All Sequential Allotment Rules Are Obviously Strategy-proof*

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Abstract: For the division problem with single-peaked preferences (Sprumont, 1991) we show that all sequential allotment rules, identified by Barberà, Jackson and Neme (1997) as the class of strategy-proof, efficient and replacement monotonic rules, are also obviously strategy-proof. Although obvious strategy-proofness is in general more restrictive than strategy-proofness, this is not the case in this setting.

Keywords: Obvious Strategy-proofness; Sequential Allotment Rules; Division Problem; Single-peaked Preferences.

JEL *Classification*: D71.

1 Introduction

The purpose of this paper is to show that, in the division problem with single-peaked preferences, all sequential allotment rules are obviously strategy-proof. We do it by constructing an algorithm that, applied to each sequential allotment rule, defines an extensive game form for which truth-telling is an obviously dominant strategy and it induces the rule.

A rule maps profiles of agents' preferences into alternatives. A rule is strategy-proof if, for each agent, truth-telling is always optimal, regardless of the preferences declared by the other agents. Namely, at each profile of preferences and for each agent, to declare the

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true preferences is a weakly dominant strategy in the game in normal form induced by the rule at the given (true) profile. To deal with the concern of how easy it is for an agent to identify that truth-telling is indeed weakly dominant (*i.e.*, how much contingent reasoning is required to do so), Li (2017) proposes the stronger notion of obvious strategy-proofness (OSP).

In this paper we show that, for the discrete division problem with single-peaked preferences, the set of sequential allotment rules coincides with the class of all obviously strategy-proof, efficient and restricted monotonic rules. And hence, under efficiency and restricted monotonicity, the requirements of strategy-proofness and obvious strategy-proofness do coincide.

The continuous division problem with single-peaked preferences, proposed and studied by Sprumont (1991), is as follows. A given amount of a perfectly divisible good has to be allotted among a set of agents that have single-peaked preferences over the set of possible shares.[1] Sprumont (1991) characterizes the uniform allocation rule as the unique one satisfying strategy-proofness, efficiency and anonymity. Barberà, Jackson and Neme (1997) studies non-anonymous rules and shows that the class of all strategy-proof, efficient and replacement monotonic rules coincides with the large family of sequential allotment rules.[2]

In general, a rule is obviously strategy-proof if there is an extensive game form, whose set of players is the set of agents and its outcomes are the alternatives, with two additional features. First, for any given profile of preferences, one can identify a profile of truth-telling behavioral strategies such that, if played, the outcome of the game is the alternative that the rule would select at the profile of preferences. Second, for each profile of preferences, truth-telling is an obviously dominant strategy; namely, for each agent, for each non truthful behavioral strategy, and for each information set that is an earliest point of departure with the truth-telling strategy (i) the agent evaluates the consequence of behaving according to truth-telling in a pessimistic way (thinking that the worse possible outcome will happen), (ii) the agent evaluates the consequence of behaving according to the non truthful strategy in an optimistic way (thinking that the best possible outcome will happen), and (iii) the pessimistic alternative associated to the truth-telling strategy is at least as good as the optimistic alternative associated to the non truthful strategy. Hence, whenever the agent has to play along the game, the action prescribed by the truth-telling strategy appears as

[1] For instance, situations where the good is a fixed number of assets in a bankruptcy, or rationed consumption bundles traded at fixed prices, or a surplus of a joint project, or a cost of a public project, or a total working time required to complete a task, and so on. In the discrete version of the division problem the amount of the good to be allotted comes in indivisible units and agents can only receive integer amounts. In the Final Remarks section at the end of the paper we explain the technical reason why we consider here the discrete version of the continuous Sprumont (1991)'s model.

[2] A rule is replacement monotonic if whenever the allotment of an agent changes after a change in his/her reported preferences, then all the other agents' allotments change in the same direction: either all increase or all decrease.

unmistakably optimal; *i.e.* obviously dominant. Mackenzie (2018) gives a general revelation principle like result for obviously strategy-proof implementation, upon which we will base our result.[3] Mackenzie (2018) shows that when looking for an extensive game form to OSP-implement a particular rule, one can restrict attention without loss of generality to the class of round table mechanisms. Those are extensive game forms with perfect information in which each agent, when playing along the game, is required to publicly reveal partial information about his/her preferences. Moreover, it is sufficient to require that, in the round table mechanism, truth-telling be a weakly dominant strategy.

At the light of the extreme behavioral criterion used to evaluate truth-telling, it is not surprising that the literature has already identified settings for which either none of the strategy-proof rules are obviously strategy-proof or only a very special and small subset of them satisfy the stronger requirement. For instance, in the complete impossibility case, Li (2017) already shows that the rule associated to the top-trading cycles algorithm in the house allocation problem of Shapley and Scarf (1974) is not obviously strategy-proof. Ashlagi and Gonczarowski (2018) shows that the rule associated to the deferred acceptance algorithm is not obviously strategy-proof for the agents belonging to the offering side. In the partial (or total) possibility case, Li (2017) characterizes the monotone price mechanisms (generalizations of ascending auctions) as those that implement all obviously strategy-proof rules on the domain of quasi-linear preferences. Li (2017) also shows that, for online advertising auctions, the rule induced by the mechanism that selects the efficient allocation and the Vickrey-Clarke-Groves payment is obviously strategy-proof. Ashlagi and Gonczarowski (2018) shows however that the rule associated to the deferred acceptance algorithm becomes obviously strategy-proof on the restricted domain of acyclic preferences introduced by Ergin (2002).[4]

Despite the fact that in many settings obvious strategy-proofness becomes significantly more restrictive than just strategy-proofness, we surprisingly find that, for the discrete division problem with single-peaked preferences, each sequential allotment rule (*i.e.*, each strategy-proof, efficient and replacement monotonic rule) is indeed obviously strategy-proof. We show it by exhibiting, for each sequential allotment rule, the extensive game form with perfect information that OSP-implements the rule. The construction of the game is done by an algorithm with two phases.

The specific phase of the algorithm only deals with sequential allotment rules that also satisfy individual rationality with respect to a reference allotment $q = (q_1, \ldots, q_n)$, where n is the number of agents and q (which depends on the rule) is feasible (*i.e.*, $q_1 + \cdots + q_n = k$, where k is the total number of units of the good that has to be allotted). This property

[3] Ashlagi and Gonczarowski (2018) first noted and used this revelation principal in the context of matching rules.

[4] For other partially positive or revelation principle like results see also Arribillaga, Massó and Neme (2018), Bade and Gonczarowski (2017), Pycia and Troyan (2018) and Troyan (2018).

requires that, at every profile, the rule should assign to each agent i an allotment that is at least as preferred as i's reference allotment q_i. Consider a sequential allotment rule that is individually rational with respect to q. Barberà, Jackson and Neme (1997) shows that q is the allotment chosen by the rule at the two extreme profiles where either all agents want to receive k units or all want to receive 0. But the reference allotment q will not be efficient at most profiles. The proposed extensive game form that OSP-implements a given sequential allotment rule is designed to identify (in a parsimonious way, in order to guarantee obvious strategy-proofness) sequences of Pareto improvements that require only changes with opposite signs of one unit of the good for two agents. And this is done by means of the specific algorithm that has two stages. In Stage 1, and given the reference allotment q, agents are asked sequentially (and have to answer publicly) whether they would like to receive an allotment that is smaller, equal or larger than their corresponding reference allotment. If one of the sets of agents who want to receive more or want to receive less is empty, then the game ends with outcome q (an efficient allotment, if agents tell the truth). Otherwise (namely, when both sets are non-empty), the algorithm moves to Stage 2 with input q and the partition of the set of agents into the subsets of agents that want to receive more, less or equal amount of the good than that of their reference allotment. The purpose of each generic step in Stage 2 is to identify two agents, agent i who wants to receive more than q_i and agent j who wants to receive less than q_j, and a new allotment q' (output of the step) that constitutes a Pareto improvement with respect to q (if agents tell the truth) with the property that $q'_i = q_i + 1$, $q'_j = q_j - 1$ and all the other components of q' are equal to those in q. Along the different steps in Stage 2, the reference allotment and the agents' partition are updated appropriately. The game ends, with the current reference allotment as its outcome, at the step where one of the sets of agents who want to receive more or want to receive less is empty (and so, no Pareto improvement is possible).

The general phase of the algorithm, used to deal with sequential allotment rules that are not individually rational with respect to any reference allotment, contains a procedure to identify a single reference allotment q in an iterative way. It starts from two extreme allotments, the one that the rule would select if all agents wanted to receive k units and the the other if all wanted to receive 0. Then, and once this allotment q is identified, the game proceeds according to the specific phase of the algorithm using q as the reference allotment.

This paper contributes to the obviously strategy-proof implementation literature by showing that, in a well known setting with a large the class of well behaved strategy-proof rules, all of them are in addition obviously strategy-proof. Namely, although obvious strategy-proofness is more restrictive than just strategy-proofness, this may not be always the case. Thus, the perception that obvious strategy-proofness is not a very appealing notion because it is too restrictive, should be taken with a certain caution.

The paper is organized as follows. Section 2 contains the basic notation and definitions. Section 3 presents the notion of obvious strategy-proofness. Section 4 contains, for each sequential allotment rule, the definition of the extensive game form that OSP-implements it and an example illustrating the algorithm. Sections 5 contains three final remarks. Appendix 1 in Section 6 collects the proofs omitted in the main text and Appendix 2 in Section 7 presents the formal definition of a sequential allotment rule and an example illustrating its definition.

2 Preliminaries

Agents are the elements of a finite set $N = \{1, \ldots, n\}$, where $n \geq 2$. They have to share $k \in \mathbb{N}_+$ units of a good, where $\mathbb{N}_+$ is the set of positive integers, each unit is indivisible and $k \geq 2$. An *allotment* is a vector $x = (x_1, \ldots, x_n) \in \{0, \ldots, k\}^n$ such that $\sum_{i=1}^{n} x_i = k$. Let X be the set of allotments. Each agent $i \in N$ has complete *preferences* R_i over $\{0, \ldots, k\}$, the set of i's possible allotments. Let P_i be the strict preferences associated with R_i. The preferences R_i are *single-peaked* if they have a unique most-preferred share $\tau(R_i) \in \{0, \ldots, k\}$, the *top* of R_i, such that $\tau(R_i) P_i x_i$ for all $x_i \in \{0, \ldots, k\} \backslash \{\tau(R_i)\}$ and for any pair $y_i, x_i \in \{0, \ldots, k\}$, $y_i < x_i < \tau(R_i)$ or $\tau(R_i) < x_i < y_i$ implies $x_i P_i y_i$. We assume that agents have single-peaked preferences. Often, the relevant information of R_i will be its top $\tau(R_i)$ and if R_i is obvious from the context we will refer to it by τ_i. We denote by $\mathbf{0}$, $\mathbf{1}$ and $\mathbf{k}$ the vectors $(0, \ldots, 0), (1, \ldots, 1), (k, \ldots, k) \in \{0, \ldots, k\}^n$ and, given $S \subset N$, by $\mathbf{0}_S$, $\mathbf{1}_S$ and $\mathbf{k}_S$ the corresponding subvectors where all agents in S receive the allotment 0, 1 and k. Moreover, for a vector $x = (x_1, \ldots, x_n) \in \{0, \ldots, k\}^n$ we denote by x_S the subvector $(x_i)_{i \in S}$ and by $(x - \mathbf{1})_S$ the subvector $(x_i - 1)_{i \in S}$.

Let $\mathcal{R}$ the set of all single-peaked preferences over $\{0, \ldots, k\}$. *Profiles* (*of preferences*) are n-tuples of single-peaked preferences over $\{0, \ldots, k\}$, one for each agent, and they are denoted by $R = (R_1, \ldots, R_n) \in \mathcal{R}^n$. When we want to stress the role of agent i's preferences we will represent a profile R by (R_i, R_{-i}) and by (R_S, R_{-S}) if we want to stress the role of the preferences of agents in S.

A discrete *division problem* is a pair (N, k), where N is the set of agents, k is the number of units of the good that have to be allotted among the agents in N, and agents have single-peaked preferences over $\{0, \ldots, k\}$.

A solution of the discrete division problem (N, k) is a rule that selects, for each profile, an allotment. Namely, a *rule* is a function $\Phi : \mathcal{R}^n \to X$; that is, $\sum_{i \in N} \Phi_i(R) = k$ for all $R \in \mathcal{R}^n$.

A desirable requirement on rules is *efficiency* in the sense that, for each profile, the selected allotment should be efficient. Namely, for each $R \in \mathcal{R}^n$, there is no $y \in X$ such that $y_i R_i \Phi_i(R)$ for all $i \in N$ and $y_j P_j \Phi_j(R)$ for at least one $j \in N$. It is easy to check that, by single-peakedness, efficiency of Φ is equivalent to the property of *same-sidedness* which requires that, for each $R \in \mathcal{R}^n$,

$$\text{if } \sum_{j \in N} \tau(R_j) \geq k \text{ then } \Phi_i(R) \leq \tau(R_i) \text{ for all } i \in N \tag{1}$$

and

$$\text{if } \sum_{j \in N} \tau(R_j) \leq k \text{ then } \Phi_i(R) \geq \tau(R_i) \text{ for all } i \in N. \tag{2}$$

Rules require agents to report single-peaked preferences over $\{0, \ldots, k\}$. A rule is strategy-proof if it is always in the best interest of agents to truthfully reveal their preferences; namely, for each profile and each agent, the true preferences are weakly dominant in the normal form game induced by the rule at the profile. Namely, a rule $\Phi : \mathcal{R}^n \to X$ is *strategy-proof* if for all $R \in \mathcal{R}^n$, all $i \in N$ and all $R'_i \in \mathcal{R}$,

$$\Phi_i(R_i, R_{-i}) R_i \Phi_i(R'_i, R_{-i}).$$

We will say that agent $i \in N$ manipulates the rule $\Phi : \mathcal{R}^n \to X$ at $R \in \mathcal{R}^n$ with $R'_i \in \mathcal{R}$ if

$$\Phi_i(R'_i, R_{-i}) P_i \Phi_i(R_i, R_{-i}).$$

Obviously, a rule $\Phi : \mathcal{R}^n \to X$ is *strategy-proof* if no agent can manipulate it.

We will restrict our attention to rules that satisfy the simplicity requirement of depending only on the profile of top allotments. Formally, a rule $\Phi : \mathcal{R}^n \to X$ is *tops-only* if for all $R, R' \in \mathcal{R}^n$ such that $\tau(R_i) = \tau(R'_i)$ for all $i \in N$, $\Phi(R) = \Phi(R')$. Hence, and abusing notation, a tops-only rule $\Phi : \mathcal{R}^n \to X$ can be written as $\Phi : \{0, \ldots, k\}^n \to X$. Accordingly, we will often use the notation $\Phi(\tau_1, \ldots, \tau_n)$ interchangeably with $\Phi(R_1, \ldots, R_n)$.

A rule is individually rational with respect to an allotment $q \in X$ if it guarantees that each agent i receives an allotment that is weakly preferred to q_i. Namely, a rule $\Phi : \mathcal{R}^n \to X$ is *individually rational* with respect to an allotment $q \in X$ if, for all $R \in \mathcal{R}^n$ and $i \in N$,

$$\Phi_i(R) R_i q_i.$$

Sprumont (1991) defines the continuous version of the division problem, where the good is completely divisible, the total amount to be allotted is a *real* number $k > 0$, agents have single-peaked preferences over the interval of real numbers $[0, k]$ and the set of allocations is

$$\bar{X} = \{x = (x_1, \ldots, x_n) \in [0, k]^n \mid \sum_{i=1}^n x_i = k\}.$$

The definition of a rule in the discrete division problem, as well as the properties defined above, can be straightforwardly translated to the continuous case. However, the property of equal treatment of equals (implied by anonymity) is not useful in the discrete case. For instance, for $k = 3$, $n = 2$ and $\tau(R_1) = \tau(R_2)$ there is no allotment with integer values where the two agents receive the same amount.[5]

[5] Herrero and Martínez (2011) proposes and studies the property of balancedness, as a constrained anonymity notion. A rule is balanced if whenever two agents report the same preferences their allotments differ at most by one unit.

Replacement monotonicity imposes a minimal symmetry property on how rules treat agents. It requires that if an agent, by changing the revealed preferences, obtains a different allotment then all other agents' allotments should change in the same direction. Namely, a rule $\Phi : \mathcal{R}^n \to X$ is *replacement monotonic* if, for all $R \in \mathcal{R}^n$, $i \in N$, and $R'_i \in \mathcal{R}$,

$$\Phi_i(R_i, R_{-i}) \leq \Phi_i(R'_i, R_{-i}) \text{ implies } \Phi_j(R_i, R_{-i}) \geq \Phi_j(R'_i, R_{-i}) \text{ for all } j \neq i.$$

For the continuous division problem, Barberà, Jackson and Neme (1997) characterizes the set of all sequential allotment rules as the class of all strategy-proof, efficient and replacement monotonic rules.[6] The proof of this statement can be adapted to the discrete division problem and for further reference, we state this characterization (and the one adding individual rationality) as Proposition 1.[7]

Proposition 1 (Barberà, Jackson and Neme (1997)) *Let (N, k) be a discrete division problem. Then, a rule $\Phi : \mathcal{R}^n \to X$ is strategy-proof, efficient and replacement monotonic if and only if Φ is a sequential allotment rule. Moreover, a rule $\Phi : \mathcal{R}^n \to X$ is strategy-proof, efficient, replacement monotonic and individually rational with respect to q if and only if Φ is a sequential allotment rule such that $\Phi(\mathbf{0}) = \Phi(\mathbf{k}) = q$.*

3 Obviously Strategy-proof Implementation

We briefly describe the notion of obvious strategy-proofness. Li (2017) proposes this notion with the aim of reducing the contingent reasoning that agents have to carry out to identify, given a rule and a profile, that truth-telling is a weakly dominant strategy in the direct revelation mechanism (*i.e.*, the game in normal form associated to the rule and the profile). A rule Φ is obviously strategy-proof if there exists an extensive game form with two properties. First, for each profile $R = (R_1, \ldots, R_n) \in \mathcal{R}^n$ one can identify a behavioral strategy profile, associated to truth-telling, such that if agents play the game according to such strategy the outcome is $\Phi(R)$, the allotment selected by the rule Φ at R; that is, the game induces Φ. Second, whenever agent i with preferences R_i has to play in the game, i evaluates the consequence of choosing the action prescribed by i's truth-telling strategy according to the *worse* possible outcome, among all outcomes that may occur as an effect of later actions made by agents along the rest of the game. In contrast, i evaluates the consequence of

[6] Appendix 2 in Section 7 at the end of the paper contains a formal definition of a sequential allotment rule. Our implementation result will not use explicitly the definition of Φ as a sequential allotment rule, but rather it will use its properties and the images of Φ at some specific profiles.

[7] The proof in Barberà, Jackson and Neme (1997) is based in Sprumont (1991)'s Lemma 1 stating that efficiency and strategy-proofness imply continuity, and the fact that strategy-proofness and continuity imply tops-onlyness. It is possible to show that in the discrete division problem efficiency, strategy-proofness and replacement monotonicity imply directly tops-onlyness. And then, the proof for the discrete division problem proceeds as in the continuous case.

choosing an action different from the one prescribed by i's truth-telling strategy according to the *best* possible outcome, among all outcomes that may occur again as an effect of later actions along the rest of the game. Then, i's truth-telling strategy is obviously dominant in the game if, whenever i has to play, its pessimistic outcome is at least as preferred as the optimistic outcome associated to any other strategy. If the game induces Φ and for each agent truth-telling is obviously dominant, then Φ is obviously strategy-proof.

For our context, two important simplifications related to obvious strategy-proofness have been identified in the literature that follows Li (2017). First, without loss of generality we can assume that the extensive game form that induces the rule has perfect information (see Ashlagi and Gonczarowski (2018) and Mackenzie (2018)). Second, the new notion of obvious strategy-proofness can be fully captured by the classical notion of strategy-proofness applied to extensive form games with perfect information. This last observation follows from the fact that, the best possible outcome obtained when agent i chooses an action different from the one prescribed by i's truth-telling strategy and the worst possible outcome obtained when agent i chooses the action prescribed by i's truth-telling strategy, are both obtained with only one behavioral strategy profile of the other agents, because the perfect information implies that all information sets are singleton sets (and each one belongs either to the subgame that follows the truth-telling choice or else to the subgame that follows the alternative choice).[8] Then, for easy presentation and following this literature, we will say that a rule is obviously strategy-proof if it is implemented by an extensive game form with perfect information for which truth-telling is a weakly dominant strategy (see Definition 1 below). We present the notion of extensive game form that will be used here to state and prove our results.

An *extensive game form* with perfect information associated to a discrete division problem (N, k) consists of the following elements.

1. A finite and partially ordered set of histories $(H, \prec)$, where:

   (a) $\emptyset \in H$ is the empty history for which $\emptyset \prec h$ for all $h \in H \backslash \{\emptyset\}$.

   (b) For each $h \in H \backslash \{\emptyset\}$, there is a unique $h' \in H$, the immediate predecessor of h, such that $h' \prec h$ and there is no $\bar{h}$ such that $h' \prec \bar{h} \prec h$. We write it as $h' \prec^{im} h$. A history h such that $\emptyset \prec^{im} h$ has length 1, and the integer $t \geq 2$ is the length of history h if there is a sequence $h^1, \ldots, h^{t-1} \in H$ such that $\emptyset \prec^{im} h^1 \prec^{im} \cdots \prec^{im} h^{t-1} \prec^{im} h$.

   (c) H can be partitioned into two sets, the set of terminal histories $H_T = \{h \in H \mid \text{there is no } \bar{h} \in H \text{ such that } h \prec \bar{h}\}$ and the set of non-terminal histories $H_{NT} = \{h \in H \mid \text{there is } \bar{h} \in H \text{ such that } h \prec \bar{h}\}$.

[8] Mackenzie (2018) formally proves this statement for a special class of extensive form games with perfect information, called round table mechanisms, but the proof can be adapted to any extensive game form with perfect information.

2. A mapping $\mathcal{N} : H_{NT} \to N$ that assigns to each non-terminal history $h \in H_{NT}$ the agent $\mathcal{N}(h)$ that has to play at h. Given $i \in N$, define $H_i = \{h \in H_{NT} \mid \mathcal{N}(h) = i\}$ as the set of histories at which i has to play. For $h \in H_{NT}$, we will often denote by i^h the agent $\mathcal{N}(h) = i$ that plays at h.

3. A set of actions A and a correspondence $\mathcal{A} : H_{NT} \twoheadrightarrow A \backslash \{\emptyset\}$ where, for each $h \in H_{NT}$, $\mathcal{A}(h)$ is the non-empty set of actions available to player $\mathcal{N}(h)$ at h.

4. An outcome function $o : H_T \to X$ that assigns an allotment $o(h) \in X$ to each terminal history $h \in H_T$.

An extensive game form with perfect information (or simply, a *game*) associated to the discrete division problem (N, k) is a six-tuple $\Gamma = (N, k, (H, \prec), \mathcal{N}, \mathcal{A}, o)$ with the above properties.[9] The set of agents N and the integer k will be fixed throughout the paper. Let $\mathcal{G}$ be the class of all games satisfying conditions 1 to 4 above.[10]

Fix a game $\Gamma \in \mathcal{G}$ and an agent $i \in N$. A (behavioral and pure) *strategy* of i in Γ is a function $\sigma_i : H_i \to A$ such that, for each $h \in H_i$, $\sigma_i(h) \in \mathcal{A}(h)$; namely, σ_i selects at each history h where i has to play one of i's available actions at h. Let Σ_i be the set of i's strategies in Γ. A strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n) \in \Sigma_1 \times \cdots \times \Sigma_n = \Sigma$ is an ordered list of strategies, one for each agent. Given $i \in N$, $\sigma \in \Sigma$ and $\sigma'_i \in \Sigma_i$ we often write (σ'_i, σ_{-i}) to denote the strategy profile where σ_i is replaced in σ by σ'_i. Let $h^\Gamma(h, \sigma)$ be the terminal history that results in Γ when agents start playing at $h \in H_{NT}$ according to $\sigma \in \Sigma$.

Fix a game $\Gamma \in \mathcal{G}$ and preferences $R_i \in \mathcal{R}$. A strategy σ_i is *weakly dominant* in Γ at R_i if, for all σ_{-i} and all σ'_i,

$$o_i(h^\Gamma(\emptyset, \sigma)) R_i o_i(h^\Gamma(\emptyset, (\sigma'_i, \sigma_{-i}))).$$

We are now ready to define obvious strategy-proofness in the context of the discrete division problem.

Definition 1 Let (N, k) be given. A rule $\Phi : \mathcal{R}^n \to X$ is *obviously strategy-proof* if there is an extensive game form $\Gamma \in \mathcal{G}$ associated to (N, k) such that:

(i) for each $R \in \mathcal{R}^n$, there exists a strategy profile $\sigma^R = (\sigma_1^{R_1}, \ldots, \sigma_n^{R_n}) \in \Sigma$ such that $\Phi(R) = o(h^\Gamma(\emptyset, \sigma^R))$,

[9] Note that the set of actions A is embedded in the definition of $\mathcal{A}$. Moreover, Γ is not yet a game in extensive form because agents' preferences over allotments are still missing. But given a game Γ and a profile R, the pair (Γ, R) is the game in extensive form where each agent i uses R_i to evaluate i's allotments, associated to terminal histories, induced by strategy profiles.

[10] According to Mackenzie (2018) a game $\Gamma \in \mathcal{G}$ is a *round table mechanism* if the set of actions A is the family of non-empty subsets of preferences $2^{\mathcal{R}}$ and (a) the set of actions at any history are disjoint subsets of preferences, (b) when a player has to play for the first time the set of actions is a partition of $\mathcal{R}$, and (c) later, the union of actions at history h is the intersection of the actions taken by agent $\mathcal{N}(h)$ at all predecessors that lead to h.

(ii) for all $i \in N$ and all $R_i \in \mathcal{R}$, $\sigma_i^{R_i}$ is weakly dominant in Γ at R_i.

When (i) holds we say that Γ *induces* Φ. When (i) and (ii) hold we say that Γ OSP-*implements* Φ and refer to $\sigma_i^{R_i}$ as i's *truth-telling* strategy.

Our main result here is to show that in the two statements of Proposition 1 strategy-proofness can be replaced by obvious strategy-proofness. But more importantly, we do it by constructing, for each sequential allotment rule $\Phi : \mathcal{R}^n \to X$, the extensive game form Γ that OSP-implements Φ.

4 The Extensive Game Form

Given a sequential allotment rule Φ, our construction of the game Γ is based on an algorithm with two phases. The specific phase has to be used when Φ is individually rational with respect to some $q \in X$, and so $\Phi(\mathbf{0}) = \Phi(\mathbf{k}) = q$. The general phase, to be used when $\Phi(\mathbf{0}) = \underline{q} \neq \overline{q} = \Phi(\mathbf{k})$, starts by modifying unit by unit $\underline{q}$ and $\overline{q}$ until they converge (in a finite number of steps) to a unique q. Then, the specific phase is applied, after performing a small adjustment to incorporate information about agents preferences that have already been disclosed along the process of modifying the two reference allotments $\underline{q}$ and $\overline{q}$ towards the unique reference allotment q.

4.1 The Individually Rational Case

Let $\Phi : \mathcal{R}^n \to X$ be a sequential allotment rule satisfying individual rationality with respect to the allotment $q \in X$. The extensive game form Γ that OSP-implements Φ is constructed by the specific phase of the algorithm that has two stages.

Stage 1 uses the reference allotment q as input information. Its objective is to partition the set of agents into three subsets (up to two of them could be empty). To do so, agents are asked sequentially (given a predetermined linear order $<$ on N), and answer publicly, whether they would like to receive strictly more than their allotment in q (denote the set of such agents by N_u, where u refers to "up"), strictly less than their allotment in q (denote the set of such agents by N_d, where d refers to "down") or exactly their allotment in q (denote the set of such agents by N_s, where s refers to "stay"). If at the end of **Stage 1** one of the two subsets N_u or N_d is empty, then the game ends with outcome q.

Otherwise (namely, if at the end of **Stage 1** the subsets N_u and N_d are both non-empty), the algorithm moves to **Stage 2** with the history of actions chosen by agents along **Stage 1** as input, and q and the partition $N = N_u \cup N_d \cup N_s$ as input information. The objective of each generic step t in **Stage 2** is to identify two agents, $i'' \in N_u$ and $i' \in N_d$, and a new allotment q' that would constitute a Pareto improvement (if agents told the truth) with respect to q, an input information of the step.[11] The identification of these

[11] To simplify the verbal argument, we only describe here the case $t = 1$.

two agents is done sequentially but in a linked way: the choice of $i' \in N_d$ depends on the previously chosen agent $i'' \in N_u$. Moreover, they are identified by evaluating Φ at two somehow extreme profiles that will be defined below. Along the different steps in **Stage 2**, the reference allotment q is updated to q', by setting

$$q'_i = \begin{cases} q_i + 1 & \text{if } i = i'' \\ q_i - 1 & \text{if } i = i' \\ q_i & \text{if } i \notin \{i'', i'\}, \end{cases}$$

and the sets N_u, N_d and N_s are also updated depending on the actions taken by i'' and i'. The algorithm stops at the step where one of the two subsets N_u or N_d is empty with the current allotment as its outcome. We now describe in detail the specific phase of the algorithm.

4.1.1 The Specific Phase of the Algorithm

Input: A sequential allotment rule Φ satisfying individual rationality with respect to $q \in X$ and a linear order $<$ on the set of agents that, without loss of generality, we assume $1 < \cdots < n$.

Set $h' = \emptyset$ and $q^\emptyset = q$. Go to **Stage 1** with input $h' = \emptyset$ and collect $q^\emptyset$ and $N^\emptyset = N_u^\emptyset = N_d^\emptyset = N_s^\emptyset = \emptyset$ as input information.

Stage 1: Go to the initialization step, **Step 1.0**.

Step 1.0:

Input of **Step 1.0:** The empty history $h' = \emptyset$.

Input information of **Step 1.0:** $q^\emptyset$ and $N^\emptyset = N_u^\emptyset = N_d^\emptyset = N_s^\emptyset = \emptyset$.

Define $i^\emptyset = 1$.

Agent $i^\emptyset$ has to choose an action from the set

$$A_{i^\emptyset}^\emptyset = \{d, q_{i^\emptyset}^\emptyset, u\}.$$[12]

Let $\bar{a}_{i^\emptyset} \in A_{i^\emptyset}^\emptyset$ be the choice of $i^\emptyset$. To homogenize the description of the sets of actions in different steps of the algorithm, and in order to interpret correctly the action $\bar{a}_{i^\emptyset}$, we identify (as being the same actions) u with k if $q_{i^\emptyset}^\emptyset = k$ and d with 0 if $q_{i^\emptyset}^\emptyset = 0$.[13] To do so, define

$$a_{i^\emptyset} = \begin{cases} k & \text{if } q_{i^\emptyset}^\emptyset = k \text{ and } \bar{a}_{i^\emptyset} \in \{u, q_{i^\emptyset}^\emptyset\} \\ 0 & \text{if } q_{i^\emptyset}^\emptyset = 0 \text{ and } \bar{a}_{i^\emptyset} \in \{d, q_{i^\emptyset}^\emptyset\} \\ \bar{a}_{i^\emptyset} & \text{otherwise.} \end{cases}$$

[12] The set of available actions $A_{i^\emptyset}^\emptyset$ (when $i^\emptyset$ plays for the first time) can be seen as a partition of $\mathcal{R}$ by identifying action u as the subset $\{R_{i^\emptyset} \in \mathcal{R} \mid \tau(R_{i^\emptyset}) > q_{i^\emptyset}^\emptyset\}$, action d as the subset $\{R_{i^\emptyset} \in \mathcal{R} \mid \tau(R_{i^\emptyset}) < q_{i^\emptyset}^\emptyset\}$ and action $q_{i^\emptyset}^\emptyset$ as the subset $\{R_{i^\emptyset} \in \mathcal{R} \mid \tau(R_{i^\emptyset}) = q_{i^\emptyset}^\emptyset\}$. Observe that when either $q_{i^\emptyset}^\emptyset = k$ or $q_{i^\emptyset}^\emptyset = 0$ this partition has only two subsets: $\{R_{i^\emptyset} \in \mathcal{R} \mid \tau(R_{i^\emptyset}) = k\}$ and $\{R_{i^\emptyset} \in \mathcal{R} \mid \tau(R_{i^\emptyset}) < k\}$ when $q_{i^\emptyset}^\emptyset = k$, or $\{R_{i^\emptyset} \in \mathcal{R} \mid \tau(R_{i^\emptyset}) = 0\}$ and $\{R_{i^\emptyset} \in \mathcal{R} \mid \tau(R_{i^\emptyset}) > 0\}$ when $q_{i^\emptyset}^\emptyset = 0$.

[13] This identification also ensures that the sets of actions satisfy the conditions of a round table mechanism.

Set $h = (\emptyset, a_{i^\emptyset})$ and define

$$N_u^h = \begin{cases} \{i^\emptyset\} & \text{if } a_{i^\emptyset} = u \\ \emptyset & \text{otherwise,} \end{cases} \quad N_d^h = \begin{cases} \{i^\emptyset\} & \text{if } a_{i^\emptyset} = d \\ \emptyset & \text{otherwise,} \end{cases} \quad N_s^h = \begin{cases} \{i^\emptyset\} & \text{if } a_{i^\emptyset} = q_{i^\emptyset}^\emptyset \\ \emptyset & \text{otherwise,} \end{cases}$$

$$N^h = N_u^h \cup N_d^h \cup N_s^h, \quad \text{and} \quad q^h = q^\emptyset.$$

Set $h' := h$ as output of **Step 1.0**. Go to **Step 1.1** with input h' and collect $q^{h'}$ and $N^{h'}$, $N_u^{h'}$, $N_d^{h'}$, $N_s^{h'}$ as input information.

Step 1.t (t$\geq$ 1):

Input of **Step 1.t:** h', the output of **Step 1.t-1** or **Stage B** (of the generic phase of the algorithm, to be defined later).

Input information of **Step 1.t:** $q^{h'}$ and $N^{h'}$, $N_u^{h'}$, $N_d^{h'}$, $N_s^{h'}$ (with $N^{h'} \neq N$).

Define $i^{h'} = \min_<\{i \in N \mid i \notin N^{h'}\}$.

Agent $i^{h'}$ has to choose an action from the set

$$A_{i^{h'}}^{h'} = \{d, q_{i^{h'}}^{h'}, u\}.$$

Let $\bar{a}_{i^{h'}} \in A_{i^{h'}}^{h'}$ be the choice of $i^{h'}$ and, as we have done in **Step 1.0**, define

$$a_{i^{h'}} = \begin{cases} k & \text{if } q_{i^{h'}}^{h'} = k \text{ and } \bar{a}_{i^{h'}} \in \{u, q_{i^{h'}}^{h'}\} \\ 0 & \text{if } q_{i^{h'}}^{h'} = 0 \text{ and } \bar{a}_{i^{h'}} \in \{d, q_{i^{h'}}^{h'}\} \\ \bar{a}_{i^{h'}} & \text{otherwise.} \end{cases}$$

Set $h = (h', a_{i^{h'}})$ and define

$$N_u^h = \begin{cases} N_u^{h'} \cup \{i^{h'}\} & \text{if } a_{i^{h'}} = u \\ N_u^{h'} & \text{otherwise,} \end{cases} \quad N_d^h = \begin{cases} N_d^{h'} \cup \{i^{h'}\} & \text{if } a_{i^{h'}} = d \\ N_d^{h'} & \text{otherwise,} \end{cases}$$

$$N_s^h = \begin{cases} N_s^{h'} \cup \{i^{h'}\} & \text{if } a_{i^{h'}} = q_{i^{h'}}^{h'} \\ N_s^{h'} & \text{otherwise,} \end{cases} \quad N^h = N_u^h \cup N_d^h \cup N_s^h, \quad \text{and} \quad q^h = q^{h'}.$$

If $N^h \neq N$, set $h' := h$ as the output of **Step 1.t**. Go to **Step 1.t+1** with input h' and collect $q^{h'}$ and $N^{h'}$, $N_u^{h'}$, $N_d^{h'}$, $N_s^{h'}$ as input information. Proceed until at some **Step 1.t$'$**, $N^h = N$ holds.

If $N^h = N$ and one of the two subsets N_u^h or N_d^h is empty, then the history h is terminal and the allotment q^h is the outcome associated to h (*i.e.*, set $o(h) = q^h$).

If $N^h = N$, $N_u^h \neq \emptyset$ and $N_d^h \neq \emptyset$, set $h'' := h$ as the output of **Stage 1**. Go to **Stage 2** with input h'' and collect $q^{h''}$ and $N_u^{h''}$, $N_d^{h''}$, $N_s^{h''}$ as input information.

Stage 2: Go to **Step 2.1.a**.

Step 2.t.a (t$\geq$ 1):

Input of **Step 2.t.a:** h''. If **t**$=1$, h'' is the output of either **Stage 1** or **Stage A** (in the generic phase of the algorithm, to be defined later). If **t**>1, h'' is the output of **Step 2.t-1.b**.
Input information of **Step 2.t.a:** $q^{h''}$ and $N_u^{h''}$, $N_d^{h''}$, $N_s^{h''}$ (with $N_u^{h''} \neq \emptyset$ and $N_d^{h''} \neq \emptyset$). Notice that $i \in N_u^{h''}$ implies $q_i^{h''} < k$ and $i \in N_d^{h''}$ implies $0 < q_i^{h''}$.

Define

$$i^{h''} = \min_{<}\{j \in N_u^{h''} \mid \Phi_j(\mathbf{k}_{N_u^{h''}}, (q^{h''} - \mathbf{1})_{N_d^{h''}}, q^{h''}_{N_s^{h''}}) \geq q_j^{h''} + 1\}.^{14}$$

Agent $i^{h''}$ has to choose an action from the set

$$A^{h''}_{i^{h''}} = \{q^{h''}_{i^{h''}} + 1, u\}.$$

Let $\bar{a}_{i^{h''}} \in A^{h''}_{i^{h''}}$ be the choice of $i^{h''}$ and, as we have done previously, define

$$a_{i^{h''}} = \begin{cases} k & \text{if } q^{h''}_{i^{h''}} + 1 = k \\ \bar{a}_{i^{h''}} & \text{otherwise.} \end{cases}$$

Set $h' = (h'', a_{i^{h''}})$ and define

$$N_u^{h'} = \begin{cases} N_u^{h''} & \text{if } a_{i^{h''}} = u \\ N_u^{h''} \backslash \{i^{h''}\} & \text{otherwise,} \end{cases} \qquad N_d^{h'} = N_d^{h''}, \quad N_s^{h'} = N \backslash (N_u^{h'} \cup N_d^{h'}) \quad \text{and} \quad q^{h'} = q^{h''}.$$

Let h' be the output of **Step 2.t.a**. Go to **Step 2.t.b** with input h' and collect $q^{h'}$, $i^{h''}$ and $N_u^{h'}$, $N_d^{h'}$, $N_s^{h'}$ as input information.[15]

Step 2.t.b (**t**≥ 1)**:**

Input of **Step 2.t.b:** h', output of **Step 2.t.a**.
Input information of **Step 2.t.b:** $q^{h'}$, $i^{h''}$ and $N_u^{h'}, N_d^{h'}, N_s^{h'}$ (with $N_d^{h'} \neq \emptyset$). Notice that $i \in N_d^{h'}$ implies $0 < q_i^{h'}$.

Define

$$i^{h'} = \min_{<}\{j \in N_d^{h'} \mid \Phi_j(q^{h'}_{i^{h''}} + 1, \mathbf{0}_{N_d^{h'}}, q^{h'}_{-N_d^{h'} \cup \{i^{h''}\}}) \leq q_j^{h'} - 1\}.^{16}$$

Agent $i^{h'}$ has to choose an action from the set

$$A^{h'}_{i^{h'}} = \{d, q^{h'}_{i^{h'}} - 1\}.$$

[14] The proof of Statement 1.1 in Appendix 1 guarantees that, as a consequence of the efficiency of Φ, $i^{h''}$ is well defined.

[15] Notice that **Step 2.t.b** (which identifies the agent that would like to receive strictly less than his reference allotment at h') will require the previous identification of $i^{h''}$ in **Step 2.t.a**, the agent that would like to receive strictly more than his reference allotment at h''.

[16] The proof of Statement 1.1 in Appendix 1 guarantees that, as a consequence of the efficiency of Φ, $i^{h'}$ is well defined.

Let $\bar{a}_{i^{h'}} \in A^{h'}_{i^{h'}}$ be the choice of $i^{h'}$ and, as we have done previously, define

$$a_{i^{h'}} = \begin{cases} 0 & \text{if } q^{h'}_{i^{h'}} - 1 = 0 \\ \bar{a}_{i^{h'}} & \text{otherwise.} \end{cases}$$

Set $h = (h', a_{i^{h'}})$ and define

$$N^h_u = N^{h'}_u, \quad N^h_d = \begin{cases} N^{h'}_d & \text{if } a_{i^{h'}} = d \\ N^{h'}_d \backslash \{i^{h'}\} & \text{otherwise,} \end{cases} \quad N^h_s = N \backslash (N^h_u \cup N^h_d), \quad \text{and}$$

$$q^h_j = \begin{cases} q^{h'}_j + 1 & \text{if } j = i^{h''} \\ q^{h'}_j - 1 & \text{if } j = i^{h'} \\ q^{h'}_j & \text{if } j \in N \backslash \{i^{h''}, i^{h'}\}. \end{cases}$$

Let h be the output of **Step 2.t.b** and collect q^h and N^h_u, N^h_d, N^h_s as output information.

If $N^h_u \neq \emptyset$ and $N^h_d \neq \emptyset$, set $h'' := h$. Go to **Step 2.t+1.a** with input h'' and collect $q^{h''}$ and $N^{h''}_u$, $N^{h''}_d$, $N^{h''}_s$ as input information.

Proceed until at some **Step 2.t′.b** one of the two subsets N^h_u or N^h_d is empty. Then, the history h is terminal and the allotment q^h is the outcome associated to h (*i.e.*, set $o(h) = q^h$).

End of the Specific Phase of the Algorithm.

Let Φ be a sequential allotment rule satisfying individual rationality with respect to q. We denote by $\Gamma^\Phi = (N, k, (H, \prec), \mathcal{N}, \mathcal{A}, o)$ the extensive game form obtained from the specific phase of the algorithm, where $((H, \prec), \mathcal{N}, \mathcal{A}, o)$ are defined in an obvious way.

Let $R_i \in \mathcal{R}$ be arbitrary and let $\tau_i = \tau(R_i)$. We define i's *truth-telling* strategy $\sigma^{R_i}_i \in \Sigma_i$ relative to R_i in Γ^Φ by looking at each history h at which i plays at h; that is, $i = i^h$ (*i.e.*, $\mathcal{N}(h) = i$).

If h is a history in **Stage 1**, then $A^h_i = \{d, q^h_i, u\}$ and

$$\sigma^{R_i}_i(h) = \begin{cases} u & \text{if } \tau_i > q^h_i \\ q^h_i & \text{if } \tau_i = q^h_i \\ d & \text{if } \tau_i < q^h_i, \end{cases} \tag{3}$$

where q^h is an input information of one step in **Stage 1**, and it remains constant and equal to q.

If i plays at h in **Stage 2** and $i \in N^h_u$, then $A^h_i = \{q^h_i + 1, u\}$ and

$$\sigma^{R_i}_i(h) = \begin{cases} u & \text{if } \tau_i > q^h_i + 1 \\ q^h_i + 1 & \text{if } \tau_i \leq q^h_i + 1, \end{cases} \tag{4}$$

where q^h is an input information of one **Step 2.t.a**.

If i plays at h in **Stage 2** and $i \in N^h_d$, then $A^h_i = \{d, q^h_i - 1\}$ and

$$\sigma^{R_i}_i(h) = \begin{cases} q^h_i - 1 & \text{if } \tau_i \geq q^h_i - 1 \\ d & \text{if } \tau_i < q^h_i - 1, \end{cases} \tag{5}$$

where q^h is an input information of one **Step 2.t.b**.

When we say that Γ^Φ OSP-implements Φ, condition (i) in Definition 1 (applied to a profile $R \in \mathcal{R}^n$) refers to the truth-telling strategy profile $\sigma^R = (\sigma_1^{R_1}, \ldots, \sigma_n^{R_n}) \in \Sigma$ defined in (3), (4) and (5) above.

Now we can state our result for individually rational allotment rules.

Theorem 1 *Let Φ be a sequential allotment rule satisfying individual rationality with respect to q. Then, the extensive game form Γ^Φ OSP-implements Φ.*

Proof See Subsection 6.1 in Appendix 1.

The proof of Theorem 1 consists of establishing that three statements are true. In the proof of Statement 1.1 we show that Γ^Φ is finite and that all agents that are called to play along the specific phase of the algorithm are uniquely identified and well defined. In the proof of Statement 2.1 we show that Γ^Φ induces Φ by verifying that, for any arbitrary profile of tops τ, $\Phi(\tau)$ coincides with the final reference allotment q^{h^τ}, outcome of Γ^Φ. We do so by establishing that, for every i, there is a relationship between τ_i and $q_i^{h^\tau}$ that depends on the final subset of agents to which i belongs to: if $i \in N_s^{h^\tau}$ then $\tau_i = q_i^{h^\tau}$, if $i \in N_u^{h^\tau}$ then $\tau_i > q_i^{h^\tau}$ and if $i \in N_d^{h^\tau}$ then $\tau_i < q_i^{h^\tau}$. In the proof of Statement 3.1 we show that truth-telling is weakly dominant in Γ^Φ. We do so by showing that whenever i is called to play, at history h with reference allotment q_i, i's truth-telling action triggers the right evolution of i's future reference allotment $q_i' : q_i' = q_i$ if i chooses q_i and enters N_s^h, $q_i' > q_i$ if i chooses up and enters or remains in N_u^h and $q_i' < q_i$ if i chooses down and enters or remains in N_d^h; these arguments are specific to whether h is an input of **Step 1.t**, **Step 2.t.a** or **Step 2.t.b**.

4.2 The General Case

Let $\Phi : \mathcal{R}^n \to X$ be a sequential allotment rule. If $\Phi(\mathbf{0}) = \Phi(\mathbf{k}) = q$, then Φ is individually rational with respect to the allotment q and, to define the game Γ^Φ, we use the specific phase of the algorithm defined in the previous subsection. Otherwise, the algorithm consists of a general phase with two stages, **Stage A** and **Stage B**. **Stage A** transforms the two allotments $\underline{q} = \Phi(\mathbf{0})$ and $\overline{q} = \Phi(\mathbf{k})$ into a unique allotment q by updating them along a finite sequence of steps. Agents that play along these steps are finally classified into three subsets, the subset of agents that would like to receive strictly more than their allotment in q (denoted by N_u), strictly less than their allotment in q (denoted by N_d) or exactly their allotment in q (denoted by N_s). Observe that the union of these three subsets may be a strict subset of N because some agents may not play along **Stage A**; the convergence of $\underline{q}$'s and $\overline{q}$'s to q does not require it (for example if $\underline{q}_i = \overline{q}_i$, agent i does not play in **Stage A**). To sort out in one of the three subsets the agents that have not played yet, **Stage B** proceeds as **Stage 1** of the specific phase of the algorithm (starting at some **Step 1.t**, where **t** is equal to the number of agents that have already played in **Stage A**) for the case of a rule that is individually rational with respect to q. We now describe in detail the general phase of the algorithm.

4.2.1 The General Phase of the Algorithm

Input: A sequential allotment rule Φ and a linear order $<$ on the set of agents that, without loss of generality, we assume $1 < \cdots < n$.

Define

$$\underline{q} = \Phi(\mathbf{0}) \quad \text{and} \quad \overline{q} = \Phi(\mathbf{k}).$$

If $\underline{q} = \overline{q}$, set $q = \underline{q} = \overline{q}$. Then, Φ is individually rational with respect to q and proceed by applying to Φ and q the specific phase of the algorithm for individually rational rules.

If $\underline{q} \neq \overline{q}$, set $h = \emptyset$, $\underline{q}^{\emptyset} = \underline{q}$, and $\overline{q}^{\emptyset} = \overline{q}$. Go to **Stage A** with input $h = \emptyset$ and collect $(\underline{q}^{\emptyset}, \overline{q}^{\emptyset})$ and $N^{\emptyset} = N_d^{\emptyset} = N_u^{\emptyset} = N_s^{\emptyset} = \emptyset$ as input information.

Stage A: Go to the initialization step, **Step A.0**.

Step A.0:

Input of **Step A.0:** The empty history $h = \emptyset$.
Input information of **Step A.0:** $(\underline{q}^{\emptyset}, \overline{q}^{\emptyset})$ and $N^{\emptyset} = N_d^{\emptyset} = N_u^{\emptyset} = N_s^{\emptyset} = \emptyset$.

Define $i^{\emptyset} = \min_{<}\{i \in N \mid \underline{q}_i^{\emptyset} < \overline{q}_i^{\emptyset}\}$.

Agent $i^{\emptyset}$ has to choose an action from the set

$$A_{i^\emptyset}^{\emptyset} = \{d, \underline{q}_{i^\emptyset}^{\emptyset}, \ldots, \overline{q}_{i^\emptyset}^{\emptyset}, u\}.$$[17]

Let $\bar{a}_{i^\emptyset} \in A_{i^\emptyset}^{\emptyset}$ be the choice of $i^{\emptyset}$. Similarly, as we did in the specific phase of the algorithm, we identify (as being the same actions) u with k if $\overline{q}_{i^\emptyset}^{\emptyset} = k$ and d with 0 if $\underline{q}_{i^\emptyset}^{\emptyset} = 0$. To do so, define

$$a_{i^\emptyset} = \begin{cases} k & \text{if } \overline{q}_{i^\emptyset}^{\emptyset} = k \text{ and } \bar{a}_{i^\emptyset} \in \{u, \overline{q}_{i^\emptyset}^{\emptyset}\} \\ 0 & \text{if } \underline{q}_{i^\emptyset}^{\emptyset} = 0 \text{ and } \bar{a}_{i^\emptyset} \in \{d, \underline{q}_{i^\emptyset}^{\emptyset}\} \\ \bar{a}_{i^\emptyset} & \text{otherwise.} \end{cases}$$

Set $h = (\emptyset, a_{i^\emptyset})$ and define

$$N_u^h = \begin{cases} \{i^{\emptyset}\} & \text{if } a_{i^\emptyset} = u \\ \emptyset & \text{otherwise,} \end{cases} \qquad N_d^h = \begin{cases} \{i^{\emptyset}\} & \text{if } a_{i^\emptyset} = d \\ \emptyset & \text{otherwise,} \end{cases}$$

$$N_s^h = \begin{cases} \{i^{\emptyset}\} & \text{if } a_{i^\emptyset} \in \{\underline{q}_{i^\emptyset}^{\emptyset}, \ldots, \overline{q}_{i^\emptyset}^{\emptyset}\} \\ \emptyset & \text{otherwise,} \end{cases} \qquad \text{and} \quad N^h = N_u^h \cup N_d^h \cup N_s^h.$$

[17]The set of available actions $A_{i^\emptyset}^{\emptyset}$ (when $i^{\emptyset}$ plays for the first time) can be seen as a partition of $\mathcal{R}$ by identifying action u as the subset $\{R_{i^\emptyset} \in \mathcal{R} \mid \tau(R_{i^\emptyset}) > \overline{q}_{i^\emptyset}^{\emptyset}\}$, action d as the subset $\{R_{i^\emptyset} \in \mathcal{R} \mid \tau(R_{i^\emptyset}) < \underline{q}_{i^\emptyset}^{\emptyset}\}$ and each action $\bar{a}_{i^\emptyset} \in \{\underline{q}_{i^\emptyset}^{\emptyset}, \ldots, \overline{q}_{i^\emptyset}^{\emptyset}\}$ as the subset $\{R_{i^\emptyset} \in \mathcal{R} \mid \tau(R_{i^\emptyset}) = \bar{a}_{i^\emptyset}\}$. Observe that when either $\overline{q}_{i^\emptyset}^{\emptyset} = k$ or $\underline{q}_{i^\emptyset}^{\emptyset} = 0$ the first or the second set is empty.

Notice that $a_{i^\emptyset} \in \{0, k\}$ implies $i^\emptyset \in N_s^h$, $i \in N_u^h$ implies $\overline{q}_i^\emptyset < k$ and $i \in N_d^h$ implies $0 < \underline{q}_{i^\emptyset}^\emptyset$.

Define two vectors $\underline{m}^h, \overline{m}^h \in \{0, \ldots, k\}^n$ as follows. For each $j \in N$,

$$\underline{m}_j^h = \begin{cases} 0 & \text{if } j \in N_d^h \cup (N \backslash N^h) \\ \overline{q}_{i^\emptyset}^\emptyset & \text{if } j \in N_u^h \text{ and } j = i^\emptyset \\ a_{i^\emptyset} & \text{if } j \in N_s^h \text{ and } j = i^\emptyset \end{cases}$$

and[18]

$$\overline{m}_j^h = \begin{cases} k & \text{if } j \in N_u^h \cup (N \backslash N^h) \\ \underline{q}_{i^\emptyset}^\emptyset & \text{if } j \in N_d^h \text{ and } j = i^\emptyset \\ a_{i^\emptyset} & \text{if } j \in N_s^h \text{ and } j = i^\emptyset. \end{cases}$$

Define

$$\Phi(\underline{m}^h) = \underline{q}^h \quad \text{and} \quad \Phi(\overline{m}^h) = \overline{q}^h. \tag{6}$$

Let h be the output of **Step A.0** and collect $(\underline{q}^h, \overline{q}^h)$ and N^h, N_u^h, N_d^h, N_s^h as output information.

If $\underline{q}^h = \overline{q}^h$ set $q^h = \underline{q}^h = \overline{q}^h$. Go to **Stage B** with input h and collect q^h and N_u^h, N_d^h, N_s^h as input information.

If $\underline{q}^h \neq \overline{q}^h$ set $h' := h$ as the output of **Step A.0**. Go to **Step A.1** with input h' and collect $(\underline{q}^{h'}, \overline{q}^{h'})$ and $N_u^{h'}, N_d^{h'}, N_s^{h'}$ as input information.

Step A.t (**t**≥ 1)**:**

Input of **Step A.t:** h', output of **Step A.t-1**.

Input information of **Step A.t:** $(\underline{q}^{h'}, \overline{q}^{h'})$ and $N^{h'}, N_u^{h'}, N_d^{h'}, N_s^{h'}$ (with $N^{h'} \neq N$).

Define $i^{h'} = \min_<\{i \in N \backslash N_s^{h'} \mid \underline{q}_i^{h'} < \overline{q}_i^{h'}\}$.[19]

Agent $i^{h'}$ has to choose an action from the set

$$A_{i^{h'}}^{h'} = \begin{cases} \{d, \underline{q}_{i^{h'}}^{h'}, \ldots, \overline{q}_{i^{h'}}^{h'}, u\} & \text{if } i^{h'} \notin N^{h'} \\ \{\underline{q}_{i^{h'}}^{h'} + 1, \ldots, \overline{q}_{i^{h'}}^{h'}, u\} & \text{if } i^{h'} \in N_u^{h'} \\ \{d, \underline{q}_{i^{h'}}^{h'}, \ldots, \overline{q}_{i^{h'}}^{h'} - 1\} & \text{if } i^{h'} \in N_d^{h'}. \end{cases}$$

Let $\bar{a}_{i^{h'}} \in A_{i^{h'}}^{h'}$ be the choice of $i^{h'}$ and, as we have done previously, define

$$a_{i^{h'}} = \begin{cases} k & \text{if } \overline{q}_{i^{h'}}^{h'} = k \text{ and } \bar{a}_{i^{h'}} \in \{u, \overline{q}_{i^{h'}}^{h'}\} \\ 0 & \text{if } \underline{q}_{i^{h'}}^{h'} = 0 \text{ and } \bar{a}_{i^{h'}} \in \{d, \underline{q}_{i^{h'}}^{h'}\} \\ \bar{a}_{i^{h'}} & \text{otherwise.} \end{cases}$$

Set $h = (h', a_{i^{h'}})$ and define

$$N_u^h = \begin{cases} N_u^{h'} \cup \{i^{h'}\} & \text{if } a_{i^{h'}} = u \\ N_u^{h'} & \text{otherwise,} \end{cases} \qquad N_d^h = \begin{cases} N_d^{h'} \cup \{i^{h'}\} & \text{if } a_{i^{h'}} = d \\ N_d^{h'} & \text{otherwise,} \end{cases}$$

[18] In this case (**Step A.0**), the condition $j = i^\emptyset$ is redundant if $j \in N_d^h \cup N_s^h \cup N_u^h$. We adopt this notation in order to be consistent with the next steps and help the reader to better understand the definitions of $\underline{m}^h$ and $\overline{m}^h$ for a generic **Step A.t**, with **t**≥ 1.

[19] The proof of Theorem 2 shows that agent $i^{h'}$ is well defined.

$$N_s^h = \begin{cases} N_s^{h'} \cup \{i^{h'}\} & \text{if } a_{i^{h'}} \in \{\underline{q}_{i^{h'}}^{h'}, \ldots, \overline{q}_{i^{h'}}^{h'}\} \\ N_s^{h'} & \text{otherwise} \end{cases} \quad \text{and} \quad N^h = N_u^h \cup N_d^h \cup N_s^h.$$

Notice that $a_{i^{h'}} \in \{0, k\}$ implies $i^{h'} \in N_s^h$, $i \in N_u^h$ implies $\overline{q}_i^{h'} < k$ and $i \in N_d^h$ implies $0 < \underline{q}_i^{h'}$.

Define two vectors $\underline{m}^h, \overline{m}^h \in \{0, \ldots, k\}^n$ as follows. For each $j \in N$,

$$\underline{m}_j^h = \begin{cases} 0 & \text{if } j \in N_d^h \cup (N \backslash N^h) \\ \overline{q}_j^{h'} & \text{if } j \in N_u^h \text{ and } j = i^{h'} \\ \underline{q}_j^{h'} & \text{if } j \in N_u^h \cup N_s^h \text{ and } j \neq i^{h'} \\ a_{i^{h'}} & \text{if } j \in N_s^h \text{ and } j = i^{h'} \end{cases}$$

and

$$\overline{m}_j^h = \begin{cases} k & \text{if } j \in N_u^h \cup (N \backslash N^h) \\ \underline{q}_j^{h'} & \text{if } j \in N_d^h \text{ and } j = i^{h'} \\ \overline{q}_j^{h'} & \text{if } j \in N_d^h \cup N_s^h \text{ and } j \neq i^{h'} \\ a_{i^{h'}} & \text{if } j \in N_s^h \text{ and } j = i^{h'}. \end{cases}$$

Define

$$\Phi(\underline{m}^h) = \underline{q}^h \quad \text{and} \quad \Phi(\overline{m}^h) = \overline{q}^h. \tag{7}$$

Let h be the output of **Step A.t** and collect $(\underline{q}^h, \overline{q}^h)$ and N^h, N_u^h, N_d^h, N_s^h as output information.

If $\underline{q}^h \neq \overline{q}^h$ set $h' := h$. Go to **Step A.t+1** with input h' and collect $(\underline{q}^{h'}, \overline{q}^{h'})$ and $N^{h'}, N_u^{h'}, N_d^{h'}, N_s^{h'}$ as input information.

Proceed until **Step A.t′**, with output h, where $\underline{q}^h = \overline{q}^h$ holds and set $q^h = \underline{q}^h = \overline{q}^h$. Then, h is the output of **Stage A**. Go to **Stage B** with input h and collect q^h and N^h, N_u^h, N_d^h, N_s^h as input information.

Stage B follows steps of **Stage 1** and **Stage 2** (if needed) in the specific phase of the algorithm as follows. If $N^h \neq N$, go to **Step 1.t**, where **t**$= |N^h|$, with input h and input information q^h and N^h, N_u^h, N_d^h, N_s^h (respectively, output and output information of **Step A.t-1**).[20] If $N^h = N$ and one of the two sets N_u^h or N_d^h is empty, the history h is terminal and the allotment q^h is the outcome associated to h (*i.e.*, set $o(h) = i$). If $N^h = N$ and $N_u^h \neq \emptyset$ and $N_d^h \neq \emptyset$, set $h'' := h$ and go to **Step 2.1.a** with input h'' and input information $q^{h''}$ and $N^{h''}, N_u^{h''}, N_d^{h''}, N_s^{h''}$ (respectively, output and output information of **Step A.t′**, the last of **Stage A**).

End of the General Phase of the Algorithm.

Let Φ be a sequential allotment rule. We denote by $\Gamma^\Phi = (N, k, (H, \prec), \mathcal{N}, \mathcal{A}, o)$ the extensive game form defined by the general phase of the algorithm, where $((H, \prec), \mathcal{N}, \mathcal{A}, o)$ are defined accordingly in an obvious way.

[20] The reason of going to **Stage 1** is to sort out the agents that have not played in **Stage A** into the three subsets of agents (those that would like to go up, down or stay).

Remark 1 Let h be the input of **Stage B** and q^h and N_u^h, N_d^h, N_s^h be its input information. Consider the subdomain of profiles

$$\mathcal{D}^h = \mathcal{D}_1^h \times \cdots \times \mathcal{D}_n^h \subset \mathcal{R}^n,$$

where, for each $i \in N$,

$$\mathcal{D}_i^h = \begin{cases} \{R_i \in \mathcal{R} \mid \tau(R_i) \in \{q_i^h, \ldots, k\}\} & \text{if } i \in N_u^h \\ \{R_i \in \mathcal{R} \mid \tau(R_i) \in \{0, \ldots, q_i^h\}\} & \text{if } i \in N_d^h \\ \{R_i \in \mathcal{R} \mid \tau(R_i) = q_i^h\} & \text{if } i \in N_s^h \\ \mathcal{R} & \text{if } i \notin N^h, \end{cases}$$

and denote by $\Phi^h : \mathcal{D}^h \to X$ the restriction of $\Phi : \mathcal{R}^n \to X$ to $\mathcal{D}^h$. Since Φ is tops-only, Φ^h is also tops-only and so, abusing with the notation, we can write it as $\Phi^h : D^h \to X$ where

$$D^h = \left([q_i^h, k]\right)_{i \in N_u^h} \times \left([0, q_i^h]\right)_{i \in N_d^h} \times \{q_i^h\}_{i \in N_s^h} \times ([0, k])_{i \notin N^h}.$$

Moreover, since Φ is strategy-proof, efficient and replacement monotonic, Φ^h is strategy-proof, efficient, replacement monotonic, and[21]

$$\Phi^h \left(\mathbf{k}_{N_u^h \cup (N \setminus N^h)}, {q^h}_{N_d^h \cup N_s^h}\right) = \Phi^h \left(\mathbf{0}_{N_d^h \cup (N \setminus N^h)}, {q^h}_{N_u^h \cup N_s^h}\right) = q^h. \tag{8}$$

Let Γ^{Φ^h} be the extensive game form defined by **Stage B** of the generic phase of the algorithm with input h, applied to the rule $\Phi^h : \mathcal{D}^h \to X$. Theorem 1 implies that Γ^{Φ^h} OSP-implements Φ^h. □

Let $R_i \in \mathcal{R}$ be arbitrary and let $\tau_i = \tau(R_i)$. We define i's *truth-telling* strategy $\sigma_i^{R_i} \in \Sigma_i$ relative to R_i in Γ^Φ by looking at each history h at which i plays at h; that is, $i = i^h$ (*i.e.*, $\mathcal{N}(h) = i$).

Let h be a history in **Stage A**, and let $(\underline{q}_i^h, \overline{q}_i^h)$ is an input information of the **Step A.t**, for which h is an input.

If h is a history in **Stage A** and $i \notin N^h$, then $A_i^h = \{d, \underline{q}_i^h, \ldots, \overline{q}_i^h, u\}$ and

$$\sigma_i^{R_i}(h) = \begin{cases} u & \text{if } \tau_i > \overline{q}_i^h \\ \tau_i & \text{if } \tau_i \in \{\underline{q}_i^h, \ldots, \overline{q}_i^h\} \\ d & \text{if } \tau_i < \underline{q}_i^h. \end{cases}$$

If h is a history in **Stage A** and $i \in N_u^h$, then $A_i^h = \{\underline{q}_i^h + 1, \ldots, \overline{q}_i^h, u\}$ and

$$\sigma_i^{R_i}(h) = \begin{cases} u & \text{if } \tau_i > \overline{q}_i^h \\ \tau_i & \text{if } \tau_i \in \{\underline{q}_i^h + 1, \ldots, \overline{q}_i^h\}. \end{cases}$$

[21] The fact that h is the outcome of **Stage A** implies that $\Phi(\underline{m}^h) = \Phi(\overline{m}^h) = q^h$. Then, the definitions of $\underline{m}^h$ and $\overline{m}^h$ and Lemmata 3 and 6 (stated and proved in Appendix 2 in Subsection 6.2) imply that the equality in (8) holds.

If h is a history in **Stage A** and $i \in N_d^h$, then $A_i^h = \{d, \underline{q}_i^h, \ldots, \overline{q}_i^h - 1\}$ and

$$\sigma_i^{R_i}(h) = \begin{cases} d & \text{if } \tau_i < \underline{q}_i^h \\ \tau_i & \text{if } \tau_i \in \{\underline{q}_i^h, \ldots, \overline{q}_i^h - 1\}. \end{cases}$$

If h is a history in **Stage B** the definition of $\sigma_i^{R_i}$ is as in (3), (4) and (5) at the end of the specific phase of the algorithm described in the previous Subsection 4.1.1.

Now we can state our general and main result.

Theorem 2 *Let Φ be a sequential allotment rule. Then, the extensive game form Γ^Φ* OSP-*implements Φ.*

Proof See Subsection 6.2 in Appendix 1.

The main difficulty of the proof of Theorem 2 is to show that, if the rule Φ is not individually rational, the process of transforming the two reference allotments $\underline{q} = \Phi(\mathbf{0})$ and $\overline{q} = \Phi(\mathbf{k})$ into a unique reference allotment q in **Stage A** is well defined and it has the right properties. In particular, the monotonicity property of the sequence of each i's reference allotments depends on the subset of agents to which i is classified throughout **Stage A** (see the statements and proofs of Lemmata 3 to 8). Then, once the algorithm enters **Stage B**, adapted arguments to those already used in the proof of Theorem 1 can be used. They are presented in the proofs of Statements 2.1, 2.2 and 2.3.

4.3 Outline of the Algorithm

In this subsection we give an outline of the algorithm.

Input: A sequential allotment rule $\Phi : \mathcal{R}^n \to X$.

If $\Phi(\mathbf{0}) = \underline{q} \neq \overline{q} = \Phi(\mathbf{k})$, go to **Stage A**.

Stage A:

Step A.t: (t≥ 0)

Input: h', output of **Step A.t-1** if **t**≥ 1 or $h' = \emptyset$ if **t**$= 0$.

Input information: $(\underline{q}^{h'}, \overline{q}^{h'})$ and $N^{h'}, N_u^{h'}, N_d^{h'}, N_s^{h'}$, with $N^{h'} \neq N$ and $N^{h'} = N_u^{h'} = N_d^{h'} = N_s^{h'} = \emptyset$ if **t**$= 0$.

Define $i^{h'}$, $a_{i^{h'}}$, $h = (h', a_{i^{h'}})$, N^h, N_u^h, N_d^h, N_s^h and $\underline{m}^h, \overline{m}^h$.

Set $\underline{q}^h = \Phi(\underline{m}^h)$ and $\overline{q}^h = \Phi(\overline{m}^h)$.

If $\underline{q}^h \neq \overline{q}^h$ set $h' := h$ and go to **Stage A.t+1** with input h'.

If $\underline{q}^h = \overline{q}^h$ set $q^h = \underline{q}^h = \overline{q}^h$ and go to **Stage B**.

Stage B:

Input: h, output of **Step A.t**.

Input information: q^h and N^h, N_u^h, N_d^h, N_s^h.

If $N^h \neq N$ go to **Step 1.t** of **Stage 1** with

t$= \left|N^h\right|$

input: h'

input information: $q^{h'}$ and $N^{h'}, N_u^{h'}, N_d^{h'}, N_s^{h'}$, with $N^{h'} \neq N$.

If $N^h = N$ and $N_u^h = \emptyset$ or $N_d^h = \emptyset$ stop with $o(h) = q^h$.

If $N^h = N$ and $N_u^h \neq \emptyset$ and $N_d^h \neq \emptyset$ set $h'' := h$ and go to **Step 2.1.a** in **Stage 2**, with

input: h''

input information: $q^{h''}$ and $N_u^{h''}, N_d^{h''}, N_s^{h''}$.

If $\Phi(\mathbf{0}) = \Phi(\mathbf{k}) = q$, go to **Stage 1**.

Stage 1:

Step 1.t: (t≥ 0)

Input: h', output of **Step 1.t-1** or **Stage B** if **t**≥ 1 or $h' = \emptyset$ if **t**$= 0$.

Input information: $q^{h'} = q$ and $N^{h'}, N_u^{h'}, N_d^{h'}, N_s^{h'}$, with $N^{h'} \neq N$, and $N^{h'} = N_u^{h'} = N_d^{h'} = N_s^{h'} = \emptyset$ if **t**$= 0$.

Define $i^{h'}$, $a_{i^{h'}}$, $h = (h', a_{i^{h'}})$, N^h, N_u^h, N_d^h, N_s^h and $q^h = q^{h'}$.

If $N^h \neq N$, set $h' := h$ and go to **Step 1.t+1** with input h'.

If $N^h = N$ and $N_u^h = \emptyset$ or $N_d^h = \emptyset$, stop with $o(h) = q^h$.

If $N^h = N$ and $N_u^h \neq \emptyset$ and $N_d^h \neq \emptyset$, set $h'' := h$ and go to **Step 2.1.a** in **Stage 2** with input h''.

Stage 2:

Step 2.t.a: (t≥ 1)

Input: h'', output of **Stage 1** or **Stage A**.

Input information: $q^{h''}$ and $N_u^{h''}, N_d^{h'}, N_s^{h''}$ with $N_u^{h''} \neq \emptyset$ and $N_d^{h''} \neq \emptyset$.

Define $i^{h''}$, $a_{i^{h''}}$, $h' = (h'', a_{i^{h''}})$, $N_u^{h'}, N_d^{h'}, N_s^{h'}$ and $q^{h'} = q^{h''}$.

Go to **Step 2.t.b** with input h'.

Step 2.t.b: (t≥ 1)

Input: h', output of **Step 2.t.a**.

Input information: $q^{h'}$, $i^{h''}$ and $N_u^{h'}, N_d^{h'}, N_s^{h'}$ with $N_d^{h'} \neq \emptyset$.

Define $i^{h'}$, $a_{i^{h'}}$, $h = (h', a_{i^{h'}})$, N_u^h, N_d^h, N_s^h and q^h.

If $N_u^h \neq \emptyset$ and $N_d^h \neq \emptyset$, set $h'' := h$ and go to **Step 2.t+1.a** with input h''.

If $N^h = \emptyset$ or $N_d^h = \emptyset$, stop with $o(h) = q^h$.

4.4 Example

In this subsection we illustrate the algorithm defined in the previous subsections with one example.

Let $N = \{1,2,3,4\}$ and $k = 12$. Consider the sequential allotment rule Φ such that

$$\Phi(0,0,0,0) = (4,4,4,0) = \underline{q} \neq \overline{q} = (0,0,0,12) = \Phi(12,12,12,12). \qquad (9)$$

Moreover, suppose that Φ is such that

$$\Phi(12,12,12,2) = \Phi(0,0,0,2) = (3,3,4,2),$$

$$\Phi(12,2,12,2) = (4,2,4,2),$$

$$\Phi(4,0,4,2) = (5,1,4,2),$$

$$\Phi(4,1,12,2) = \Phi(4,0,5,2) = (4,1,5,2),$$

and

$$\Phi(4,0,12,2) = \Phi(4,0,6,2) = (4,0,6,2).$$

Observe that those values (that will be used in what follows) are consistent with the existence of a rule obtained by means of Definition 3 (in Appendix 2, Section 7) and satisfying strategy-proofness, efficiency and replacement monotonicity.

We apply the general phase of the algorithm to the rule Φ. Since (9) holds, Φ is not individually rational with respect to any allotment. Go to **Stage A**.

Step A.0: The input history is $h = \emptyset$ and the input information is $\underline{q}^{\emptyset} = (4,4,4,0)$, $\overline{q}^{\emptyset} = (0,0,0,12)$ and $N^{\emptyset} = N_u^{\emptyset} = N_d^{\emptyset} = N_s^{\emptyset} = \emptyset$. Since 4 is the unique agent in $\{i \in N \mid \underline{q}_i^{\emptyset} < \overline{q}_i^{\emptyset}\}$, set $i^{\emptyset} = 4$ and $A_4^{\emptyset} = \{d, 0, \ldots, 12, u\}$. Assume agent 4 chooses $\bar{a}_4 = 2$ (this would happen if $\tau_4 = 2$), and so $a_4 = 2$. Set $h = (\emptyset, a_4 = 2)$ and define $N_u^h = N_d^h = \{\emptyset\}$ and $N^h = N_s^h = \{4\}$. Then,

$$\underline{m}^h = (0,0,0,2) \quad \text{and} \quad \overline{m}^h = (12,12,12,2),$$

and we assumed that the sequential allotment rule is such that $\Phi(\overline{m}^h) = \Phi(\underline{m}^h) = (3,3,4,2) = \underline{q}^h = \overline{q}^h$. Set $q^h = (3,3,4,2)$. Go to **Stage B** at **Step 1.1** of **Stage 1** with input $h' := h = (\emptyset, a_4 = 2)$.

Step 1.1: The input history is $h' = (\emptyset, a_4 = 2)$ and the input information is $q^{h'} = (3,3,4,2)$, $N_u^{h'} = N_d^{h'} = \{\emptyset\}$ and $N^{h'} = N_s^{h'} = \{4\}$. Since 1 is the agent with the smallest index in $\{i \in N \mid i \notin N^{h'}\}$, set $i^{h'} = 1$ and $A_1^{h'} = \{d, 3, u\}$. Assume agent 1 chooses $\bar{a}_1 = u$ (this would happen if $\tau_1 > 3$), and so $a_1 = u$. Set $h = (\emptyset, a_4 = 2, a_1 = u)$ and define $N_u^h = \{1\}$, $N_d^h = \{\emptyset\}$, $N_s^h = \{4\}$ and $N^h = \{1,4\}$. Set $q^h = (3,3,4,2)$ and since $N^h \neq N$ go to **Step 1.2** with input $h' := h = (\emptyset, a_4 = 2, a_1 = u)$.

Step 1.2: The input history is $h' = (\emptyset, a_4 = 2, a_1 = u)$ and the input information is $q^{h'} = (3,3,4,2)$, $N_u^{h'} = \{1\}$, $N_d^{h'} = \{\emptyset\}$, $N_s^{h'} = \{4\}$ and $N^{h'} = \{1,4\}$. Since 2 is the agent with the smallest index in $\{i \in N \mid i \notin N^{h'}\}$, set $i^{h'} = 2$ and $A_2^{h'} = \{d, 3, u\}$. Assume agent 2 chooses $\bar{a}_2 = d$ (this would happen if $\tau_2 < 3$), and so $a_2 = d$. Set $h = (\emptyset, a_4 = 2, a_1 = u, a_2 = d)$ and define $N_u^h = \{1\}$, $N_d^h = \{2\}$, $N_s^h = \{4\}$ and $N^h = \{1,2,4\}$. Set $q^h = (3,3,4,2)$ and since $N^h \neq N$ go to **Step 1.3** with input $h' := h = (\emptyset, a_4 = 2, a_1 = u, a_2 = d)$.

Step 1.3: The input history is $h' = (\emptyset, a_4 = 2, a_1 = u, a_2 = d)$ and the input information is $q^{h'} = (3,3,4,2)$, $N_u^{h'} = \{1\}$, $N_d^{h'} = \{2\}$, $N_s^{h'} = \{4\}$ and $N^{h'} = \{1,2,4\}$. Since 3 is the unique agent in $\{i \in N \mid i \notin N^{h'}\}$, set $i^{h'} = 3$ and $A_3^{h'} = \{d, 4, u\}$. Assume agent 3 chooses $\bar{a}_3 = u$ (this would happen if $\tau_3 > 4$), and so $a_3 = u$. Set $h = (\emptyset, a_4 = 2, a_1 = u, a_2 = d, a_3 = u)$ and define $N_u^h = \{1,3\}$, $N_d^h = \{2\}$, $N_s^h = \{4\}$, and $N^h = \{1,2,3,4\}$. Set $q^h = (3,3,4,2)$ and since $N^h = N$, $N_u^h \neq \emptyset$ and $N_d^h \neq \emptyset$, go to **Stage 2** at **Step 2.1.a** with input $h'' := h = (\emptyset, a_4 = 2, a_1 = u, a_2 = d, a_3 = u)$.

Step 2.1.a: The input history is $h'' = (\emptyset, a_4 = 2, a_1 = u, a_2 = d, a_3 = u)$ and the input information is $q^{h''} = (3,3,4,2)$, $N_u^{h''} = \{1,3\}$, $N_d^{h''} = \{2\}$, and $N_s^{h''} = \{4\}$. Observe that, since $q_2^{h''} - 1 = 2$, the profile of tops $(\mathbf{k}_{N_u^{h''}}, (q^{h''} - \mathbf{1})_{N_d^{h''}}, q^{h''}_{N_s^{h''}})$, used to identify $i^{h''}$, is equal to the vector $(12, 2, 12, 2)$. We assumed that the sequential allotment rule Φ is such that $\Phi(12,2,12,2) = (4,2,4,2)$. Since agent 1 is the unique agent in the set $\{j \in N_u^{h''} \mid \Phi_j(12,2,12,2) \geq q_j^{h''} + 1\}$, set $i^{h''} = 1$ and $A_1^{h''} = \{4, u\}$. Assume agent 1 chooses $\bar{a}_1 = 4$ (this would happen if $\tau_1 = 4$), and so $a_1 = 4$. Set $h' = (\emptyset, a_4 = 2, a_1 = u, a_2 = d, a_3 = u, a_1 = 4)$ and define $N_u^{h'} = \{3\}$, $N_d^{h'} = \{2\}$ and $N_s^{h'} = \{1,4\}$. Set $q^{h'} = (3,3,4,2)$ and go to **Step 2.1.b** with input h'.

Step 2.1.b: The input history is $h' = (\emptyset, a_4 = 2, a_1 = u, a_2 = d, a_3 = u, a_1 = 4)$ and the input information is $q^{h'} = (3,3,4,2)$, $i^{h''} = 1$, $N_u^{h'} = \{3\}$, $N_d^{h'} = \{2\}$, and $N_s^{h'} = \{1,4\}$. Observe that, since $q_{i^{h''}}^{h'} + 1 = 4$, the profile of tops $(q_{i^{h''}}^{h'} + 1, \mathbf{0}_{N_d^{h'}}, q^{h'}_{-N_d^{h'} \cup \{i^{h''}\}})$, used to identify $i^{h'}$, is equal to the vector $(4,0,4,2)$. We assumed that the sequential allotment rule Φ is such that $\Phi(4,0,4,2) = (5,1,4,2)$. Since agent 2 is the unique agent in the set $\{j \in N_d^{h'} \mid \Phi_j(4,0,4,2) \leq q_j^{h'} - 1\}$, set $i^{h'} = 2$ and $A_2^{h'} = \{d, 2\}$. Assume agent 2 chooses $\bar{a}_2 = d$ (this would happen if $\tau_2 < 2$), and so $a_2 = d$. Set $h = (\emptyset, a_4 = 2, a_1 = u, a_2 = d, a_3 = u, a_1 = 4, a_2 = d)$ and define $N_u^h = \{3\}$, $N_d^h = \{2\}$, $N_s^h = \{1,4\}$ and $q^h = (4,2,4,2)$. Since $N_u^h \neq \emptyset$ and $N_d^h \neq \emptyset$ go to **Step 2.2.a** with input $h'' := h = (\emptyset, a_4 = 2, a_1 = u, a_2 = d, a_3 = u, a_1 = 4, a_2 = d)$.

Step 2.2.a: The input history is $h'' = (\emptyset, a_4 = 2, a_1 = u, a_2 = d, a_3 = u, a_1 = 4, a_2 = d)$ and the input information is $q^{h''} = (4,2,4,2)$, $N_u^{h''} = \{3\}$, $N_d^{h''} = \{2\}$ and $N_s^{h''} = \{1,4\}$. Observe that, since $q_2^{h''} - 1 = 1$, the profile of tops $(\mathbf{k}_{N_u^{h''}}, (q^{h''} - \mathbf{1})_{N_d^{h''}}, q^{h''}_{N_s^{h''}})$, used to identify $i^{h''}$, is equal to the vector $(4,1,12,2)$. We assumed that the sequential allotment rule is such that $\Phi(4,1,12,2) = (4,1,5,2)$. Since agent 3 is the unique agent in the set $\{j \in N_u^{h''} \mid \Phi_j(4,1,12,2) \geq q_j^{h''} + 1\}$, set $i^{h''} = 3$ and $A_3^{h''} = \{5, u\}$. Assume agent 3 chooses $\bar{a}_3 = u$ (this would happen if $\tau_3 > 5$), and so $a_3 = u$. Set $h' = (\emptyset, a_4 = 2, a_1 = u, a_2 = d, a_3 = u, a_1 = 4, a_2 = d, a_3 = u)$ and define $N_u^{h'} = \{3\}$, $N_d^{h'} = \{2\}$ and $N_s^{h'} = \{1,4\}$. Set $q^{h'} = (4,2,4,2)$ and go to **Step 2.2.b** with input h'.

Step 2.2.b: The input history is $h' = (\emptyset, a_4 = 2, a_1 = u, a_2 = d, a_3 = u, a_1 = 4, a_2 = d, a_3 = u)$ and the input information is $q^{h'} = (4,2,4,2)$, $i^{h''} = 3$, $N_u^{h'} = \{3\}$, $N_d^{h'} = \{2\}$, and $N_s^{h'} = \{1,4\}$. Observe that, since $q_{i^{h''}}^{h'} + 1 = 5$, the profile of tops $(q_{i^{h''}}^{h'} + 1, \mathbf{0}_{N_d^{h'}}, q^{h'}_{-N_d^{h'} \cup \{i^{h''}\}})$, used to identify $i^{h'}$, is equal to the vector $(4, 0, 5, 2)$. We assumed that the sequential allotment rule Φ is such that $\Phi(4, 0, 5, 2) = (4, 1, 5, 2)$. Since agent 2 is the unique agent in the set $\{j \in N_d^{h'} \mid \Phi_j(4, 0, 5, 2) \leq q_j^{h'} - 1\}$, set $i^{h'} = 2$ and $A_2^{h'} = \{d, 1\}$. Assume agent 2 chooses $\bar{a}_2 = 1$ (this would happen if $\tau_2 = 1$), and so $a_2 = 1$. Set $h = (\emptyset, a_4 = 2, a_1 = u, a_2 = d, a_3 = u, a_1 = 4, a_2 = d, a_3 = u, a_2 = 1)$ and define $N_u^h = \{3\}$, $N_d^h = \emptyset$ and $N_s^h = \{1, 2, 4\}$ and $q^h = (4, 1, 5, 2)$. Since $N_d^h = \emptyset$ the history h is terminal and the allotment $o(h) = q^h = (4, 1, 5, 2)$ is the outcome associated to h, and the outcome of the algorithm. Observe that $\Phi(\tau_1, \tau_2, \tau_3, \tau_4) = (4, 1, 5, 2)$ for any profile of top allotments $(\tau_1, \tau_2, \tau_3, \tau_4)$ such that $\tau_1 = 4$, $\tau_2 = 1$, $\tau_3 \geq 6$ and $\tau_4 = 2$, which are those consistent with the truth-telling choices made by agents along the play of the game.

5 Final Remarks

We finish the paper with three remarks.

First, our implementation result requires that the rule Φ satisfies replacement monotonicity. Example 1 below contains an instance of a discrete division problem where there is a tops-only, strategy-proof, efficient and non replacement monotonic rule that is not obviously strategy-proof.

Example 1 Consider the discrete division problem where $N = \{1, 2, 3\}$ and $k = 2$. Let $\Psi : \mathcal{R}^3 \to X$ be the tops-only rule such that, for every $\tau = (\tau_1, \tau_2, \tau_3) \in \{0, 1, 2\}^3$, the top of agent 1 determines the ordering between agents 2 and 3 to successively select the allotment they wish, if available, and agent 1's allotment is equal to the remainder, if any. Namely,

$$\Psi(\tau_1, \tau_2, \tau_3) = \begin{cases} (2 - \tau_2 - \min\{2 - \tau_2, \tau_3\}, \tau_2, \min\{2 - \tau_2, \tau_3\}) & \text{if } \tau_1 \in \{0, 1\} \\ (2 - \tau_3 - \min\{2 - \tau_3, \tau_2\}, \min\{2 - \tau_3, \tau_2\}, \tau_3) & \text{if } \tau_1 = 2. \end{cases}$$

It is easy to check that Ψ is strategy-proof and efficient. To see that it is not replacement monotonic, consider $\tau = (\tau_1, \tau_2, \tau_3) = (0, 1, 2)$ and $\tau' = (\tau_1', \tau_2, \tau_3) = (2, 1, 2)$. Then, $\Psi(\tau) = (0, 1, 1)$ and $\Psi(\tau') = (0, 0, 2)$. Since $\Psi_1(\tau) = \Psi_1(\tau')$, $\Psi_2(\tau) > \Psi_2(\tau')$ and $\Psi_3(\tau) < \Psi_3(\tau')$, Ψ is not replacement monotonic.

To obtain a contradiction, assume that Ψ is obviously strategy-proof on $\mathcal{R}^3$, and let Γ be the game that OSP-implements Ψ. Given a profile of tops τ, let $\sigma^\tau = (\sigma_1^{\tau_1}, \ldots, \sigma_n^{\tau_n})$ be the truth-telling strategy profile (namely, for each $i \in N$ and $R_i \in \mathcal{R}$, $\sigma_i^{\tau_i} = \sigma_i^{R_i}$ where $\tau_i = \tau(R_i)$). Since Γ induces Ψ, there must exists $h \in H_{NT}$ such that h is one of the shortest histories with the property that agent $\mathcal{N}(h) = i$ has available at least two different actions. Denote those actions by $a^1, a^2 \in \mathcal{A}(h)$. Assume first that $\mathcal{N}(h) = 1$ and consider the two profiles of tops $\tau = (1, 0, 0)$ and $\tau' = (2, 1, 0)$. Then, since Γ induces Ψ, $o_1(h^\Gamma(h, \sigma^\tau)) = 2 = \Psi_1(\tau)$ and $o_1(h^\Gamma(h, \sigma^{\tau'})) = 1 = \Psi_1(\tau')$. Consider σ_2 and σ_3 with the following properties: (i) they coincide respectively with $\sigma_2^{\tau_2}$ and $\sigma_3^{\tau_3}$ at all histories that follow (h, a^1), (ii) they coincide respectively with $\sigma_2^{\tau_2'}$ and $\sigma_3^{\tau_3'}$ at all histories that follow (h, a^2) and (iii) $h \prec h^\Gamma(\emptyset, (\sigma_1^{\tau_1}, \sigma_2, \sigma_3))$ and $h \prec h^\Gamma(\emptyset, (\sigma_1^{\tau'_1}, \sigma_2, \sigma_3))$. Note that such strategies do exist since h was selected to be one of the shortest histories with $a^1, a^2 \in \mathcal{A}(h)$. Since $\tau(P_1) = 1$ and

$$o_1(h^\Gamma(\emptyset, (\sigma_1^{\tau'_1}, \sigma_2, \sigma_3))) = 1P_1 2 = o_1(h^\Gamma(\emptyset, (\sigma_1^{\tau_1}, \sigma_2, \sigma_3))),$$

the strategy $\sigma_1^{\tau_1}$ is not weakly dominant in Γ, a contradiction with the assumption that Γ OSP-implements Ψ. Assume now that $\mathcal{N}(h) = 2$ and consider the two profiles of tops $\tau = (2, 1, 2)$ and $\tau' = (1, 1, 2)$. Then, since Γ induces Ψ, $o_2(h^\Gamma(h, \sigma^\tau)) = 0 = \Psi_2(\tau)$ and $o_2(h^\Gamma(h, \sigma^{\tau'})) = 1 = \Psi_2(\tau')$. Consider σ_1 and σ_3 with the following properties: (i) they coincide respectively with $\sigma_1^{\tau_1}$ and $\sigma_3^{\tau_3}$ at all histories that follow (h, a^1), (ii) they coincide respectively with $\sigma_1^{\tau'_1}$ and $\sigma_3^{\tau'_3}$ at all histories that follow (h, a^2) and (iii) $h \prec h^\Gamma(\emptyset, (\sigma_1, \sigma_2^{\tau_2}, \sigma_3))$ and $h \prec h^\Gamma(\emptyset, (\sigma_1, \sigma_2^{\tau'_2}, \sigma_3))$. Note that such strategies do exist since h was selected to be one of the shortest histories with $a^1, a^2 \in \mathcal{A}(h)$. Since $\tau(P_2) = 1$ and

$$o_2(h^\Gamma(\emptyset, (\sigma_1, \sigma_2^{\tau'_2}, \sigma_3))) = 1P_2 0 = o_1(h^\Gamma(\emptyset, (\sigma_1, \sigma_2^{\tau_2}, \sigma_3))),$$

the strategy $\sigma_2^{\tau_2}$ is not weakly dominant in Γ, a contradiction with the assumption that Γ OSP-implements Ψ. A similar argument can be used to obtain a contradiction when $\mathcal{N}(h) = 3$. □

Second, Pycia and Troyan (2018) proposes a strengthening of obvious strategy-proofness called strong obvious strategy-proofness. The key difference with Li (2017)'s notion is that, when comparing the possible outcomes of the truth-telling strategy with the possible outcomes of any deviation at an earliest point of departure, the truth-telling strategy may change along the subsequent play of the game, instead of being fixed as in Li (2017). It is easy to see that not all sequential allotment rules satisfy this stronger requirement. However, the subclass of sequential dictators do (and they can be described as sequential allotment rules), since agents play only once along the game. We conjecture that, at the light of Theorem 4 in Pycia and Troyan (2018), the class of all efficient, restricted monotonic and strong obviously strategy-proof rules coincides with class of all sequential dictator rules.

Third, our extensive game form is based on the discrete version of Sprumont (1991)'s continuous model. An OSP-implementation of any sequential allotment rule in the continuous version of the model should deal with the technical difficulties that may arise in extensive game forms where agents play in a continuous fashion (see for instance Alós-Ferrer and Ritzberger (2013)). For simplicity, we have decided to undertake our analysis in the discrete version of the model, first studied by Herrero and Martínez (2011).

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6 Appendix 1: Proofs

We will intensively use the following notation. Given $\Gamma \in \mathcal{G}$, $h \in H$ and $i \in N$ we will denote by $\widehat{h}$ the history at which i has played for the last time before h; namely, $\widehat{h}$ is such that (i) $\widehat{h} \prec h$, (ii) $i^{\widehat{h}} = i$ and (iii) there is no $\widetilde{h} \neq \widehat{h}$ such that $\widehat{h} \prec \widetilde{h} \prec h$ and $i^{\widetilde{h}} = i^{\widehat{h}}$. Of course, $\widehat{h}$ depends on i but since i will be clear from the context, we will omit its reference when denoting this earlier history.

6.1 Theorem 1

Let Φ be a sequential allotment rule satisfying individually rational with respect to $q \in X$ and let Γ^{Φ} be the extensive game form obtained from the specific phase of the algorithm defined in Subsection 4.1.1.

We first present preliminary results that we will use in the proof of Theorem 1.

Lemma 1 *Let $h = (h'', a_{i^{h''}}, a_{i^{h'}})$ be the output of* **Step 2.t.b** *(for* **t**$\geq$*1), with $h' = (h'', a_{i^{h''}})$, and let* q^h and N_u^h, N_d^h, N_s^h *be its output information. Then,*
(1.1) $\Phi(\mathbf{0}_{N_d^h}, q^h_{N\backslash N_d^h}) = q^h$ *and*
(1.2) *if* $N_u^h \neq \emptyset$, *then*

$$\Phi_j(\mathbf{k}_{N_u^h}, (q^{h''} - \mathbf{1})_{N_d^h}, q^h_{N_s^h}) = \begin{cases} q_j^{h''} - 1 & \text{if } j \in N_d^h \\ q_j^h & \text{if } j \in N_s^h. \end{cases}$$

Proof of Lemma 1

(1.1) If $N_d^h = \emptyset$, the statement follows from efficiency of Φ and feasibility of q^h. Consider now the case $N_d^h \neq \emptyset$. We proceed by induction on **t**.

Suppose **t**$= 1$. Hence, h'' is the output of **Stage 1**, $N^{h''} = N$ and $q^{h''} = q$. By strategy-proofness and efficiency of Φ and $\Phi(\mathbf{0}) = q^{h''}$,

$$\Phi(\mathbf{0}_{N_d^{h''}}, q^{h''}_{N_u^{h''}}, q^{h''}_{N_s^{h''}}) = q^{h''}. \tag{10}$$

Then,

$$\Phi_i(\mathbf{0}_{N_d^{h''}}, q^{h''}_{N_u^{h''} \backslash \{i^{h''}\}}, q^{h''}_{i^{h''}} + 1, q^{h''}_{N_s^{h''}}) \leq \begin{cases} q_i^{h''} + 1 & \text{if } i = i^{h''} \\ q_i^{h''} - 1 & \text{if } i = i^{h'} \\ q_i^{h''} & \text{if } i \in N \backslash \{i^{h''}, i^{h'}\}. \end{cases} \tag{11}$$

The inequality in the first raw of (11) follows from strategy-proofness, single-peakedness, (10) and the fact that $i^{h''} \in N_u^{h''}$. The inequality in the second raw follows from the definition of $i^{h'}$ and $q^{h'} = q^{h''}$. The inequality in the third raw follows from replacement monotonicity and (10). Since $q^{h''}$ is a feasible allotment, the inequality in (11) can be replaced by an equality, and since

$$q_i^h = \begin{cases} q_i^{h''} + 1 & \text{if } i = i^{h''} \\ q_i^{h''} - 1 & \text{if } i^{h''} = i^{h'} \\ q_i^{h''} & \text{if } i \in N \backslash \{i^{h''}, i^{h'}\} \end{cases}$$

and $i^{h'} \in N_d^{h''}$, we obtain

$$\Phi(\mathbf{0}_{N_d^{h''}}, q^h_{N \backslash N_d^{h''}}) = q^h. \tag{12}$$

To prove that the statement in (1.1) holds, we proceed by distinguishing between two cases, depending on the action $a_{i^{h'}} \in A^{h'}_{i^{h'}} = \{d, q^{h'}_{i^{h'}} - 1\}$.

<u>Case 1</u>*:* Suppose $a_{i^{h'}} = d$. Then, $N_d^h = N_d^{h''}$ and, by (12),

$$\Phi(\mathbf{0}_{N_d^h}, q^h_{N \backslash N_d^h}) = q^h.$$

Case 2: Suppose $a_{i^{h'}} = q^{h'}_{i^{h'}} - 1$. Then, $N^h_d = N^{h'}_d \backslash \{i^{h'}\} = N^{h''}_d \backslash \{i^{h'}\}$, where the second equality follows from $N^{h'}_d = N^{h''}_d$. Hence, by (12), strategy-proofness of Φ implies

$$\Phi_{i^{h'}}(\mathbf{0}_{N^h_d}, q^h_{N \backslash N^h_d}) = q^h_{i^{h'}}. \tag{13}$$

Then, by (12), (13) and replacement monotonicity,

$$\Phi(\mathbf{0}_{N^h_d}, q^h_{N \backslash N^h_d}) = q^h.$$

This finishes the proof of (1.1) for the case **t**$= 1$.

Induction hypothesis: for **t**> 1, if $h'' = (h'''', a_{i^{h''''}}, a_{i^{h'''}})$ is the input of **Step 2.t-1.a**, then

$$\Phi(\mathbf{0}_{N^{h''}_d}, q^{h''}_{N \backslash N^{h''}_d}) = q^{h''}. \tag{14}$$

Observe that in the proof of (1.1) for the case **t**$= 1$, (10) can be replaced by (14) and using the same argument, it follows that

$$\Phi(\mathbf{0}_{N^h_d}, q^h_{N \backslash N^h_d}) = q^h.$$

(1.2) Assume $N^h_u \neq \emptyset$. By condition (1) in the characterization of efficient rules,

$$\Phi_j(\mathbf{k}_{N^h_u}, (q^{h''} - \mathbf{1})_{N^h_d}, q^h_{N^h_s}) \leq \begin{cases} q^{h''}_j - 1 & \text{if } j \in N^h_d \\ q^h_j & \text{if } j \in N^h_s. \end{cases} \tag{15}$$

In order to show that the other inequality holds as well, we proceed by induction on **t**.

Suppose **t**$= 1$. Hence, h'' is the output of **Stage 1** and $N^{h''} = N$, $N^{h''}_u \neq \emptyset$, $N^{h''}_d \neq \emptyset$, and $q^{h''} = q$. By strategy-proofness and efficiency of Φ and $\Phi(\mathbf{k}) = q^{h''}$,

$$\Phi(\mathbf{k}_{N^{h''}_u}, q^{h''}_{N^{h''}_d}, q^{h''}_{N^{h''}_s}) = q^{h''}. \tag{16}$$

Let $i_1 \in N^{h''}_d$. By strategy-proofness (for i_1), single-peakedness, replacement monotonicity and (16),

$$\Phi_j(\mathbf{k}_{N^{h''}_u}, q^{h''}_{N^{h''}_d \backslash \{i_1\}}, q^{h''}_{i_1} - 1, q^{h''}_{N^{h''}_s}) \geq \begin{cases} q^{h''}_j - 1 & \text{if } j = i_1 \\ q^{h''}_j & \text{if } j \in N^{h''}_s \cup (N^{h''}_d \backslash \{i_1\}). \end{cases} \tag{17}$$

Let $i_2 \in N^{h''}_d \backslash \{i_1\}$, if any. By strategy-proofness (for i_2), single-peakedness, replacement monotonicity and (17),

$$\Phi_j(\mathbf{k}_{N^{h''}_u}, q^{h''}_{N^{h''}_d \backslash \{i_1, i_2\}}, (q^{h''} - \mathbf{1})_{\{i_1, i_2\}}, q^{h''}_{N^{h''}_s}) \geq \begin{cases} q^{h''}_j - 1 & \text{if } j \in \{i_1, i_2\} \\ q^{h''}_j & \text{if } j \in N^{h''}_s \cup (N^{h''}_d \backslash \{i_1, i_2\})). \end{cases}$$

If we continue in the same way, we obtain that $N^{h''}_d \backslash \{i_1, \ldots, i_T\} = \emptyset$ for a $T \geq 1$, and

$$\Phi_j(\mathbf{k}_{N^{h''}_u}, (q^{h''} - \mathbf{1})_{N^{h''}_d}, q^{h''}_{N^{h''}_s}) \geq \begin{cases} q^{h''}_j - 1 & \text{if } j \in N^{h''}_d \\ q^{h''}_j & \text{if } j \in N^{h''}_s. \end{cases} \tag{18}$$

Furthermore, by definition of $i^{h''}$,

$$\Phi_{i^{h''}}(\mathbf{k}_{N_u^{h''}}, (q^{h''} - \mathbf{1})_{N_d^{h''}}, q^{h''}_{N_s^{h''}}) \geq q^{h''}_{i^{h''}} + 1. \tag{19}$$

From (18) and (19), and since $q_j^h = q_j^{h''}$ for all $j \in N_s^{h''}$, $q^{h''}_{i^{h''}} + 1 = q^h_{i^{h''}}$, $q^{h''}_{i^{h'}} - 1 = q^h_{i^{h'}}$ and $i^{h'} \in N_d^{h''}$,

$$\Phi_j(\mathbf{k}_{N_u^{h''}}, (q^{h''} - \mathbf{1})_{N_d^{h''}}, q^h_{N_s^{h''}}) \geq \begin{cases} q_j^{h''} - 1 & \text{if } j \in N_d^{h''} \\ q_j^h & \text{if } j \in N_s^{h''} \cup \{i^{h''}, i^{h'}\}. \end{cases} \tag{20}$$

Since $N_d^{h''} = N_d^h$ or $N_d^h = N_d^{h''} \backslash \{i^{h'}\}$ with $i^{h'} \in N_s^h$ and $q^{h''}_{i^{h'}} - 1 = q^h_{i^{h'}}$ and $N_s^{h''} \subset N_s^h \subset N_s^{h''} \cup \{i^{h''}, i^{h'}\}$,

$$\Phi_j(\mathbf{k}_{N_u^{h''}}, (q^{h''} - \mathbf{1})_{N_d^h}, q^h_{N_s^h \backslash \{i^{h''}\}}) \geq \begin{cases} q_j^{h''} - 1 & \text{if } j \in N_d^h \\ q_j^h & \text{if } j \in N_s^h \cup \{i^{h''}\}. \end{cases} \tag{21}$$

Since $N_u^{h''} = N_u^h$ and $N_s^h \backslash \{i^{h''}\} = N_s^h$ or $N_u^h = N_u^{h''} \backslash \{i^{h''}\}$ with $i^{h''} \in N_s^h$ and $q^h_{i^{h''}} \leq k$, by strategy-proofness (for $i^{h''}$) and replacement monotonicity,

$$\Phi_j(\mathbf{k}_{N_u^h}, (q^{h''} - \mathbf{1})_{N_d^h}, q^h_{N_s^h}) \geq \begin{cases} q^{h''} - 1 & \text{if } j \in N_d^h \\ q_j^h & \text{if } j \in N_s^h. \end{cases} \tag{22}$$

This, together with (15), finishes the proof of (1.2) for the case **t**$= 1$.

Induction hypothesis: for **t**> 1, if $h'' = (h'''', a_{i^{h''''}}, a_{i^{h'''}})$ is the input of the **Step 2.t-1.a**, then

$$\Phi_j(\mathbf{k}_{N_u^{h''}}, (q^{h''''} - \mathbf{1})_{N_d^{h''}}, q^{h''}_{N_s^{h''}}) = \begin{cases} q_j^{h''''} - 1 & \text{if } j \in N_d^{h''} \\ q_j^{h''} & \text{if } j \in N_s^{h''}. \end{cases} \tag{23}$$

We now show that (23) holds as well when h'''' is replaced by h'' and h'' by h. We first prove that (23) implies (18). Then, to obtain (22), the proof follows the same argument used in the case **t**$= 1$.

If $i^{h'''} \notin N_d^{h''}$, then $q_j^{h''''} = q_j^{h''}$ for all $j \in N_d^{h''}$. Therefore (23) implies (18) and the proof follows as in the case **t**$= 1$.

If $i^{h'''} \in N_d^{h''}$, then $q_j^{h''''} - 1 = q_j^{h''} - 1$ if $j \in N_d^{h''} \backslash \{i^{h'''}\}$ and $q_j^{h''''} - 1 = q_j^{h''}$ if $j = i^{h'''}$. Then, by strategy-proofness (for $i^{h'''}$) and (23),

$$\Phi_{i^{h'''}}(\mathbf{k}_{N_u^{h''}}, (q^{h''} - \mathbf{1})_{N_d^{h''}}, q^{h''}_{N_s^{h''}}) \geq q^{h''}_{i^{h'''}} - 1, \tag{24}$$

which is (18) for $j = i^{h'''}$. Now, since $q^{h''''}_{i^{h'''}} - 1 \geq q^{h''}_{i^{h'''}} - 1$, by replacement monotonicity and (23),

$$\Phi_j(\mathbf{k}_{N_u^{h''}}, (q^{h''} - \mathbf{1})_{N_d^{h''}}, q^{h''}_{N_s^{h''}}) \geq \begin{cases} q_j^{h''} - 1 & \text{if } j \in N_d^{h''} \backslash \{i^{h'''}\} \\ q_j^{h''} & \text{if } j \in N_s^{h''}. \end{cases} \tag{25}$$

Therefore, (24) and (25) imply (18) and the proof of (1.2) follows as in the case **t**$= 1$. ■

Lemma 2 *Let $h \in H_T$ be a terminal history (and hence, $N_u^h = \emptyset$ or $N_d^h = \emptyset$). Then,*
(2.1) *if $N_u^h = \emptyset$, $\Phi(\mathbf{0}_{N_d^h}, q_{N_s^h}^h) = q^h$ and*
(2.2) *if $N_u^h \neq \emptyset$, $\Phi(\mathbf{k}_{N_u^h}, q_{N_s^h}^h) = q^h$.*

Proof of Lemma 2 Suppose h is the output of **Stage 1**. Then, $q^h = q = \Phi(\mathbf{0}) = \Phi(\mathbf{k})$ and, by strategy-proofness and replacement monotonicity,

$$\Phi(\mathbf{k}_{N_u^h}, q_{N \setminus N_u^h}^h) = \Phi(\mathbf{0}_{N_d^h}, q_{N \setminus N_d^h}^h) = q^h. \tag{26}$$

Then, (2.1) and (2.2) follow from (26).

Now suppose that $h = (h'', a_{i^{h''}}, a_{i^{h'}})$ is the output of **Step 2.t.b**. Then, (2.1) follows from (1.1) in Lemma 1. To show that (2.2) holds, assume $N_u^h \neq \emptyset$, and so $N_d^h = \emptyset$. We now show that

$$\Phi(\mathbf{k}_{N_u^h}, q_{N_s^h}^h) = q^h.$$

By (1.2) in Lemma 1 and the fact that $N_d^h = \emptyset$, for all $j \in N_s^h$,

$$\Phi_j(\mathbf{k}_{N_u^h}, q_{N_s^h}^h) = q_j^h. \tag{27}$$

Let $j \in N_u^h$. We show that

$$\Phi_j(\mathbf{k}_{N_u^h}, q_{N_s^h}^h) \geq q_j^h$$

holds by distinguishing between two cases.

Case 1: Suppose j has not played along **Stage 2**. Then, by replacement monotonicity and the definition of q_j^h,

$$\Phi_j(\mathbf{k}_{N_u^h}, q_{N_s^h}^h) \geq q_j^h = q_j = \Phi_j(\mathbf{k}).$$

Case 2: Suppose j has played along **Stage 2**. Since $j \in N_u^h$, there exists $\widetilde{h} = (\widetilde{h}'', a_{i^{\widetilde{h}''}}, a_{i^{\widetilde{h}'}}) \prec h$, the output of **Step 2.t.b** for some $t \geq 1$, such that $j = i^{\widetilde{h}''}$, and $\widetilde{h}''$ is the last history at which j has played along **Stage 2**. By definition of $i^{\widetilde{h}''}$ and q_j^h,

$$\Phi_j(\mathbf{k}_{N_u^{\widetilde{h}''}}, (q^{\widetilde{h}''} - \mathbf{1})_{N_d^{\widetilde{h}''}}, q_{N_s^{\widetilde{h}''}}^{\widetilde{h}''}) \geq q_j^{\widetilde{h}''} + 1 = q_j^h. \tag{28}$$

If $i \in N_s^{\widetilde{h}''}$, then $i \in N_s^h$ and by the definition of q_i^h,

$$q_i^h = q_i^{\widetilde{h}''}.$$

If $i \in N_d^{\widetilde{h}''}$, and since $N_d^h = \emptyset$ implies $i \in N_s^h$, we have that i is called to play at least once at some history $\bar{h}'$ such that $\widetilde{h}'' \preceq \bar{h}' \prec h$ (and so, $i^{\bar{h}'} = i$). Then, by definition of $q_i^{\bar{h}}$ and the fact that $i \in N_d^{\widetilde{h}''}$,

$$q_i^h \leq q_i^{\bar{h}} \leq q_i^{\widetilde{h}''} - 1.$$

By replacement monotonicity and the fact that $j \in N_u^h \subset N_u^{\widetilde{h}''}$,

$$\Phi_j(\mathbf{k}_{N_u^{\widetilde{h}''}}, (q^{\widetilde{h}''} - \mathbf{1})_{N_d^{\widetilde{h}''}}, q_{N_s^{\widetilde{h}''}}^{\widetilde{h}''}) \leq \Phi_j(\mathbf{k}_{N_u^h}, q_{N_s^h}^h). \tag{29}$$

Thus, (28) and (29) imply that

$$q_j^h \leq \Phi_j(\mathbf{k}_{N_u^h}, q_{N_s^h}^h). \tag{30}$$

By (27), (30) and feasibility of q^h,

$$\Phi(\mathbf{k}_{N_u^h}, q_{N_s^h}^h) = q^h.$$

This concludes the proof of Lemma 2. ■

Proof of Theorem 1 It follows from the three statements that we will present and prove successively.

Statement 1.1 *Let Φ be an individually rational sequential allotment rule. Then, the extensive game form Γ^Φ* is *well defined and finite.*

Proof of Statement 1.1 We first argue that the agents that are called to play along the specific phase of the algorithm are uniquely identified and well defined. This is clear in any **Step 1.t** for **t**≥ 1.

Consider now **Step 2.t.a** and **Step 2.t.b**, for **t**≥ 1, with corresponding inputs h'' and h', where h'' is not a terminal history. Since $i \in N_d^{h''}$ implies $0 < q_i^{h''}$,

$$\Phi_j(\mathbf{k}_{N_u^{h''}}, (q^{h''} - \mathbf{1})_{N_d^{h''}}, q_{N_s^{h''}}^{h''}) \tag{31}$$

is well defined for all $j \in N$. Moreover, the sum of the components of the vector where Φ_j is applied in (31) is larger or equal than k. Hence, by efficiency,

$$\sum_{j \notin N_u^{h''}} \Phi_j(\mathbf{k}_{N_u^{h''}}, (q^{h''} - \mathbf{1})_{N_d^{h''}}, q_{N_s^{h''}}^{h''}) \leq \sum_{j \in N_s^{h''}} q_j^{h''} + \sum_{j \in N_d^{h''}} (q_j^{h''} - 1) < \sum_{j \notin N_u^{h''}} q_j^{h''}.$$

By feasibility of $q^{h''}$,

$$\sum_{j \in N_u^{h''}} \Phi_j(\mathbf{k}_{N_u^{h''}}, (q^{h''} - \mathbf{1})_{N_d^{h''}}, q_{N_s^{h''}}^{h''}) > \sum_{j \in N_u^{h''}} q_j^{h''}.$$

Hence, $\min_{<}\{j \in N_u^{h''} \mid \Phi_j(\mathbf{k}_{N_u^{h''}}, (q^{h''} - \mathbf{1})_{N_d^{h''}}, q_{N_s^{h''}}^{h''}) \geq q_j^{h''} + 1\}$ is well defined and so is $i^{h''}$. Similarly, and since $i \in N_u^{h''}$ implies $q_i^{h''} < k$,

$$\Phi_j(q_{i^{h''}}^{h'} + 1, \mathbf{0}_{N_d^{h'}}, q_{-N_d^{h'} \cup \{i^{h''}\}}^{h'}) \tag{32}$$

is well defined for all $j \in N$. Moreover, the sum of the components of the vector where Φ_j is applied in (32) is smaller or equal than k. Hence, by efficiency,

$$\sum_{j \notin N_d^{h'}} \Phi_j(q_{i^{h''}}^{h'} + 1, \mathbf{0}_{N_d^{h'}}, q_{-N_d^{h'} \cup \{i^{h''}\}}^{h'}) \geq \sum_{j \in N \setminus (N_d^{h'} \cup \{i^{h''}\})} q_j^{h'} + q_{i^{h''}}^{h'} + 1 > \sum_{j \notin N_d^{h'}} q_j^{h'}.$$

By feasibility of $q^{h'}$,

$$\sum_{j \in N_d^{h'}} \Phi_j(q_{i^{h''}}^{h'} + 1, \mathbf{0}_{N_d^{h'}}, q_{-N_d^{h'} \cup \{i^{h''}\}}^{h'}) < \sum_{j \in N_d^{h'}} q_j^{h'}.$$

Hence, $\min_{<}\{j \in N_d^{h'} \mid \Phi_j(q_{i^{h''}}^{h'}+1, \mathbf{0}_{N_d^{h'}}, q^{h'}_{-N_d^{h'}\cup\{i^{h''}\}}) \leq q_j^{h'}-1\}$ is well defined and so is $i^{h'}$.

Second, the sets of agents defined along the specific phase of the algorithm are well defined because they evolve, for any $h \succ h'$, as follows.

(a) If $i \in N_u^{h'}$ for some h', then $i \notin N_d^h$ and $i \in N_u^h \cup N_s^h$.

(b) If $i \in N_d^{h'}$ for some h', then $i \notin N_u^h$ and $i \in N_d^h \cup N_s^h$.

(c) If $i \in N_s^{h'}$ for some h', then $i \in N_s^h$.

(d) If $N_u^{h'}, N_s^{h'}, N_d^{h'}$ is a partition of N, then N_u^h, N_s^h, N_d^h is also a partition of N.

To see that the statements (a) to (d) hold, let $h \succ h'$ be arbitrary. Assume $i \in N_u^{h'}$. If i is not called to play anymore, $i \in N_u^h$. Suppose i is called to play at h (*i.e.*, $i^h = i$). Then, $d \notin A_i^h$. Hence, either $i \in N_u^h$ or $i \in N_s^h$. Thus, (a) holds. Symmetrically if $i \in N_d^{h'}$. Suppose now that $i \in N_s^{h'}$. Then i will not be called to play anymore. Hence, $N_s^h \supseteq N_s^{h'}$ which implies that $i \in N_s^h$. Thus, (c) holds. The proof of (d) follows immediately from the definitions of N_u^h, N_s^h and N_d^h.

Now, we argue that the extensive game form defined by the algorithm is finite.

Stage 1 ends because $N\backslash N^{h'} \supsetneq N\backslash N^h$ holds, where $h = (h', a_{i^{h'}})$, and so, for some history $\overline{h}$, $N^{\overline{h}} = N$. In **Stage 2**, if h is an output history of some **Step 2.t.b** and $N_u^h = \emptyset$ or $N_d^h = \emptyset$ then h is a terminal history and the game ends. If $N_u^h \neq \emptyset$ and $N_d^h \neq \emptyset$, then $h = (h'', a_{i^{h''}}, a_{i^{h'}})$ with $h' = (h'', a_{i^{h''}})$, $N_s^{h''} \subset N_s^h \subset N_s^{h''} \cup \{i^{h''}, i^{h'}\}$ and $q_{i^{h''}}^h = q_{i^{h''}}^{h''} + 1$, $q_{i^{h'}}^h = q_{i^{h'}}^{h''} - 1$ and $q_i^h = q_i^{h''}$ for all $i \notin \{i^{h''}, i^{h'}\}$ (since by definition, $q_i^{h'} = q_i^{h''}$). Then, the algorithm stops at some **Step 2.t.b** with output h because $N_u^h = \emptyset$ or $N_d^h = \emptyset$ (recall that for any history $\overline{h}$ of **Stage 2**, with h^* being its immediate successor, $q_{i^{\overline{h}}}^{\overline{h}} = k$ implies $i^{\overline{h}} \in N_s^{h^*}$ and $q_{i^{\overline{h}}}^{\overline{h}} = 0$ implies $i^{\overline{h}} \in N_s^{h^*}$). ■

Let H^1 and H^2 be the sets of histories that are outputs of some steps in **Stage 1** and **Stage 2** of Γ^Φ, respectively. In addition, let H^* be the set of histories that are output of **Stage 1** or **Step 2.t.b**, for some **t**≥ 1. Note that $H^1 \subseteq H^*$. By the definition of the specific phase of the algorithm, the set of terminal histories H_T of Γ^Φ can be written as

$$H_T = \{h \in H^* \mid N_u^h = \emptyset \text{ or } N_d^h = \emptyset\}.$$

For each terminal history $h \in H_T$ the output of the game Γ^Φ is $o(h) = q^h$.

Statement 2.1 *Let Φ be an individually rational sequential allotment rule. Then, Γ^Φ induces Φ; namely, for all $R \in \mathcal{R}^n$,*

$$\Phi(R) = o(h^\Gamma(\varnothing, \sigma^R)).$$

Proof of Statement 2.1 Let $R \in \mathcal{R}^n$ be arbitrary, let $\tau = (\tau(R_1), \ldots, \tau(R_n)) \in \{0, \ldots, k\}^n$ be the profile of tops at R and set $h^\tau := h^\Gamma(\varnothing, \sigma^R)$. We first state and prove three claims.

Claim 1: *If $i \in N_s^{h^\tau}$, then $\tau_i = q_i^{h^\tau}$.*

Proof of Claim 1: Let $i \in N_s^{h^\tau}$. Then, by the definitions of the set N_s^h at any h and of the history $\widehat{h}$, $i = i^{\widehat{h}}$ and $\sigma_i^{R_i}(\widehat{h}) \notin \{u, d\}$. If $\widehat{h} \in H^1$, it follows that $\tau_i = \sigma_i^{R_i}(\widehat{h}) = q_i^{\widehat{h}} = q_i^{h^\tau}$. If $\widehat{h} \in H^2$, and since $\widehat{h} \preceq h^\tau$ and h^τ is the history induced by the profile of truth-telling strategies, $\tau_i \geq q_i^{\widehat{h}} + 1$ if $i \in N_u^{\widehat{h}}$ and $\tau_i \leq q_i^{\widehat{h}} - 1$ if $i \in N_d^{\widehat{h}}$. Then, since $\sigma_i^{\tau_i}(\widehat{h}) \notin \{u, d\}$,

$$\tau_i = \begin{cases} q_i^{\widehat{h}} + 1 & \text{if } i \in N_u^{\widehat{h}} \\ q_i^{\widehat{h}} - 1 & \text{if } i \in N_d^{\widehat{h}}. \end{cases} \tag{33}$$

Then,

$$\tau_i = q_i^{h^\tau},$$

where the equality follows from (33) and the definition of $q_i^{h^\tau} = q_i^{\overline{h}}$, where $\widehat{h} \prec^{im} \overline{h}$. □

Claim 2: *If $i \in N_d^{h^\tau}$, then $\tau_i < q_i^{h^\tau}$.*

Proof of Claim 2: Let $i \in N_d^{h^\tau}$. Then, by the definitions of the set N_d^h at any h and the history $\widehat{h}$, $i = i^{\widehat{h}}$ and $\sigma_i^{R_i}(\widehat{h}) = d$. If $\widehat{h} \in H^1$, it follows that $\tau_i < q_i^{\widehat{h}} = q_i^{h^\tau}$. If $\widehat{h} \in H^2$, and since $\widehat{h} \preceq h^\tau$ and $i \in N_d^{\widehat{h}}$, by the definition of $\sigma_i^{R_i}$, we have that $\tau_i < q_i^{\widehat{h}} - 1$. But since $q_i^{\widehat{h}} - 1 = q_i^{h^\tau}$ and h^τ is a terminal history of **Stage 2**, it follows that $\tau_i < q_i^{h^\tau}$. □

Claim 3: *If $i \in N_u^{h^\tau}$, then $\tau_i > q_i^{h^\tau}$.*

Proof of Claim 3: Let $i \in N_u^{h^\tau}$. Then, by the definitions of the set N_u^h at any h and the history $\widehat{h}$, $i = i^{\widehat{h}}$ and $\sigma_i^{R_i}(\widehat{h}) = u$. If $\widehat{h} \in H^1$, it follows that $\tau_i > q_i^{\widehat{h}} = q_i^{h^\tau}$. If $\widehat{h} \in H^2$, and since $\widehat{h} \preceq h^\tau$ and $i \in N_u^{\widehat{h}}$, by the definition of $\sigma_i^{R_i}$, we have $\tau_i > q_i^{\widehat{h}} + 1$. But since $q_i^{\widehat{h}} + 1 = q_i^{h^\tau}$ and h^τ is a terminal history of **Stage 2**, it follows that $\tau_i > q_i^{h^\tau}$. □

We proceed with the proof of Statement 2.1 by distinguishing between two cases.

Case 1: Assume $N_u^{h^\tau} = \emptyset$. By (2.1) in Lemma 2 and Claim 1,

$$\Phi(\mathbf{0}_{N_d^{h^\tau}}, \tau_{N_s^{h^\tau}}) = q^{h^\tau}.$$

By Claim 2, $0 \leq \tau_i < q_i^{h^\tau}$ for every $i \in N_d^{h^\tau}$. Therefore, by strategy-proofness and replacement monotonicity,

$$\Phi(\tau_{N_d^{h^\tau}}, \tau_{N_s^{h^\tau}}) = q^{h^\tau}.$$

Hence, $\Phi(R) = o(h^\Gamma(\varnothing, \sigma^R))$.

Case 2: Assume $N_u^{h^\tau} \neq \emptyset$. Then, $N_d^{h^\tau} = \emptyset$ and, by (2.2) in Lemma 2 and Claim 1,

$$\Phi\left(\mathbf{k}_{N_u^{h^\tau}}, \tau_{N_s^{h^\tau}}\right) = q^{h^\tau}.$$

By Claim 3, $\tau_i > q_i^{h^\tau}$ for every $i \in N_u^{h^\tau}$. Therefore, by strategy-proofness and replacement monotonicity,

$$\Phi\left(\tau_{N_u^{h^\tau}}, \tau_{N_s^{h^\tau}}\right) = q^{h^\tau}.$$

Hence, $\Phi(R) = o(h^\Gamma(\varnothing, \sigma^R))$. ■

Statement 3.1 *Let Φ be an individually rational sequential allotment rule and let $R \in \mathcal{R}^n$ be a profile. Then, for all $i \in N$, the truth-telling strategy $\sigma_i^{R_i}$ is weakly dominant in Γ^Φ.*

Proof of Statement 3.1 Consider agent i with preferences R_i and top allotment τ_i. Let $\sigma_i^{R_i}$ be i's truth-telling strategy relative to R_i and let σ_i' be any other strategy. We want to show that, for all σ_{-i},

$$x_i = o_i(h^{\Gamma^\Phi}(\emptyset, (\sigma_i^{R_i}, \sigma_{-i})))R_i o_i(h^{\Gamma^\Phi}(\emptyset, (\sigma_i', \sigma_{-i}))) = x_i' \tag{34}$$

holds. Let σ_{-i} be arbitrary. Condition (34) holds trivially if $x_i = x_i'$. Assume $x_i \neq x_i'$. Let h' be the earliest history at which $\sigma_i^{R_i}(h') \neq \sigma_i'(h')$ along the equal play induced by both $(\sigma_i^{R_i}, \sigma_{-i})$ and (σ_i', σ_{-i}) up to h'. We proceed by distinguishing among several cases, depending on the step of the specific phase of the algorithm for which the history h' is an input of, and agent i is called to play at h'.

Case 1: The history h' is an input of some **Step 1.t** (*i.e.*, $i^{h'} = i$). Then, $i \in N \backslash N^{h'}$. Let q be the allotment that is part of the input information at this step. We distinguish among three cases.

Subcase 1.1: $\tau_i = q_i$. Then, $\sigma_i^{R_i}(h') = \tau_i$. Hence, $i \in N_s^h$ for all $h \succeq (h', \sigma_i'(h'))$, agent i is not called to play anymore and $x_i = \tau_i$. Thus, (34) holds.

Subcase 1.2: $\tau_i > q_i$. Then, $\sigma_i^{R_i}(h') = u$. Setting $\bar{h} = (h', \sigma_i^{R_i}(h'))$, we have that $i \in N_u^{\bar{h}}$. Then, for all $\bar{h} \preceq h$, by definition of q_i^h,

$$q_i \leq q_i^h. \tag{35}$$

Denote by $h^\Gamma(h', (\sigma_i^{R_i}, \sigma_{-i}))$ the terminal history obtained when agents play starting at h' according to $(\sigma_i^{R_i}, \sigma_{-i})$. Because $\bar{h} \preceq h^\Gamma(h', (\sigma_i^{R_i}, \sigma_{-i}))$,

$$q_i^{h'} \leq x_i. \tag{36}$$

Since $\bar{h} = (h', \sigma_i^{R_i}(h'))$ and h' is an input of some **Step 1.t**, $\bar{h}$ is a history of **Stage 1** or **Stage 2**. Hence, $i \in N_u^{\bar{h}}$ implies that $i \in N_u^{h^{\Gamma^\Phi}(h', (\sigma_i^{R_i}, \sigma_{-i}))} \cup N_s^{h^{\Gamma^\Phi}(h', (\sigma_i^{R_i}, \sigma_{-i}))}$. Therefore, by the definition of $\sigma_i^{R_i}$,

$$x_i \leq \tau_i. \tag{37}$$

By (35), (36) and (37),

$$q_i \leq x_i \leq \tau_i. \tag{38}$$

On the other hand, since $\sigma_i'(h') \neq \sigma_i^{R_i}(h') = u$, we have that $\sigma_i'(h') \in \{q_i, d\}$. Then, by setting $\check{h} = (h', \sigma_i'(h'))$, $i \in N_d^{\check{h}} \cup N_s^{\check{h}}$ which means that $q_i^h \leq q_i$ for all $\check{h} \preceq h$. Therefore, since $\check{h} \preceq h^\Gamma(h', (\sigma_i', \sigma_{-i}))$,

$$x_i' \leq q_i. \tag{39}$$

By (38) and (39), single-peakedness of R_i implies that $x_i R_i x_i'$. Thus, (34) holds.

Subcase 1.3: $\tau_i < q_i$. Then, $\sigma_i^{R_i}(h') = d$. With a symmetric argument to the one used in Subcase 1.2, we obtain that $\tau_i \leq x_i \leq q_i \leq x_i'$, and single-peakedness of R_i implies that $x_i R_i x_i'$. Thus, (34) holds.

Case 2: The history h' is an input of some **Step 2.t.a**, and to make the notation consistent with the one used to define such step, set $h'' := h'$ and so $i^{h''} = i$. Then, $i \in N_u^{h''}$. Let $q_i^{h''}$ be i's allotment that is part of the input information at this step. Since $i \in N_u^{h''}$ and h'' is a history in the path starting at the empty history $\emptyset$ when agents play according to $(\sigma_i^{R_i}, \sigma_{-i})$, agent i has chosen u before h''. Hence, by definition of $\sigma_i^{R_i}$,

$$q_i^{h''} + 1 \leq \tau_i. \tag{40}$$

We distinguish between two subcases.

Subcase 2.1: $\tau_i = q_i^{h''} + 1$. Then, $\sigma_i^{R_i}(h'') = q_i^{h''} + 1 = \tau_i$. Hence, $i \in N_s^h$ for all $h \succeq (h'', \sigma_i^{R_i}(h''))$, agent i is not called to play anymore, and so $x_i = \tau_i$. Thus, (34) holds.

Subcase 2.2: $\tau_i > q_i^{h''} + 1$. Then, $\sigma_i^{R_i}(h'') = u$ and $\sigma_i'(h'') = q_i^{h''} + 1$. Hence, $i \in N_s^h$ for all $h \succeq (h'', \sigma_i'(h''))$, agent i is not called to play anymore, and so $x_i' = q_i^{h''} + 1$. Furthermore, using a similar argument to the one used in Subcase 1.2, it is easy to see that $q_i^{h''} + 1 \leq x_i \leq \tau_i$. Since $q_i^{h''} + 1 = x_i'$, single-peakedness of R_i implies that $x_i R_i x_i'$. Thus, (34) holds.

Case 3: The history h' is an input of some **Step 2.t.b** (*i.e.*, $i^{h'} = i$). Then, $i \in N_d^{h'}$. Let $q_i^{h'}$ be i's allotment that is part of the input information at this step. Since $i \in N_d^{h'}$ and h' is a history in the path starting at the empty history $\emptyset$ when agents play according to $(\sigma_i^{R_i}, \sigma_{-i})$, agent i has chosen d before h'. Hence, by the definition of $\sigma_i^{R_i}$,

$$\tau_i \leq q_i^{h'} - 1. \tag{41}$$

We distinguish between two cases.

Subcase 3.1: $\tau_i = q_i^{h'} - 1$. Then, $\sigma_i^{R_i}(h') = q_i^{h'} - 1 = \tau_i$. Hence, $i \in N_s^h$ for all $h \succeq (h', \sigma_i^{R_i}(h'))$, agent i is not called to play anymore, and so $x_i = \tau_i$. Thus, (34) holds.

Subcase 3.2: $\tau_i < q_i^{h'} - 1$. Then, $\sigma_i^{R_i}(h') = d$ and $\sigma_i'(h') = q_i^{h'} - 1$. Hence, $i \in N_s^h$ for all $h \succeq (h', \sigma_i'(h'))$, agent i is not called to play anymore, and so $x_i' = q_i^{h'} - 1$. Furthermore, using a similar argument to the one used in Subcase 2.2, it is easy to see that $\tau_i \leq x_i \leq q_i^{h'} - 1$. Since $q_i^{h'} - 1 = x_i'$, single-peakedness of R_i implies that $x_i R_i x_i'$. Thus, (34) holds. ■

6.2 Theorem 2

Before starting with the proof of Theorem 2, we present an alternative (and equivalent) way of defining, along **Stage A**, the vectors $\overline{q}^h$ and $\underline{q}^h$ as images of Φ evaluated respectively at $\overline{m}^h$ and $\underline{m}^h$, according to either (6) or (7). Remember that the definitions of $\overline{m}^h$ and $\underline{m}^h$ use respectively the vectors $\overline{q}^{h'}$ and $\underline{q}^{h'}$, where h' is the immediate predecessor of h. It will be convenient to define instead two vectors $\overline{p}^h$ and $\underline{p}^h$ as images of Φ evaluated respectively at $\overline{l}^h$ and $\underline{l}^h$ which depend not only on h' (and its embedded information) but also on all subhistories up to h (and their embedded information). In Lemma 6 we will show that $\overline{p}^h = \overline{q}^h$, $\underline{p}^h = \underline{q}^h$, $\overline{l}^h = \overline{m}^h$ and $\underline{l}^h = \underline{m}^h$ hold, and so both descriptions are equivalent. Therefore, all histories, sets and vectors defined along **Stage A** using $\overline{m}^h$ and $\underline{m}^h$ (according to the original definition of the algorithm in Subsection 4.2.1) coincide with those that would have been obtained from having used instead the vectors $\overline{l}^h$ and $\underline{l}^h$. However, the more involved description using all subhistories up to h is more helpful for the proofs of some of the statements required to prove Theorem 2.

For the empty history $\emptyset$, define

$$\overline{p}^{\emptyset} = \Phi(\mathbf{k}) \quad \text{and} \quad \underline{p}^{\emptyset} = \Phi(\mathbf{0}),$$

and, for any history $h \succ \emptyset$, define

$$\overline{l}_j^h = \begin{cases} k & \text{if } j \in N_u^h \cup (N \backslash N^h) \\ \underline{p}_{i^{\widehat{h}}}^{\widehat{h}} & \text{if } j \in N_d^h \text{ and } j = i^{\widehat{h}} \\ a_{i^{\widehat{h}}} & \text{if } j \in N_s^h \text{ and } j = i^{\widehat{h}} \end{cases}$$

and

$$\underline{l}_j^h = \begin{cases} 0 & \text{if } j \in N_d^h \cup (N \backslash N^h) \\ \overline{p}_{i^{\widehat{h}}}^{\widehat{h}} & \text{if } j \in N_u^h \text{ and } j = i^{\widehat{h}} \\ a_{i^{\widehat{h}}} & \text{if } j \in N_s^h \text{ and } j = i^{\widehat{h}}, \end{cases}$$

where, given j, $\widehat{h}$ is the longest subhistory of h at which j has played at (*i.e.*, $j = i^{\widehat{h}}$ and, for all $\widehat{h} \prec \widetilde{h} \prec h$, $j \neq i^{\widetilde{h}}$). Then, define

$$\overline{p}^h = \Phi(\overline{l}^h) \quad \text{and} \quad \underline{p}^h = \Phi(\underline{l}^h).$$

Let Hm be the set of histories obtained by evaluating Φ at vectors $\overline{m}^h$ and $\underline{m}^h$ for histories h in **Stage A** (as defined in Subsection 4.2.1) and let Hl be the set of histories that would be obtained by evaluating Φ at vectors $\overline{l}^h$ and $\underline{l}^h$ for histories h in **Stage A** (as in Subsection 4.2.1, using $\overline{l}^h$ and $\underline{l}^h$ instead of $\overline{m}^h$ and $\underline{m}^h$, respectively).

Remark 2 We want to emphasize that any history h contains all information needed to recover the sets N_u^h, N_d^h and N_s^h as well as all choices made by agents along h (specifically, $a_{i^{h'}}$ and $a_{i^{\widehat{h}}}$) regardless of the vectors used to evaluate Φ; for instance, from the history $h' = (\emptyset, a_4 = 2, a_1 = u, a_2 = d, a_3 = u, a_1 = 4, a_2 = d, a_3 = u)$, input of **Step 2.2.b** in the example of Subsection 4.3, we can obtain in an unambiguous way the sets $N_u^{h'} = \{3\}$, $N_d^{h'} = \{2\}$, and $N_s^{h'} = \{1, 4\}$ and all choices made by agents along h (for instance, $a_2^{(\emptyset, a_4=2, a_1=u)} = d$). In particular, for $h \in Hm \cup Hl$, the corresponding sets N_u^h, N_d^h and N_s^h as well as all choices made by agents along h, can be obtained from h itself, independently of whether the vectors $\overline{m}^h$ and $\underline{m}^h$ or $\overline{l}^h$ and $\underline{l}^h$ have been used to generate h.

Claim 1 *Let $h \in Hl$ be the output of* **Step A.t** *and let $h' \prec^{im} h$ (i.e., $h = (h', a_{i^{h'}})$). Then,*

(C1.1) *if $i^{h'} \in N_u^h$, $\overline{l}^h = \overline{l}^{h'}$ and $\underline{l}^h = (\underline{l}^{h'}_{-i^{h'}}, \overline{p}^{h'}_{i^{h'}})$,* (C1.2) *if* $i^{h'} \in N^h_d$, $\bar{l}^h = (\bar{l}^{h'}_{-i^{h'}}, \underline{p}^{h'}_{i^{h'}})$ *and* $\underline{l}^h = \underline{l}^{h'}$,

(C1.3) *if* $i^{h'} \in N^h_s$, $\bar{l}^h = (\bar{l}^{h'}_{-i^{h'}}, a_{i^{h'}})$ *and* $\underline{l}^h = (\underline{l}^{h'}_{-i^{h'}}, a_{i^{h'}})$.

Proof of Claim 1

(C1.1) Assume $i^{h'} \in N^h_u$. Then, $a_{i^{h'}} = u$ and so, by the definition of $A_{i^{h'}}$, either $i^{h'} \in N^{h'}_u$ or $i^{h'} \notin N^{h'}$. In both cases, $\bar{l}^h_{i^{h'}} = \bar{l}^{h'}_{i^{h'}} = k$. Since $\bar{l}^h$ and $\bar{l}^{h'}$ only differ in the $i^{h'}$-th component, $\bar{l}^h = \bar{l}^{h'}$. By definition, $\underline{l}^h_{i^{h'}} = \bar{p}^{h'}_{i^{h'}}$. Since $\underline{l}^h$ and $\underline{l}^{h'}$ only differ in the $i^{h'}$-th component, $\underline{l}^h = (\underline{l}^{h'}_{-i^{h'}}, \bar{p}^{h'}_{i^{h'}})$.

(C1.2) Assume $i^{h'} \in N^h_d$. Then, $a_{i^{h'}} = d$ and so, by the definition of $A_{i^{h'}}$, either $i^{h'} \in N^{h'}_d$ or $i^{h'} \notin N^{h'}$. In both cases, $\underline{l}^h_{i^{h'}} = \underline{l}^{h'}_{i^{h'}} = 0$. Since $\underline{l}^h$ and $\underline{l}^{h'}$ only differ in the $i^{h'}$-th component, $\underline{l}^h = \underline{l}^{h'}$. By definition, $\bar{l}^h_{i^{h'}} = \underline{p}^{h'}_{i^{h'}}$. Since $\bar{l}^h$ and $\bar{l}^{h'}$ only differ in the $i^{h'}$-th component, $\bar{l}^h = (\bar{l}^{h'}_{-i^{h'}}, \underline{p}^{h'}_{i^{h'}})$.

(C1.3) Assume $i^{h'} \in N^h_s$. Then, $a_{i^{h'}} \in \{\underline{p}^{h'}_{i^{h'}}, \ldots, \bar{p}^{h'}_{i^{h'}}\}$. Hence, $\bar{l}^h_{i^{h'}} = \underline{l}^h_{i^{h'}} = a_{i^{h'}}$. Since $\bar{l}^h$ and $\bar{l}^{h'}$ and $\underline{l}^h$ and $\underline{l}^{h'}$ differ only in the $i^{h'}$-th component, which is equal to $a_{i^{h'}}$, we have $\bar{l}^h = (\bar{l}^{h'}_{-i^{h'}}, a_{i^{h'}})$ and $\underline{l}^h = (\underline{l}^{h'}_{-i^{h'}}, a_{i^{h'}})$. □

Lemma 3 *Let* $h = (h', a_{i^{h'}}) \in Hl$ *be the output of* **Step A.t** *and assume* $N^h \neq N$. *Then,*

(3.1) $\sum_{i \in N} \bar{l}^h_i \geq k$,

(3.2) $\bar{p}^{h'}_i \leq \bar{p}^h_i$ *for all* $i \neq i^{h'}$,

(3.3) *if* $i^{h'} \in N^h_s$, $a_{i^{h'}} = \bar{p}^h_{i^{h'}}$,

(3.4) *if* $i^{h'} \in N^h_u$, $\bar{p}^{h'}_{i^{h'}} = \bar{p}^h_{i^{h'}}$,

(3.5) *if* $i \in N^h_s \backslash \{i^{h'}\}$, $a_{i^{\widehat{h}}} = \bar{p}^{h'}_{i^{\widehat{h}}} = \bar{p}^h_{i^{\widehat{h}}}$ *where* $\widehat{h}$ *is such that* $i = i^{\widehat{h}}$,

(3.6) $\sum_{i \in N} \underline{l}^h_i \leq k$,

(3.7) $\underline{p}^h_i \leq \underline{p}^{h'}_i$ *for all* $i \neq i^{h'}$,

(3.8) *if* $i^{h'} \in N^h_s$, $a_{i^{h'}} = \underline{p}^h_{i^{h'}}$,

(3.9) *if* $i^{h'} \in N^h_d$, $\underline{p}^{h'}_{i^{h'}} = \underline{p}^h_{i^{h'}}$,

(3.10) *if* $i \in N^h_s \backslash \{i^{h'}\}$, $a_{i^{\widehat{h}}} = \underline{p}^{h'}_{i^{\widehat{h}}} = \underline{p}^h_{i^{\widehat{h}}}$ *where* $\widehat{h}$ *is such that* $i = i^{\widehat{h}}$.

Proof of Lemma 3

(3.1) Since $\bar{l}^h_i = k$ for all $i \in N \backslash N^h \neq \emptyset$, the definition of $\bar{l}^h$ implies

$$\sum_{i \in N} \bar{l}^h_i \geq k.$$

(3.2) Let $i \neq i^{h'}$. By definition of $i^{h'}$, $i^{h'} \notin N^{h'}_s$. We distinguish between two cases.

<u>Case 1</u>*:* Assume $i^{h'} \in N^{h'}_u \cup (N \backslash N^{h'})$. Then, $\bar{l}^{h'}_{i^{h'}} = k \geq \bar{l}^h_{i^{h'}}$. Since, by Claim 1, $\bar{l}^h_{-i^{h'}} = \bar{l}^{h'}_{-i^{h'}}$ holds, by replacement monotonicity and the definitions of $\bar{p}^h$ and $\bar{p}^{h'}$,

$$\bar{p}^{h'}_i \leq \bar{p}^h_i \text{ for all } i \neq i^{h'}.$$

Case 2: Assume $i^{h'} \in N_d^{h'}$. Then, $i^{h'} \in N_d^h \cup N_s^h$ and by definition of $\bar{l}^h_{i^{h'}}$,

$$\bar{l}^h_{i^{h'}} = \begin{cases} \underline{p}^{h'}_{i^{h'}} & \text{if } i^{h'} \in N_d^h \\ a_{i^{h'}} & \text{if } i^{h'} \in N_s^h. \end{cases}$$

Hence, by efficiency and (3.1), $\bar{l}^h_{i^{h'}} \leq \overline{p}^{h'}_{i^{h'}} = \Phi_{i^{h'}}(\bar{l}^{h'}) \leq \bar{l}^{h'}_{i^{h'}}$. Since, by Claim 1, $\bar{l}^h_{-i^{h'}} = \bar{l}^{h'}_{-i^{h'}}$ holds, by replacement monotonicity and the definitions of $\overline{p}^h$ and $\overline{p}^{h'}$,

$$\overline{p}^{h'}_i \leq \overline{p}^h_i \text{ for all } i \neq i^{h'}.$$

(3.3) By the definition of $\bar{l}^h$, $i^{h'} \in N_s^h$ implies $\bar{l}^h_{i^{h'}} = a_{i^{h'}}$. By efficiency of Φ and (3.1),

$$\overline{p}^h_{i^{h'}} = \Phi_{i^{h'}}(\bar{l}^h) \leq a_{i^{h'}}. \tag{42}$$

By Claim 1 and since $a_{i^{h'}}$ belongs to $\{\underline{p}^{h'}_{i^{h'}}, \ldots, \overline{p}^{h'}_{i^{h'}}\}$, $\bar{l}^h_{-i^{h'}} = \bar{l}^{h'}_{-i^{h'}}$, $\bar{l}^h_{i^{h'}} = a_{i^{h'}} \leq \overline{p}^{h'}_{i^{h'}} = \Phi_{i^{h'}}(\bar{l}^{h'}) \leq \bar{l}^{h'}_{i^{h'}}$, where the last inequality holds by efficiency and (3.1). By strategy-proofness,

$$\overline{p}^h_{i^{h'}} = \Phi_{i^{h'}}(\bar{l}^h) \geq a_{i^{h'}}. \tag{43}$$

By (42) and (43),

$$a_{i^{h'}} = \overline{p}^h_{i^{h'}}.$$

(3.4) Assume $i^{h'} \in N_u^h$. By Claim 1, $\bar{l}^{h'} = \bar{l}^h$ and so $\overline{p}^{h'} = \overline{p}^h$.

(3.5) Let $i \in N_s^h$ and $i \neq i^{h'}$. Let $\widehat{h}$ be the history at which i has played for the last time before h and let $\overline{h}$ be the immediate successor of $\widehat{h}$ on the path towards h (*i.e.*, $\widehat{h} \prec^{im} \overline{h} \preceq^{im} \cdots \preceq^{im} h' \prec^{im} h$). Then, $i = i^{\widehat{h}} \in N_s^{\overline{h}}$. Therefore, by (3.3),

$$a_{i^{\widehat{h}}} = \overline{p}^{\overline{h}}_{i^{\widehat{h}}}. \tag{44}$$

By (3.2), and since for any history h^* such that $\overline{h} \prec h^* \prec h$, $i^{\widehat{h}} \in N_s^{h^*}$ and $i^{\widehat{h}} \neq i^{h^*}$,

$$\underline{l}^h_{i^{\widehat{h}}} = \bar{l}^h_{i^{\widehat{h}}} = a_{i^{\widehat{h}}} = \overline{p}^{\overline{h}}_{i^{\widehat{h}}} \leq \overline{p}^{h'}_{i^{\widehat{h}}} \leq \overline{p}^h_{i^{\widehat{h}}}, \tag{45}$$

where the first equality follows from the definitions of $\underline{l}^h$ and $\bar{l}^h$, the second equality from (44), and the two inequalities follow from (3.2), successively applied in the case of the first inequality. Furthermore, by efficiency of Φ and (3.1),

$$\underline{l}^h_{i^{\widehat{h}}} = \bar{l}^h_{i^{\widehat{h}}} \geq \Phi_{i^{\widehat{h}}}(\bar{l}^h) = \overline{p}^h_{i^{\widehat{h}}}.$$

Therefore, by (45),

$$a_{i^{\widehat{h}}} = \overline{p}^{h'}_{i^{\widehat{h}}} = \overline{p}^h_{i^{\widehat{h}}}.$$

(3.6) By definition of $\underline{l}^h$, we have that, for all $i \in N_d^h \cup (N \backslash N^h)$,

$$\underline{l}^h_i = 0. \tag{46}$$

Let $i \in N_u^h$. We distinguish between two cases. First, assume there exists $\widehat{h} \prec h'$ such that $i \in N_u^{\widehat{h}}$, $i \in N_u^{h^*}$ for all $\widehat{h} \prec h^* \preceq h$ and i does not play at h^*. Then,

$$\underline{l}_i^h = \overline{p}_i^{\widehat{h}} \leq \overline{p}_i^h, \tag{47}$$

where the equality follows from the definition of $\underline{l}^h$ and the inequality from (3.4) and (3.2). Assume now that $\widehat{h} = h'$ (because $i = i^{h'}$). Then, by the definition of $\underline{l}^h$ and (3.4),

$$\underline{l}_i^h = \overline{p}_i^{h'} = \overline{p}_i^h. \tag{48}$$

Let $i \in N_s^h$. We distinguish between two cases. First, assume $i \neq i^{h'}$ and let $\widehat{h}$ be such that $i^{\widehat{h}} = i$. Then,

$$\underline{l}_i^h = a_{i^{\widehat{h}}} = \overline{p}_i^h, \tag{49}$$

where the first equality follows from the definition of $\underline{l}^h$ and the second one from (3.5). Assume now that $i = i^{h'}$. Then, since $\widehat{h} = h'$, the definition of $\underline{l}^h$ and (3.3) imply that

$$\underline{l}_i^h = a_{i^{h'}} = \overline{p}_i^h. \tag{50}$$

Therefore, by (46) to (50), for all $i \in N$,

$$\underline{l}_i^h \leq \overline{p}_i^h.$$

Hence, by feasibility of $\overline{p}^h$,

$$\sum_{i \in N} \underline{l}_i^h \leq k.$$

The proofs of (3.7), (3.8), (3.9) and (3.10) are symmetric to the proofs of (3.2), (3.3), (3.4) and (3.5), and so they are omitted. ■

Lemma 4 *Let $h = (h', a_{i^{h'}}) \in Hl$ be the output of* **Step A.t** *and assume $N^h \neq N$. Then,*
(4.1) if $i^{h'} \in N_u^h$, $\underline{p}_{i^{h'}}^h = \overline{p}_{i^{h'}}^h = \overline{p}_{i^{h'}}^{h'}$,
(4.2) if $i^{h'} \in N_d^h$, $\underline{p}_{i^{h'}}^h = \overline{p}_{i^{h'}}^h = \underline{p}_{i^{h'}}^{h'}$,
(4.3) if $i \in N_s^h$, $\underline{p}_i^h = \overline{p}_i^h = a_{i^{\widehat{h}}}$ where $\widehat{h}$ is such that $i = i^{\widehat{h}}$,
(4.4) if $i \in N_d^{h'} \setminus \{i^{h'}\}$, then $\overline{p}_i^h = \overline{p}_i^{h'}$,
(4.5) if $i \in N_u^{h'} \setminus \{i^{h'}\}$, then $\underline{p}_i^h = \underline{p}_i^{h'}$.

Proof of Lemma 4

(4.1) Let $i^{h'} \in N_u^h$. Then, by (C1.1) in Claim 1,

$$\overline{l}^h = \overline{l}^{h'} \text{ and } \underline{l}^h = (\underline{l}_{-i^{h'}}^{h'}, \overline{p}_{i^{h'}}^{h'}). \tag{51}$$

By definition of $\overline{p}^h$ and $\overline{p}^{h'}$,

$$\overline{p}^h = \overline{p}^{h'}. \tag{52}$$

By (3.6) in Lemma 3, (51), and efficiency of Φ,

$$\Phi_{i^{h'}}(\underline{l}^h) \geq \overline{p}^{h'}_{i^{h'}}.$$

If $\Phi_{i^{h'}}(\underline{l}^h) > \overline{p}^{h'}_{i^{h'}}$, then

$$\Phi_{i^{h'}}(\underline{l}^{h'}_{-i^{h'}}, \overline{p}^{h'}_{i^{h'}}) = \Phi_{i^{h'}}(\underline{l}^h) > \overline{p}^{h'}_{i^{h'}} > \underline{p}^{h'}_{i^{h'}} = \Phi_{i^{h'}}(\underline{l}^{h'}),$$

where the second strict inequality follows from the definition of $i^{h'}$. But, by single-peakedness, it contradicts strategy-proofness of Φ. Then, by (52) and the definition of $\underline{p}^h$,

$$\underline{p}^h_{i^{h'}} = \Phi_{i^{h'}}(\underline{l}^h) = \overline{p}^{h'}_{i^{h'}} = \overline{p}^h_{i^{h'}},$$

which is the statement in (4.1).

(4.2) The proof proceeds as in (4.1), using symmetric arguments.

(4.3) Let $i \in N^h_s$ and let $\widehat{h}$ be such that $i = i^{\widehat{h}}$. If $\widehat{h} \neq h'$ the proof follows from (3.5) and (3.10) in Lemma 3. If $\widehat{h} = h'$ the proof follows from (3.3) and (3.8) in Lemma 3.

(4.4) The proof is by induction on the length of the histories.

Assume that $h = (\emptyset, a_{i\emptyset})$ is a history of length 1. Then, $h' = \emptyset$ and $N^{h'}_d = \emptyset$. Therefore, (4.4) holds trivially.

Assume that (4.4) holds for all $\overline{h} \prec h$. We prove that it holds for h.

Let $i \in N^{h'}_d \backslash \{i^{h'}\}$. Then, by the definition of $\overline{l}^{h'}$, $\overline{l}^{h'}_i = \underline{p}^{\widehat{h}'}_i$ where $\widehat{h}'$ is such that $i = i^{\widehat{h}'}$. Let $\overline{h}$ be such that $\overline{h}' = \widehat{h}'$ and $\overline{h} \prec h$ (*i.e.*, $\widehat{h}' \prec^{im} \overline{h} \prec h$). By (4.2),

$$\underline{p}^{\widehat{h}'}_i = \underline{p}^{\overline{h}'}_i = \underline{p}^{\overline{h}}_i = \overline{p}^{\overline{h}}_i. \tag{53}$$

By the induction hypothesis and the definition of $\overline{h}$, and since $i \in N^{h^*}_d$ and $i \neq i^{h^*}$ for all $\overline{h} \preceq h^* \preceq h'$,

$$\overline{p}^{\overline{h}}_i = \overline{p}^{h'}_i. \tag{54}$$

Hence,

$$\overline{l}^h_i = \overline{l}^{h'}_i = \overline{p}^{\overline{h}}_i = \overline{p}^{h'}_i, \tag{55}$$

where the first equality follows from $i \neq i^{h'}$, the second follows from (53) and $\overline{l}^{h'}_i = \underline{p}^{\widehat{h}'}_i$, and the third from (54). Therefore, by (3.2) in Lemma 3, efficiency of Φ and (3.1) in Lemma 3, and (55),

$$\overline{p}^{h'}_i \leq \overline{p}^h_i \leq \overline{l}^h_i = \overline{p}^{h'}_i.$$

Thus,

$$\overline{p}^h_i = \overline{p}^{h'}_i.$$

(4.5) The proof proceeds as in (4.4), using symmetric arguments. ■

Lemma 5 *Let $h \in Hl$ be the output of* **Step A.t**. *Then, $N^h \neq N$.*

Proof of Lemma 5 If $h = \emptyset$ the statement follows trivially. Assume that h is a non-empty history and, to obtain a contradiction, that $N^h = N$. Then, by the definition of **Stage A**, there exists $\overline{h} \prec h$ such that $N^{\overline{h}} = N_u^{\overline{h}} \cup N_s^{\overline{h}} \cup N_d^{\overline{h}} = N \backslash \{i^{\overline{h}}\}$. By definition of $i^{\overline{h}}$

$$\underline{p}_{i^{\overline{h}}}^{\overline{h}} < \overline{p}_{i^{\overline{h}}}^{\overline{h}}.$$

Without loss of generality, we can assume that agent $i^{\overline{h}}$ is not called to play between $\overline{h}$ and h. Then, since $\overline{p}^{\overline{h}}$ is feasible, there exists $i \neq i^{\overline{h}}$ such that

$$\overline{p}_i^{\overline{h}} < \underline{p}_i^{\overline{h}}.$$

Since $N_u^{\overline{h}} \cup N_s^{\overline{h}} \cup N_d^{\overline{h}} = N/\{i^{\overline{h}}\}$, $i \in N_u^{\overline{h}} \cup N_s^{\overline{h}} \cup N_d^{\overline{h}}$. We can apply now Lemmata 3 and 4 to $\overline{h}$, because $N^{\overline{h}} \neq N$.

<u>Case 1</u>: If $i \in N_s^{\overline{h}}$ we obtain a contradiction with (4.3).

<u>Case 2</u>: If $i \in N_u^{\overline{h}}$ we obtain a contradiction with either (4.1) or (4.1), (4.5) and (3.2).

<u>Case 3</u>: If $i \in N_d^{\overline{h}}$ we obtain a contradiction with either (4.2) or (4.2), (4.4) and (3.7). ■

Observe that by Lemma 5 the hypothesis that $N^h \neq N$ in Lemmata 3 and 4 is without loss of generality. Hence, those two lemmata apply to any history in **Stage A**, or output of **Stage A**. Note that in the proof of Lemma 5 we have used Lemmata 3 and 4, applied to history $\overline{h}$, where $N^{\overline{h}} \neq N$.

Lemma 6 *Let h be a history such that $h \in Hm \cap Hl$. Then,*

$$\overline{l}^h = \overline{m}^h \text{ and } \underline{l}^h = \underline{m}^h.$$

Moreover, $Hm = Hl$.

Proof of Lemma 6 The proof is by induction on the length of the histories. The induction hypothesis is that for all $t \geq 1$, the set of histories of length t in Hm and Hl coincide and if $h \in Hm$ has length t, then

$$\overline{l}^h = \overline{m}^h \text{ and } \underline{l}^h = \underline{m}^h.$$

If $t = 1$, then the induction hypothesis holds trivially from the definitions.

Assume that the induction hypothesis holds for all $t < \bar{t}$. We will prove that it holds for $\bar{t}$.

Let $h = (h', a_{i^{h'}})$ be a history of length $\bar{t}$ in Hm (or Hl). Since h' has length $\bar{t} - 1$, by the induction hypothesis, h' is a history in Hm and Hl and

$$\overline{l}^{h'} = \overline{m}^{h'} \text{ and } \underline{l}^{h'} = \underline{m}^{h'},$$

which means that

$$\overline{q}^{h'} = \overline{p}^{h'} \text{ and } \underline{q}^{h'} = \underline{p}^{h'}.$$

Then, $h = (h', a_{i^{h'}})$ is a history of length $\bar{t}$ in Hl (or Hm).

Now, we prove that $\bar{l}^h = \overline{m}^h$ holds.

If $i \in N_u^h \cup (N \backslash N^h)$, $\bar{l}^h = \overline{m}^h$ holds by their definitions.

If $i \in N_s^h \backslash \{i^{h'}\}$, by (3.5) in Lemma 3 and Lemma 5, $a_{i^{\widehat{h}}} = \overline{p}_{i^{\widehat{h}}}^{h'}$ where $i = i^{\widehat{h}}$. Therefore,

$$\overline{m}_i^h = \overline{q}_i^{h'} = \overline{p}_i^{h'} = a_{i^{\widehat{h}}} = \bar{l}_i^h.$$

If $i \in N_d^h \backslash \{i^{h'}\}$, $\overline{q}_i^{h'} = \overline{p}_i^{h'}$ by the induction hypothesis. Moreover, $i \in N_d^{h'} \backslash \{i^{h'}\}$ and by the definition of $\widehat{h}$, $i = i^{\widehat{h}}$ and $i \in N_d^{\widehat{h}} \backslash \{i^{h'}\}$. By (4.4) in Lemma 4 and Lemma 5, $\overline{p}_i^h = \overline{p}_i^{h'}$. By (4.2) in Lemma 4 applied to $i = i^{\widehat{h}}$, $\overline{p}_{i^{\widehat{h}}}^{h'} = \underline{p}_{i^{\widehat{h}}}^{\widehat{h}}$. Therefore,

$$\overline{m}_i^h = \overline{q}_i^{h'} = \overline{p}_i^{h'} = \underline{p}_i^{\widehat{h}} = \bar{l}_i^h.$$

If $i = i^{h'}$ and $i \in N_s^h$, then $\widehat{h} = h'$ and, by their definitions,

$$\overline{m}_i^h = a_{i^{h'}} = a_{i^{\widehat{h}}} = \bar{l}_i^h.$$

If $i = i^{h'}$ and $i \in N_d^h$, then $\widehat{h} = h'$ and, by their definitions,

$$\overline{m}_i^h = \underline{q}_i^{h'} = \underline{p}_i^{h'} = \underline{p}_{i^{\widehat{h}}}^{\widehat{h}} = \bar{l}_i^h.$$

Therefore,

$$\bar{l}^h = \overline{m}^h.$$

The proof that $\underline{l}^h = \underline{m}^h$ holds as well proceeds as the proof of $\bar{l}^h = \overline{m}^h$, using symmetric arguments.

Therefore, the sets of actions available to agent $i^{h'}$ if we use $\bar{l}$ and $\underline{l}$ or $\overline{m}$ and $\underline{m}$ coincide. Hence, $h = (h', a_{i^{h'}})$ is the same history and belongs to Hl and Hm. ■

Lemma 7 *Let $\overline{h} \prec h$ be two histories in* **Stage A** *(h may be itself the output of* **Stage A***). Then,*

(7.1) *if $i \in N_u^{\overline{h}}$ and $i = i^{\overline{h}'} = i^{h'}$, $\overline{q}_i^{\overline{h}} < \overline{q}_i^h$,*

(7.2) *if $i \in N_d^{\overline{h}}$ and $i = i^{\overline{h}'} = i^{h'}$ $\underline{q}_i^{\overline{h}} > \underline{q}_i^h$,*

(7.3) *if $i \in N_s^{\overline{h}}$, $\underline{q}_i^{\overline{h}} = \overline{q}_i^{\overline{h}} = \underline{q}_i^h = \overline{q}_i^h$.*

Proof of Lemma 7

(7.1) Assume $i \in N_u^{\overline{h}}$ and $i = i^{\overline{h}'} = i^{h'}$. Without loss of generality we can assume that $i = i^{\overline{h}'}$ does not play between $\overline{h}'$ and h'. By Lemma 6,

$$\underline{q}_i^{\overline{h}} = \overline{q}_i^{\overline{h}} = \underline{q}_i^{h'} < \overline{q}_i^{h'} = \overline{q}_i^h,$$

where the first equality follows from (4.1) in Lemma 4 and $i = i^{\overline{h}'}$, the second follows from iterate application of (4.5) in Lemma 4 (if needed) and $i \neq i^{\widetilde{h}}$ for all $\overline{h} \prec \widetilde{h} \prec h'$ (if any), the strict inequality follows from the definition of $i^{h'}$ and $i = i^{h'}$ and the last equality follows from (4.1) in Lemma 4 and $i = i^{h'}$.

(7.2) The proof is symmetric to the proof of (7.1).

(7.3) The proof follows from (4.3) in Lemma 4. □

Lemma 8 *Let h be the output of* **Stage A** *and let $h^* \preceq h$. Then,*

(8.1) $q_i^h = a_{i\widehat{h}}$ *for all $i \in N_s^h$ where $\widehat{h}$ is such that $i = i^{\widehat{h}}$,*

(8.2) $q_i^h = \overline{q}_i^{\widehat{h}}$ *for all $i \in N_u^h$ where $\widehat{h}$* is such that $i = i^{\widehat{h}}$,

(8.3) $q_i^h = \underline{q}_i^{\widehat{h}}$ *for all $i \in N_d^h$ where $\widehat{h}$* is such that $i = i^{\widehat{h}}$,

(8.4) $q_i^h \geq \overline{q}_i^{h^*}$ *for all $i \in N_u^h \cup N_s^h$ where $i = i^{h^*}$ and $a_{i^{h^*}} = u$,*

(8.5) $q_i^h \leq \underline{q}_i^{h^*}$ *for all $i \in N_d^h \cup N_s^h$ where $i = i^{h^*}$ and $a_{i^{h^*}} = d$.*

Proof of Lemma 8

(8.1) The proof follows from Lemma 6 and (4.3) in Lemma 4.

(8.2) Let $i = i^{\widehat{h}} \in N_u^h$ and let $\overline{h}$ be such that $\widehat{h} \prec^{im} \overline{h} \preceq h$. Since $i^{\widehat{h}} \in N_u^h$ and $i^{\widehat{h}} \in N_u^{\overline{h}}$, by (4.1) and (4.5) in Lemma 4, and by Lemma 6,

$$\overline{q}_i^{\widehat{h}} = \overline{q}_i^{\overline{h}} = \underline{q}_i^{\overline{h}} = \underline{q}_i^{h}.$$

Now, since h is the output of **Stage A**, $\underline{q}_i^h = q_i^h$. Hence,

$$q_i^h = \overline{q}_i^{\widehat{h}}.$$

(8.3) The proof proceeds as the proof of (8.2), using symmetric arguments.

(8.4) Let $i \in N_u^h \cup N_s^h$, where $i = i^{h^*}$ and $a_{i^{h^*}} = u$. We distinguish between the two sets to which i can belong to.

<u>Case 1</u>: $i \in N_u^h$. By (8.1), we can assume without loss of generality that h^* is the last history at which agent i has played; namely, $h^* = \widehat{h}$. Then, $q_i^h = \overline{q}_i^{h^*}$ follows from (8.2) in Lemma 8.

<u>Case 2</u>: $i \in N_s^h$. Then, since $i = i^{h^*}$ and $a_{i^{h^*}} = u$, $h^* \prec \widehat{h} \prec h$. Consider $\overline{h}$ such that $h^* \prec^{im} \overline{h} \preceq \widehat{h}$. Then,

$$\overline{q}_i^{h^*} = \overline{q}_i^{\overline{h}} = \underline{q}_i^{\overline{h}} \leq \underline{q}_i^{\widehat{h}},$$

where the first two equalities follow from (4.1) in Lemma 4, $i = i^{h^*}$ and Lemma 6, and the inequality follows directly from Lemma 6 and (4.5) in Lemma 4 if i does not play between h^* and $\widehat{h}$, and from (4.5) in Lemma 4 and (8.1) in Lemma 8 otherwise (perhaps, after applying them iteratively). Since $i \in N_s^h$, by Lemma 6 and (4.3) in Lemma 4,

$$q_i^h = a_i^{\widehat{h}} \in \{\underline{q}_i^{\widehat{h}}, \ldots, \overline{q}_i^{\widehat{h}}\}.$$

Thus,

$$\overline{q}_i^{h^*} \leq q_i^h.$$

(8.5) The proof proceeds as the proof of (8.4), using symmetric arguments. □

Proof of Theorem 2 It follows from the three statements that we will present and prove successively. Along the proof we will use the statement of Lemma 6.

Statement 1.2 *Let Φ be a sequential allotment rule. Then, the extensive game form Γ^{Φ} is well defined and finite.*

Proof of Statement 1.2 By Statement 1.1 it will be sufficient to show that **Stage A** of the game is well defined and finite. We first argue that the agents that are called to play along **Stage A** are uniquely identified and well defined.

Consider **Step A.t**, with $\mathbf{t} \geq 0$, with input h' and $\underline{q}^{h'} \neq \overline{q}^{h'}$. If $\mathbf{t}=0$ and so $h' = \emptyset$, $i^{\emptyset} = \min_{<}\{i \in N \mid \underline{q}_i^{\emptyset} < \overline{q}_i^{\emptyset}\}$ is well defined. If $\mathbf{t} \geq 1$ and so $h' \neq \emptyset$, $\underline{m}^{h'} \neq \overline{m}^{h'}$. Hence, $N \backslash N_s^{h'} \neq \emptyset$. Since $\underline{q}^{h'}$ and $\overline{q}^{h'}$ are feasible, $\sum_{i \in N} \underline{q}_i^{h'} = \sum_{i \in N} \overline{q}_i^{h'} = k$. Hence, by (7.3) in Lemma 7, there exists at least one $i \in N \backslash N_s^{h'}$ such that $\underline{q}_i^{h'} < \overline{q}_i^{h'}$, and so $i^{h'} = \min_{<}\{i \in N \backslash N_s^{h'} \mid \underline{q}_i^{h'} < \overline{q}_i^{h'}\}$ is well defined.

As in the proof of Statement 1.1, it is easy to see that the sets of agents N_u^h, N_d^h and N_s^h are well defined. The proof that **Stage A** is finite follows from Lemma 8 and the following two facts: (i) if $\overline{q}_i^{h'} = k$ and $i = i^{h'} \in N_u^{h'}$, then $i \in N_s^h$ and (ii) if $\underline{q}_i^{h'} = 0$ and $i = i^{h'} \in N_d^{h'}$, then $i \in N_s^h$. ■

We now proceed to state and prove that Γ^{Φ} OSP-implements Φ.

Fix Φ and let Γ^{Φ} be the game defined in Section 4. We decompose a behavioral strategy profile σ in Γ^{Φ} into $\sigma = (\sigma_A, \sigma_B)$, where σ_A is the restriction of σ to histories in **Stage A** and σ_B is the restriction of σ to histories in **Stage B**.

Statement 2.2 *Let Φ be a sequential allotment rule. Then, Γ^{Φ} induces Φ; namely, for all $R \in \mathcal{R}^n$,*

$$\Phi(R) = o(h^{\Gamma}(\emptyset, \sigma^R)).$$

Proof of Statement 2.2 Let $R \in \mathcal{R}^n$ be arbitrary and let $\tau = (\tau(R_1), \ldots, \tau(R_n)) \in \{0, \ldots, k\}^n$ be the profile of tops at R. Let $h^{A\tau}$ be the output of **Stage A** under σ_A^R and set $A\tau := h^{A\tau}$, so that $q^{A\tau} := q^{h^{A\tau}}$. Then, by definition of σ_A^R, $\tau_i = q_i^{A\tau}$ for all $i \in N_s^{A\tau}$, $\tau_i > q_i^{A\tau}$ for all $i \in N_u^{A\tau}$ and $\tau_i < q_i^{A\tau}$ for all $i \in N_d^{A\tau}$. Hence, $R \in \mathcal{D}^{A\tau}$, where $\mathcal{D}^{A\tau}$ is defined as in Remark 1. Thus, $\tau \in D^{A\tau}$ and

$$\Phi(R) = \Phi^{A\tau}(R) = o(h^{\Gamma^{\Phi^{A\tau}}}(h^{A\tau}, \sigma_B^R)) = o(h^{\Gamma}(\varnothing, \sigma^R)),$$

where the second equality follows from Remark 1 and Statement 2.1, after identifying $h^{A\tau}$ with the input history of **Stage B**. □

Statement 3.2 *Let Φ be a sequential allotment rule and let $R \in \mathcal{R}^n$ be a profile. Then, for all $i \in N$, the truth-telling strategy $\sigma_i^{R_i}$ is weakly dominant in Γ^{Φ}.*

Proof of Statement 3.2 Consider agent i with preferences R_i and top allotment τ_i. Let $\sigma_i^{R_i}$ be i's truth-telling strategy relative to R_i and let σ_i' be any other strategy. We want to show that, for all σ_{-i},

$$x_i = o_i(h^{\Gamma^\Phi}(\emptyset, (\sigma_i^{R_i}, \sigma_{-i}))R_i o_i(h^{\Gamma^\Phi}(\emptyset, (\sigma'_i, \sigma_{-i})) = x'_i \tag{56}$$

holds. Let σ_{-i} be arbitrary. Condition (56) holds trivially if $x_i = x'_i$. Assume $x_i \neq x'_i$. Let h' be the earliest history at which $\sigma_i^{R_i}(h') \neq \sigma'_i(h')$ along the equal play induced by both $(\sigma_i^{\tau_i}, \sigma_{-i})$ and (σ_i, σ_{-i}) up to h'. We proceed by distinguishing among several cases, depending on the step of the algorithm for which the history h' is an input of, and agent i is called to play at h'.

Case 1: The history h' is an input of some **Step A.t** (*i.e.*, $i^{h'} = i$). Then, $\underline{q}_i^{h'} < \overline{q}_i^{h'}$. We distinguish among three subcases.

Subcase 1.1: $\underline{q}_i^{h'} \leq \tau_i \leq \overline{q}_i^{h'}$. Then, $\sigma_i^{R_i}(h') = \tau_i$. Hence, $i \in N_s^h$ for all $h \succeq (h', \sigma_i^{R_i}(h'))$, i is not called to play anymore, and $x_i = \tau_i$. Thus, (56) holds.

Subcase 1.2: $\overline{q}_i^{h'} < \tau_i$. Then, $\sigma_i^{R_i}(h') = u$. Let $\bar{h}$ be the output of **Stage A** when $(\sigma_i^{\tau_i}, \sigma_{-i})$ is played starting at h'. Therefore, $(h', \sigma_i^{R_i}(h')) \preceq \bar{h}$ and $i \in N_u^{\bar{h}} \cup N_s^{\bar{h}}$. By (8.4) in Lemma 8, $\overline{q}_i^{h'} \leq q_i^{\bar{h}}$. Then, $\overline{q}_i^{h'} \leq q_i^{\bar{h}} \leq q_i^{h^{\Gamma^\Phi}(h', (\sigma_i^{R_i}, \sigma_{-i}))} = x_i$,[22] where the last inequality holds because $q_i^{\widetilde{h}'} \leq q_i^{\widetilde{h}}$ for all $i \in N_u^{\widetilde{h}'} \cup N_s^{\widetilde{h}'}$ if $\widetilde{h}'$ is a history in **Stage B**. Since $\sigma_i^{R_i}(h') = u$, $i \in N_u^{h^{\Gamma^\Phi}(h', (\sigma_i^{R_i}, \sigma_{-i}))} \cup N_s^{h^{\Gamma^\Phi}(h', (\sigma_i^{R_i}, \sigma_{-i}))}$. Therefore, by the definition of $\sigma_i^{R_i}$, $x_i \leq \tau_i$. Thus,

$$\overline{q}_i^{h'} \leq x_i \leq \tau_i. \tag{57}$$

On the other hand, and since $\sigma'_i(h') \neq \sigma_i^{R_i}(h')$, we have $\sigma'_i(h') \in \{\underline{q}_i^{h'}, \ldots, \overline{q}_i^{h'}\}$ or $\sigma'_i(h') = d$. Let $\widetilde{h}$ be the output of **Stage A** when (σ'_i, σ_{-i}) is played starting at h'. Therefore, $(h', \sigma'_i(h')) \preceq \widetilde{h}$, and $i \in N_d^{\widetilde{h}} \cup N_s^{\widetilde{h}}$. Then, by (8.1) and (8.5) in Lemma 8, $q_i^{\widetilde{h}} \leq \underline{q}_i^{h'} \leq \overline{q}_i^{h'}$. Then,

$$x'_i = q_i^{h^{\Gamma^\Phi}(h', (\sigma_i^{R_i}, \sigma_{-i}))} \leq q_i^{\widetilde{h}} \leq \overline{q}_i^{h'}, \tag{58}$$

where the first inequality holds because $q_i^{\widetilde{h}} \leq q_i^{\widetilde{h}'}$ for all $i \in N_u^{\widetilde{h}'} \cup N_s^{\widetilde{h}'}$ if $\widetilde{h}'$ is a history in **Stage B**. Therefore, by (57) and (58),

$$x'_i \leq \overline{q}_i^{h'} \leq x_i \leq \tau_i.$$

Therefore, by single-peakedness, $x_i R_i x'_i$. Thus, (56) holds.

Subcase 1.3: $\tau_i < \underline{q}_i^{h'}$. With a symmetric argument to the one used in Subcase 1.2, we obtain that $\tau_i \leq x_i \leq \underline{q}_i^{h'} \leq x'_i$. By single-peakedness, $x_i R_i x'_i$. Thus, (56) holds.

Case 2: The history h' is an input of some step at **Stage B**. In this case the proof is as the one used to prove Statement 3.1. ■

[22] Remember that $h^{\Gamma^{GF}}(h', (\sigma_i^{R_i}, \sigma_{-i}))$ denotes the terminal history that follows when agents play $(\sigma_i^{R_i}, \sigma_{-i})$ in the game Γ^Φ starting at h'.

7 Appendix 2: Definition of Sequential Allotment Rules

We define the class of sequential allotment rules for the discrete division problem as the natural extension of its definition for the continuous division problem, presented in Barberà, Jackson and Neme (1997). A sequential allotment rule uses reference allotments to sequentially allocate the good similar to how the uniform allocation rule does with the egalitarian allotment as reference.[23] We closely follow the description of Barberà, Jackson and Neme (1997), where the reader will find detailed explanations of a sequential allotment rule.

Definition 2 The function $g : X \times \mathcal{R}^n \to X \times \mathcal{R}^n$ is a *sequential adjustment function* relative to $q^L \in X$ and $q^H \in X$ if the following are true for any $(q^t, R) \in X \times \mathcal{R}^n$ such that $(q^t, R) = g(q^{t-1}, R) =: g^t(q^0, R)$ for some $t \geq 1$,[24] where $q^0 = q^H$ if $\sum_{j \in N} \tau(R_j) \geq k$ and $q^0 = q^L$ if $\sum_{j \in N} \tau(R_j) < k$:
(i) $q_i^t = \tau(R_i)$ if $(k - \sum_{j \in N} \tau(R_j)) \left(q_i^{t-1} - \tau(R_i)\right) \leq 0$.
(ii) $\left(q_i^t - q_i^{t-1}\right)(k - \sum_{j \in N} \tau(R_j)) \leq 0$ if $(k - \sum_{j \in N} \tau(R_j)) \left(q_i^{t-1} - \tau(R_i)\right) > 0$.
(iii) If $\min\{\tau(R_i'), \tau(R_i)\} > q_i^{t-1}$ and $\sum_{j \in N} \tau(R_j) \geq k$ or $\max\{\tau(R_i'), \tau(R_i)\} < q_i^{t-1}$ and $\sum_{j \in N} \tau(R_j) < k$, then $g(q^{t-1}, R) = g(q^{t-1}, (R_i', R_{-i}))$.
(iv) Let $(q'^n, (R_i', R_{-i})) = g^n(q^0, (R_i', R_{-i}))$ and $(q^n, R) = g^n(q^0, R))$. Then,
if $\tau(R_i') \leq \tau(R_i)$ and $\sum_{j \in N} \tau(R_j) \geq k$, then $q_{i'}'^n \geq q_{i'}^n$ for $i' \neq i$,
if $\tau(R_i') \geq \tau(R_i)$ and $\sum_{j \in N} \tau(R_j) < k$, then $q_{i'}'^n \leq q_{i'}^n$ for $i' \neq i$.

Definition 3 A rule $\Phi : \mathcal{R}^n \to X$ is a *sequential allotment rule* if there exist $q^L, q^H \in X$ and a sequential adjustment function $g : X \times \mathcal{R}^n \to X \times \mathcal{R}^n$ relative to q^L, q^H such that:

$$(\Phi(R), R) = \begin{cases} g^n(q^H, R) & \text{if } \sum_{j \in N} \tau(R_j) \geq k \\ g^n(q^L, R) & \text{if } \sum_{j \in N} \tau(R_j) < k. \end{cases}$$

A sequential allotment rule follows a procedure of at most n steps, where each step uses two reference allotments that may differ, depending on whether the sum of the tops is smaller or larger than the amount to be allotted. Along the procedure, the reference allotments evolve according to the iterative application of the sequential adjustment function g.

Let $\Phi : \mathcal{R}^n \to X$ be a sequential allotment rule and let $R \in \mathcal{R}^n$. Then, the allotment $\Phi(R)$ can be obtained as the outcome of the following algorithm.

Step 1: *Input:* $k \in \mathbb{N}_+$ and profile $R \in \mathcal{R}^n$ with a vector of tops $\tau = (\tau(R_1), \ldots, \tau(R_n))$. Set $\underline{q}^0 = \Phi(\mathbf{0})$ and $\overline{q}^0 = \Phi(\mathbf{k})$.

If $\sum_{j \in N} \tau_j = k$ set $\Phi(R) = \tau$. Stop.

If $\sum_{j \in N} \tau_j > k$ set $\Phi_i(R) = \tau_i$ for all $i \in \{j \in N \mid \tau_j \leq \overline{q}_i^0\} := S_1$. Compute $K_1 = k - \sum_{j \in S_1} \Phi_j(R)$. If $S_1 = \emptyset$ stop and set $\Phi(R) = \tau$. Otherwise, go to **Step 2**.

[23]For the continuous division problem, Sprumont (1991) characterizes the uniform allocation rule as the unique rule that satisfies strategy-proofness, efficiency and anonymity.

[24]The notation g^t denotes g composed with itself t-times, with $g^0(q, R) = (q, R)$.

If $\sum_{j\in N}\tau_j < k$ set $\Phi_i(R)=\tau_i$ for all $i\in\{j\in N \mid \tau_j \geq \underline{q}_i^0\} := S_1$. Compute $K_1 = k-\sum_{j\in S_1}\Phi_j(R)$. If $S_1=\emptyset$ stop and set $\Phi(R)=\tau$. Otherwise, go to **Step 2**.

Step t+1: *Input:* $K_t\in\mathbb{R}_{++}$, $S_t \subsetneq N$, and profile $R_{N\backslash S_t}$ with a vector of tops $\tau_{N\backslash S_t}$. Set $\underline{q}^t=\Phi(\tau_{S_t},\mathbf{0}_{N\backslash S_t})$ and $\overline{q}^t=\Phi(\tau_{S_t},\mathbf{k}_{N\backslash S_t})$.

If $\sum_{j\in S_t}\tau_j > K_t$ set $\Phi_i(R)=\tau_i$ for all $i\in\{j\in N\backslash S_t \mid \tau_j\leq \overline{q}_i^t\} := S_{t+1}$. Compute $K_{t+1}=K_t-\sum_{j\in S_{t+1}}\Phi_j(R)$. If $S_{t+1}=\emptyset$ stop and set

$$\Phi_i(R)=\begin{cases}\tau_i & \text{if } i\in S_t\\ \overline{q}_i^t & \text{if } i\in N\backslash S_t.\end{cases}$$

Otherwise, go to **Step t+2**.

If $\sum_{j\in S_t}\tau_j < K_t$ set $\Phi_i(R)=\tau_i$ to all $i\in\{j\in N\backslash S_t \mid \tau_j\geq \underline{q}_i^t\} = S_{t+1}$. Compute $K_{t+1}=K_t-\sum_{j\in S_{t+1}}\Phi_j(R)$. If $S_{t+1}=\emptyset$ stop and set

$$\Phi_i(R)=\begin{cases}\tau_i & \text{if } i\in S_t\\ \underline{q}_i^t & \text{if } i\in N\backslash S_t.\end{cases}$$

Otherwise, go to **Step t+2**.

Observe that the procedure stops at some Step t' such that $S_{t'}=\emptyset$. Example 2 illustrates this procedure.

Example 2: Let $n=5$, $k=35$, and Φ be given. Let $\underline{q}^0=\Phi(\mathbf{0})=(1,1,7,10,16)$ and $\overline{q}^0=\Phi(\mathbf{k})=(4,6,8,9,8)$.

Consider first the profile of tops $\tau=(0,2,6,9,12)$. Then, $\sum_{j\in N}\tau_j=29<35$. In **Step 1**, $S_1=\{j\in N \mid \tau_j\geq \underline{q}_j^0\}=\{2\}$, $\Phi_2(R)=2$, and $K_1=33$. In **Step 2**, set $\underline{q}^1=\Phi(0,2,0,0,0)$ and $\overline{q}^1=\Phi(35,2,35,35,35)$, and assume $\underline{q}^1=(0,2,7,10,16)$. Since $\tau_1+\tau_3+\tau_4+\tau_5=27<33$, $S_2=\{j\in\{1,3,4,5\} \mid \tau_j\geq \underline{q}_j^1\}=\{1\}$, $\Phi_1(R)=0$, and $K_2=33$. In **Step 3**, set $\underline{q}^2=\Phi(0,2,0,0,0)$ and $\overline{q}^2=\Phi(0,2,35,35,35)$, and assume $\underline{q}^2=(0,2,7,10,16)$. Since $\tau_3+\tau_4+\tau_5=27<33$, $S_3=\{j\in\{3,4,5\} \mid \tau_j\geq \underline{q}_j^1\}=\emptyset$ and stop with $\Phi(R)=(0,2,7,10,16)$.

Consider now the profile of tops $\tau=(2,8,5,12,17)$. Then, $\sum_{j\in N}\tau_j=44>35$. In **Step 1**, $S_1=\{j\in N \mid \tau_j\leq \overline{q}_j^0\}=\{1,3\}$, $\Phi_1(R)=2$, $\Phi_3(R)=5$ and $K_1=28$. In **Step 2**, set $\overline{q}^1=\Phi(2,35,5,35,35)$ and $\underline{q}^1=\Phi(2,0,5,0,0)$, and assume $\overline{q}^1=(2,7,5,10,11)$. Since $\tau_2+\tau_4+\tau_5=37>28$, $S_2=\{j\in\{2,4,5\} \mid \tau_j\leq \overline{q}_j^1\}=\emptyset$ and stop with $\Phi(R)=(2,7,5,10,11)$. □