On strategy-proofness and semilattice single-peakedness*

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Abstract

We study social choice rules defined on the domain of semilattice single-peaked preferences. Semilattice single-peakedness has been identified as the necessary condition that a set of preferences must satisfy so that the set can be the domain of a strategy-proof, tops-only, anonymous and unanimous rule. We characterize the class of all such rules on that domain and show that they are deeply related to the supremum of the underlying semilattice structure.

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1 Introduction

We characterize the class of all strategy-proof and simple rules defined on the domain of semilattice single-peaked preferences. A rule is a systematic procedure to select

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an alternative as a function of the declared profile of agents' preferences. A rule is strategy-proof if for each agent truthful declaration is optimal regardless of the other agents' declared preferences. A rule is simple if it is tops-only (the selected alternative depends only on the profile of top alternatives), unanimous (whenever all agents agree that an alternative is the most preferred one, the rule selects it), and anonymous (agents' identities do not play any role).

The notion of semilattice single-peakedness constitutes a weakening of the classical notion of single-peakedness, identified by Black (1948) as a natural and meaningful restriction on preferences whenever the set of alternatives is linearly ordered.[1] A preference is single-peaked if there is a unique most preferred alternative (the top), and alternatives further away from the top, in each of the two possible directions of the linear order, are less preferred than alternatives closer to the top. Moulin (1980) characterizes the family of all strategy-proof and tops-only rules on the domain of single-peaked preferences.

Semilattice single-peakedness requires that the underlying order structure on the set of alternatives is a (join-)semilattice and that the preference is decreasing further away from the top, *only* in the increasing direction of the semilattice.[2]

There are at least three reasons to study the domain of semilattice single-peaked preferences.

First, semilattice single-peakedness may represent a substantial weakening of the single-peaked condition, meaningful whenever agents' preferences are restricted (by the underlying partial order structure on the set of alternatives) only along a particular direction of the partial order, and not in others. Since a strategy-proof and simple rule on a domain of preferences remains strategy-proof and simple in any of its subdomains, characterizations of strategy-proof and simple rules on larger domains are useful because they identify rules that will remain strategy-proof and simple even if the rule would have to operate only on any of its subdomains.

Second, many of the domain restrictions identified in the literature as domains admitting strategy-proof rules satisfy weak versions of single-peakedness.[3] Thus, one

[1]A set is linearly ordered if there is a complete, antisymmetric and transitive binary relation over it.

[2]A partial order over a set is a reflexive, antisymmetric and transitive binary relation. A partial order is a (join-)semillatice if every pair of elements in the set has a least upper bound, named the supremum of the pair.

[3]Among others, Demange (1982), Border and Jordan (1983), Barberà, Sonnenschein and Zhou (1991), Barberà, Gül and Stacchetti (1993), Danilov (1994), Barberà, Massó and Neme (1997),

may read these results as identifying single-peakedness as a sufficient condition to be satisfied by a domain in order to guarantee that the domain admits the possibility of designing on it strategy-proof rules. However, all those domains are subsets of semilattice single-peaked preferences.

Third, semilattice single-peakedness is roughly the necessary condition that a rich domain has to satisfy in order to admit a strategy-proof and simple rule.[4] Some recent papers have tried to identify the key property of a domain of preferences admitting strategy-proof and simple rules. Chatterji, Sanver and Sen (2013) identifies a structure on the set of alternatives for which the domain of preferences over this set has to be semi single-peaked, an extended weak version of single-peakedness for finite sets of alternatives with a tree structure induced by a connectedness property of the domain. That is, if a domain is rich and connected then semi single-peakedness is a necessary condition that any set of preferences has to satisfy in order to admit a strategy-proof and simple rule, whenever the number of agents is even. Chatterji and Massó (2018) takes a similar approach to the one taken by Chatterji, Sanver and Sen (2013) but without assuming any *a priori* structure on the set of alternatives nor a connectedness property of the domain. They show that when the number of agents is even, if a rich domain admits a strategy-proof and simple rule then the domain has to be a subset of the set of semilattice single-peaked preferences, where the semilattice from which semilattice single-peakedness is defined is obtained from the rule. They also show that given any set of alternatives, partially ordered by a semilattice (denoted by $\succeq$), the rule that selects at each preference profile the supremum (according to $\succeq$) of the set of tops is strategy-proof and simple on the domain of semilattice single-peaked preferences, regardless of whether the number of agents is even or odd. We refer to this rule as the *supremum* rule and denote it by $\sup_{\succeq}$.

Our aim in this paper is to identify the class of *all* strategy-proof and simple rules defined on a semilattice single-peaked domain. Specifically, assume that the set of alternatives A is partially ordered by a semilattice $\succeq$. Consider the set of all

Peremans, Peters, van der Stel and Storcken (1997), Barberà, Massó and Serizawa (1998), Schummer and Vohra (2002), Barberà, Massó and Neme (2005) and Weymark (2008) study the family of strategy-proof rules (or some of its subfamilies) in settings where the set of alternatives has a more complex structure than just a linear order. In each of these generalizations, single-peakedness still appears as a natural and meaningful domain restriction.

[4]Observe that if the domain of preferences is not rich (*i.e.* too small in a precise sense), strategy-proofness loses all of its bite, and then arbitrary (and not rich) domains may trivially appear as being necessary for the possibility of admitting strategy-proof and simple rules.

semilattice single-peaked preferences relative to the semilattice $(A, \succeq)$. Theorem 1 characterizes the family of all strategy-proof and simple rules on this domain.[5] One of the main consequences of Theorem 1 is the following. Assume that the set of alternatives A does not have a supremum according to the semilattice $\succeq$.[6] Then, the supremum rule is the unique strategy-proof and simple rule on the domain of semilattice single-peaked preferences on $(A, \succeq)$. To understand this result remember that in Chatterji and Massó (2018) the property of semilattice single-peakedness is defined relative to the partial order induced by the rule itself. Hence, in this case the domain is so tailor-made to the rule that it only admits the supremum rule. And the supremum rule is strategy-proof and simple even when the set of alternatives has a supremum, denoted by α (*i.e.*; $\alpha \succ x$ for all x in A different from α). However, in this case other rules may also be strategy-proof and simple, and we identify all of them by the following procedure. Consider the set of alternatives $A^\star(\succeq)$ with the property that each alternative in the set is (strictly) related by the semilattice only to α. If the set $A^\star(\succeq)$ is empty, then the $\sup_\succeq$ is again the unique strategy-proof and simple rule on the domain of semilattice single-peaked preferences over A (note that $\alpha \notin A^\star(\succeq)$). If this set is non-empty, for each alternative x belonging to $A^\star(\succeq)$ and each integer $1 \leq q^x < n$, where n is the number of agents, define a rule f on the domain of semilattice single-peaked preferences on $(A, \succeq)$ as follows. Consider any profile of semilattice single-peaked preferences R. If the number of agents with top on x is larger or equal to q^x, set $f(R) = x$; otherwise, $f(R)$ is equal to the supremum (according to $\succeq$) of the set of top alternatives in the profile R. A rule f that can be described in this way is named a *quota-supremum* rule.

Our main result says that the class composed by the supremum of the original semilattice $\succeq$, the $\sup_\succeq$ rule, and the family of all quota-supremum rules, one for each pair composed by $x \in A^\star(\succeq)$ and $1 \leq q^x < n$, coincides with the class of all strategy-proof and simple rules on the domain of semilattice single-peaked preferences on $(A, \succeq)$. We also obtain additional results related to the quota-supremum rules and show that the domain of semilattice single-peaked preferences on $(A, \succeq)$ is maximal for the $\sup_\succeq$ rule; namely, the $\sup_\succeq$ rule is not strategy-proof on any domain of preferences strictly containing the set of semilattice single-peaked preferences on $(A, \succeq)$.

[5] We already know that this family is non-empty because the $\sup_\succeq$ rule is one of them.

[6] The fact that $(A, \succeq)$ is a semilattice guarantees that any pair of alternatives has supremum (the least upper bound of the pair of alternatives), but if A is not finite it may or it may not have, as a set, supremum (a least upper bound of A).

The paper is organized as follows. Section 2 contains basic notation and definitions, the definition of semilattice single-peakedness, and preliminary results. Section 3 contains the main result of the paper. In Section 4 we illustrate our result by applying it to the setting of two well-known restricted domains of preferences: the domain of single-peaked preferences over a subset of real numbers and the domain of separable preferences over the family of all subsets of a given set of objects. Section 5 contains the definition of rich domain, the statement that any semilattice single-peaked domain is rich, and some additional results. Section 6 concludes with two final remarks. The Appendix at the end of the paper contains the statements and proofs of some remarks and lemmata that will be used in the proof of the main result, and collects the proofs that are omitted in the text.

2 Preliminaries

2.1 Basic notation and definitions

Let $N = \{1, \ldots, n\}$ be the finite set of agents, with $n \geq 2$, and A be any set of alternatives. The set A can be finite or infinite. We assume throughout the paper that $|A| > 2$.[7] Each agent $i \in N$ has a preference (relation) $R_i \in \mathcal{D}$ over A, where $\mathcal{D}$ is an arbitrary and given subset of complete and transitive binary relations over A.[8] The set $\mathcal{D}$ is referred to as the domain of preferences. For any $x, y \in A$, xR_iy means that agent i considers alternative x to be at least as good as alternative y. Let P_i and I_i denote the strict and indifference relations induced by R_i over A, respectively.[9] We assume that for each $R_i \in \mathcal{D}$ there exists $t(R_i) \in A$, the top of R_i, such that $t(R_i)P_iy$ for all $y \in A\backslash\{t(R_i)\}$. For $x \in A$, we denote by R_i^x an arbitrary preference in $\mathcal{D}$ with $t(R_i^x) = x$. Let $\mathcal{R}$ denote the universal domain of preferences over A with a unique top. We also assume that for each $x \in A$ the domain $\mathcal{D}$ contains at least one preference R_i^x. A profile $R = (R_1, \ldots, R_n) \in \mathcal{D}^n$ is an $n-$tuple of preferences, one for each agent. To emphasize the role of agent i or subset of agents S we will often write the profile R as (R_i, R_{-i}) or as (R_S, R_{-S}).

[7]The cardinality of any given set X is denoted by $|X|$. The case $|A| = 2$ is very especial and discussed in the Final Remarks section at the end of the paper.

[8]A binary relation $\succeq$ over A is *complete* if for all $x, y \in A$ either $x \succeq y$ or $y \succeq x$ and it is *transitive* if for all $x, y, z \in A$, $x \succeq y$ and $y \succeq z$ imply $x \succeq z$.

[9]Namely, for any $x, y \in A$, xP_iy if and only if xR_iy and yR_ix does not hold, and xI_iy if and only if xR_iy and yR_ix.

A *(social choice) rule* is a mapping $f : \mathcal{D}^n \to A$ that assigns to every profile $R \in \mathcal{D}^n$ an alternative $f(R) \in A$.

A rule $f : \mathcal{D}^n \to A$ is *tops-only* if for all $R, R' \in \mathcal{D}^n$ such that $t(R_i) = t(R'_i)$ for all $i \in N$, $f(R) = f(R')$. Hence, a tops-only rule $f : \mathcal{D}^n \to A$ can be written as $f : A^n \to A$. Accordingly, whenever f be tops-only we will use the notation $f(t(R_1), \ldots, t(R_n))$ interchangeably with $f(R_1, \ldots, R_n)$.

A rule $f : \mathcal{D}^n \to A$ is *unanimous* if for all $R \in \mathcal{D}^n$ and $x \in A$ such that $t(R_i) = x$ for all $i \in N$, $f(R) = x$.

To define an anonymous rule on $\mathcal{D}^n$ define, for every profile $R \in \mathcal{D}^n$ and every one-to-one mapping $\sigma : N \to N$, the profile $R^\sigma = (R_{\sigma(1)}, \ldots, R_{\sigma(n)})$ as the $\sigma-$permutation of R, where for all $i \in N$, $R_{\sigma(i)}$ is the preference that agent $\sigma(i)$ had in the profile R. Observe that the domain $\mathcal{D}^n$ is closed under permutations, since it is the Cartesian product of the same set $\mathcal{D}$. A rule $f : \mathcal{D}^n \to A$ is *anonymous* if for all one-to-one mappings $\sigma : N \to N$ and all $R \in \mathcal{D}^n$, $f(R^\sigma) = f(R)$.

Preferences are idiosyncratic and agents' private information, and they have to be elicited by means of a rule. A rule is strategy-proof if for every agent at every preference profile truth-telling is a weakly dominant strategy in the direct revelation game induced by the rule at the preference profile. Formally, a rule $f : \mathcal{D}^n \to A$ is *strategy-proof* if for all $R \in \mathcal{D}^n$, all $i \in N$ and all $R'_i \in \mathcal{D}$,

$$f(R) R_i f(R'_i, R_{-i}).$$

We say that agent i can manipulate f at R if there exists $R'_i \in \mathcal{D}$ such that $f(R'_i, R_{-i}) P_i f(R)$.

We are interested in strategy-proof rules satisfying in addition unanimity, anonymity and tops-onlyness. Unanimity is a natural and weak form of efficiency: if all agents consider an alternative as being the most-preferred one, the rule should select it. Anonymity imposes that the rule treats all agents equally: the social outcome is selected without paying attention to the identities of the agents. Tops-onlyness constitutes a basic simplicity requirement. We will refer to a tops-only, anonymous and unanimous rule as a *simple* rule.

We will assume that the set A is partially ordered by a binary relation $\succeq$ that is a semilattice over A; namely, for all $x, y, z \in A$, (i) $x \succeq x$ (reflexivity), (ii) $x \succeq y$ and $y \succeq x$ imply $x = y$ (antisymmetry), and (iii) $x \succeq y$ and $y \succeq z$ imply $x \succeq z$ (transitivity). The partial order $\succeq$ is a *(join-)semilattice* over A if for each pair $x, y \in A$ the least upper bound of $\{x, y\}$ exists; in this case it is named the *supremum* of x and y, and it is denoted by $\sup_{\succeq}\{x,y\}$.[10] A semilattice $\succeq$ over A will often be denoted by $(A,\succeq)$. Given a semilattice $(A,\succeq)$ and $x,y\in A$, we write $x\succ y$ if $x\succeq y$ and $x\neq y$.

Fix a binary relation $\succeq$ over A. Given two alternatives $x,y\in A$ with $y\succeq x$, define the set $[x,y]$ as

$$[x,y]=\{z\in A\mid y\succeq z \text{ and } z\succeq x\}.$$

If x and y are two distinct alternatives and related by $\succeq$ as $y\succeq x$, then the set $[x,y]$ is obtained by adding to the set $\{x,y\}$ all alternatives in A that "lie between" x and y according to $\succeq$. For $y\not\succeq x$ define $[x,y]=\emptyset$.

2.2 Semilattice single-peakedness

Chatterji and Massó (2018) shows that if a rich domain of preferences admits a strategy-proof and simple rule $f:\mathcal{D}^n\to A$, for n even, then all preferences in the domain $\mathcal{D}$ have to be semilattice single-peaked, where the semilattice from which semilattice single-peakedness is defined is identified from the rule. As we have already said in the Introduction, this notion constitutes a weakening of single-peakedness and contains as particular cases many of the domains identified in the restricted domain literature; for instance, those studied in Moulin (1980), Demange (1982), Border and Jordan (1983), Barberà, Sonnenschein and Zhou (1991), Barberà, Gül and Stacchetti (1993), Danilov (1994), Peremans, Peters, van der Stel and Storcken (1997), Barberà, Massó and Serizawa (1998), Schummer and Vohra (2001), Weymark (2008) and Chatterji, Sanver and Sen (2013).[11] Following Chatterji and Massó, we now define semilattice single-peaked preferences on a semilattice $(A,\succeq)$.

Definition 1 *Let $\succeq$ be a semilattice over A and let $x\in A$. The preference $R_i^x\in\mathcal{D}$ is* ***semilattice single-peaked*** *on $(A,\succeq)$ if, for all $y,z\in A$, $\sup_{\succeq}\{x,y\}R_i^x\sup_{\succeq}\{z,y\}$.*

Example 1 at the end of this section illustrates this definition. Given a semilattice $\succeq$ over A, denote by $\mathcal{SSP}(\succeq)$ the domain of semilattice single-peaked preferences on $(A,\succeq)$.

[10] Given $x,y\in A$, an element $z\in A$ is an *upper bound* of $\{x,y\}$ if $z\succeq x$ and $z\succeq y$. An element $t\in A$ is the *supremum* of $\{x,y\}$ if and only if (i) t is an upper bound of $\{x,y\}$ and (ii) $w\succeq t$ for all upper bound w of $\{x,y\}$ (*i.e.*, t is the least upper bound of $\{x,y\}$).

[11] Chatterji and Massó (2018) carefully describes some of these inclusions.

2.3 Preliminary results and example

Chatterji and Massó (2018) characterizes semilattice single-peakedness by means of two properties, the first one more directly related to the notion of single-peakedness. Remark 1 states this result, which will be very useful in the sequel.

Remark 1 *Let $\succeq$ be a semilattice over A and let $x \in A$. The preference $R_i^x \in \mathcal{D}$ is semilattice single-peaked on $(A, \succeq)$ if and only if the following two properties hold:*
(i) for all $y, z \in A$ such that $x \preceq y \preceq z$, $yR_i^x z$;
(ii) for all $w \in A$ such that $x \npreceq w$, $\sup_{\succeq}\{x, w\} R_i^x w$.

The well-known notion of single-peakedness on a linearly ordered set can be naturally extended to a set that is partially ordered by a semilattice.

Definition 2 *Let $\succeq$ be a semilattice over A and let $x \in A$. The preference $R_i^x \in \mathcal{D}$ is **single-peaked** on $(A, \succeq)$ if the following two properties hold:*
(i) for all $y, z \in A$ such that $x \preceq y \prec z$ or $z \prec y \preceq x$, $yP_i^x z$;
(ii) for all $w \in A$ such that $x \npreceq w$, $\sup_{\succeq}\{x, w\} P_i^x w$.

Given a semilattice $\succeq$ over A, denote by $\mathcal{SP}(\succeq)$ the domain of single-peaked preferences on $(A, \succeq)$.

Remark 2 *For every semilattice $\succeq$ over A, $\mathcal{SP}(\succeq) \subset \mathcal{SSP}(\succeq)$ holds immediately by their definitions. To see that the inclusion may be strict, consider the set of alternatives $A = \{x, y, z\}$ and let $\succeq$ be such that $x \succeq y \succeq z$. Then, $\mathcal{SP}(\succeq) \subsetneq \mathcal{SSP}(\succeq)$ because R_i^x such that $zP_i^x y$ belongs to $\mathcal{SSP}(\succeq)$ but not to $\mathcal{SP}(\succeq)$.*

Remark 3 *When the semilattice $\succeq$ is a linear order over A, condition (i) in Definition 2 still generalizes the usual definition of single-peakedness. Moreover, if condition (i) holds then condition (ii) is redundant since $x \npreceq w$ implies, by completeness of $\succeq$, that $x \succ w$ and so $\sup_{\succeq}\{x, w\} = xP_i^x w$ follows trivially.*

Example 1 illustrates the notion of semilattice single-peakedness.

Example 1 Let $A = \{x_1, \ldots, x_9\}$ be a set with nine alternatives and let $\succeq$ be the semilattice represented in Figure 1 where, for any $x_k, x_{k'} \in A$, an arrow from x_k to $x_{k'}$ means $x_{k'} \succ x_k$, and arrows that would follow from transitivity are deleted.

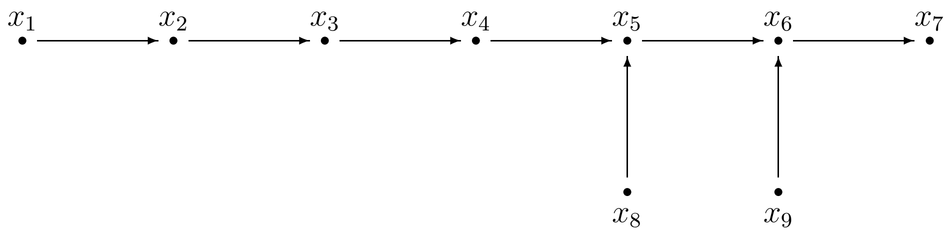

Figure 1

The set of $\mathcal{SSP}(\succeq)$ is large but restricted; in particular, the set of strict preferences listed in Table 1 are semilattice single-peaked on $(A, \succeq)$, while the ones listed in Table 2 are not. Each column represents a strict preference with the convention that an alternative located in a row is strictly preferred to all alternatives located in lower rows.[12]

$P_i^{x_3}$	$\widehat{P}_i^{x_3}$	$P_i^{x_8}$	$\widehat{P}_i^{x_8}$	$P_i^{x_5}$
x_3	x_3	x_8	x_8	x_5
x_4	x_2	x_5	x_5	x_3
x_5	x_1	x_6	x_6	x_8
x_1	x_4	x_9	x_7	x_4
x_2	x_5	x_7	x_9	x_6
x_6	x_8	x_1	x_4	x_7
x_7	x_6	x_3	x_3	x_9
x_9	x_9	x_2	x_2	x_2
x_8	x_7	x_4	x_1	x_1

Table 1

$\overline{P}_i^{x_3}$	$\widetilde{P}_i^{x_3}$	$\overline{P}_i^{x_8}$	$\widetilde{P}_i^{x_8}$
x_3	x_3	x_8	x_8
x_5	x_2	x_6	x_4
x_4	x_1	x_9	x_6
x_1	x_4	x_5	x_7
x_2	x_8	x_7	x_9
x_6	x_5	x_1	x_5
x_7	x_6	x_3	x_2
x_9	x_9	x_2	x_3
x_8	x_7	x_4	x_1

Table 2

Preferences $P_i^{x_3}$, $\widehat{P}_i^{x_3}$, $P_i^{x_8}$, $\widehat{P}_i^{x_8}$ and $P_i^{x_5}$ belong to $\mathcal{SSP}(\succeq)$ because, for instance, according to Definition 1, and Remark 1,

- $\sup_{\succeq}\{x_3, x_4\} = x_4 P_i^{x_3} x_5 = \sup_{\succeq}\{x_5, x_4\}$, $\sup_{\succeq}\{x_3, x_8\} = x_5 P_i^{x_3} x_6 = \sup_{\succeq}\{x_9, x_8\}$,
  - $x_3 \prec x_4 \prec x_5$ and $x_4 P_i^{x_3} x_5$, and $\sup_{\succeq}\{x_3, x_8\} = x_5 P_i^{x_3} x_8$.
- $\sup_{\succeq}\{x_8, x_6\} = x_6 P_i^{x_8} x_7 = \sup_{\succeq}\{x_7, x_6\}$, $\sup_{\succeq}\{x_8, x_3\} = x_5 P_i^{x_8} x_6 = \sup_{\succeq}\{x_9, x_3\}$,
  - $x_8 \prec x_6 \prec x_7$ and $x_6 P_i^{x_8} x_7$, and $\sup_{\succeq}\{x_8, x_9\} = x_6 P_i^{x_3} x_9$.

[12]To ease the exposition in this and all examples that follow we only consider strict preferences.

Preferences $\overline{P}_i^{x_3}$, $\widetilde{P}_i^{x_3}$, $\overline{P}_i^{x_8}$ and $\widetilde{P}_i^{x_8}$ do not belong to $\mathcal{SSP}(\succeq)$ because, for instance, according to Definition 1, and Remark 1,

- $\sup_{\succeq}\{x_3, x_4\} = x_4$ and $x_5 = \sup_{\succeq}\{x_5, x_4\}$, but $x_5\overline{P}_i^{x_3}x_4$,
  - $x_3 \prec x_4 \prec x_5$ and $x_5\overline{P}_i^{x_3}x_4$.
- $\sup_{\succeq}\{x_8, x_5\} = x_5$ and $x_6 = \sup_{\succeq}\{x_5, x_9\}$, but $x_6\overline{P}_i^{x_8}x_5$,
  - $x_8 \prec x_5 \prec x_6$ and $x_6\overline{P}_i^{x_8}x_5$.

However, preference $\widehat{P}_i^{x_3}$ belongs to $\mathcal{SP}(\succeq)$ but preference $P_i^{x_3}$ does not because, for instance, according to Definition 2,

- $x_1 \prec x_2 \prec x_3$ and $x_2\widehat{P}_i^{x_3}x_1$, but $x_1P_i^{x_3}x_2$. □

3 Main result

Our goal in this paper is to characterize, for any semilattice $(A, \succeq)$, the class of all strategy-proof and simple rules on the domain of semilattice single-peaked preferences $\mathcal{SSP}(\succeq)$. We start by defining, given a semilattice $(A, \succeq)$, the supremum rule $\sup_{\succeq}$ on the domain $\mathcal{SSP}(\succeq)$. We then argue that the supremum rule is simple and strategy-proof on $\mathcal{SSP}(\succeq)$.

Let $(A, \succeq)$ be a semilattice and let $\mathcal{SSP}(\succeq)$ be the set of semilattice single-peaked preferences on $(A, \succeq)$. The *supremum rule*, denoted as $\sup_{\succeq} : \mathcal{SSP}(\succeq)^n \to A$, is defined by setting, for each profile $R = (R_1, \ldots, R_n) \in \mathcal{SSP}(\succeq)^n$,

$$\sup_{\succeq}(R_1, \ldots, R_n) = \sup_{\succeq}\{t(R_1), \ldots, t(R_n)\}.$$

The supremum rule $\sup_{\succeq}$ is unanimous, anonymous and tops-only by definition, and so it is simple. To see that $\sup_{\succeq}$ is strategy-proof first define $t(R) = \{t(R_i) : i \in N\}$ as the set of (different) tops at profile R and $t(R_{-i}) = \{t(R_j) : j \in N\backslash\{i\}\}$. Next, consider any profile $R \in \mathcal{SSP}(\succeq)^n$, agent $i \in N$ and preference $R_i' \in \mathcal{SSP}(\succeq)$. By semilattice single-peakedness,

$$\sup_{\succeq}(R) = \sup_{\succeq}\{t(R_i), \sup_{\succeq}t(R_{-i})\}\ R_i\ \sup_{\succeq}\{t(R_i'), \sup_{\succeq}t(R_{-i})\} = \sup_{\succeq}(R_i', R_{-i}).$$

Thus, $\sup_{\succeq}$ is strategy-proof.

However, there are semilattices $(A, \succeq)$ for which $\sup_{\succeq}$ is not the unique strategy-proof and simple rule on $\mathcal{SSP}(\succeq)$. Theorem 1 below identifies the class of all strategy-proof and simple rules on the domain of semilattice single-peaked preferences. To state this characterization we need the following notation and definitions. Given $R \in \mathcal{SSP}(\succeq)^n$ and $x \in A$, define $N(R, x) = \{i \in N : t(R_i) = x\}$ as the set of agents whose top is x at profile R. Assume A has a supremum, denoted as $\sup_{\succeq} A = \alpha$.[13] Namely, there is $\alpha \in A$ such that $\alpha \succ x$ for all $x \in A \backslash \{\alpha\}$. Let

$$A^{\star}(\succeq) = \{x \in A : \text{ for each } y \in A \setminus \{\alpha\}, x \not\succeq y \text{ and } y \not\succeq x\}$$

be the set of alternatives that, according to $\succeq$, are not related to any other alternative but α. Observe that $\alpha \notin A^{\star}(\succeq)$; for instance, for $A = \{x, y, z\}$ and the semilattice $x \succ y$, $x \succ z$, $y \not\succ z$ and $z \not\succ y$, $\sup_{\succeq} A = x$ and $A^{\star}(\succeq) = \{y, z\}$. Moreover, $A^{\star}(\succeq)$ may be empty; for instance, in the case of the semilattice considered in Example 1.

Definition 3 *Let $\succeq$ be a semilattice over A such that $\sup_{\succeq} A$ exists. The rule $f : \mathcal{SSP}(\succeq)^n \to A$ is a* ***quota-supremum*** *rule if there are $x \in A^{\star}(\succeq)$ and integer q^x with $1 \leq q^x < n$ such that, for every $R \in \mathcal{SSP}(\succeq)^n$,*

$$f(R) = \begin{cases} x & \text{if } |N(R, x)| \geq q^x \\ \sup_{\succeq} t(R) & \text{otherwise.} \end{cases}$$

We refer to the alternative $x \in A^{\star}(\succeq)$, whose existence makes f to be a quota-supremum rule, as the *alternative associated to f*, and to q^x as the *quota* of x. Given $x \in A^{\star}(\succeq)$ we can generate a new semilattice, the one obtained from the original one by declaring x to be above α, and maintaining all the other binary comparisons not involving x as in the original semilattice. Example 2 below illustrates this procedure.

Denote by $\mathcal{Q}(\succeq)$ the set of quota-supremum rules defined on $\mathcal{SSP}(\succeq)$, and by $\mathcal{F}(\succeq)$ the set of strategy-proof and simple rules defined on $\mathcal{SSP}(\succeq)$. We know that $\mathcal{F}(\succeq) \neq \emptyset$ since we have just argued that $\sup_{\succeq}$ is strategy-proof and simple. The main result of the paper is the following.

Theorem 1 *Let $\succeq$ be a semilattice over A. The rule $f : \mathcal{SSP}(\succeq)^n \to A$ is strategy-proof and simple if and only if $f = \sup_{\succeq}$ or f is a quota-supremum; i.e.,*

$$\mathcal{F}(\succeq) = \{\sup_{\succeq}\} \cup \mathcal{Q}(\succeq).$$

Proof See Appendix 7.5. ■

[13] We are abusing a bit the notation and use $\sup_{\succeq}$ to denote the supremum rule and $\sup_{\succeq} A$ to denote the supremum of the set A.

Corollary 1 *Let $\succeq$ be a semilattice over A such that* $\sup_{\succeq} A$ *does not exist. Then,* $\mathcal{Q}(\succeq) = \emptyset$ *and* $\mathcal{F}(\succeq) = \{\sup_{\succeq}\}$.

Example 2 illustrates the content of Theorem 1 when $n = 2$ and indicates a procedure to identify for this case, the full class of strategy-proof and simple rules on the domain $\mathcal{SSP}(\succeq)$; namely, all quota-supremum rules.

Example 2 Let $A = \{x, y, z\}$ be the set of alternatives and let $N = \{1, 2\}$ be the set of agents. Consider the semilattice $(A, \succeq_1)$, where $x \succ_1 y$, $x \succ_1 z$, $y \not\succ_1 z$ and $z \not\succ_1 y$. Obviously, $\sup_{\succeq_1} A = x$ and the supremum rule $\sup_{\succeq_1} = f_1$ can be described by

$$\begin{aligned} f_1(x,y) &= x \\ f_1(x,z) &= x \\ f_1(y,z) &= x \end{aligned}$$

together with the corresponding choices required by unanimity and anonymity. Observe that $A^\star(\succeq_1) = \{y, z\}$. Then, the set of (strict) semilattice single-peaked preferences $\mathcal{SSP}(\succeq_1)$ on $(A, \succeq_1)$ contains the four preferences listed in Table 3 where, for any pair $a, b \in \{x, y, z\}$, P_i^{ab} refers to the unique strict preference $aP_i^{ab}bP_i^{ab}c$.

P_i^{xy}	P_i^{xz}	P_i^{yx}	P_i^{zx}
x	x	y	z
y	z	x	x
z	y	z	y

Table 3

Consider now the two semilattices $(A, \succeq_2)$ and $(A, \succeq_3)$, where $\succeq_2$ and $\succeq_3$ are obtained from $\succeq_1$ by setting $y \succ_2 x$ and $z \succ_3 x$, respectively, and adding the relations implied by transitivity. Figure 2 represents $\succeq_1$, $\succeq_2$ and $\succeq_3$, where an arrow between two alternatives points to their supremum, and arrows that would follow by transitivity are omitted.

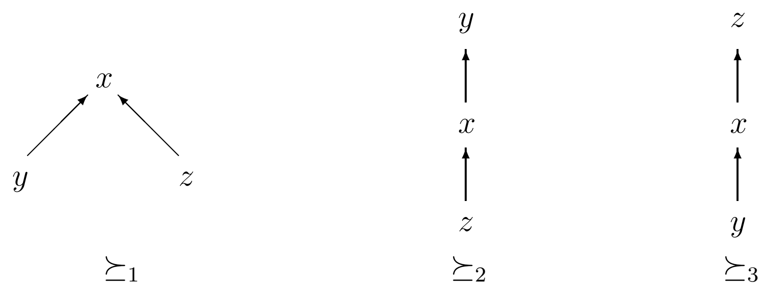

Figure 2

Hence, the rules $\sup_{\succeq_2} = f_2$ and $\sup_{\succeq_3} = f_3$ can be described by

$$\begin{aligned} f_2(x,y) &= y \\ f_2(x,z) &= x \\ f_2(y,z) &= y \end{aligned} \qquad \text{and} \qquad \begin{aligned} f_3(x,y) &= x \\ f_3(x,z) &= z \\ f_3(y,z) &= z \end{aligned}$$

together with the corresponding choices required by unanimity and anonymity. Observe that $A^\star(\succeq_2) = A^\star(\succeq_3) = \emptyset$ and $f_2, f_3 \in \mathcal{Q}(\succeq_1)$, where f_2 and f_3 can also be seen as the quota-supremum rules associated respectively to y and z (in this case, and since there are only two agents, the only admissible quota for any quota-supremum rule is equal to 1). Given the semilattice $\succeq_1$, the set of strategy-proof and simple rules on $\mathcal{SSP}(\succeq_1)$ is the set $\mathcal{F}(\succeq_1) = \{f_1, f_2, f_3\}$. $\square$

For any $n \geq 2$, a quota-supremum rule with associated alternative $x \in A^\star(\succeq)$ and quota $q^x = 1$ can also be described as the supremum of a slightly modified semilattice. Let $(A, \succeq)$ be a semilattice and let $x \in A^\star(\succeq)$. Define the semilattice $\succeq^x$ on A obtained from $\succeq$ by setting $x \succ^x y$ for all $y \in A\backslash\{x\}$ and, for all $y, z \in A\backslash\{x\}$, $y \succ^x z$ if and only if $y \succ z$. If $f : \mathcal{SSP}(\succeq)^n \to A$ is a quota-supremum rule with associated alternative x and quota $q^x = 1$, then $f(R) = \sup_{\succeq^x} t(R)$ for all $R \in \mathcal{SSP}(\succeq)^n$. Asking for the support of at least one agent for x, and otherwise selecting the supremum of all tops according to $\succeq$, is equivalent to obtaining directly the supremum of all tops, according to $\succeq^x$.

To see that all quota-supremum rules with quota one are strategy-proof, fix a quota-supremum rule $f : \mathcal{SSP}(\succeq)^n \to A$, where $x \in A^\star(\succeq)$ is its associated alternative and $q^x = 1$. Let $R_i^y \in \mathcal{SSP}(\succeq)$ and $R_{-i} \in \mathcal{SSP}(\succeq)^{n-1}$ be arbitrary. If $y = x$ then $f(R_i^y, R_{-i}) = y$, and i cannot manipulate f at (R_i^y, R_{-i}). If $y \neq x$, we distinguish between two cases. First, suppose $x \in t(R_i^y, R_{-i})$; then, $f(R_i^y, R_{-i}) = x$ and $f(R_i', R_{-i}) = x$ for all $R_i' \in \mathcal{SSP}(\succeq)$, and i cannot manipulate f at (R_i^y, R_{-i}). Second, suppose $x \notin t(R_i^y, R_{-i})$; then, $f(R_i^y, R_{-i}) = \sup_{\succeq} t(R_i^y, R_{-i})$. If $\sup_{\succeq} t(R_i^y, R_{-i}) = y$, agent i cannot manipulate f at (R_i^y, R_{-i}). If $\sup_{\succeq} t(R_i^y, R_{-i}) \neq y$ we distinguish between two subcases. First, suppose $t(R_i') \neq x$; then,

$$y \prec f(R_i^y, R_{-i}) \preceq \sup_{\succeq} t(R_i', R_{-i}) = f(R_i', R_{-i}),$$

but by semilattice single-peakedness, $f(R_i^y, R_{-i}) R_i^y f(R_i', R_{-i})$, and i cannot manipulate f at (R_i^y, R_{-i}) with R_i'. Second, suppose $t(R_i') = x$; then $f(R_i', R_{-i}) = x$ and since

$$y \prec f(R_i^y, R_{-i}) \prec \sup_{\succeq} t(R_i', R_{-i}) = \sup_{\succeq} A$$

and $y \not\preceq f(R_i', R_{-i}) = x$, by (ii) in Remark 1 characterizing semilattice single-peakedness and transitivity of R_i^y, $f(R_i^y, R_{-i}) R_i^y f(R_i', R_{-i}) = x$, and i cannot manipulate f at (R_i^y, R_{-i}) with R_i'. Thus, f is strategy-proof and trivially simple. In the proof of Theorem 1, in Appendix 7.5, we extend this argument from the case $q^x = 1$ to any quota $1 < q^x < n$, and show that any strategy-proof and simple rule $f : \mathcal{SSP}(\succeq)^n \to A$ is either the $\sup_{\succeq}$ rule if $\sup_{\succeq} A$ does not exist or else a quota-supremum rule.

4 Two examples

In this section we illustrate our result by applying it to the setting of two well-known restricted domains of preferences. The example in Subsection 4.1 corresponds to the domain of single-peaked preferences over a subset of real numbers. The example in Subsection 4.2 corresponds to the domain of separable preferences over the family of all subsets of a given set of objects. In each of the two examples we identify the semilattices and their corresponding rules that, among the class of all strategy-proof and simple rules on the single-peaked or separable domains, remain strategy-proof on the larger domain of semilattice single-peaked preferences. Moreover we show how, according to our result, these rules can be described either as the supremum of the underlying semilattice or as quota-supremum rules.

4.1 Median voters

The set of alternatives A is a subset of real numbers. We distinguish between two cases, depending on whether A is equal to the real line $\mathbb{R}$ (as in Moulin (1980)) or A is finite (as the one-dimensional case in Barberà, Gül and Stachetti (1993)).

4.1.1 Moulin (1980)

The set A is the set of real numbers $\mathbb{R}$, linearly ordered by the binary relation $\geq$ ("to be larger or equal than"), and agents preferences are single-peaked (relative to $\geq$) over $\mathbb{R}$.[14] We first consider the case where the semilattice on $\mathbb{R}$ coincides with the binary relation $\geq$. Then, the semilattice $(\mathbb{R}, \geq)$ does not have supremum and the set of semilattice single-peaked preferences on $(\mathbb{R}, \geq)$ is

$$\mathcal{SSP}(\geq) = \{R_i \in \mathcal{R} : t(R_i) = x \text{ for some } x \in \mathbb{R} \text{ and } x < y < z \text{ implies } yR_iz\}.$$

That is, a semilattice single-peaked preference $R_i \in \mathcal{SSP}(\geq)$ is decreasing on the right of its top alternative, and it is unrestricted between pairs of alternatives that are either on the left of the top or on different sides of the top. Figure 3 depicts a semilattice single-peaked preference $R_i \in \mathcal{SSP}(\geq)$ with top on alternative x.[15]

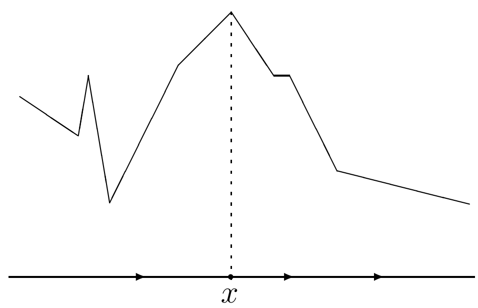

Figure 3

Since $A^\star(\geq) = \emptyset$, Theorem 1 says that the unique strategy-proof and simple rule on $\mathcal{SSP}(\geq)^n$ is the $\sup_\geq$ rule, which corresponds in Moulin (1980)'s characterization of all strategy-proof and simple rules on the domain of single-peaked preferences (relative to $\geq$) to the median voter rule where $n-1$ fixed votes are located at $+\infty$; that is, for each $R \in \mathcal{SSP}(\geq)^n$,

$$\begin{aligned} \sup_\geq(R) &= \sup_\geq\{t(R_1), \ldots, t(R_n)\} \\ &= median_\geq\{t(R_1), \ldots, t(R_n), \underbrace{+\infty, \ldots, +\infty}_{(n-1)\text{-times}}\}. \end{aligned}$$

[14] Namely, the preference R_i over $\mathbb{R}$ is *single-peaked* (*relative to* $\geq$) if (i) R_i has a unique top alternative $t(R_i)$ and (ii) $y < x < t(R_i)$ or $t(R_i) < x < y$ implies xR_iy.

[15] Preferences in $\mathcal{SSP}(\geq)$ would arise naturally when agents are willing to consume the public good only at higher levels of their top alternatives. For instance, if the linear order $\geq$ represents the flow of a river, semilattice single-peakedness would reflect the fact that agents are able to move (*i.e.*, consume the good) using only the flow of the river. Semilattice single-peakedness does not impose restrictions on how agents order alternatives upstream of their top alternatives but it requires monotonicity downstream.

We now consider other cases where the semilattice over $\mathbb{R}$ does not coincide with the binary relation $\geq$ and the supremum rule can be identified with a median voter rule. For any fixed $\alpha \in \mathbb{R}$, consider the semilattice $\succeq$ where $y < x \leq \alpha$ or $\alpha \leq x < y$ implies $x \succ y$ and $x < \alpha < y$ implies $x \not\succeq y$ and $y \not\succeq x$. Thus, $\sup_{\succeq} = \alpha$ and the set of semilattice single-peaked preferences on $(\mathbb{R}, \succeq)$ is

$$\begin{aligned} \mathcal{SSP}(\succeq) = \quad & \{R_i \in \mathcal{R} : t(R_i) = x \text{ for some } x \in \mathbb{R}, \\ & t(R_i) < y < z \leq \alpha \text{ or } \alpha \leq z < y < t(R_i) \text{ implies } yR_iz \text{ and} \\ & t(R_i) \not\succeq y \text{ and } y \not\succeq t(R_i) \text{ imply } \sup_{\succeq}\{t(R_i), y\} = \alpha R_i y\}. \end{aligned}$$

That is, a semilattice single-peaked preference $R_i \in \mathcal{SSP}(\succeq)$ is decreasing from $t(R_i)$ towards α and the preference between pairs of alternatives after α, all according to $\succeq$, are unrestricted but none is strictly preferred to α. Figure 4 represents this semilattice (as arrows pointing to α) and depicts a semilattice single-peaked preference $R_i \in \mathcal{SSP}(\succeq)$ on $(\mathbb{R}, \succeq)$.

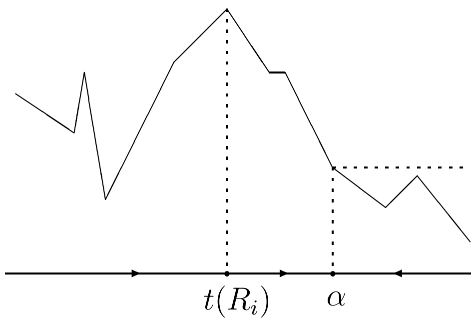

Figure 4

Since $A^\star(\succeq) = \emptyset$ also holds, Theorem 1 says that the unique strategy-proof and simple rule on $\mathcal{SSP}(\succeq)^n$ is the $\sup_{\succeq}$, which corresponds, in the previously mentioned Moulin (1980)'s characterization, to the median voter rule (relative to $\geq$) where $n-1$ fixed votes are located at α; that is, for each $R \in \mathcal{SSP}(\succeq)^n$,

$$\begin{aligned} \sup_{\succeq}(R) &= \sup_{\succeq}\{t(R_1), \ldots, t(R_n)\} \\ &= \mathit{median}_{\geq}\{t(R_1), \ldots, t(R_n), \underbrace{\alpha, \ldots, \alpha}_{(n-1)\text{-times}}\}. \end{aligned}$$

4.1.2 Barberà, Gül and Stachetti (1993)

The set A is a finite subset of real numbers. Without loss of generality suppose that $A = \{x_1, \ldots, x_M\}$, where $x_1 < \cdots < x_M$. We distinguish between two types of semilattices over A, depending on whether or not the semilattice has the property that there is one alternative that is related only to the supremum.

First consider any semilattice $\succeq$ with the property that, for some m such that $1 \leq m \leq M$, $\sup_{\succeq} A = x_m$ and $A^{\star}(\succeq) = \emptyset$; for instance the case $m = M$ and $x_1 \prec \ldots \prec x_M$ or the case $2 < m < M-1$ and $x_1 \prec x_2 \prec \cdots \prec x_m$ and $x_M \prec x_{M-1} \prec \cdots \prec x_m$. Then, the set $\mathcal{SSP}(\succeq)$ of semilattice single-peaked preferences on $(A, \succeq)$ is the natural extension of the set for the continuous case to this discrete setting. Again, Theorem 1 says that the unique strategy-proof and simple rule on $\mathcal{SSP}(\succeq)^n$ is the $\sup_{\succeq}$ rule. This rule can also be represented as the $median_{\geq}$, according to the ordering $\geq$, of the set of the profile of tops and $(n-1)$ fixed votes located at x_m.[16] Namely, for each $R \in \mathcal{SSP}(\succeq)^n$,

$$\begin{aligned} \sup_{\succeq}(R) &= \sup_{\succeq}\{t(R_1), \ldots, t(R_n)\} \\ &= median_{\geq}\{t(R_1), \ldots, t(R_n), \underbrace{x_m, \ldots, x_m}_{(n-1)\text{-times}}\}. \end{aligned}$$

Suppose now that $M \geq 4$ and consider the semilattice such that and $x_1 \prec x_2 \prec \cdots \prec x_{M-1}$ and $x_M \prec x_{M-1}$. Figure 5 depicts the case $M = 6$.

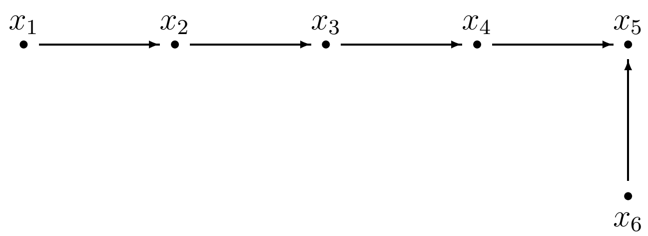

Figure 5

Then, $\sup_{\succ} A = x_{M-1}$, $A^{\star}(\succeq) = x_M$ and the set of semilattice single-peaked preferences on $(\mathbb{R}, \succeq)$ is

$$\begin{aligned} \mathcal{SSP}(\succeq) = \quad & \{R_i \in \mathcal{R} : t(R_i) = x_m \text{ for some } x_m \in A, \\ & \text{if } m = M \text{ then } x_{M-1} R_i x_j \text{ for all } j < M-1 \text{ and} \\ & \text{if } m < M-1 \text{ then } x_m \prec x_j \prec x_{j'} \preceq x_{M-1} \text{ implies } x_j R_i x_{j'} \text{ and } x_{M-1} R_i x_M\}. \end{aligned}$$

Observe that when $t(R_i) = x_{M-1}$, R_i is unrestricted. It is easy to see that the family of quota-supremum rules on the domain $\mathcal{SSP}(\succeq)^n$ with associated alternative x_M

[16] Moreover, and according to the description of median voters by means of left-coalition systems used in Barberà, Gül and Stachetti (1993), this rule corresponds to the case (using also the ordering $x_1 < \ldots < x_M$ to define the left-coalition system) where the family of winning coalitions for all alternatives $x_j < x_m$ is equal to N, while the family of winning coalitions for all alternatives $x_j \geq x_m$ is equal to $2^N \backslash \{\emptyset\}$.

and quota $1 \leq q < n$, denoted by $\{f^q\}_{q=1}^{q=n-1}$, can be described as median voter rules, where the $n-1$ fixed votes $\alpha_1, \ldots, \alpha_{n-1}$ are given in Table 4, and by setting for each $R \in \mathcal{SSP}(\succeq)^n$,

$$f^q(R) = median_{\geq}\{t(R_1, \ldots, t(R_n), \alpha_1, \ldots, \alpha_{n-1}\},$$

where the fixed votes $\alpha_1, \ldots \alpha_{n-1}$ are given by the row corresponding to quota q in Table 4.

Quota	Fixed votes at x_{M-1}	Fixed votes at x_M
$q = 1$	none	$\alpha_1 = \cdots = \alpha_{n-1} = x_M$
$q = 2$	$\alpha_1 = x_{M-1}$	$\alpha_2 = \cdots = \alpha_{n-1} = x_M$
$\vdots$	$\vdots$	$\vdots$
$q = k$	$\alpha_1 = \cdots = \alpha_{k-1} = x_{M-1}$	$\alpha_k = \cdots = \alpha_{n-1} = x_M$
$\vdots$	$\vdots$	$\vdots$
$q = n-1$	$\alpha_1 = \cdots = \alpha_{n-2} = x_{M-1}$	$\alpha_{n-1} = x_M$

Table 4

4.2 Voting by committees

Consider the special case of voting by quota studied by Barberà, Sonnenschein and Zhou (1991) with two objects.[17] A society has to choose a subset from a given set of two objects $\{a, b\}$. Namely, the set of alternatives is $A = \{\{\emptyset\}, \{a\}, \{b\}, \{a, b\}\}$ which can be identified with the unit vertices of the two-dimensional cube $A = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$. Moreover, agents' preferences are separable: adding an object to a set that does not contain the object makes the set strictly preferred to the original set if and only the added object is strictly preferred to the empty set (as a singleton set, the object is good). Denote by $\mathcal{S}$ the set of all strict and separable preferences over A. Let $\succeq$ be the semilattice represented in Figure 5, where an arrow between two alternatives points to their supremum, and the arrow that would follow from transitivity is omitted. It holds that $\mathcal{S} \subsetneq \mathcal{SSP}(\succeq)$; for instance, the preference $P_i^{(1,1)}$ where $(1, 1)P_i^{(1,1)}(0, 0)P_i^{(1,1)}(1, 0)P_i^{(1,1)}(0, 1)$ is not separable over the two-dimensional cube but it is semilattice single-peaked over $(A, \succeq)$.

[17]For the illustration that we want to make here two objects is enough; the extension to the case with three or more objects is straightforward.

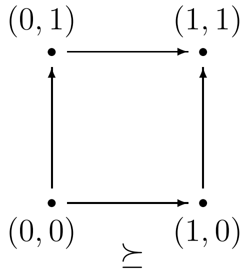

Figure 5

Let k^a and k^b be two integers such that $1 \leq k^a \leq n$ and $1 \leq k^b \leq n$. According to the terminology in Barberà, Sonnenschein and Zhou (1991), applied to the case with two objects, a rule $f : \mathcal{D}^n \rightarrow A$ is *voting by quota* (k^a, k^b) if, for all $P \in \mathcal{D}^n$, object $x \in \{a, b\}$ is an element of the chosen set at P (*i.e.*, $x \in f(P)$) if and only if $|N(R, x)| \geq k^x$. Barberà, Sonnenschein and Zhou (1991) show that $f : \mathcal{S}^n \rightarrow A$ is strategy-proof, anonymous and onto if and only if f is voting by quota.

For the semilattice depicted in Figure 5, and since $A^\star(\succeq) = \emptyset$, the unique strategy-proof and simple rule on $\mathcal{SSP}(\succeq)^n$ is voting by quota $(k^a, k^b) = (1, 1)$, which corresponds to the $\sup_{\succeq}$ rule. Figure 6 depicts the other three semilattices over A.

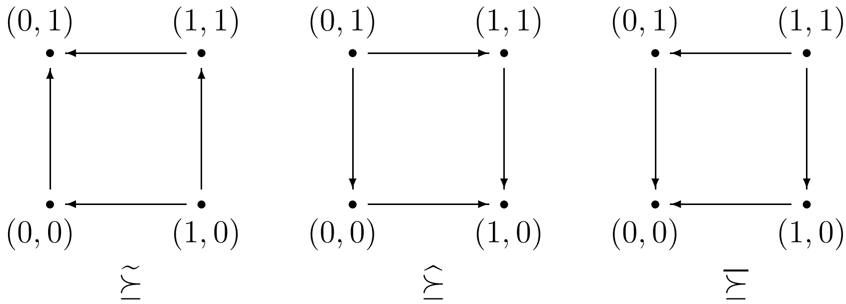

Figure 6

Since $A^\star(\widetilde{\succeq}) = A^\star(\widehat{\succeq}) = A^\star(\overline{\succeq}) = \emptyset$, the unique strategy-proof and simple rule in each of the three cases is the one that corresponds to voting by quota $(\widetilde{k}^a, \widetilde{k}^b) = (n, 1)$, $(\widehat{k}^a, \widehat{k}^b) = (1, n)$ and $(\overline{k}^a, \overline{k}^b) = (n, n)$, or to $\sup_{\widetilde{\succeq}}$, $\sup_{\widehat{\succeq}}$ and $\sup_{\overline{\succeq}}$, respectively, each corresponding to the unique strategy-proof and simple rule on the respective domains $\mathcal{SSP}(\widetilde{\succeq})^n$, $\mathcal{SSP}(\widehat{\succeq})^n$ and $\mathcal{SSP}(\overline{\succeq})^n$.

5 Rich domains and additional results

Chatterji and Massó (2018) shows that semilattice single-peakedness is a necessary condition of a domain $\mathcal{D}$ admitting a strategy-proof and simple rule $f : \mathcal{D}^n \rightarrow A$, provided that n is even and $\mathcal{D}$ is rich.[18] Here, we first present the Chatterji and Massó (2018) notion of rich domain and show that, given any semilattice $(A, \succeq)$, the domain $\mathcal{SP}(\succeq)$ is indeed rich, and hence $\mathcal{SSP}(\succeq)$ is rich as well (richness is a property inherited by supersets).

Definition 4 *Let $\succeq$ be a binary relation over A. The domain $\mathcal{D}$ is **rich** on $(A, \succeq)$ if, for all $x, y, z \in A$ with $z \notin [x, y] \neq \emptyset$, there exist $R_i^x, R_i^y \in \mathcal{D}$ such that $yP_i^x z$ and $xP_i^y z$.*

To illustrate richness return to Example 1 and consider, for instance, alternatives $x_3, x_5, x_8 \in A$ for which $x_8 \notin [x_3, x_5] \neq \emptyset$. In this case, $P_i^{x_3}, P_i^{x_5} \in \mathcal{SSP}(\preceq)$ are such that $x_5 P_i^{x_3} x_8$ and $x_3 P_i^{x_5} x_8$.

Well-known domains of preferences satisfying generalized notions of single-peakedness studied in the literature are rich (see Chatterji and Massó (2018)). However, subsets of single-peaked domains may not be rich, if they are substantially restricted; for example, the Euclidean preference domain is not rich.[19] Nevertheless, the set of all single-peaked preferences $\mathcal{SP}(\succeq)$ is rich on $(A, \succeq)$. This result is stated as Lemma 1.

Lemma 1 *Let $\succeq$ be a semilattice over A. Then, the domain $\mathcal{SP}(\succeq)$ is rich on $(A, \succeq)$.*

Proof Let $x, y, z \in A$ be such that $z \notin [x, y] \neq \emptyset$. Note that $x \neq z$ and $y \neq z$. We show that there is $R_i^x \in \mathcal{SP}(\succeq)$ such that $yP_i^x z$ (to show that there is $R_i^y \in \mathcal{SP}(\succeq)$ such that $xP_i^y z$ follows an analogous argument and it is omitted). If $x = y$ then all preferences $R_i^x \in \mathcal{SP}(\succeq)$ trivially fulfill $yP_i^x z$, so assume $x \neq y$. There are three cases to consider:

1. $\boldsymbol{y \prec z}$. Then, $x \prec y \prec z$. By assumption, there exists at least one $R_i^x \in \mathcal{SP}(\succeq)$, and by part (i) of Definition 2, $yP_i^x z$.

2. $\boldsymbol{z \prec y}$. If $z \prec x$, then $z \prec x \prec y$ and there is $R_i^x \in \mathcal{SP}(\succeq)$ such that $yP_i^x z$, since Definition 2 imposes no restriction on how R_i^x should order y and z. Otherwise, when $z \not\prec x$, if $y = \sup_{\succeq}\{x, z\}$ we have $yP_i^x z$ by part (ii) of Definition 2, whereas

[18] Note that our results in this paper hold for any $n \geq 2$, independently on whether n is odd or even.

[19] Let A be a subset of an Euclidean space, let $\|\cdot\|$ be the Euclidean norm and let $x \in A$. We say that R_i^x is an *Euclidean* preference if, for all $y, z \in A$, $yR_i^x z$ if and only if $\|x - y\| \leq \|x - z\|$. Euclidean domains are not rich: for each $x \in A$, there is only one Euclidean preference whose top is x.

if $y \neq \sup_{\succeq}\{x, z\}$ Definition 2 imposes no restriction on how R_i^x should order y and z, and so we can always take $R_i^x \in \mathcal{SP}(\succeq)$ such that $yP_i^x z$.

3. $\boldsymbol{y \not\prec z}$ **or** $\boldsymbol{z \not\prec y}$. Notice that, in this case, $y \neq \sup_{\succeq}\{x, z\}$. Again, Definition 2 imposes no restriction on how R_i^x should order y and z, and so we can always take $R_i^x \in \mathcal{SP}(\succeq)$ such that $yP_i^x z$.

Hence, the domain $\mathcal{SP}(\succeq)$ is rich on $(A, \succeq)$. ■

As we have already said, richness is a property inherited by larger domains. Hence, we obtain, as a consequence of Lemma 1, that the domain $\mathcal{SSP}(\succeq)$ is rich on $(A, \succeq)$.

Corollary 2 *Let $\succeq$ be a semilattice over A. Then, the domain $\mathcal{SSP}(\succeq)$ is rich on $(A, \succeq)$.*

Our approach in this paper takes as given a semilattice $(A, \succeq)$ relative to which the domain of semilattice single-peaked preferences $\mathcal{SSP}(\succeq)$ is defined. Our main result characterizes the class of all strategy-proof and simple rules on $\mathcal{SSP}(\succeq)^n$.

The approach in Chatterji and Massó (2018) first takes as given a simple rule $f : \mathcal{D}^2 \to A$, defined for two agents on an arbitrary domain $\mathcal{D}$. They define the binary relation $\succeq^f$ over A induced by f as follows: for all $x, y \in A$,

$$x \succeq^f y \text{ if and only if } f(x, y) = x.$$

They first show that $\succeq^f$ is reflexive (since f is unanimous), antisymmetric (since f is anonymous) and transitive whenever $\mathcal{D}$ is rich on $(A, \succeq^f)$. Moreover, the supremum (according to $\succeq^f$) of every pair of alternatives always exists. Hence, the partial order $\succeq^f$ is a (join-)semilattice over the set of alternatives. Now, suppose that an arbitrary simple rule $g : \mathcal{D}^n \to A$, for n even, is given. Define $f^g : \mathcal{D}^2 \to A$ by setting, for all $x, y \in A$,

$$f^g(x, y) = g(\underbrace{x, \ldots, x}_{\frac{n}{2}\text{-times}}, \underbrace{y, \ldots, y}_{\frac{n}{2}\text{-times}}).$$

Chatterji and Massó (2018)'s main result says that if $g : \mathcal{D}^n \to A$ is strategy-proof and simple, n is even and $\mathcal{D}$ is rich on $(A, \succeq^{f^g})$, then $\mathcal{D}$ is a domain of semilattice single-peaked preferences on $(A, \succeq^{f^g})$.

Following the point of view of Chatterji and Massó (2018) for $n = 2$, and extending it to the case of any $n \geq 2$, we define the binary relation $\succeq^{f,k}$ over A induced by a simple rule $f : \mathcal{D}^n \to A$ and some quota $1 \leq k < n$ as follows.

Definition 5 *Let $f : \mathcal{D}^n \to A$ be a simple rule and let k be an integer such that $1 \leq k < n$. The binary relation $\succeq^{f,k}$ (over A)* ***induced by f with quota k*** *is defined by setting, for each pair $x, y \in A$,*

$$x \succeq^{f,k} y \text{ if and only if } f(R) = x$$

for any $R \in \mathcal{D}^n$ such that $t(R) = \{x, y\}$ and $|N(R, x)| \geq k$.

Namely, $x \succeq^{f,k} y$ holds if and only if f selects x whenever at least k agents have the top at x and all other agents have the top at y. Propositions 1 and 2 relate the two different approaches. Proposition 1 for quota-supremum rules with any quota and Proposition 2 for any strategy-proof and simple rule and quota one.

Proposition 1 states that if $(A, \succeq)$ is a semilattice and $f : \mathcal{SSP}(\succeq)^n \to A$ is a quota-supremum rule with associate alternative x and quota $q^x = k$, then $\sup_{\succeq^{f,k}} A = x$ and $\succeq^{f,k}$ coincides with $\succeq$ except for the comparisons involving x.

Proposition 1 *Let $\succeq$ be a semilattice over A, let $f : \mathcal{SSP}(\succeq)^n \to A$ be a quota-supremum rule with associated alternative $x \in A^{\star}(\succeq)$ and quota $q^x = k$, and let $\succeq^{f,k}$ be the binary relation induced by f over A with quota k. Then,*

(i) $x \succeq^{f,k} y$ for all $y \in A \setminus \{x\}$,

(ii) $y \succeq^{f,k} z$ if and only if $y \succeq z$ for all distinct $y, z \in A \setminus \{x\}$.

PROOF See Appendix 7.2. ■

Proposition 2 generalizes the main point illustrated in Example 2, by establishing that a strategy-proof and simple rule $f : \mathcal{SSP}(\succeq)^n \to A$ coincides with the $\sup_{\succeq}$ rule if and only if $\succeq$ coincides with $\succeq^{f,1}$.

Proposition 2 *Let $\succeq$ be a semilattice over A, let $f : \mathcal{SSP}(\succeq)^n \to A$ be a strategy-proof and simple rule and let $\succeq^{f,1}$ be the binary relation induced by f over A with quota 1. Then, $f = \sup_{\succeq}$ if and only if $\succeq$ is equal to $\succeq^{f,1}$.*

PROOF See Appendix 7.3. ■

We finish this section by showing that the domain $\mathcal{SSP}(\succeq)$ is a maximal domain for the $\sup_{\succeq}$ rule.

Definition 6 *Let $\mathcal{D} \subseteq \mathcal{R}$ be a domain and let $f : \mathcal{R}^n \to A$ be a simple rule such that $f : \mathcal{D}^n \to A$ is strategy-proof. The domain $\mathcal{D}$ is* **maximal** *for f if for any $\widetilde{\mathcal{D}}$ such that $\mathcal{D} \subsetneq \widetilde{\mathcal{D}} \subseteq \mathcal{R}$, the rule $f : \widetilde{\mathcal{D}}^n \to A$ is not strategy-proof.*

Proposition 3 *Let $\succeq$ be a semilattice over A and let* $\sup_{\succeq} : \mathcal{SSP}(\succeq)^n \to A$ *be the supremum rule. Then,* $\mathcal{SSP}(\succeq)$ *is maximal for* $\sup_{\succeq}$.

PROOF See Appendix 7.4. ■

Remark 4 *When $n = 2$ the converse of Proposition 3 is also true: the only strategy-proof and simple rule for which $\mathcal{SSP}(\succeq)$ is maximal for it is the supremum rule. Example 2 below illustrates this fact.*

Example 2 (continued) The domain $\mathcal{SSP}(\succeq_1)$ is not maximal for f_2 and f_3, but either rule can be represented as the supremum of their respective induced semilattice (with quota 1); namely, $f_2 = \sup_{\succeq_2}$ and $f_3 = \sup_{\succeq_3}$, and $\succeq^{f_2,1} = \succeq_2$ and $\succeq^{f_3,1} = \succeq_3$. Moreover, the maximal domains for the rules f_2 and f_3, that coincide with their associated semilattice single-peaked domains, are obtained by adding to $\mathcal{SSP}(\succeq_1)$ the preference P_i^{yz} for rule f_2 and the preference P_i^{zy} for rule f_3; namely,

$$\begin{aligned}
\mathcal{SSP}(\succeq_1) &= \{P_i^{xy}, P_i^{xz}, P_i^{yx}, P_i^{zx}\} \\
\mathcal{SSP}(\succeq_2) &= \{P_i^{xy}, P_i^{xz}, P_i^{yx}, P_i^{zx}, P_i^{yz}\} = \mathcal{SSP}(\succeq_1) \cup \{P_i^{yz}\} \\
\mathcal{SSP}(\succeq_3) &= \{P_i^{xy}, P_i^{xz}, P_i^{yx}, P_i^{zx}, P_i^{zy}\} = \mathcal{SSP}(\succeq_1) \cup \{P_i^{zy}\}.
\end{aligned}$$

Of course, Propositions 2 and 3 are satisfied by $f_1 : \mathcal{SSP}(\succeq_1)^2 \to A$. It is easy to check that $f_1(R) = \sup_{\succeq_1} t(R)$ for all $R \in \mathcal{SSP}(\succeq_1)^2$ and that $\succeq_1 = \succeq^{f_1}$ holds. Moreover, $\mathcal{SSP}(\succeq_1)$ is a maximal domain for f_1 but $\mathcal{SSP}(\succeq_2)$ is not because $f_1 : \mathcal{SSP}(\succeq_2)^2 \to A$ is not strategy-proof since

$$z = f_1(P_1^{zx}, P_2^{zx}) P_1^{yz} f_1(P_1^{yz}, P_2^{zx}) = x,$$

which means that agent 1 can manipulate f_1 at (P_1^{yz}, P_2^{zx}) by declaring P_1^{zx} instead of P_1^{yz}. Observe that $P_1^{yz} \notin \mathcal{SSP}(\succeq_1)$ and indeed, $f_1(P_1^{yz}, P_2^{zx}) = x \neq y = \sup_{\succeq_2}\{y, z\}$ where $\{y, z\} = t(P_1^{yz}, P_2^{zx})$ and $\succeq_1 \neq \succeq^{f_2}$. Similarly, $\mathcal{SSP}(\succeq_3)$ is not a maximal domain for $f_1 : \mathcal{SSP}(\succeq_3)^2 \to A$ because it is not strategy-proof since

$$y = f_1(P_1^{yx}, P_2^{yx}) P_1^{zy} f_1(P_1^{zy}, P_2^{yx}) = x,$$

which means that agent 1 can manipulate f_1 at (P_1^{zy}, P_2^{yx}) by declaring P_1^{yx} instead of P_1^{zy}. Observe that $P_1^{zy} \notin \mathcal{SSP}(\succeq_1)$ and indeed, $f_1(P_1^{zy}, P_2^{yx}) = x \neq z = \sup_{\succeq_3}\{y, z\}$ where $\{y, z\} = t(P_1^{zy}, P_2^{yx})$ and $\succeq_1 \neq \succeq^{f_3}$. □

The next example exhibits a simple and strategy-proof rule $f : \mathcal{SSP}(\succeq)^3 \to A$ such that $\mathcal{SSP}(\succeq)$ is maximal for f but $f \neq \sup_{\succeq}$, so the converse of Proposition 3 does not hold for $n > 2$.

Example 3 Let $\succeq_1$ be as in Example 2. Consider the rule $f_4 : \mathcal{SSP}(\succeq_1)^3 \to A$ described by

$$f_4(y,y,z) = y \qquad f_4(x,x,y) = x$$
$$f_4(x,y,y) = y \quad \text{and} \quad f_4(x,y,z) = x$$
$$f_4(y,z,z) = x \qquad f_4(x,z,z) = x$$
$$f_4(x,x,z) = x$$

together with the corresponding choices required by unanimity and anonymity. It is easy to check that $f_4 : \mathcal{SSP}(\succeq_1)^3 \to A$ is strategy-proof. Then, $f_4 \neq \sup_{\succeq_1}$ but $\mathcal{SSP}(\succeq_1)$ is maximal for f_4, since the addition in the domain of either P_i^{yz} or P_i^{zy} would induce a manipulation by some of the agents. To see that, suppose P_i^{yz} or P_i^{zy} is available to agent 1. Then,

$$z = f_4(P_1^{zx}, P_2^{zx}, P_3^{zx}) P_1^{yz} f_4(P_1^{yz}, P_2^{zx}, P_3^{zx}) = x$$

or

$$y = f_4(P_1^{yx}, P_2^{yx}, P_3^{zx}) P_1^{zy} f_4(P_1^{zy}, P_2^{yx}, P_3^{zx}) = x,$$

contradicting strategy-proofness. Hence, the converse of Proposition 3 does not hold for $n > 2$. □

6 Final remarks

We finish the paper with two remarks.

First, we have assumed, and intensively used from the very beginning, that $|A| \geq 3$. The case $A = \{x, y\}$ is special since there are only two semilattices over A, $x \succ^x y$ or $y \succ^y x$, and they are somehow equivalent since the two corresponding semilattice single-peaked domains coincide with the universal domain of strict preferences over A; namely, $\mathcal{SSP}(\succ^x) = \mathcal{SSP}(\succ^y) = \{P_i^x, P_i^y\} = \mathcal{R}$. It is well-known that when the set of alternatives has cardinality two, the class of all strategy-proof and simple rules is voting by quota $(t^x, t^y) \in \{1, \ldots, n\}^2$, where $t^x + t^y = n + 1$.[20] When x is used as the reference alternative, and y is identified with the empty set ("x is not elected"),

[20] This is equivalent to the one-object case of voting by committees in Barberà, Sonnenschein and Zhou (1991), after identifying either y with the empty set ("x is not elected") or x with the empty set ("y is not elected"). Then, the domain of separable (and additive) preferences coincides with the universal domain of strict preferences.

for all $R \in \mathcal{R}^n$,

$$f^x(R) = \begin{cases} x & \text{if } |N(R,x)| \geq t^x \\ y & \text{otherwise,} \end{cases} \tag{1}$$

or when y is used as a references alternative, and x is identified with the empty set ("y is not elected"), for all $R \in \mathcal{R}^n$,

$$f^y(R) = \begin{cases} y & \text{if } |N(R,y)| \geq t^y \\ x & \text{otherwise.} \end{cases} \tag{2}$$

Since $t^x + t^y = n + 1$, the two rules coincide because

$$\begin{aligned} |N(R,x)| &\geq t^x \\ &\iff |N(R,x)| + t^y \geq n+1 = |N(R,x)| + |N(R,y)| + 1 \\ &\iff t^y \geq |N(R,y)| + 1 \\ &\iff t^y > |N(R,y)|. \end{aligned}$$

Hence, for all $R \in \mathcal{R}^n$, $f^x(R) = f^y(R)$. For the semilattice $\succ^x$, $A^\star(\succ^x) = \{y\}$ and $f^y : \mathcal{R}^n \to \{x, y\}$ in (2) is the description of voting by quota (t^x, t^y) as a quota-supremum rule for y with quota t^y, and for the semilattice $\succ^y$, $A^\star(\succ^y) = \{x\}$ and $f^x : \mathcal{R}^n \to \{x, y\}$ in (1) is the description of voting by quota (t^x, t^y) as a quota-supremum rule for x with quota t^x. Hence, the characterization in Theorem 1 remains valid even when $|A| = 2$.

Second, one may wonder whether our analysis of strategy-proof rules on semilattice single-peaked domains could be straightforwardly extended to the analysis of group strategy-proof rules on the same domain.[21] The answer to this query is negative. The domain of semilattice single-peaked preferences does not satisfy indirect sequential inclusion, the weaker condition on a domain identified by Barberà, Berga and Moreno (2010) as being sufficient for the equivalence of individual and group strategy-proofness. To see that in general strategy-proofness is strictly weaker than group strategy-proofness on semilattice single-peaked domains, consider again the special case of voting by quota studied by Barberà, Sonnenschein and Zhou (1991) with two objects (see Subsection 4.2). Then, voting by quota 1 for the two objects is not efficient since $\sup_{\succeq}\{(1,0),(0,1)\} = (1,1)$ but there are separable preferences $P_1^{(1,0)}$

[21] If this would have been the case, some arguments in the proofs of our results would be simpler, since instead of replacing the preferences of agents sequentially they could be replaced simultaneously all together.

and $P_2^{(0,1)}$ for which $(0,0)P_1^{(1,0)}(1,1)$ and $(0,0)P_2^{(0,1)}(1,1)$. Hence, the $\sup_{\succeq}$ rule although strategy-proof on the domain $\mathcal{SSP}(\succeq)$ it is not efficient, and so it is not group strategy-proof, even on the set of separable preferences $\mathcal{S}$ over the two-dimensional cube, a subdomain of semilattice single-peaked preferences on $(A, \succeq)$.

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7 Appendix

7.1 Preliminary remarks and lemmata

Remarks 5 and 6 identify circumstances under which, given any three different alternatives $x, y, z \in A$, the set $\mathcal{SSP}(\succeq)$ contains preferences with top on x that may or may not freely order y and z. The two remarks will be useful in some of the proofs that follow.

Remark 5 *Let $x, y, z \in A$ be three different alternatives and let $\succeq$ be a semilattice over A. If $x \succ y$, then*
(i) there exists $R_i^x \in \mathcal{SSP}(\succeq)$ such that $zP_i^x y$,
(ii) there exists $R_i^x \in \mathcal{SSP}(\succeq)$ such that $yP_i^x z$.

Proof Suppose $x, y, z \in A$ are three different alternatives with $x \succ y$. We first show that part (i) of the Remark holds. If $z \in [y, x]$, then $y \notin [z, x] \neq \emptyset$ and the result follows from richness. If $z \notin [y, x]$, we will show that neither condition (i) nor condition (ii) in Remark 1 characterizing *semilattice single-peakedness* imply $yR_i^x z$. First, notice that since $x \succ y$, condition (i) in Remark 1 cannot be applied, and hence the ordering between y and z is unrestricted. Second, notice that although $y \not\succeq x$, condition (ii) in Remark 1 only implies $x = \sup_{\succeq}\{x, y\} R_i^x y$, and hence the ordering between y and z is also unrestricted. Thus, there exists $R_i^x \in \mathcal{SSP}(\succeq)$ such that $zP_i^x y$. To show that part (ii) holds, notice that if $z \notin [y,x]$ the result follows from *richness*; whereas if $z \in [y,x]$ it follows that $x \succ z \succ y$ and parts (i) and (ii) of Remark 1 can not be applied, and hence the ordering between y and z is unrestricted. Thus, there exists $R_i^x \in \mathcal{SSP}(\succeq)$ such that $yP_i^x z$. ■

Remark 6 *Let $x, y, z \in A$ be three different alternatives and let $\succeq$ be a semilattice over A. If $x \not\succ y$, $y \not\succ x$ and $z \notin [x, \sup_{\succeq}\{x,y\}]$, then there is $R_i^x \in \mathcal{SSP}(\succeq)$ such that $yP_i^x z$.*

Proof Let $x, y, z \in A$ be three different alternatives and suppose $x \not\succ y$, $y \not\succ x$ and $z \notin [x, \sup_{\succeq}\{x,y\}]$. We will show that neither condition (i) nor condition (ii) in Remark 1 characterizing *semilattice single-peakedness* imply that, for each $R_i^x \in \mathcal{SSP}(\succeq)$, $zR_i^x y$ holds. First, notice that since $x \not\prec y$, condition (i) in Remark 1 cannot be applied. Second, notice that since $x \not\prec y$, condition (ii) in Remark 1 only implies that $\sup_{\succeq}\{x,y\}R_i^x y$. But since $z \neq \sup_{\succeq}\{x,y\}$, there exists $R_i^x \in \mathcal{SSP}(\succeq)$ such that $yP_i^x z$. ■

To illustrate the content of Remarks 5 and 6 return again to Example 1 and observe, for instance, that

- $x_3 \succ x_1$ and there exist $P_i^{x_3}, \widehat{P}_i^{x_3} \in \mathcal{SSP}(\succeq)$ such that $x_1 P_i^{x_3} x_2$ and $x_2 \widehat{P}_i^{x_3} x_1$.
- $x_8 \not\succ x_9$, $x_9 \not\succ x_8$, $\sup_{\succeq}\{x_8, x_9\} = x_6$ and $x_4 \notin [x_8, x_6]$ and there is $P_i^{x_8} \in \mathcal{SSP}(\succeq)$ such that $x_9 P_i^{x_8} x_4$.

We next state and prove several lemmata that will be used in the proofs of the Propositions and Theorem 1.

Lemma 2 *Let $\succeq$ be a semilattice over A, let $f : \mathcal{SSP}(\succeq)^n \to A$ be a strategy-proof and simple rule, and let $R \in \mathcal{SSP}(\succeq)^n$. Assume $x, y \in A$ are such that $x \succ y$ or $y \succ x$ and $t(R) = \{x, y\}$. Then, $f(R) \in \{x, y\}$.*

Proof Let the hypothesis of the Lemma hold. Without loss of generality, assume $x \succ y$. The proof is by induction on the cardinality of $N(R,x)$. Consider first $N(R,x) = \{i\}$ (this implies $N(R,y) = N \setminus \{i\}$) and assume $f(R) = z \notin \{x,y\}$. By Remark 5 (ii), there is $R_i^x \in \mathcal{SSP}(\succeq)$ such that $yP_i^x z$. By *tops-onlyness*, $f(R_i^x, R_{-i}) = f(R) = z$. Furthermore, by *unanimity*, $f(R_i^y, R_{-i}) = y$ for any $R_i^y \in \mathcal{SSP}(\succeq)$. Then,

$$f(R_i^y, R_{-i}) = yP_i^x z = f(R_i^x, R_{-i}),$$

contradicting *strategy-proofness.* Hence, $f(R) \in \{x,y\}$. Now, suppose $R \in \mathcal{SSP}(\succeq)^n$ is such that, for all $1 \leq k \leq n-2$, $|N(R,x)| = k$ and $f(R) = \{x,y\}$. We want to see that if $R' \in \mathcal{SSP}(\succeq)^n$ is such that $|N(R',x)| = k+1$, then $f(R') \in \{x,y\}$. To obtain a contradiction, suppose $f(R') = z \notin \{x,y\}$. Let $i \in N(R',x)$. By Remark 5 (ii), there is $R_i^x \in \mathcal{SSP}(\succeq)$ such that $yP_i^x z$. By *tops-onlyness,* $f(R_i^x, R'_{-i}) = f(R') = z$. Furthermore, by the inductive hypothesis and *tops-onlyness*, $f(R_i^y, R'_{-i}) \in \{x,y\}$ for any $R_i^y \in \mathcal{SSP}(\succeq)$. Then, in both case,

$$f(R_i^y, R'_{-i}) P_i^x z = f(R_i^x, R'_{-i}),$$

contradicting *strategy-proofness.* Hence, $f(R') \in \{x,y\}$. ■

Lemma 3 *Let $\succeq$ be a semilattice over A, let $f : \mathcal{SSP}(\succeq)^n \to A$ be a strategy-proof and simple rule and let k be such that $1 \leq k < n$. Assume $x, y \in A$ are such that $x \succ y$ and there is $R \in \mathcal{SSP}(\succeq)^n$ such that $t(R) = \{x,y\}$, $|N(R,y)| = k$ and $f(R) = y$. Then, $f(\widetilde{R}) = y$ for each $\widetilde{R} \in \mathcal{SSP}(\succeq)^n$ such that $|N(\widetilde{R},y)| \geq k$.*

Proof Let the hypothesis of the Lemma hold. Let $\widetilde{R} \in \mathcal{SSP}(\succeq)^n$ be such that $|N(\widetilde{R},y)| \geq k$. We want to show that $f(\widetilde{R}) = y$. Let $S \subset N$ be such that $S \subseteq N(\widetilde{R},y)$ and $|S| = k$. By *anonymity,* we can assume that $N(R,y) = S$. By *tops-onlyness*, $f(\widetilde{R}_S, R_{-S}) = y$. Let $i \in N \setminus S$. We claim that $f(\widetilde{R}_{S\cup\{i\}}, R_{-(S\cup\{i\})}) = y$. If not, $f(\widetilde{R}_{S\cup\{i\}}, R_{-(S\cup\{i\})}) = w \neq y$ and, as $x \succ y$, by Remark 5 (i) there is $R_i^x \in \mathcal{SSP}(\succeq)$ such that $wP_i^x y$. Since $N(R,y) = S$, $t(R_i) = x$. By *tops-onlyness*, $f(\widetilde{R}_S, R_i^x, R_{-(S\cup\{i\})}) = f(\widetilde{R}_S, R_{-S})$. Therefore,

$$f(\widetilde{R}_{S\cup\{i\}}, R_{-(S\cup\{i\})}) = wP_i^x y = f(\widetilde{R}_S, R_i^x, R_{-(S\cup\{i\})}),$$

contradicting *strategy-proofness.* Hence, $f(\widetilde{R}_{S\cup\{i\}}, R_{-(S\cup\{i\})}) = y$. Continuing in the same way, we can successively change the preferences of each of the remaining agents in $N \setminus S$ to finally obtain $f(\widetilde{R}) = y$. ■

Lemma 4 *Let $\succeq$ be a semilattice over A and let $f : \mathcal{SSP}(\succeq)^n \to A$ be a strategy-proof and simple rule. Then, for each $R \in \mathcal{SSP}(\succeq)^n$,*

$$f(R) \in \bigcup_{i\in N} [t(R_i), \sup_{\succeq} t(R)].$$

Proof Let the hypothesis of the Lemma hold. The proof is by induction on the cardinality of $t(R)$. If $R \in \mathcal{SSP}(\succeq)^n$ is such that $|t(R)| = 1$, then the result is trivially true by *unanimity.* Suppose that the result holds for any $R \in \mathcal{SSP}(\succeq)^n$ such that $|t(R)| = k$ and $1 \leq k < n$. Consider now a profile $R \in \mathcal{SSP}(\succeq)^n$ such that $t(R) = \{x_1, \ldots, x_{k+1}\}$ and assume $f(R) = y \notin \cup_{j=1}^{k+1}[x_j, \sup_{\succeq} t(R)]$. There are two cases to consider.

1. **$\sup_{\succeq} t(R) \in t(R)$**. Without loss of generality, assume $\sup_{\succeq} t(R) = x_{k+1}$. Then,

$$\cup_{j=1}^{k+1}[x_j, \sup_{\succeq} t(R)] = \cup_{j=1}^{k}[x_j, x_{k+1}].$$

If $|N(R, x_{k+1})| > 1$, take any $i \in N(R, x_{k+1})$ and consider $R'_i \in \mathcal{SSP}(\succeq)$ such that $t(R'_i) \in t(R) \setminus \{x_{k+1}\}$. We claim that $f(R'_i, R_{-i}) = z \notin \cup_{j=1}^{k}[x_j, x_{k+1}]$. If $z = y$ this is obvious, by the contradiction hypothesis, so assume $z \neq y$ and, to obtain a contradiction, suppose $z \in \cup_{j=1}^{k}[x_j, x_{k+1}]$. We now show that $z \neq x_{k+1}$, because otherwise,

$$f(R'_i, R_{-i}) = z = x_{k+1} P_i^{x_{k+1}} y = f(R_i^{x_{k+1}}, R_{-i}),$$

contradicting *strategy-proofness.* Thus, $x_{k+1} \succ z$. By Remark 5 (ii) there is $R_i^{x_{k+1}} \in \mathcal{SSP}(\succeq)$ such that $z P_i^{x_{k+1}} y$. Notice that, by *tops-onlyness*, $f(R_i^{x_{k+1}}, R_{-i}) = f(R) = y$. Therefore,

$$f(R'_i, R_{-i}) = z P_i^{x_{k+1}} y = f(R_i^{x_{k+1}}, R_{-i}),$$

contradicting *strategy-proofness.* Hence, $f(R'_i, R_{-i}) \notin \cup_{j=1}^{k}[x_j, x_{k+1}]$. If necessary (*i.e.*, if there are still several agents with top at x_{k+1}), we repeat this process until we obtain a profile $\widetilde{R}$ such that $t(\widetilde{R}) = \{x_1, \ldots, x_{k+1}\}$, $|N(\widetilde{R}, x_{k+1})| = 1$, and $f(\widetilde{R}) = t \notin \cup_{j=1}^{k}[x_j, x_{k+1}]$. Hence, let $N(\widetilde{R}, x_{k+1}) = \{i\}$ and consider $R'_i \in \mathcal{SSP}(\succeq)$ such that $t(R'_i) \in t(\widetilde{R}) \setminus \{x_{k+1}\}$. Let $f(R'_i, \widetilde{R}_{-i}) = w$ and $\widetilde{x} = \sup_{\succeq}\{x_1, \ldots, x_k\}$. Then, by the inductive hypothesis, $w \in \cup_{j=1}^{k}[x_j, \widetilde{x}]$. Therefore, $x_{k+1} \succ \widetilde{x} \succeq w$ and, by Remark 5 (ii), there is $R_i^{x_{k+1}} \in \mathcal{SSP}(\succeq)$ such that $w P_i^{x_{k+1}} t$. By *tops-onlyness*, $f(R_i^{x_{k+1}}, \widetilde{R}_{-i}) = f(\widetilde{R}) = t$. Therefore,

$$f(R'_i, \widetilde{R}_{-i}) = w P_i^{x_{k+1}} t = f(R_i^{x_{k+1}}, \widetilde{R}_{-i}),$$

contradicting *strategy-proofness.*

2. **$\sup_{\succeq} t(R) \notin t(R)$**. Let $\widetilde{x} = \sup_{\succeq} t(R)$. Consider any $x^\star \in t(R)$ and let $t^\star(R) = \{x \in t(R) : x \succeq x^\star \text{ or } x^\star \succeq x\}$. Notice that, since $\sup_{\succeq} t(R) \notin t(R)$, $1 \leq |t^\star(R)| < |t(R)|$. Let $N^\star = \{i \in N : t(R_i) \in t^\star(R)\}$ and let $i^\star \in N(R, x^\star)$.

Claim: Let $\boldsymbol{i \in N^\star}$ and let $\boldsymbol{R'_i \in \mathcal{SSP}(\succeq)}$ be such that $\boldsymbol{t(R'^\star_i(R)}$. Then, $\boldsymbol{f(R'_i, R_{-i}) \notin \cup_{j=1}^{k+1}[x_j, \widetilde{x}]}$.

Let $i \in N^\star$, $x_i = t(R_i)$, and $z = f(R'_i, R_{-i})$. If $z = y$ this is obvious, by the contradiction hypothesis, so assume $z \neq y$ and, to obtain a contradiction, suppose $z \in \cup_{j=1}^{k+1}[x_j, \widetilde{x}]$. Then, there is $R_i^{x_i} \in \mathcal{SSP}(\succeq)$ such that

$$zP_i^{x_i}y, \tag{3}$$

where, remember, $y = f(R)$. To see that (3) holds, there are two cases to consider. First, assume that $z \succeq x_i$. As $y \notin \cup_{j=1}^{k+1}[x_j, \widetilde{x}]$, it follows that $y \notin [x_i, z]$ and (3) is implied by *richness.* Second, assume that $z \not\succeq x_i$. If $x_i \succeq z$, (3) follows from Remark 5 (ii). If $x_i \not\succeq z$, and since $y \notin [x_i, \widetilde{x}]$, (3) follows from Remark 6. By *tops-onlyness*, $f(R_i^{x_i}, R_{-i}) = f(R) = y$. Therefore,

$$f(R'_i, R_{-i}) = zP_i^{x_i}y = f(R_i^{x_i}, R_{-i}),$$

contradicting *strategy-proofness.* Hence, $f(R'_i, R_{-i}) \notin \cup_{j=1}^{k+1}[x_j, \widetilde{x}]$ and the Claim is proved.

Using the Claim, we proceed by changing the preferences of all the members of $N^\star$ except $i^\star$. We obtain a new profile $\widetilde{R} \in \mathcal{SSP}(\succeq)$ such that $t(\widetilde{R}) \subset t(R)$, $N(\widetilde{R}, x^\star) = \{i^\star\}$, and $f(\widetilde{R}) \notin \cup_{j=1}^{k+1}[x_j, \widetilde{x}]$.

To finish the proof, consider $R'_{i^\star} \in \mathcal{SSP}(\succeq)$ such that $t(R'_{i^\star}) \in t(R) \setminus t^\star(R)$. Let $t = f(\widetilde{R})$, $w = f(R'_{i^\star}, \widetilde{R}_{-i^\star})$ and $\widehat{x} = \sup_{\succeq}[t(R) \setminus t^\star(R)]$. Then, by the inductive hypothesis, $w \in \bigcup_{x_j \in t(R)\setminus t^\star(R)}[x_j, \widehat{x}]$. Since $\sup_{\succeq}\{x^\star, w\} = \widetilde{x}$ and $t \notin [x^\star, \widetilde{x}]$, by Remark 6, there is $R_{i^\star}^{x^\star} \in \mathcal{SSP}(\succeq)$ such that $wP_{i^\star}^{x^\star}t$. By *tops-onlyness*, $f(R_{i^\star}^{x^\star}, \widetilde{R}_{-i}) = f(\widetilde{R}) = t$. Therefore,

$$f(R'_{i^\star}, \widetilde{R}_{-i^\star}) = wP_{i^\star}^{x^\star}t = f(R_{i^\star}^{x^\star}, \widetilde{R}_{-i^\star}),$$

contradicting *strategy-proofness.*

Hence, $f(R) \in \cup_{j=1}^{k+1}[x_j, \sup_{\succeq} t(R)]$. ■

Lemma 5 *Let $\succeq$ be a semilattice over A and let $f : \mathcal{SSP}(\succeq)^n \to A$ be a strategy-proof and simple rule. Let $R \in \mathcal{SSP}(\succeq)^n$, $x \in A$, $i \in N$ and $R_i^x \in \mathcal{SSP}(\succeq)$. If $x \succ t(R_i)$, $x \succ f(R)$ and $f(R) \notin [t(R_i), x]$, then*

$$f(R_i^x, R_{-i}) = f(R).$$

Proof Let the hypothesis of the Lemma hold. First, notice that

$$f(R_i^x, R_{-i}) \in \{f(R), x\}. \tag{4}$$

Otherwise, since $x \succ f(R)$, by Remark 5 (ii) there is $\widetilde{R}_i^x \in \mathcal{SSP}(\succeq)$ such that $f(R)\widetilde{P}_i^x f(R_i^x, R_{-i})$. By *tops-onlyness,* $f(R_i^x, R_{-i}) = f(\widetilde{R}_i^x, R_{-i})$. Therefore,

$$f(R)\widetilde{P}_i^x f(\widetilde{R}_i^x, R_{-i}),$$

contradicting *strategy-proofness.* Second, assume $f(R_i^x, R_{-i}) = x$. Since $f(R) \notin [t(R_i), x]$, by *richness* there is $\widetilde{R}_i \in \mathcal{SSP}(\succeq)$ such that $t(\widetilde{R}_i) = t(R_i)$ and $x\widetilde{P}_i f(R)$. By *tops-onlyness,* $f(R) = f(\widetilde{R}_i, R_{-i})$. Therefore,

$$f(R_i^x, R_{-i})\widetilde{P}_i f(\widetilde{R}_i, R_{-i}),$$

contradicting *strategy-proofness.* To conclude, as (4) holds and $f(R_i^x, R_{-i}) \neq x$, the result follows. ■

Lemma 6 *Let $\succeq$ be a semilattice over A and let $f : \mathcal{SSP}(\succeq)^n \to A$ be a strategy-proof and simple rule. If $R \in \mathcal{SSP}(\succeq)^n$ is such that $\sup_{\succeq} t(R) \notin t(R)$ and $\sup_{\succeq} t(R) \succ f(R)$, then there is $\widetilde{R} \in \mathcal{SSP}(\succeq)^n$ such that $\sup_{\succeq} t(\widetilde{R}) = \sup_{\succeq} t(R)$, $\sup_{\succeq} t(\widetilde{R}) \in t(\widetilde{R})$, and $f(\widetilde{R}) = f(R)$.*

Proof Let the hypothesis of the Lemma hold. Take $i \in N$ such that $f(R) \notin [t(R_i), \sup_{\succeq} t(R)]$ (such i exists, because otherwise $f(R) = \sup_{\succeq} t(R)$). Let $R_i' \in \mathcal{SSP}(\succeq)$ be such that $t(R_i') = \sup_{\succeq} t(R)$ and consider the profile $\widetilde{R} = (R_i', R_{-i})$. Then, $\sup_{\succeq} t(\widetilde{R}) = \sup_{\succeq} t(R) \in t(\widetilde{R})$ and since $f(R) \notin [t(R_i), \sup_{\succeq} t(R)]$ holds, by Lemma 5, $f(\widetilde{R}) = f(R)$. ■

Lemma 7 *Let $\succeq$ be a semilattice over A, let $f : \mathcal{SSP}(\succeq)^n \to A$ be a strategy-proof and simple rule and let k be such that $1 \leq k < n$. Assume $x, y \in A$ are such that $x \succ y$ and there is $R \in \mathcal{SSP}(\succeq)^n$ such that $t(R) = \{x, y\}$, $|N(R, y)| = k$ and $f(R) = y$. Then, $f(\widetilde{R}) \in \{y, \sup_{\succeq} t(\widetilde{R})\}$ for all $\widetilde{R} \in \mathcal{SSP}(\succeq)^n$ such that $|N(\widetilde{R}, y)| < k$.*

Proof Let the hypothesis of the Lemma hold. Let $\widetilde{R} \in \mathcal{SSP}(\succeq)^n$ be such that $|N(\widetilde{R}, y)| = k - 1$. If $k = 1$ then $t(\widetilde{R}) = \{x\}$, and by *unanimity,* $f(\widetilde{R}) = x = \sup_{\succeq} t(\widetilde{R})$ and the statement holds trivially. Suppose $k > 1$. Let $i \in N(R, y)$ and set $S \equiv N(R, y) \setminus \{i\}$. By *anonymity,* we can assume $S = N(\widetilde{R}, y)$. By *tops-onlyness,* $f(\widetilde{R}_S, R_{-S}) = f(R)$. Let $j \in N(R, x)$. First, we show that

$$f(\widetilde{R}_{S\cup\{j\}}, R_{-(S\cup\{j\})}) = y. \tag{5}$$

Otherwise, since $x \succ y$, by Remark 5 (i) there is $R_j^x \in \mathcal{SSP}(\succeq)$ such that

$$f(\widetilde{R}_{S\cup\{j\}}, R_{-(S\cup\{j\})})P_j^x y.$$

By *tops-onlyness,* $f(\widetilde{R}_S, R_j^x, R_{-(S\cup\{j\})}) = f(\widetilde{R}_S, R_{-S}) = y$. Therefore,

$$f(\widetilde{R}_{S\cup\{j\}}, R_{-(S\cup\{j\})})P_j^x(\widetilde{R}_S, R_j^x, R_{-(S\cup\{j\})}),$$

contradicting *strategy-proofness.* Thus, (5) holds. Continuing in the same way, we can change the preferences of each of the remaining agents in $N(R,x)$ to obtain

$$f(\widetilde{R}_{-i}, R_i) = y. \tag{6}$$

Next, we show that $f(\widetilde{R}) \in \{y, \sup_{\succeq} t(\widetilde{R})\}$. Let $s \equiv \sup_{\succeq} t(\widetilde{R})$ and, to get a contradiction, suppose $f(\widetilde{R}) \notin \{y, s\}$. Then, as $f(\widetilde{R}) \prec s$ by Lemma 4, we can assume that $\widetilde{R}$ is such that $s \in t(\widetilde{R})$ by Lemma 6. By *anonymity,* assume $i \in N(R, s)$. Since $s \succ f(\widetilde{R})$, by Remark 5 (i) there is $R_i^s \in \mathcal{SSP}(\succeq)$ such that $yP_i^s f(\widetilde{R})$. By *tops-onlyness,* $f(\widetilde{R}) = f(R_i^s, \widetilde{R}_{-i})$. Therefore, using (6),,

$$f(R_i, \widetilde{R}_{-i})P_i^s f(R_i^s, \widetilde{R}_{-i}),$$

contradicting *strategy-proofness.* Hence, $f(\widetilde{R}) \in \{y, s\}$.

We just proved the Lemma when $\widetilde{R}$ is such that $|N(\widetilde{R}, y)| = k - 1$. In order to prove the statement for a profile $\widehat{R}$ such that $N(\widehat{R}, y) = k - 2$ we use a similar reasoning and the fact that the result is true whenever there are $k - 1$ tops in y. We successively apply the same reasoning to profiles in which the cardinality of the set of agents with top in y is smaller. ■

Lemma 8 *Let $\succeq$ be a semilattice over A such that there is $\sup_{\succeq} A \equiv \alpha$ and let $f : \mathcal{SSP}(\succeq)^n \to A$ be a strategy-proof and simple rule. Assume $f(R) = \alpha$ whenever $\alpha \in t(R)$. Then, $f(R) = \sup_{\succeq} t(R)$ for each $R \in \mathcal{SSP}(\succeq)^n$.*

Proof Let the hypothesis of the Lemma hold. To obtain a contradiction, assume there is $R \in \mathcal{SSP}(\succeq)^n$ such that $f(R) \neq \sup_{\succeq} t(R)$. Since $f(R) \in \cup_{i\in N}[t(R_i), \sup_{\succeq} t(R)]$ by Lemma 4,

$$\sup_{\succeq} t(R) \succ f(R). \tag{7}$$

By Lemma 6, it is without loss of generality to assume that $\sup_{\succeq} t(R) \in t(R)$. Let $i \in N(R, \sup_{\succeq} t(R))$. As $\alpha \succeq \sup_{\succeq} t(R) \succ f(R)$, by *richness*, there is $R_i' \in \mathcal{SSP}(\succeq)$

such that $t(R_i') = \sup_{\succeq} t(R)$ and $\alpha P_i' f(R)$. By *tops-onlyness*, $f(R_i', R_{-i}) = f(R)$. By the hypothesis, $f(R_i^{\alpha}, R_{-i}) = \alpha$ for any $R_i^{\alpha} \in \mathcal{SSP}(\succeq)$. Therefore,

$$f(R_i^{\alpha}, R_{-i}) = \alpha P_i' f(R) = f(R_i', R_{-i}),$$

contradicting *strategy-proofness.* Hence, $f(R) = \sup_{\succeq} t(R)$. ■

Lemma 9 *Let $\succeq$ be a semilattice over A such that $\sup_{\succeq} A$ does not exist and let $f : \mathcal{SSP}(\succeq)^n \to A$ be a strategy-proof and simple rule. Then, $f(R) = \sup_{\succeq} t(R)$ for each $R \in \mathcal{SSP}(\succeq)^n$.*

Proof Let the hypothesis of the Lemma hold. Notice first that, if $|t(R)| = 1$ then the result follows by *unanimity.* To obtain a contradiction, let $R \in \mathcal{SSP}^n$, with $|t(R)| > 1$, be such that $f(R) \neq \sup_{\succeq} t(R)$. Since $f(R) \in \cup_{i \in N}[t(R_i), \sup_{\succeq} t(R)]$ by Lemma 4,

$$\sup_{\succeq} t(R) \succ f(R). \tag{8}$$

By Lemma 6, it is without loss of generality to assume that $\sup_{\succeq} t(R) \in t(R)$. Let

$$s \equiv \sup_{\succeq} t(R). \tag{9}$$

First, notice that there is $x \in A$ such that $x \succ s$. Since, by the hypothesis, there is no $\sup_{\succeq} A$, there exists $y \in A$ such that $s \not\succeq y$. If $y \succ s$, take $x = y$; whereas if $s \not\succeq y$ and $y \not\succeq s$, take $x = \sup_{\succeq}\{s, y\}$. We now proceed through several steps.

Claim 1: If $S \subseteq N(R, s)$ and $\widetilde{R}_S \in \mathcal{SSP}(\succeq)^{|S|}$ is such that $t(\widetilde{R}_i) = x$ for each $i \in S$, then $f(\widetilde{R}_S, R_{-S}) = f(R)$.

The proof is by induction on the cardinality of S. Suppose first that $S = \{i\}$ and consider any $\widetilde{R}_i \in \mathcal{SSP}(\succeq)$ such that $t(\widetilde{R}_i) = x$. As $f(R) \notin [s, x]$, by Lemma 5,

$$f(\widetilde{R}_i, R_{-i}) = f(R). \tag{10}$$

Suppose now that $f(\widetilde{R}_S, R_{-S}) = f(R)$ and consider $i \in N(R, s) \backslash S$. Let $\widetilde{R}_i \in \mathcal{SSP}(\succeq)$ be such that $t(\widetilde{R}_i) = x$. As $f(\widetilde{R}_S, R_{-S}) \notin [s, x]$, by Lemma 5,

$$f(\widetilde{R}_{S\cup\{i\}}, R_{-S\cup\{i\}}) = f(\widetilde{R}_S, R_{-S}). \tag{11}$$

Thus, $f(\widetilde{R}_{S\cup\{i\}}, R_{-S\cup\{i\}}) = f(R)$ and the Claim holds.

Claim 2: If $S \supseteq N(R, s)$ and $\widetilde{R}_S \in \mathcal{SSP}(\succeq)^{|S|}$ is such that $t(\widetilde{R}_i) = x$ for each $i \in S$, then $f(\widetilde{R}_S, R_{-S}) = f(R)$.

The proof is by induction on the cardinality of S. Let $\widetilde{N} \equiv N(R,s)$, $i \notin \widetilde{N}$ and consider $\widetilde{R}_{\widetilde{N}} \in \mathcal{SSP}(\succeq)^{|\widetilde{N}|}$ with $t(\widetilde{R}_j) = x$ for each $j \in \widetilde{N}$ and $R'_i \in \mathcal{SSP}(\succeq)$ with $t(R'_i) = s$. Since $s \succ f(R)$ and $f(R_{\widetilde{N}}, R_{-\widetilde{N}}) = f(R)$ by Claim 1, by Lemma 5 it follows that

$$f(R'_i, \widetilde{R}_{\widetilde{N}}, R_{-(\widetilde{N}\cup\{i\})}) = f(\widetilde{R}_{\widetilde{N}}, R_{-\widetilde{N}}).$$

Now consider $\widetilde{R}_i \in \mathcal{SSP}(\succeq)$ such that $t(\widetilde{R}_i) = x$. Since $x \succ f(R)$ and $f(\widetilde{R}_{\widetilde{N}}, R_{-\widetilde{N}}) = f(R)$ by Claim 1, by Lemma 5 it follows that

$$f(\widetilde{R}_{\widetilde{N}\cup\{i\}}, R_{-(\widetilde{N}\cup\{i\})}) = f(\widetilde{R}_{\widetilde{N}}, R_{-\widetilde{N}}).$$

To finish the proof of Claim 2, suppose next that $S \supsetneq N(R,s)$, $\widetilde{R}_S \in \mathcal{SSP}(\succeq)^{|S|}$ is such that $t(\widetilde{R}_i) = x$ for each $i \in S$ and $f(\widetilde{R}_S, R_{-S}) = f(R)$. Let $i \notin S$. First, let $R'_i \in \mathcal{SSP}(\succeq)$ such that $t(R'_i) = s$. By an analogous reasoning as the one presented to obtain (10), we can prove that $f(R'_i, \widetilde{R}_S, R_{-(S\cup i)}) = f(\widetilde{R}_S, R_{-S})$. Second, let $\widetilde{R}_i \in \mathcal{SSP}(\succeq)$ be such that $t(\widetilde{R}_i) = x$. By an analogous reasoning as the one presented to obtain (11), we can prove that $f(\widetilde{R}_{S\cup i}, R_{-(S\cup i)}) = f(\widetilde{R}_S, R_{-S})$. Therefore, $f(\widetilde{R}_{S\cup i}, R_{-(S\cup i)}) = f(R)$ and the inductive proof is complete.

Concluding. Let $\widetilde{R} \in \mathcal{SSP}(\succeq)^n$ be such that $t(\widetilde{R}_i) = x$ for each $i \in N$. By *unanimity,* $f(\widetilde{R}) = x$. Applying Claim 2 when $S = N$ we get $f(\widetilde{R}) = f(R)$. Then, $f(R) = x$. But, as $x \succ s$ and $s \succ f(R)$, it follows by (8) and (9) that $f(R) \succ f(R)$, a contradiction. Hence, $f(R) = \sup_{\succeq} t(R)$ for each $R \in \mathcal{SSP}(\succeq)^n$. ■

7.2 Proof of Proposition 1

Let the hypothesis of the Proposition hold.

(i) This is direct consequence of the fact that f is a *quota-supremum* with associated alternative x and quota k.

(ii) ($\Longrightarrow$) Assume $y \succeq^{f,k} z$. Since f is a *quota-supremum*, $\alpha = \sup_{\succeq} A$ exists. Since y and z are different and $y \neq x$, we claim that $z \neq \alpha$. Otherwise, $y \succeq^{f,k} \alpha$ implies $f(R) = y$ for $\widehat{R} \in \mathcal{SSP}^n$ such that $t(\widehat{R}) = \{y, \alpha\}$ and $|N(\widehat{R}, y)| \geq k$, contradicting the definition of *quota-supremum.* Therefore, $z \neq \alpha$. Next, suppose, to get a contradiction, that $y \not\succeq z$. First, consider the case $z \succ y$. Let $i \in N$, and let $R \in \mathcal{SSP}(\succeq)^n$ be such that $t(R_i) = z$, $|N(R,y)| = k-1$ and $|N(R,\alpha)| = n-k$. Then, by the definition of f as a *quota-supremum* rule with associate alternative and quota k, $f(R) = \alpha$. Consider any $R_i^y \in \mathcal{SSP}(\succeq)$. As $y \succeq^{f,k} z$, $z \succ y$ and profile (R_i^y, R_{-i}) reaches the quota k for alternative y, $f(R_i^y, R_{-i}) = y$ by Lemma 3. As $z \succ y$, by Remark 5 (ii), there is $R_i^z \in \mathcal{SSP}(\succeq)$ such that $yP_i^z\alpha$. By *tops-onlyness,* $f(R_i^z, R_{-i}) = f(R)$. Therefore,

$$f(R_i^y, R_{-i}) = yP_i^z\alpha = f(R_i^z, R_{-i}),$$

contradicting *strategy-proofness.* Second, consider the case $z \not\succeq y$. Let $t \equiv \sup_{\succeq}\{y,z\}$. Then $t \succ y$. Let $R \in \mathcal{SSP}(\succeq)^n$ be such that $t(R) = \{y,z\}$ and $|N(R,y)| = k$. By definition of *quota-supremum*, $f(R) = \sup_{\succeq}\{y,z\} = t \neq y$, which contradicts $y \succeq^{f,k} z$. Thus, $y \succeq z$.

($\Longleftarrow$) Assume $y \succeq z$. Let $R \in \mathcal{SSP}(\succeq)^n$ be such that $t(R) = \{y,z\}$ and $|N(R,y)| \geq k$. As f is a *quota-supremum* with associated alternative x and quota k, $f(R) = \sup_{\succeq}\{y,z\} = y$, which implies $y \succeq^{f,k} z$. ■

7.3 Proof of Proposition 2

Let the hypothesis of the Proposition hold.

($\Longrightarrow$) Let $x, y \in A$ and assume $f(R) = \sup_{\succeq} t(R)$ for each $R \in \mathcal{SSP}(\succeq)^n$. First, suppose $x \succeq y$, take $i \in N$ and $\widetilde{R} \in \mathcal{SSP}(\succeq)^n$ such that $t(\widetilde{R}_i) = x$ and $t(\widetilde{R}_j) = y$ for each $j \in N \setminus \{i\}$. Since $\sup_{\succeq} t(\widetilde{R}) = x$, then $f(\widetilde{R}) = x$. Therefore, $x \succeq^{f,1} y$. Second, suppose $x \succeq^{f,1} y$. If $\widetilde{R} \in \mathcal{SSP}(\succeq)^n$ is as before, then $x = f(\widetilde{R}) = \sup_{\succeq} t(\widetilde{R})$ and $x \succeq y$.

($\Longleftarrow$) Assume $\succeq$ is equal to $\succeq^{f,1}$ but $f \neq \sup_{\succeq}$. By Lemmata 8 and 9, there are $\alpha \in A$ such that $\alpha = \sup_{\succeq} A$ and $\widetilde{R} \in \mathcal{SSP}(\succeq)^n$ such that $\alpha \in t(\widetilde{R})$ and $f(\widetilde{R}) \neq \alpha$. Let $x \equiv f(\widetilde{R})$ and let $i \in N(\widetilde{R}, \alpha)$. Consider $j \in N \setminus \{i\}$ and take any $R_j^x \in \mathcal{SSP}(\succeq)$. By *strategy-proofness,* $f(R_j, \widetilde{R}_{-j}) = x$. Continuing in the same way, we successively change each of the preferences of the remaining agents in $N \setminus \{i\}$ to preferences with top in x obtaining, by *strategy-proofness,*

$$f(\widetilde{R}_i, R_{-i}^x) = x. \tag{12}$$

As $\alpha = \sup_{\succeq} A$, $\alpha \succeq x$. Then, by the hypothesis, $\alpha \succeq^{f,1} x$ which implies $f(\widetilde{R}_i, R_{-i}^x) = \alpha$ since $i \in N(\widetilde{R}, \alpha)$. But this contradicts (12). ■

7.4 Proof of Proposition 3

Let the hypothesis of the Proposition hold. Assume $\mathcal{SSP}(\succeq)$ is not *maximal* for $\sup_{\succeq}$. Then, there is $\widehat{R}_i \in \mathcal{R} \setminus \mathcal{SSP}(\succeq)$ such that $\sup_{\succeq} : [\mathcal{SSP}(\succeq) \cup \{\widehat{R}_i\}]^n \to A$ is *simple* and *strategy-proof.* By Remark 1 characterizing *semilattice single-peakedness*, there are two cases to consider:

1. **There are $x, y, z \in A$ with $z \succ y \succ x$, $t(\widehat{R}_i) = x$ and $z\widehat{P}_i y$.** Let $R_{-i} \in \mathcal{SSP}(\succeq)^{n-1}$ be such that $t(R_j) = y$ for each $j \in N \setminus \{i\}$ and let $R_i^z \in \mathcal{SSP}(\succeq)$ be arbitrary. As $z \succ y \succ x$,

$$\sup_{\succeq}(R_i^z, R_{-i}) = z\widehat{P}_i y = \sup_{\succeq}(\widehat{R}_i, R_{-i}),$$

contradicting *strategy-proofness.*

2. **There are $x, w \in A$ such that $w \not\succeq x$, $t(\widehat{R}_i) = x$ and $w\widehat{P}_i \sup_{\succeq}\{x, w\}$.** First, notice that $x \not\succeq w$ also. Otherwise $w\widehat{P}_i x$, contradicting the fact that $t(\widehat{R}_i) = x$. Let $t = \sup_{\succeq}\{x, w\}$ and consider $R_{-i} \in \mathcal{SSP}(\succeq)^{n-1}$ such that $t(R_j) = w$ for each $j \in N \setminus \{i\}$ and any $R_i^w \in \mathcal{SSP}(\succeq)$. Then,

$$\sup_{\succeq}(R_i^w, R_{-i}) = w\widehat{P}_i t = \sup_{\succeq}(\widehat{R}_i, R_{-i}),$$

contradicting *strategy-proofness.*

Hence, $\mathcal{SSP}(\succeq)$ is *maximal* for $\sup_{\succeq}$. ■

7.5 Proof of Theorem 1

($\Longrightarrow$) Assume $f : \mathcal{SSP}(\succeq)^n \to A$ is a *strategy-proof* and *simple* rule. If there is no $\sup_{\succeq} A$, by Lemma 9, $f(R) = \sup_{\succeq} t(R)$ for each $R \in \mathcal{SSP}(\succeq)^n$, and therefore $f = \sup_{\succeq}$. So let us assume there is $\sup_{\succeq} A$. Let $\alpha \equiv \sup_{\succeq} A$ and consider $f \neq \sup_{\succeq}$. By Lemma 8, there is $R^\star \in \mathcal{SSP}(\succeq)^n$ such that $\alpha \in t(R^\star)$ and $f(R^\star) \neq \alpha$. Let $x \equiv f(R^\star)$ and let

$$k^\star \equiv \min_k \{k = |N(R, x)| : R \in \mathcal{SSP}(\succeq)^n \text{ with } t(R) = \{\alpha, x\} \text{ and } f(R) = x\}.$$

Step 1: $x \in A^\star(\succeq)$. We need to prove that, for $y \in A \setminus \{\alpha, x\}$, $x \not\succeq y$ and $y \not\succeq x$. In order to get a contradiction, first suppose $y \succeq x$. Let $R \in \mathcal{SSP}(\succeq)^n$ be such that $t(R) = \{x, \alpha\}$ and $|N(R, x)| = k^\star$. Then, $f(R) = x$ by the definition of $k^\star$. Let $i \in N(R, x)$ and consider any $R_i^y \in \mathcal{SSP}(\succeq)$. By Lemma 7, $f(R_i^y, R_{-i}) \in \{x, \alpha\}$. There are two cases to consider:

1.1. $f(R_i^y, R_{-i}) = x$. As $y \succ x$, by Remark 5 (i) there is $\widetilde{R}_i^y \in \mathcal{SSP}(\succeq)$ such that $\alpha\widetilde{P}_i^y x$. By *tops-onlyness,* $f(\widetilde{R}_i^y, R_{-i}) = f(R_i^y, R_{-i})$. Let $R_i^\alpha \in \mathcal{SSP}(\succeq)$ be arbitrary. By the definition of $k^\star$ and Lemma 2, $f(R_i^\alpha, R_{-i}) = \alpha$. Therefore,

$$f(R_i^\alpha, R_{-i}) = \alpha\widetilde{P}_i^y x = f(\widetilde{R}_i^y, R_{-i}),$$

contradicting *strategy-proofness.*

1.2. $\boldsymbol{f(R_i^y, R_{-i}) = \alpha}$. As $y \succ x$, by Remark 5 (ii) there is $\widetilde{R}_i^y \in \mathcal{SSP}(\succeq)$ such that $x\widetilde{P}_i^y\alpha$. By *tops-onlyness,* $f(\widetilde{R}_i^y, R_{-i}) = f(R_i^y, R_{-i})$. Therefore,

$$f(R) = x\widetilde{P}_i^y\alpha = f(\widetilde{R}_i^y, R_{-i}),$$

contradicting *strategy-proofness.*

Thus, $y \not\succ x$. The case $x \succeq y$ follows a similar argument, interchanging the roles of x and y. Hence, $x \in A^\star(\succeq)$.

Step 2: $\boldsymbol{f \in \mathcal{Q}(\succeq)}$ with associated alternative $\boldsymbol{x}$ and quota $\boldsymbol{q^x = k^\star}$. Let $R \in \mathcal{SSP}(\succeq)^n$. There are two cases to consider:

2.1. $\boldsymbol{|N(R, x)| \geq k^\star}$. Then $f(R) = x$ by Lemma 3.

2.2. $\boldsymbol{|N(R, x)| < k^\star}$. We want show that $f(R) = \sup_\succeq t(R)$. Notice that, by Lemma 7, $f(R) \in \{x, \sup_\succeq t(R)\}$. In order to get a contradiction, assume $f(R) = x$. Let $N^x \equiv N(R, x)$ and $N^\alpha \equiv N(R, \alpha)$. Consider first the case $N^\alpha = \emptyset$. Let $i \in N\backslash N^x$. As $\alpha \succ t(R_i)$ and $f(R) \notin [t(R_i), \alpha]$ (since $f(R) \in A^\star(\succeq)$), by Lemma 5, $f(R_i^\alpha, R_{-i}) = f(R) = x$. Continuing in the same way, we successively change preferences of all the remaining agents in $N \setminus N^x$ to obtain $f(R_{N^x}, R_{-N^x}^\alpha) = x$. But this contradicts the definition of $k^\star$.

To finish, consider the case $N^\alpha \neq \emptyset$. Let $R_{N^\alpha}^\alpha \in \mathcal{SSP}(\succeq)^{|N^\alpha|}$. By *tops-onlyness,* $f(R_{N^\alpha}^\alpha, R_{-N^\alpha}) = f(R) = x$. Next, let $i \in N \setminus (N^x \cup N^\alpha)$, which exists because otherwise $N \setminus (N^x \cup N^\alpha) = \emptyset$ would imply $f(R) = \alpha$, a contradiction. As $\alpha \succ t(R_i)$ and $f(R_{N^\alpha}^\alpha, R_{-N^\alpha}) \notin [t(R_i), \alpha]$, by Lemma 5, $f(R_{N^\alpha \cup \{i\}}^\alpha, R_{-(N^\alpha \cup \{i\})}) = x$. Continuing in the same way, we successively change preferences of all the remaining agents in $N \setminus (N^x \cup N^\alpha)$ to obtain $f(R_{N^x \cup N^\alpha}, R_{-(N^x \cup N^\alpha)}^\alpha) = x$. But this also contradicts the definition of $k^\star$.

($\Longleftarrow$) That $\sup_\succeq$ is *strategy-proof* and *simple* has been presented in the main text (see Section 3). Let f be a *quota-supremum* rule with associated alternative x and quota q^x. By definition, f is *unanimous, anonymous* and *tops-only*, and therefore *simple.* We know show that f is also *strategy-proof.* Let $R \in \mathcal{SSP}(\succeq)^n$, $i \in N$ and $R_i' \in \mathcal{SSP}(\succeq)$ be arbitrary, and assume $f(R) \neq t(R_i)$. There are two cases to consider:

1. $\boldsymbol{f(R) = x}$. Since $x \in A^\star(\succeq)$, $|N(R, x)| \geq k$. As $t(R_i) \neq x$, $f(R_i', R_{-i}) = x$ because $|N((R_i', R_{-i}), x)| \geq |N(R, x)|$. Thus, there is no profitable manipulation from agent i.

2. $\boldsymbol{f(R) = \sup_{\succeq} t(R)}$. We want to show that

$$f(R) R_i f(R_i', R_{-i}). \tag{13}$$

If $f(R_i', R_{-i}) = \sup_{\succeq}(R_i', R_{-i})$, then by *semilattice single-peakedness* and associativity of the supremum,

$$f(R) = \sup_{\succeq}\{t(R_i), \sup_{\succeq} t(R_{-i})\} \; R_i \; \sup_{\succeq}\{t(R_i'), \sup_{\succeq} t(R_{-i})\} = f(R_i', R_{-i}),$$

so (13) holds. If $f(R_i', R_{-i}) = x$, as $x \in A^{\star}(\succeq)$, it follows that $t(R_i) \npreceq x$ and, by Remark 1 (ii),

$$\alpha = \sup_{\succeq}\{t(R_i), x\} R_i x. \tag{14}$$

Also, since $t(R_i) \preceq \sup_{\succeq} t(R) \preceq \alpha$, by Remark 1 (i),

$$\sup_{\succeq} t(R) R_i \alpha. \tag{15}$$

Therefore, by hypothesis and using (14) and (15) we obtain

$$f(R) = \sup_{\succeq} t(R) R_i x = f(R_i', R_{-i}),$$

so (13) holds.

Hence, the *quota-supremum* rule f is *strategy-proof.* ■