rightarrow p^*$, we have that

$$p(t) \cdot y(t) \rightarrow p^* \cdot y^* \tag{7}$$

Thus, (6) and (7) together imply that for t large enough

$$p(t) \cdot \hat{y} > p(t) \cdot y(t)$$

which CONTRADICTS $y(t) = y^f(p(t))$ $\square$

Aggregate Supply

Definition. *Aggregate Production Set*
The set

$$Y = \sum_{f \in \mathcal{F}} Y^f$$

is called the aggregate production set and contains all the production possibilities for the economy

Definition. *Aggregate Supply*
The aggregate supply is given by

$$Y + \{\omega\}$$

and corresponds to the set that constraints *aggregate consumption*

Proposition 1.3.2 Given prices $p \in \mathbb{R}^n_{++}$, an aggregate production vector $y^* \in Y$ maximizes aggregate profit $p \cdot y$ if and only if there exist individual production vectors $y^{*f} \in Y^f$ such that $y^{*f} = y^f(p)$ $\forall f \in \mathcal{F}$ and $\sum_{f \in \mathcal{F}} y^{*f} = y^*$

Proof.

$\boxed{\Rightarrow}$

$$y^* \in Y \Rightarrow \exists\, y^{*f} \in Y^f \quad \text{s.t.} \quad \sum_{f \in \mathcal{F}} y^{*f} = y^*$$

Now ... is $y^{*f} = y^f(p) \quad \forall f \in \mathcal{F}$??. Suppose not, then for some firm $f' \in \mathcal{F} \quad \exists\, \hat{y}^{f'}$ such that

$$p \cdot \hat{y}^{f'} > p \cdot y^{*f'}$$

Then , if the above is true, we have that

$$\hat{y} = \sum_{f \neq f'} y^{*f} + \hat{y}^{f'} \in Y \tag{8}$$

and also

$$p \cdot \hat{y} = \sum_{f \neq f'} p \cdot y^{*f} + p \cdot \hat{y}^{f'} > p \cdot y^* \tag{9}$$

(8) and (9) together are in CONTRADICTION with the assumption that y^*maximizes aggregate profit $p \cdot y$ on Y.

$\boxed{\Leftarrow}$

Let $y^* = \sum_{f \in \mathcal{F}} y^{*f}$ where $y^{*f} = y^f(p)\ \forall f \in \mathcal{F}$

Consider any $\hat{y} \in Y$. Then, there must exist $\hat{y}^f \in Y^f \quad \forall f \in \mathcal{F}$ such that $\hat{y} = \sum_{f \in \mathcal{F}} \hat{y}^f$

We know that $p \cdot y^{*f} \geq p \cdot \hat{y}^f\ \forall f \in \mathcal{F}$. Therefore

$$\sum_{f \in \mathcal{F}} p \cdot y^{*f} \geq \sum_{f \in \mathcal{F}} p \cdot \hat{y}^f$$

Hence, $p \cdot y^* \geq p \cdot \hat{y}\ \forall \hat{y} \in Y$. That is, $y^* = y(p)$ $\square$

Private Ownership Economy

- Consumers: $\mathcal{I} = \{1, \ldots, \mathcal{I}\}$ indexed by superindex ^{i}
  Each consumer $i \in \mathcal{I}$ is endowed with:
  - Preferences: $\succsim^i$ that satisfy assumption 1.2.1
  - Initial endowments: $\omega^i = (\omega_1^i, \ldots, \omega_n^i) \in \mathbb{R}^n_{++}$
  - Shares: $\theta^{if} \rightsquigarrow$ share of firm $f \in \mathcal{F}$ owned by i

$$\forall i, f \quad \theta^{if} \geq 0 \quad \text{and} \quad \sum_{i \in \mathcal{I}} \theta^{if} = 1 \quad \forall f \in \mathcal{F}$$

  $\theta^i = (\theta^{i1}, \theta^{i2}, \ldots, \theta^{iF})$ is the *portfolio* of individual i

- Firms: $\mathcal{F} = \{1, 2, \ldots, F\}$ indexed by superindex ^{f}

- Goods: n-goods indexed by subindex $_j$

A *Private ownership economy with production*, $\mathcal{P}$, is completely characterized by

$$\mathcal{P} = (\mathcal{I},\ \{\succsim^i\}_{i \in \mathcal{I}},\ \{\omega^i\}_{i \in \mathcal{I}},\ \{\theta^i\}_{i \in \mathcal{I}},\ \mathcal{F},\ \{Y^f\}_{f \in \mathcal{F}})$$

Definition. *The <u>excess demand function</u> is given by*

$$z(p) = \sum_{i \in \mathcal{I}} x^i(p) - (\sum_{i \in \mathcal{I}} \omega^i + \sum_{f \in \mathcal{F}} y^f(p))$$

where

$$\begin{array}{rl} (i) & x^i(p) = \arg\max \succcurlyeq^i \text{ on} \\ & \{x \in \mathbb{R}^n_+ \,|\, p \cdot x^i \leq p \cdot \omega^i + \sum_{f \in \mathcal{F}} \theta^{if} p \cdot y^f(p)\} \\ (ii) & y^f(p) = \arg\max p \cdot y^f \text{ on} \\ & \{y^f \in \mathbb{R}^n \,|\, y^f \in Y^f\} \end{array}$$

Proposition 1.3.3 Walras' Law

For any $p \in \mathbb{R}^n_+$, $p \cdot z(p) = 0$

Proof. Exercise □

Definition. *Walrasian Equilibrium*
Given a production economy $\mathcal{P}$, a <u>walrasian equilibrium</u> is a vector of prices $p^* \in \mathbb{R}^n_+$ such that

$$z(p^*) = 0$$

Theorem 1.3.1 Existence of Walrasian Equilibrium
Let be $\mathcal{P}$ be an *economy with production* such that

(i) $\succcurlyeq^i$ satisfy assumption 1.2.1 $\forall i \in \mathcal{I}$

(ii) Y satisfies assumption 1.3.1

(iv) $\omega^i \gg 0$

Then, there exists $p^* \in \mathbb{R}^n_+$ such that $z(p^*) = 0$

Proof. Choose M large enough so that if

$$z \in \{Y + \{\sum_{i \in \mathcal{I}} \omega^i\}$$

then

$$|z_j| < M \; \forall j = 1, 2, \ldots, n$$

Let

$$y(p) = \arg\max p \cdot y \quad \text{s.t.} \quad y \in Y$$

Since Y is compact and strictly convex (assumption 1.3.1) we know that $y(p)$ exists and is a continuous function (like in the proof of Proposition 1.3.1). Also, from

Proposition 1.3.2, $\exists\, y^f(p) \in Y^f$ such that

$$\begin{array}{rl} (i) & y^f(p) = \arg\max\, p \cdot y \quad \text{s.t.} \quad y \in Y^f \\ (ii) & y(p) = \sum_{f \in \mathcal{F}} y^f(p) \end{array}$$

Hence, $\forall p \in \mathbb{R}^n_+$ $y^f(p)$ exists and is continuous and, thus

$$p \cdot y^f(p) \quad \text{exists and is continuous}$$

Consider now the "truncated" demand function $\hat{x}(p)$ where

$$\hat{x}^i(p) = \{x \in \mathbb{R}^n_+ \,|\, x \text{ maximizes } \succsim^{\mathrm{i}} \text{ on } \hat{B}^i(p)\}$$

where

$$\begin{aligned} \hat{B}^i \quad (p) = \quad & \{x \in \mathbb{R}^n_+ \,|\, p \cdot x \leq p \cdot \omega^i + \sum_{f \in \mathcal{F}} \theta^{if} p \cdot y^f(p) \\ & \text{and} \quad x_j \leq M \,\forall j\} \end{aligned}$$

Given the assumptions used, $\hat{x}^i(p)$ exists, is unique, and is continuous (because of the continuity of $p \cdot y^f(p)$)

Consider now the "truncated" excess demand function $\hat{z}(p)$ where

$$\hat{z}(p) = \sum_{i \in \mathcal{I}} \hat{x}^i(p) - (\sum_{i \in \mathcal{I}} \omega^i + y(p))$$

By Debreu's Lemma

$$\left.\begin{array}{cc} \hat{z}(p) & \text{is continuous} \\ p \cdot \hat{z}(p) \leq 0 & \text{Walras' law} \end{array}\right\} \Rightarrow \exists p^* \quad \text{s.t.} \quad \hat{z}(p^*) \leq 0$$

By monotonicity,

$$p \cdot \hat{x}^i(p) = p \cdot \omega^i + \sum_{f \in \mathcal{F}} \theta^{if} p \cdot y^f(p)$$

hence,

$$\sum_{i \in \mathcal{I}} p \cdot \hat{x}^i(p) = \sum_{i \in \mathcal{I}} p \cdot \omega^i + \sum_{i \in \mathcal{I}} \sum_{f \in \mathcal{F}} \theta^{if} p \cdot y^f(p)$$

$$\sum_{i \in \mathcal{I}} p \cdot \hat{x}^i(p) = \sum_{i \in \mathcal{I}} p \cdot \omega^i + \sum_{f \in \mathcal{F}} \underbrace{\sum_{i \in \mathcal{I}} \theta^{if}}_{1} p \cdot y^f(p)$$

$$p \cdot (\sum_{i \in \mathcal{I}} \hat{x}(p) - (\sum_{i \in \mathcal{I}} p \cdot \omega^i + \sum_{f \in \mathcal{F}} p \cdot y^f(p))) = 0$$

$$p \cdot \hat{z}(p) = 0$$

Then, using the same technique as in Theorem 1.2.3 (Existence of Walrasian equilibrium without production), we can conclude that

$$\boxed{\exists p^* \in \mathbb{R}^n_{++} \quad \text{such that} \quad z(p^*) = 0}$$

□

Welfare Theorems

Definition. *Walrasian Equilibrium Allocations*
Let $\mathcal{P}$ be a *production economy* and let p^* be a *Walrasian Equilibrium (WE)*. Then, (x^*, y^*) is called a Walrasian Equilibrium Allocation (WEA) where

$$x^* \equiv x(p^*) = (x^1(p^*), x^2(p^*), \ldots, x^I(p^*))$$
$$y^* \equiv y(p^*) = (y^1(p^*), y^2(p^*), \ldots, y^F(p^*))$$

$\mathcal{W}(\mathcal{P})$ denotes the set of all WEA's of $\mathcal{P}$

Theorem 1.3.2 First Welfare Theorem

Let $\mathcal{P}$ be a *production economy* such that $\succsim^i$ satisfy local non-satiation $\forall i \in \mathcal{I}$. Then, every Walrasian equilibrium allocation (x^*, y^*) is Pareto efficient

Proof. First, recall that

$$x^i \succ^i x^{*i} \Rightarrow p^* \cdot x^i > p^* \cdot \omega^i + \sum_{f^{\circ} \in \mathcal{F}} \theta^{if} p^* \cdot y^{*f}$$

$$x^i \succsim^i x^{*i} \Rightarrow p^* \cdot x^i \geq p^* \cdot \omega^i + \sum_{f \in \mathcal{F}} \theta^{if} p^* \cdot y^{*f}$$

Hence,

$$\sum_{i\in\mathcal{I}} p^* \cdot x^i > \sum_{i\in\mathcal{I}} p^* \cdot \omega^i + \sum_{i\in\mathcal{I}} \sum_{f\in\mathcal{F}} \theta^{if} p^* \cdot y^{*f} \quad (10)$$

Suppose now that (x^*, y^*) is not a Pareto efficient allocation, that is, there exist $(\hat{x}, \hat{y})$ such that

$$(a) \quad \hat{x}^i \succcurlyeq^i x^{*i}\ \forall i \in \mathcal{I} \quad \text{and} \quad \hat{x}^i \succ^i x^{*i} \text{ for some } i \in \mathcal{I}$$

$$(b) \quad \sum_{i\in\mathcal{I}} \hat{x}^i = \sum_{i\in\mathcal{I}} \omega^i + \sum_{f\in\mathcal{F}} \hat{y}^f$$

From (b) we have that

$$\sum_{i\in\mathcal{I}} p^* \cdot \hat{x}^i = \sum_{i\in\mathcal{I}} p^* \cdot \omega^i + \sum_{f\in\mathcal{F}} p^* \cdot \hat{y}^f \quad (11)$$

Since y^*maximizes profits at prices p^*, we have that

$$\sum_{f\in\mathcal{F}} p^* \cdot \hat{y}^f \leq \sum_{f\in\mathcal{F}} p^* \cdot y^{*f}$$

Thus, equation (11) can be rewritten as

$$\sum_{i\in\mathcal{I}} p^* \cdot \hat{x}^i \leq \sum_{i\in\mathcal{I}} p^* \cdot \omega^i + \sum_{f\in\mathcal{F}} p^* \cdot y^{*f}$$

Now, since $\forall f \in \mathcal{F} \quad \sum_{i\in\mathcal{I}} \theta^{if} = 1$ we have

$$\sum_{i\in\mathcal{I}} p^* \cdot \hat{x}^i \leq \sum_{i\in\mathcal{I}} p^* \cdot \omega^i + \sum_{f\in\mathcal{F}} \sum_{i\in\mathcal{I}} \theta^{if} p^* \cdot y^{*f}$$

Switching summations ...

$$\sum_{i\in\mathcal{I}} p^* \cdot \hat{x}^i \leq \sum_{i\in\mathcal{I}} p^* \cdot \omega^i + \sum_{i\in\mathcal{I}} \sum_{f\in\mathcal{F}} \theta^{if} p^* \cdot y^{*f}$$

which is a CONTRADICTION with (10)

$\square$

Theorem 1.3.3 Second Welfare Theorem

Let $\mathcal{P}$ be a *production economy* such that

(i) $\succsim^i$are convex $\forall i \in \mathcal{I}$
(ii) $x \gg \hat{x} \Rightarrow x^i \succ^i \hat{x}^i$ for some $i \in \mathcal{I}$
(iii) Y is convex

(A) Then, $\exists p^* \neq 0$ such that

(A.1) $p^* \cdot y^* \geq p^* \cdot Y$ [Profit Maximization]
(A.2) $x^i \succsim^i x^{*i} \Rightarrow p^* \cdot x^i \geq p^* \cdot x^{*i}$ [Expenditure Minimization]

If in addition

(iv) $\mathbb{R}^n_- \subset Y$
(v) $x^{*i} \gg 0 \quad \forall i \in \mathcal{I}$
(vi) $\succsim^i$ are continuous $\forall i \in \mathcal{I}$

(B) Then, $\exists \hat{\theta}^{if}$, $\hat{\omega}^i$ such that

(B.1) $\sum_{i \in \mathcal{I}} \hat{\theta}^{if} = 1, \quad \hat{\theta}^{if} \geq 0, \quad \sum_{i \in \mathcal{I}} \hat{\omega}^i = \sum_{i \in \mathcal{I}} \omega^i$

(B.2) (p^*, x^*, y^*) is a Walrasian Equilibrium of $\hat{\mathcal{P}}$, where

$$\hat{\mathcal{P}} = (\mathcal{I}, \{\succsim^i\}_{i \in \mathcal{I}}, \{\hat{\omega}^i\}_{i \in \mathcal{I}}, \{\hat{\theta}^i\}_{i \in \mathcal{I}}, \mathcal{F}, \{Y^f\}_{f \in \mathcal{F}})$$

Proof. Let $\bar{u}^i(x^{*i}) = \{x \in \mathbb{R}^n_+ \,|\, x \succcurlyeq^i x^{*i}\}$. Notice that

$$
\begin{aligned}
&- \quad \bar{u}^i(x^{*i}) \quad \text{is convex} \\
&- \quad x^{*i} \in \bar{u}^i(x^{*i}) \; \forall i \in \mathcal{I}
\end{aligned}
$$

Let

$$K = \sum_{i \in \mathcal{I}} (\bar{u}^i(x^{*i}) - \{x^{*i}\}) + (\{y^*\} - Y)$$

and notice that

$$
\begin{aligned}
&- \quad K \quad \text{is convex} \\
&- \quad 0 \text{ is not interior to } K
\end{aligned}
$$

Thus, by Minkowski's

$$\exists p^* \neq 0 \quad \text{such that} \quad p^* \cdot K \geq 0$$

(A.1) Note that $0 \in \bar{u}^i(x^{*i}) - \{x^{*i}\}\ \forall i \in \mathcal{I}$. Therefore, $\forall y \in Y$

$$y^* - y = \sum_{i \in \mathcal{I}} 0 + (y^* - y) \in K$$

Thus, by Minkowski's

$$p^* \cdot (y^* - y) \geq 0 \Rightarrow p^* \cdot y^* \geq p^* \cdot y \Rightarrow \text{Profit Maximization}$$

(A.2) $x^i \succcurlyeq^i x^{*i} \Rightarrow x^i \in \bar{u}^i(x^{*i})$. Therefore

$$x^i - x^{*i} = \sum_{h \neq i} 0 + (x^i - x^{*i}) + 0 \in K$$

Thus, by Minkowski's

$$p^* \cdot (x^i - x^{*i}) \geq 0 \Rightarrow p^* \cdot x^i \geq p^* \cdot x^{*i} \Rightarrow \text{Expend. Min.}$$

(B.1) Since $\mathbb{R}^n_- \subset Y \Rightarrow \mathbb{R}^n_+ \subset K \Rightarrow p^* \geq 0$. Also, since $x^{*i} \gg 0$, we have that $p^* \cdot x^{*i} > 0\ \forall i \in \mathcal{I}$

Let

$$\alpha^i = \frac{p^* \cdot x^{*i}}{\sum_{h \in \mathcal{I}} p^* \cdot x^{*h}}$$

Notice that

$$- \quad \alpha^i > 0$$
$$- \quad \sum_{i \in \mathcal{I}} \alpha^i = 1$$

Let

$$\hat{\omega}^i = \alpha^i \sum_{h \in \mathcal{I}} \omega^i \quad \text{and} \quad \hat{\theta}^{if} = \alpha^i \quad \forall i \in \mathcal{I}$$

Notice that

$$- \quad \sum_{i \in \mathcal{I}} \hat{\omega}^i = \sum_{i \in \mathcal{I}} \omega^i$$
$$- \quad \sum_{i \in \mathcal{I}} \hat{\theta}^{if} = 1$$
$$- \quad \hat{\theta}^{if} \geq 0$$

(B.2) Note that (x^*, y^*) is a feasible allocation for $\hat{\mathcal{P}}$ because it is feasible in $\hat{\mathcal{P}}$.
Is it a Walrasian equilibrium allocation of $\hat{\mathcal{P}}$?

Notice first that for any $f \in \mathcal{F}$, y^{*f} maximizes profits because of **(A.1)**
So, we need to show that for every $i \in \mathcal{I}$, x^{*i} maximizes $\succcurlyeq^i$ over the *new* Budget Set

$$\hat{B}^i(p^*) = \{x \in \mathbb{R}^n_+ \,|\, p^* \cdot x \leq p \cdot \hat{\omega}^i + \sum_{f \in \mathcal{F}} \hat{\theta}^{if} p^* \cdot y^{*f}\}$$

Suppose not, that is, suppose that for some $i \in \mathcal{I}$ there exists some alternative allocation $\hat{x}^i$ such that $\hat{x}^i \succ^i x^{*i}$ and

$$\begin{aligned} p^* \cdot \hat{x}^i \leq p^* \cdot \hat{\omega}^i + \sum_{f \in \mathcal{F}} \hat{\theta}^{if} p^* \cdot y^{*f} = \\ = p^* \cdot \alpha^i \sum_{h \in \mathcal{I}} \omega^h + p^* \cdot \alpha^i \sum_{f \in \mathcal{F}} y^{*f} = \\ = p^* \cdot \alpha^i (\sum_{h \in \mathcal{I}} \omega^h + y^*) = \\ = \alpha^i p^* \sum_{h \in \mathcal{I}} x^{*h} = p^* x^{*i} \end{aligned}$$

From **(A.2)** we know that for any $x^i \succcurlyeq^i x^{*i}$ it must be the case that $p^* \cdot x^i \geq p^* \cdot x^{*i}$. Thus, it must be the case that

$$p^* \cdot \hat{x}^i = p^* \cdot x^{*i}$$

Consider then $x_\alpha = \alpha \hat{x}^i$ with $\alpha < 1$ but close enough to 1 so that $x_\alpha \succ^i x^{*i}$. Then,

$$p^* \cdot x_\alpha = \alpha p^* \cdot \hat{x}^i = \alpha p^* \cdot x^{*i} < p^* \cdot x^{*i}$$

which is a CONTRADICTION with **(A.2)**

Hence, (x^*, y^*) is a Walrasian equilibrium allocation of $\hat{\mathcal{P}}$ $\square$

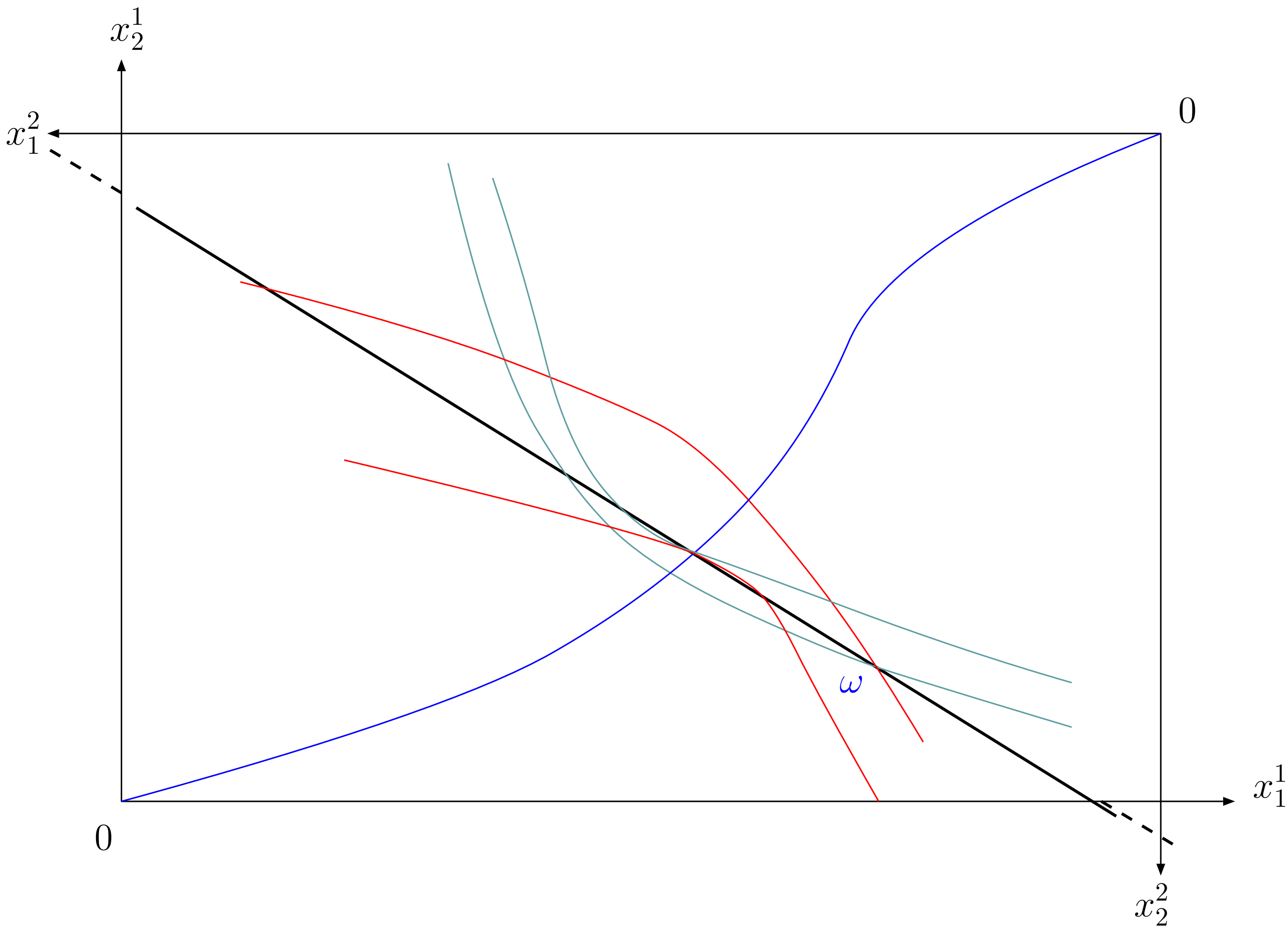
1.4 Core and Equilibria

We know that

$$\mathcal{W}(\mathcal{E}) \subset \mathcal{C}(\mathcal{E})$$

but, in general,

$$\mathcal{W}(\mathcal{E}) \neq \mathcal{C}(\mathcal{E})$$



Edgeworth (1881) conjectured that if the economy *grew large* then the core would "shrink" and (eventually) would coincide with the set of Walrasian equilibria. We will make the economy *grow large* in a very specific manner

Definition. *Replica Economies (Debreu-Scarf, 1964)* Let $\mathcal{E} = (\mathcal{I}, \{\succcurlyeq^i\}_{i\in\mathcal{I}}, \{\omega^i\}_{i\in\mathcal{I}})$ be an exchange economy. Then, $\mathcal{E}_r$ is called the r^{th}-replica of $\mathcal{E}$ and consists of a economy with r "replicas" of each individual $i \in \mathcal{I}$, each with the same preferences $\succcurlyeq^i$ and initial endowments ω^i as its "original". That is,

$$\mathcal{E}_r = \{\underbrace{\mathcal{I} \cup \mathcal{I} \cup \cdots \cup \mathcal{I}}_{r-times}, \{\succcurlyeq^i\}_{i\in\cup_1^r\mathcal{I}}, \{\omega^i\}_{i\in\cup_1^r\mathcal{I}})$$

- Each individual in $\mathcal{E}_r$ is indexed with the superindex ^{iq}, where $i \in \mathcal{I}$ and $q \in \{1, 2, \ldots, r\}$

- For each individual in $\mathcal{E}_r$, x^{iq} denotes the allocation of the i^{th} individual in the q^{th} replica, Therefore, an allocation of $\mathcal{E}_r$ is a vector in $\mathbb{R}_+^{Irn}$ of the form

$$x = (\underbrace{x^{11}, x^{21}, \ldots, x^{I1}}_{1}, \underbrace{x^{12}, \ldots, x^{I2}}_{2}, \ldots, \underbrace{x^{1r}, \ldots, x^{Ir}}_{r})$$

- An allocation x is *feasible* in $\mathcal{E}_r$ if

$$\sum_{q=1}^{r}\sum_{i=1}^{I} x^{iq} \leq r \cdot \sum_{i=1}^{I} \omega^i$$

Proposition 1.4.1 Walrasian equilibria in replica economies

$$x^* \in \mathcal{W}(\mathcal{E}) \Leftrightarrow (x^*, x^*, \ldots, x^*) \in \mathcal{W}(\mathcal{E}_r)$$

Proof. Obvious $\square$

Proposition 1.4.2 Core equilibria in replica economies

$$x^* \in \mathcal{C}(\mathcal{E}) \Leftarrow (x^*, x^*, \ldots, x^*) \in \mathcal{C}(\mathcal{E}_r)$$

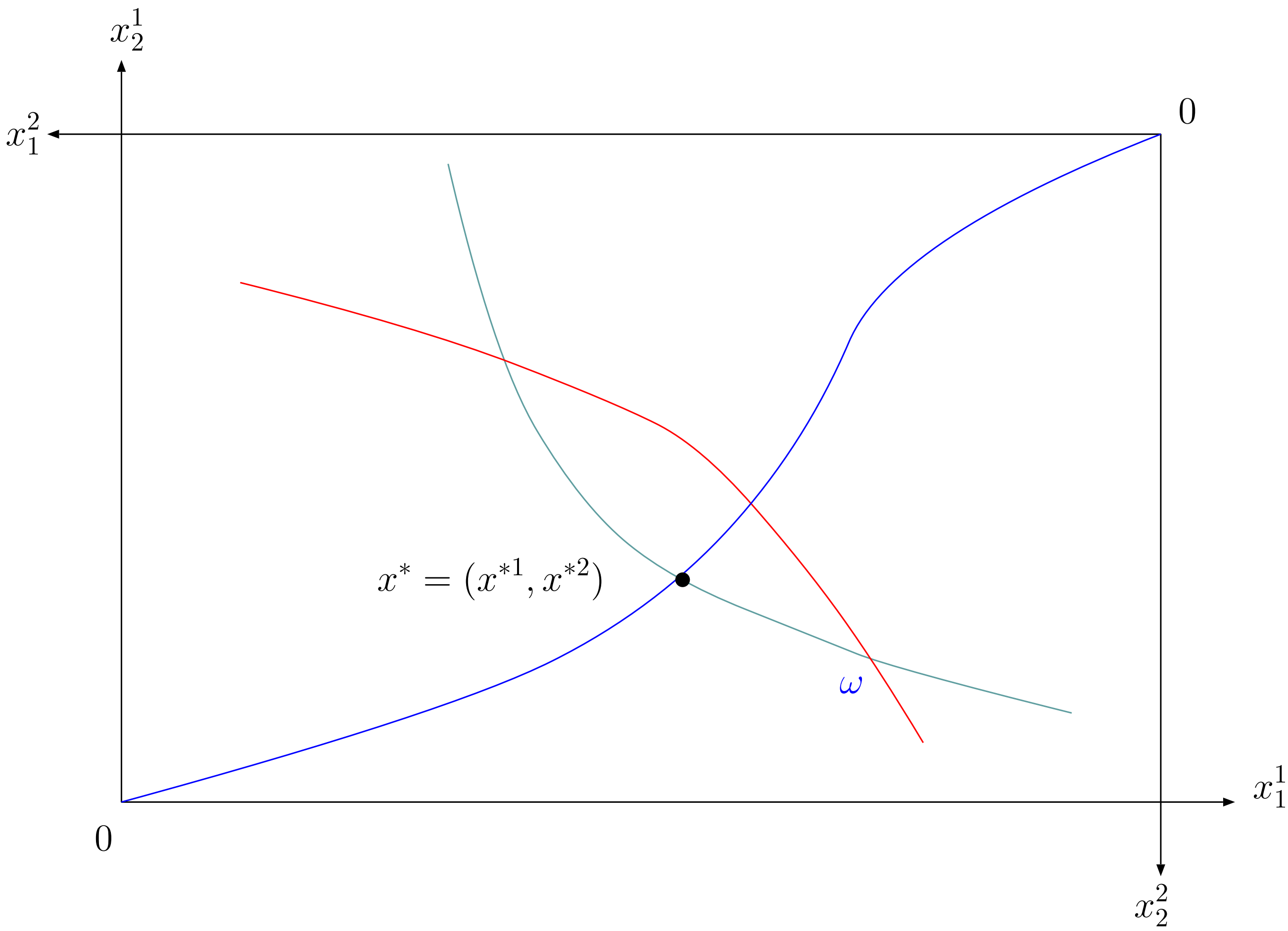
Proof. Suppose not, that is, suppose that there exists some coalition $S \subset \mathcal{I}$ together with some allocation x' (that is feasible for S) so that S blocks x^*

But then, since the same coalition S is present in the $\mathcal{E}_r$ economy, it can also block $(x^*, x^*, \ldots, x^*)$ with $(x', x^*, \ldots, x^*)$, which CONTRADICTS $(x^*, x^*, \ldots, x^*) \in \mathcal{C}(\mathcal{E}_r)$ $\square$

Example: To show that

$$x^* \in \mathcal{C}(\mathcal{E}) \not\Rightarrow (x^*, x^*, \ldots, x^*) \in \mathcal{C}(\mathcal{E}_r)$$

Consider $x^* = (x^{*1}, x^{*2})$ as in the picture below



Clearly, $x^* \in \mathcal{C}(\mathcal{E})$. We will show that $(x^*, x^*) \notin \mathcal{C}(\mathcal{E}_2)$

Consider now the coalition $S \subset \mathcal{I} \cup \mathcal{I}$ composed of individuals $^{11}, {}^{21}, \text{and}\, {}^{12}$ and the alternative allocation $(\hat{x}^{11}, \hat{x}^{21}, \hat{x}^{12})$ for S, where

$$\hat{x}^{11} = \hat{x}^{12} = \frac{1}{2}\omega^1 + \frac{1}{2}x^{*1}$$
$$\hat{x}^{21} = x^{*2}$$

It is feasible for S since

$$\begin{aligned} \hat{x}^{11} + \hat{x}^{12} + \hat{x}^{21} &= 2(\frac{1}{2}\omega^1 + \frac{1}{2}x^{*1}) + x^{*2} = \\ \omega^1 + x^{*1} + x^{*2} &= \omega^1 + \omega^1 + \omega^2 \end{aligned}$$

Finally, because of strict convexity of the preferences, it is clear that

$$\hat{x}^{11} \succ^{11} x^{*1}$$
$$\hat{x}^{21} \sim^{21} x^{*2}$$
$$\hat{x}^{12} \succ^{12} x^{*1}$$

Hence, S blocks the allocation (x^*, x^*)

Proposition 1.4.3 Equal Treatment in the Core

$$x \in \mathcal{C}(\mathcal{E}_r) \Rightarrow x^{iq} = x^{iq'}$$

$$\forall i \in \mathcal{I},\ \forall q,\ q' \in \{1, 2, \ldots, r\}$$

Proof. Let $I = 2$ and $r = 2$ (can be generalized)

Let $x = (x^{11}, x^{21}, x^{12}, x^{22}) \in \mathcal{C}(\mathcal{E}_2)$

Proceed by contradiction.

Assume WLOG that $x^{11} \neq x^{12}$

Since preferences are complete, we have that

$$\begin{array}{rl} \text{either} & x^{11} \succsim^1 x^{12} \\ \text{or} & x^{12} \succsim^1 x^{11} \end{array}$$

assume WLOG that $x^{11} \succsim^1 x^{12}$

Similarly, assume WLOG that $x^{21} \succsim^2 x^{22}$

So, we have that

$$^{12} \quad \text{is worse of than} \quad ^{11}$$

$$^{22} \quad \text{is worse of than} \quad ^{21}$$

Consider

$$\bar{x}^{12} = \frac{x^{11} + x^{12}}{2}$$
$$\bar{x}^{22} = \frac{x^{21} + x^{22}}{2}$$

Consider the coalition $S = \{\,^{12},\,^{22}\}$.
Clearly, by strict convexity

$$\bar{x}^{12} \succ^{1} x^{12}$$
$$\bar{x}^{22} \succcurlyeq^{2} x^{22}$$

Is $(\bar{x}^{12}, \bar{x}^{22})$ feasible for S ?

$$\begin{aligned}\bar{x}^{12} + \bar{x}^{22} &= \frac{x^{11} + x^{12}}{2} + \frac{x^{21} + x^{22}}{2} = \\ &= \frac{x^{11} + x^{12} + x^{21} + x^{22}}{2} = \\ &= \frac{2\omega^1 + 2\omega^2}{2} = \omega^1 + \omega^2\end{aligned}$$

So, S blocks x, which contradicts being in $\mathcal{C}(\mathcal{E}_2)$ $\square$

Therefore, if $x \in \mathcal{C}(\mathcal{E}_r)$ then x must be of the form

$$x = (\underbrace{x^1, \ldots, x^I}_{1}, \underbrace{x^1, \ldots, x^I}_{2}, \ldots, \underbrace{x^1, \ldots, x^I}_{r})$$

Notation

$$C_r = \{x = (x^1, \ldots, x^I) \in \mathbb{R}_+^{nI} \,|\, (x, x, \ldots, x) \in \mathcal{C}(\mathcal{E}_r)$$

Proposition 1.4.4 The core shrinks

$$C_1 \supset C_2 \supset C_3 \supset \cdots \supset C_r \supset \cdots$$

Proof. By induction. It is enough to prove that if $r > 1$ then $C_r \subset C_{r-1}$

Let $x = (x^1, \ldots, x^I) \in C_r$. Them it must be the case that $x \in C_{r-1}$. Otherwise, if there exists a coalition S in $\mathcal{E}_{r-1}$ such that blocks x with another allocation x' that is feasible for them, then such coalition S will also be present in $\mathcal{E}_r$ and will also block x with the same alternative x' $\square$

So far, we have the following relationships

$$x^* \in C_r \Leftrightarrow (x^*, \dots, x^*) \in \mathcal{C}(\mathcal{E}_r) \underset{\text{Proposition 1.4.2}}{\overset{\text{Example Page 96}}{\overset{\not\Leftarrow}{\Rightarrow}}} x^* \in \mathcal{C}(\mathcal{E})$$

$$\text{Figure Page 93} \; \not\Downarrow \Uparrow \; \text{Theorem 1.2.4} \qquad \text{Figure Page 93} \; \not\Downarrow \Uparrow \; \text{Theorem 1.2.4}$$

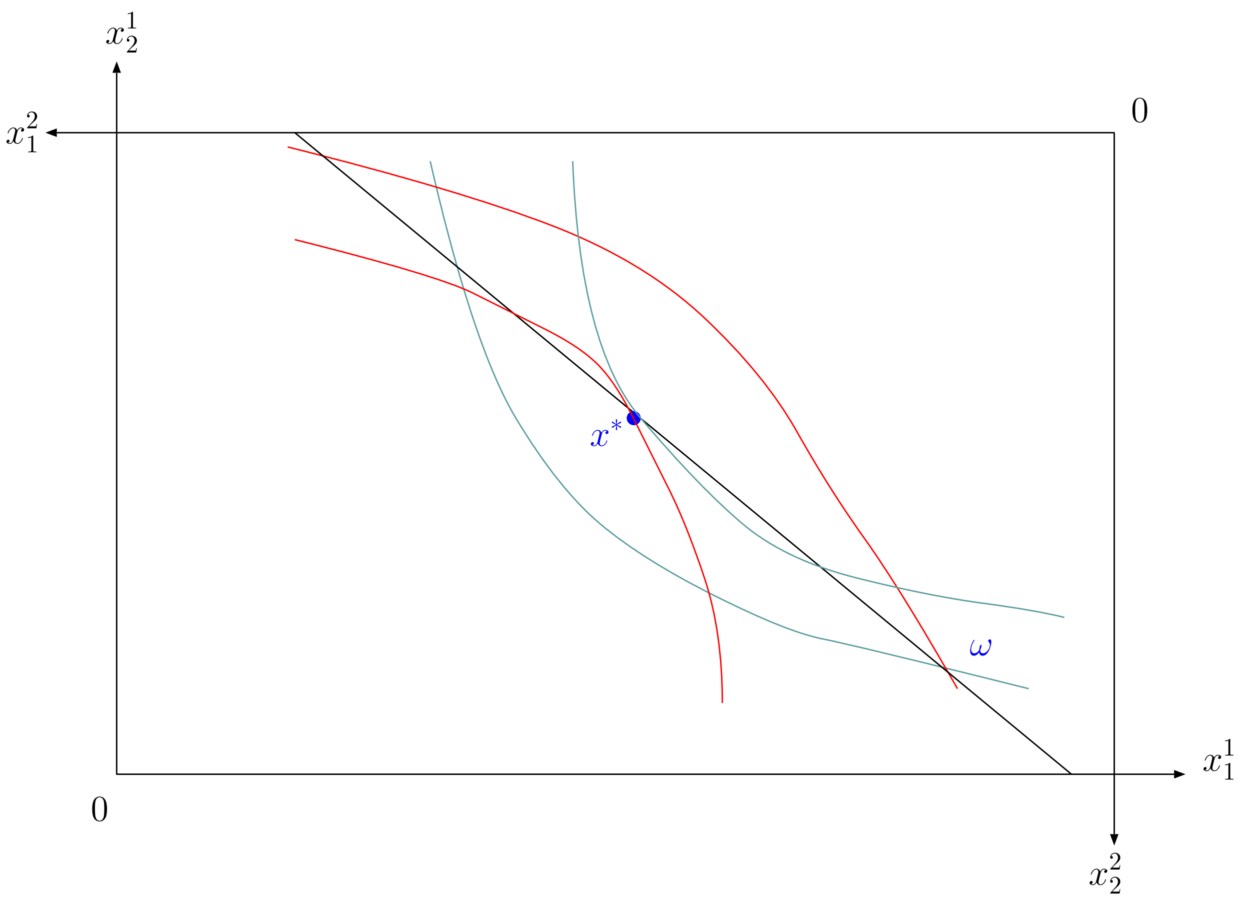
$$(x^*, \dots, x^*) \in \mathcal{W}(\mathcal{E}_r) \underset{\text{Proposition 1.4.1}}{\Leftrightarrow} x^* \in \mathcal{W}(\mathcal{E})$$

Theorem 1.4.1 The Limit Theorem of the Core (Debreu-Scarf 1963)

If $x \in C_r$ for any $r \geq 1$, then $x \in \mathcal{W}(\mathcal{E})$

Proof. (2 individuals. Can be generalized)

By contradiction. Suppose $x^* \in C_r$ for $r = 1, 2, \ldots$ but $x^* \notin \mathcal{W}(\mathcal{E})$.
In particular, this means that $x^* \in C_1 = \mathcal{C}(\mathcal{E})$. Hence, we have a situation as in the following picture



where:

- The indifference curves of the two individuals must be tangent to each other (otherwise x^* would not be in the core)

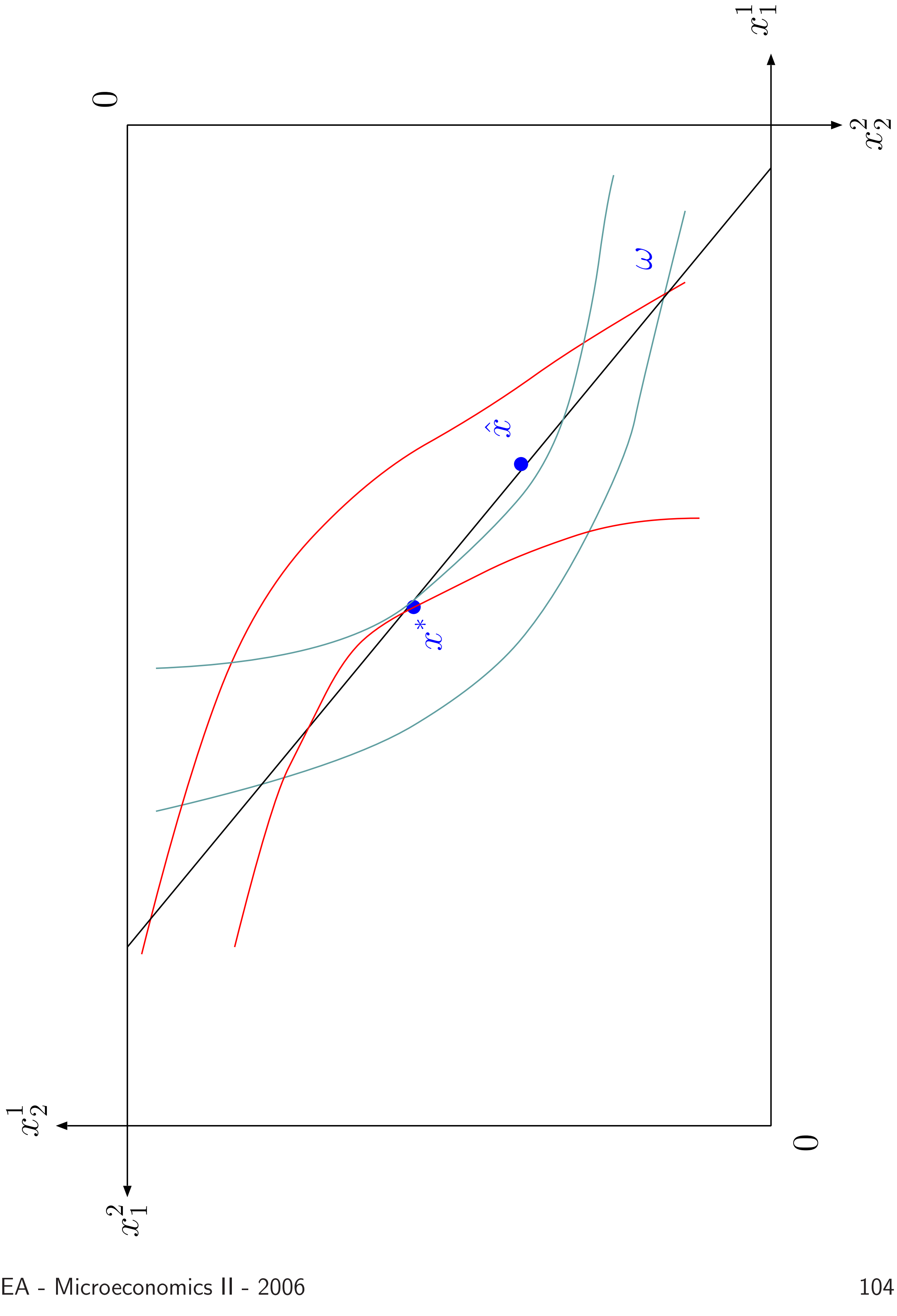
- The budget line that goes through x^* must NOT be tangent to (at least one of) the indifference curves (otherwise x^* would be a Walrasian equilibrium)

That is,

- $MRS^1(x^{*1}) = MRS^2(x^{*2})$

- Either $\frac{p_1}{p_2} \neq MRS^1(x^{*1})$ or $\frac{p_1}{p_2} \neq MRS^2(x^{*2})$

Assume, wlog, that $\frac{p_1}{p_2} \neq MRS^1(x^{*1})$ and consider the allocation $\hat{x}$ as in the following picture



0
x_1^1
x_2^2
ω
$\hat{x}$
x^*
x_2^1
x_1^2
0

For the first individual, $\hat{x}^1$ can be written as

$$\hat{x}^1 = \frac{1}{q}\omega^1 + \frac{q-1}{q}x^{*1}$$

Clearly, by strict convexity

$$\hat{x}^1 \succ^1 x^{*1}$$

Since we are assuming that $x^* \in C_r$ for any $r = 1, 2, \ldots$ we have that $(x^*, x^*, \ldots, x^*) \in \mathcal{C}(\mathcal{E}_r)$ for any $r = 1, 2, \ldots$

Consider the following coalition in the $\mathcal{E}_q$ economy

$$\begin{aligned} S &= \{ ^{11}, ^{12}, \ldots, ^{1q}, ^{21}, ^{22}, \ldots, ^{2(q-1)} \} \\ &= \underbrace{\mathcal{I} \cup \mathcal{I} \cup \cdots \cup \mathcal{I}}_{q-times} - \{ ^{2q} \} \end{aligned}$$

Give $\hat{x}^1$ to each "type 1" individual and leave all "type 2" individuals with their x^{*2} allocations. That is,

$$\hat{x}^{11} = \hat{x}^{12} = \cdots = \hat{x}^{1q} = \hat{x}^{1} = \frac{1}{q}\omega^1 + \frac{q-1}{q}x^{*1} \succ^1 x^{*1}$$
$$\hat{x}^{21} = x^{*2}$$
$$\hat{x}^{22} = x^{*2}$$
$$\vdots$$
$$\hat{x}^{2(q-1)} = x^{*2} \qquad (12)$$

This allocation is feasible ...

$$\begin{aligned} q\hat{x}^1 + (q-1)\hat{x}^2 &= q(\frac{1}{q}\omega^1 + \frac{q-1}{q}x^{*1}) + (q-1)x^{*2} = \\ &= \omega^1 + (q-1)(x^{*1} + x^{*2}) \end{aligned}$$

we know that $x^{*1} + x^{*2} = \omega^1 + \omega^2$ (by assumption, $x^* \in \mathcal{C}(\mathcal{E})$), hence

$$\begin{aligned} q\hat{x}^1 + (q-1)\hat{x}^2 &= \omega^1 + (q-1)(\omega^1 + \omega^2) = \\ &= q\omega^1 + (q-1)\omega^2 \qquad (13) \end{aligned}$$

Therefore, $\hat{x}$ as defined above:

- Is feasible because of (13)
- Can be used by coalition S to block x^* because of (12)

Thus, we reach a contradiction with the assumption that $x^* \in C_r$ for any $r = 1, 2, \ldots$ $\square$