Straightening out the concept of direct and indirect input requirements

Ferran Sancho
Department of Economics
Universitat Autònoma de Barcelona
01893-Bellaterra, Catalonia, Spain
ferran.sancho@uab.cat

Abstract: Gim & Kim (1998) proposed a generalization of Jeong (1982, 1984) reinterpretation of the Hawkins-Simon condition for macroeconomic stability to off-diagonal matrix elements. This generalization is conceptually relevant for it offers a complementary view of interindustry linkages beyond final or net output influence. The extension is completely similar to the 'total flow' idea introduced by Szyrmer (1992) or the 'output-to-output' multiplier of Miller & Blair (2009). However the practical implementation of Gim & Kim is actually faulty since it confuses the appropriate order of output normalization. We provide a new and elementary solution for the correct formalization using standard interindustry accounting concepts.

Keywords: output multipliers, input multipliers, Leontief multipliers.

JEL: C67, D57, 022, R15

*Support from research projects MICINN-ECO2009-11857 and SGR2009-578 is gratefully acknowledged. The usual caveat applies. Email: ferran.sancho@uab.cat.

1. Introduction

The classical demand-driven interindustry model introduced the idea of 'multipliers' as the quantification of the endogenous response of gross output to exogenous changes in net output, or final demand. Jeong (1982, 1984), through a thorough examination of the Hawkins-Simon (1949) condition, introduced a new linkage effect between direct and indirect input requirements in response to changes in gross output itself but restricted his calculation to the own output effects, i.e. the diagonal elements of the technology matrix. Gim & Kim (1998) extended Jeong's idea to the off-diagonal elements providing researchers with the possibility of using an alternative multiplier matrix when measuring interdependence effects. The idea is simple but powerful. Changes in exogenous net output give rise, thanks to the necessary general equilibrium adjustments in production, to a new equilibrium level in gross output which is the result of the interplay of direct and indirect effects. When the change in net output is unitary, the appropriate column of the Leontief inverse provides the information on direct and indirect gross output of all goods that are required to sustain a new unit of final output. The calculation is simple and the interpretation is straightforward due to the model causality running from net output to gross output. Jeong (1982, 1984) and Gim & Kim (1998) extend this idea to calculate the total input requirements of all goods needed directly and indirectly to sustain a new unit of gross output—instead of the traditional new unit of net output. Notice however that this extension does not correspond to a causal model since gross output is, by default, the endogenous variable in the interindustry model. This does not preclude, however, the possibility of accounting for the implicit direct and indirect effects underlying the supply of a unit of gross output.

In this short note we first recall Gim & Kim (1998) main result and show how it translates in terms of the habitual variables in interindustry analysis. We then compare it to Szyrmer's (1992) 'total flow' proposal and Miller & Blair (2009) 'output-to-output' multipliers. We comment how they define the same concept but the order of matrix multiplication, which yields gross output normalization, is however the opposite, even though they depart from exactly the same direct and indirect inputs requirement matrix. Clearly then somebody has performed an incorrect output normalization. We show that Gim & Kim (1998) got it wrong and we now provide the correct answer using a simple conceptual approach based on elementary interindustry accounting.

2. Accounting rules for net and gross output.

There are two basic distinctions to bear in mind. First is the distinction between total requirements to sustain a new unit of net output or, alternatively, a new unit of gross output. The second distinction is between total output requirements and total input (direct and indirect) requirements. In the standard interindustry model production technology is represented by an $n \times n$ non-negative matrix $\mathbf{A}$. Column j=1, 2,..., n of matrix $\mathbf{A}$ is the vector of direct input requirements for producing one unit of gross output of good *j*. Thus a_{ij} is the amount of good *i* needed as a direct input to produce one gross unit of good *j*. In consequence and because of the linearity assumption, if $\mathbf{x}$ represents a vector of gross output, $\mathbf{A}\cdot\mathbf{x}$ embodies the vector of all intermediate inputs needed to produce $\mathbf{x}$. Since gross output $\mathbf{x}$ can only be demanded for intermediate demand use $\mathbf{A}\cdot\mathbf{x}$ or final demand use $\mathbf{y}$, it must be the case that in equilibrium total supply equals total demand:

$$\mathbf{x} = \mathbf{A}\cdot\mathbf{x} + \mathbf{y} \tag{1}$$

Provided some technicalities are satisfied (Nikaido, 1972) expression (1) can be non-negatively solved as:

$$\mathbf{x} = \left(\mathbf{I} - \mathbf{A}\right)^{-1}\cdot\mathbf{y} = \mathbf{L}\cdot\mathbf{y} \tag{2}$$

with $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1}$ being the so-called Leontief inverse and $\mathbf{I}$ the identity matrix. The nice thing of expression (2) is that quickly and easily yields gross output multipliers in response to changes in final demand or net output. Indeed, suppose for instance that the final demand vector $\mathbf{y}$ has a 1 in position one and zeroes elsewhere. Then equilibrium gross output $\mathbf{x}$ for such $\mathbf{y}$ coincides with the first column in the Leontief inverse $\mathbf{L}$. A new net output unit of good 1 can be sustained if the economy produces the level of gross outputs indicated by the first column in $\mathbf{L}$. Notice how the model causality in (2) runs from exogenous net output $\mathbf{y}$ to endogenous gross output $\mathbf{x}$. Notice too that matrix $\mathbf{L}$ shows gross output levels and that $\mathbf{L}$ can be expanded according to the matrix series:

$$\mathbf{L} = \mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \cdots + \mathbf{A}^{\mathrm{k}} + \cdots \tag{3}$$

As a result the matrix product $\mathbf{A}\cdot\mathbf{L}$ indicates total intermediate inputs needed for producing the gross output levels contained in the columns of $\mathbf{L}$. Since it can be seen that:

$$\mathbf{A}\cdot\mathbf{L} = \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \cdots + \mathbf{A}^{\mathrm{k}} + \cdots = \mathbf{L} - \mathbf{I} \tag{4}$$

we can verify that expression (4) corresponds to matrix $\mathbf{\Gamma}^{\mathrm{f}}$ of Gim & Kim (1998, their expressions (2) and (3)) where:

$$\mathbf{\Gamma}^{\mathrm{f}} = \mathbf{L} - \mathbf{I} = \mathbf{A}\cdot\mathbf{L} \tag{5}$$

corresponds to the direct (i.e. $\mathbf{A}$) and indirect (i.e. $\mathbf{A}^2 + \mathbf{A}^3 + \cdots$) input requirements needed to sustain unitary additions of new net output. To obtain direct and indirect input requirements $\mathbf{\Gamma}^{\mathrm{g}}$ to sustain new units of gross output, Kim and Gim normalize matrix $\mathbf{\Gamma}^{\mathrm{f}}$ by pre-multiplying it by the inverse of a diagonal matrix $\hat{\mathbf{D}}$ whose elements correspond to the diagonal elements of the Leontief inverse $\mathbf{L}$. In their case:

$$\mathbf{\Gamma}^{g} = \hat{\mathbf{D}}^{-1} \cdot \mathbf{\Gamma}^{f} \tag{6}$$

However when we look at Szyrmer (1992, his expression (9)) the matrix of 'total flows' corresponds to $\mathbf{F} = \mathbf{L} \cdot \hat{\mathbf{D}}^{-1}$ which when used to derive the corresponding direct and indirect input requirements (pre-multiplying by **A**) needed to sustain additional units of gross output yields (adopting the above notation):

$$\mathbf{\Gamma}^{g} = \mathbf{A} \cdot \mathbf{F} = \mathbf{A} \cdot \mathbf{L} \cdot \hat{\mathbf{D}}^{-1} = \mathbf{\Gamma}^{f} \cdot \hat{\mathbf{D}}^{-1} \tag{7}$$

According to Kim & Gim (1998) the input requirement matrix is normalized across rows (pre-multiplication by a diagonal matrix) whereas for Szyrmer (1992) the normalization is across columns (post-multiplication by the same diagonal matrix). Clearly both approaches cannot be correct at the same time. Miller & Blair (2009, section 6.6.2), in turn, use the notion of 'output-to-output' multipliers to measure total gross output required to sustain a unit of new gross output. It is obtained through the normalization of the Leontief inverse columns using the corresponding on-diagonal elements. If $\mathbf{L}^*$ denotes Miller & Blair (2009) normalization, using the notation here we would find:

$$\mathbf{L}^* = \mathbf{L} \cdot \hat{\mathbf{D}}^{-1} \tag{8}$$

which coincides with **F**. Since the columns of $\mathbf{L}^*$ represent normalized gross outputs we can derive the direct and indirect input requirements pre-multiplying once again by matrix **A**. In this case recalling and using expression (4) above, we find from Miller & Blair's normalized matrix the same total input requirement concept as Szyrmer's in expression (7):

$$\mathbf{A} \cdot \mathbf{L}^* = \mathbf{A} \cdot \mathbf{L} \cdot \hat{\mathbf{D}}^{-1} = \mathbf{\Gamma}^{f} \cdot \hat{\mathbf{D}}^{-1} = \mathbf{\Gamma}^{g} \tag{9}$$

3. DOSSO's numerical example.

We illustrate these different concepts using Jeong (1984) same 2×2 numerical example taken from Dorfman *et al.* (1958). From a direct coefficient matrix **A** such as:

$$\mathbf{A} = \begin{bmatrix} 0.100 & 1.458 \\ 0.160 & 0.167 \end{bmatrix}$$

we use the above developments to calculate:

a) **L**: From expression (2) we can calculate Leontief's inverse matrix, with each column indicating gross outputs needed to sustain one unit of net output:

$$\mathbf{L} = \begin{bmatrix} 1.613 & 2.823 \\ 0.310 & 1.743 \end{bmatrix}$$

b) $\mathbf{\Gamma}^{\mathrm{f}}$: From (4) we find the total input requirements matrix for net output, with each column showing all inputs needed directly and indirectly to sustain one unit of the corresponding net output:

$$\mathbf{\Gamma}^{\mathrm{f}} = \begin{bmatrix} 0.613 & 2.823 \\ 0.310 & 0.743 \end{bmatrix}$$

c) $\mathbf{\Gamma}^{\mathrm{g}}$ (Gim & Kim): From (5) we obtain their total input requirements matrix for gross output, each column indicating all inputs directly and indirectly needed to sustain one unit of the corresponding gross output:

$$\mathbf{\Gamma}^{\mathrm{g}} = \begin{bmatrix} 0.380 & 1.750 \\ 0.178 & 0.426 \end{bmatrix}$$

d) $\mathbf{\Gamma}^{\mathrm{g}}$ (Szyrmer): Same total requirements idea as in c) but now using expression (6) we obtain:

$$\mathbf{\Gamma}^{\mathrm{g}} = \begin{bmatrix} 0.380 & 1.620 \\ 0.192 & 0.426 \end{bmatrix}$$

e) **L*** (Miller & Blair): Normalized Leontief inverse or 'output-to-output' multiplier matrix, each column quantifying gross outputs of all goods needed to sustain one unit of the corresponding gross output:

$$\mathbf{L*} = \begin{bmatrix} 1.000 & 1.620 \\ 0.192 & 1.000 \end{bmatrix}$$

The question remains on which of the two $\mathbf{\Gamma}^{\mathrm{g}}$ matrices is the correct one. The approach by Gim & Kim has been used in the literature with a somewhat blind theoretical acceptance. See

Mariolis & Rodousaki (2011, their expression (3)) as an example of this. We now turn to provide an elementary answer to this question.

4. **Tracing direct and indirect input requirements**.

Consider a 2×2 matrix $\mathbf{A} = (a_{ij})$ of direct technical coefficients with good 1 representing 'iron' and good 2 being 'coal'. Thus a_{11} is 'iron' directly needed to produce one unit of 'iron', a_{21} is 'coal' directly needed to produce one unit of 'iron', and so on. We will focus, for simplicity's sake, in accounting direct and indirect requirements for a unit of gross output of 'iron'. The same considerations would apply *mutatis mutandis* to total requirements for 'coal', just reversing the order of the relevant sectoral indices.

We will start calculating al 'coal' directly and indirectly required as input to be able to supply one gross unit of 'iron'. Firstly notice that a_{21} units of 'coal' are directly needed to produce such a gross unit of 'iron'. Second, notice that the 'coal' to produce one unit of 'coal' is given by a_{22} but since only the above a_{21} units of 'coal' are involved here, the proportioned indirect requirement of 'coal' will be $a_{22} \cdot a_{21}$. This number is the 'coal' needed to produce the 'coal' needed for producing one gross unit of 'iron'. Thirdly, the quantity $a_{22} \cdot a_{21}$ is 'coal' and it needs to be produced. The 'coal' to produce a 'coal' level of $a_{22} \cdot a_{21}$ will be $a_{22} \cdot (a_{22} \cdot a_{21}) = a_{22}^2 \cdot a_{21}$. The recursive nature of indirect requirements should now be clear. A 'coal' requirement of $a_{22} \cdot (a_{22} \cdot a_{21})$ will have to be produced and the needed 'coal' can again be calculated simply by $a_{22} \cdot (a_{22} \cdot (a_{22} \cdot a_{21})) = a_{22}^3 \cdot a_{21}$, and so on.

Adding up all the 'coal' for 'iron' productive rounds we would find g_{21}, the total direct and indirect input requirements of 'coal' for a gross unit of 'iron':

$$\begin{aligned} g_{21} &= a_{21} + a_{22} \cdot a_{21} + a_{22} \cdot (a_{22} \cdot a_{21}) + \cdots + a_{22} \cdot (a_{22}^{k-1} \cdot a_{21}) + \cdots = \\ &= (1 + a_{22} + a_{22}^2 + a_{22}^3 + \cdots + a_{22}^k + \cdots) \cdot a_{21} = (1 - a_{22})^{-1} \cdot a_{21} \end{aligned} \tag{10}$$

The source of the error in Gim & Kim (1998) can be now clearly pinpointed in their expression (17). There the successive rounds of good 2 necessary to produce output of good 1 are expanded using the wrong 'iron' coefficient a_{11} instead of the correct 'coal' one a_{22}.

We now proceed to compute the direct and indirect 'iron' requirements for a gross unit of 'iron'. Directly it is immediate that a_{11} units of 'iron' are required as input for each gross unit of 'iron'. Indirectly we have to include all the 'iron' that will be activated through the 'coal' sector in response to the 'iron' initially needed to produce a unit of 'coal', i.e. a_{12}. But in total g_{21} units of 'coal' will be required for each unit of 'iron'. Since the initial need is just of a_{12}

units, the proportioned requirements will be $a_{12}\cdot g_{21}$. Direct and indirect input requirement of ‘iron’ for a gross unit of ‘iron’ will therefore be:

$$g_{11} = a_{11} + a_{12} \cdot g_{21} = a_{11} + a_{12} \cdot (1 - a_{22})^{-1} \cdot a_{21} \qquad (11)$$

We have just completed the first column of a matrix of direct and indirect input requirements for a unit of gross output of ‘iron’. Replicating the argument for the second good (‘coal’) we would derive the second column (i.e. all input requirements of ‘iron’ and ‘coal’ for a gross unit of ‘coal’) and if we denote again this matrix by $\mathbf{\Gamma}^{\mathbf{g}}$ we would have:

$$\mathbf{\Gamma}^{\mathbf{g}} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} a_{11} + a_{12} \cdot (1 - a_{22})^{-1} \cdot a_{21} & (1 - a_{11})^{-1} \cdot a_{12} \\ (1 - a_{22})^{-1} \cdot a_{21} & a_{22} + a_{21} \cdot (1 - a_{11})^{-1} \cdot a_{12} \end{bmatrix} \qquad (12)$$

Plugging in expression (12) the values from Dorfman *et al.* (1958) for matrix $\mathbf{A}$ we obtain:

$$\mathbf{\Gamma}^{\mathbf{g}} = \begin{bmatrix} 0.380 & 1.620 \\ 0.192 & 0.426 \end{bmatrix}$$

In conclusion $\mathbf{\Gamma}^{\mathbf{g}}$ as calculated here, following the inner accounting logic of the interindustry model, corresponds to the column normalization of $\mathbf{\Gamma}^{\mathbf{f}}$ and our result in (12) proves Szyrmer and Miller & Blair, each from a different conceptual perspective, got the right normalization. This makes economic sense since dividing each entry in column j of $\mathbf{\Gamma}^{\mathbf{f}}$ by a different diagonal Leontief coefficient, as Gim & Kim do, does not seem to respect the underlying homotheticity of the multisectoral production function.

5. Some concluding remarks

The extension to the $n\times n$ case could be analyzed along the above lines using partitioned matrices. If we distinguish sector 1 and group the rest of n-1 sectors in a ‘block’ called 2, the formal derivation could be extended using matrix algebra, and so on for each possible partition after suitable permutations of rows and columns are performed. The economic interpretations on dependency linkages, which are the essential part here, would however remain the same provided the appropriate block of sectors substitutes the previously isolated sector in the 2×2 case (Jeong, 1982). A further verification of the correctedness of the result in (12) could be obtained using the hypothetical extraction method, as suggested by Szyrmer (1992). In this case, when a sector is fully extracted by eliminating its column from matrix $\mathbf{A}$ and the equilibrium recomputed, the difference in output between the integrated case and the extracted one captures the direct and indirect interdependencies. When applied to Dorfman *et al.* (1958) matrix $\mathbf{A}$, the same numerical results as in d) above or from our expression (12) are obtained. The computationally equivalent extraction method, however, does not allow for an easy identification and interpretation of the bilateral interactions.

It is most important to have a correct and proper accounting of direct and indirect input requirements, be it for the traditional net of for the alternative gross units of output, since in empirical applications multipliers are routinely used by decision makers and confusion cannot be allowed since wrongly founded policy decisions are too costly to society.

REFERENCES

Dorfman, R., P.A. Samuelson, and R.M. Solow (1958). *Linear programming and economic analysis*. McGraw-Hill, New York.

Gim, H. U., and K. Kim (1998). "The general relation between two different notions of direct and indirect input requirements". *Journal of Macroeconomics*, vol. 20(1), pp. 199-208

Hawkins, D., and H.A. Simon (1949). "Note: Some conditions of macroeconomic stability". *Econometrica*, vol. 17(3), pp. 245-248.

Jeong, K. (1982). "Direct and indirect requirements: a correct economic interpretation of the Hawkins-Simon conditions". *Journal of Macroeconomics*, vol. 4 (3), pp. 349-356.

Jeong, K. (1984). "The relation between two different notions of direct and indirect input requirements". *Journal of Macroeconomics*, vol. 6 (4), pp. 473-476.

Mariolis, T., and E. Rodousaki (2011). "Total requirements for gross output and intersectoral linkages: a note on Dimitriev's contribution to the theory of profits". *Contributions to Political Economy*, vol. 30, pp. 67-75.

Miller, R., and P. Blair (2009). *Input-output analysis: foundations and extensions*. Cambridge University Press.

Nikaido, H. (1972). *Introduction to Sets and Mappings in Modern Economics*. North-Holland, Amsterdam, the Netherlands.

Szyrmer, J.M. (1992). "Input-output coefficients and multipliers from a total flow perspective". *Environment and Planning A*, vol. 24, pp. 921-937.