# A Many-to-Many 'Rural Hospital Theorem'

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**Abstract**

We show that the full version of the so-called 'rural hospital theorem' (Roth, 1986) generalizes to many-to-many matching where agents on both sides of the market have separable and substitutable preferences.

*Keywords:* matching, many-to-many, stability, rural hospital theorem.
*JEL–Numbers:* C78, D60.

## 1 Introduction

In many entry-level labor markets, workers are matched with firms through a clearinghouse. It has been shown that clearinghouses that employ so-called stable mechanisms often perform better than those that employ unstable mechanisms.[1] Stability guarantees that parties cannot profitably recontract from the matching established by the mechanism.

Taking the requirement of stability as granted, an important question is whether the choice of a particular stable mechanism affects the numerical distribution of workers; and if not, whether it affects the composition of the firms that do not fill all their vacant positions. For instance, Sudarshan and Zisook (1981) observed that in the US market of medical graduates the National Resident Matching Program (NRMP) fails to fill the posts of many rural hospitals. Roth (1984,1986) showed that the problem of the rural hospitals cannot be attributed to NRMP's particular stable mechanism. More precisely, any other stable mechanism would yield (R1) the same numerical distribution of medical interns and would assign (R2) the same interns to each rural hospital that does not fill all its posts.

Roth (1984,1986) established his two findings for many-to-one markets (i.e., each firm can hire multiple workers) with so-called responsive preferences. The two results are now known as the 'rural hospital theorem.' Under the (standard) assumption of substitutable

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[1]See, for instance, Roth (1991).

preferences,[2] several papers have extended both parts of the theorem. Concerning the many-to-one framework, Alkan (2002) extended (R1) to cardinally monotonic preferences, and Martínez et al. (2000) extended (R2) to separable preferences. The examples of Martínez et al. (2000, Example 5) and Kojima (2011, Example 1) show that the two properties do not necessarily hold if one agent has (substitutable) preferences that do not pertain to the corresponding domain.

In the many-to-many framework (where each worker can be employed by multiple firms), (R1) still holds on Alkan's (2002) domain of cardinally monotonic preferences. However, (R2) was only established for the domain of responsive preferences (Alkan, 1999). Here, we show that in fact (R2) also holds on the substantially richer domain of separable preferences (as in the many-to-one case). Our short proof is based on a strong but apparently slightly overlooked structural result regarding the set of pairwise stable matchings by Roth (1984).

In Section 2, we present the model and the preference domains. In Section 3, we discuss the literature on the rural hospital theorem and prove our result.

# 2 Model

There are two disjoint and finite sets of agents: a set of workers $W$ and a set of firms $F$. Let $A = W \cup F$ denote the set of agents. A generic worker, firm, and agent are denoted by $w$, $f$, and $a$, respectively. The preferences of an agent $a$ are given by a linear order $P_a$ over $2^{S_a}$ where $S_a \equiv F$ if $a \in W$, and $S_a \equiv W$ otherwise. For each agent $a$, let $q_a \geq 1$ (its 'quota') be the smallest positive integer $q_a \geq 1$ such that for any $S \subseteq S_a$ with $|S| > q_a$, $\emptyset \, P_a \, S$.[3] A preference profile is a tuple $P = (P_a)_{a \in A}$. For any $S \subseteq S_a$, let $Ch(S, P_a)$ denote agent $a$'s most preferred subset of $S$ according to $P_a$. Throughout, we make the following (standard) assumption on each agent $a$'s linear order $P_a$.

**Substitutability.** For any $b, c \in S \subseteq S_a$ with $b \neq c$, $[b \in Ch(S, P_a) \Rightarrow b \in Ch(S \backslash c, P_a)]$.

A matching $\mu$ is a mapping from $A$ into $2^F \cup 2^W$ such that for all $a, a' \in A$, $\mu(a) \in 2^{S_a}$ and $[a \in \mu(a') \Leftrightarrow a' \in \mu(a)]$. Matching $\mu$ is blocked by agent $a$ if $\mu(a) \neq Ch(\mu(a), P_a)$. Matching $\mu$ is blocked by a worker-firm pair $(w, f)$ if $w \notin \mu(f)$, $w \in Ch(\mu(f) \cup w, P_f)$, and $f \in Ch(\mu(w) \cup f, P_w)$. A matching is (pairwise) **stable** if it is not blocked by any agent or worker-firm pair.[4]

For any profile of substitutable preferences $P$, the set of stable matchings $S(P)$ is non-empty. Roth (1984, Theorem 2) showed the existence of a firm-optimal stable matching $\mu_F$ (which all firms like at least as well as any other stable matching) and likewise a

[2]Substitutability was introduced by Kelso and Crawford (1982) and guarantees the existence of (pairwise) stable matchings.

[3]The interpretation is that agent $a$ can definitely not work for/hire more than $q_a$ agents from the other side of the market. Note that for any $q'_a \geq |S_a|$, $[S \subseteq S_a$ with $|S| > q'_a$ implies $\emptyset \, P_a \, S]$ trivially. Hence, $q_a \leq |S_a|$.

[4]Note that we do not consider larger blocking coalitions. As a consequence, the core does not necessarily coincide with the set of stable matchings. However, Roth and Sotomayor (1990, page 157) pointed out that for certain many-to-many markets pairwise stability is still of primary importance. Giving blocking power to larger coalitions would lead to a (possibly empty) subset of pairwise stable matchings for which our result still holds.

worker-optimal stable matching $\mu_W$. In fact, the set of stable matchings satisfies the following properties (which will be key in the proof of our result).

**Theorem 1.** [Roth (1984, Theorem 2)]
*Let $P$ be substitutable. Let $\mu \in S(P)$. For all $w \in W$ and $f \in F$,*
(i). $Ch(\mu_F(f) \cup \mu(f), P_f) = \mu_F(f)$;
(ii). $Ch(\mu_W(w) \cup \mu(w), P_w) = \mu_W(w)$.

The matching literature has studied the following preference domains.

**Responsiveness.**[5]
For any $S \subseteq S_a$ with $|S| < q_a$ and for any $b, c \in (S_a \backslash S) \cup \{\emptyset\}$, $[(S \cup b) P_a (S \cup c) \Leftrightarrow b P_a c]$.

**Separability.**[6]
For any $S \subseteq S_a$ with $|S| < q_a$ and for any $b \in S_a \backslash S$, $[(S \cup b) P_a S \Leftrightarrow b P_a \emptyset]$.

**Cardinal Monotonicity.**[7]
For any $S, S' \subseteq S_a$, $[S \subseteq S' \Rightarrow |Ch(S, P_a)| \leq |Ch(S', P_a)|]$.

Figure 1 illustrates (1) the strict inclusion relationships between the latter three preference domains (responsiveness implies separability and separability implies cardinal monotonicity), and (2) the absence of inclusion relationship with respect to substitutability. The domain of separable and substitutable preferences is much richer than the domain of responsive preferences. For instance, consider the case where larger coalitions are preferred to smaller coalitions (in terms of cardinality). Then, responsiveness imposes a substantial number of restrictions on the ranking of coalitions of the same size,[8] whereas separability (together with substitutability) does not impose *any* restriction at all. The references to R2 in Figure 1 are discussed in the next section.

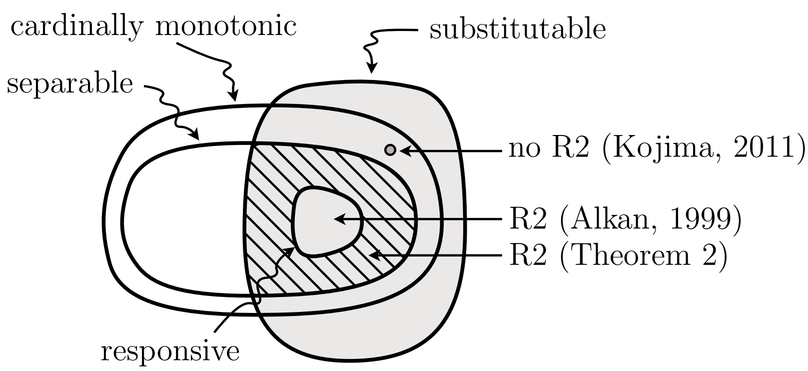


Figure 1: Preference domains

[5]Responsiveness was first formalized by Roth (1985).

[6]Separability was introduced by Martínez et al. (2000). Separability alone (i.e., without substitutability) does not guarantee the existence of a stable matching (Martínez et al. 2000, Example 1).

[7]Cardinal monotonicity was introduced by Alkan (2002). It is called size monotonicity and law of aggregate demand by Alkan and Gale (2003) and Hatfield and Milgrom (2005), respectively.

[8]In particular, by responsiveness, any two coalitions of the same size that only differ in one element have a fixed relative ranking. Also, even if responsiveness does not determine *a priori* the relative ranking of two coalitions of the same size $S$ and $S'$, choosing any relative ranking between the two will fix the relative ranking of all coalitions $S \cup T$ and $S' \cup T$ where $T \cap (S \cup S') = \emptyset$.

# 3 Result

The matching literature established the (first or both of the) next two results for different preference domains:

**Rural Hospital Theorem.**
**R1.** For all $\mu, \mu' \in S(P)$ and $a \in A$, $|\mu(a)| = |\mu'(a)|$;
**R2.** For all $\mu, \mu' \in S(P)$ and $a \in A$, $|\mu(a)| < q_a \Rightarrow \mu(a) = \mu'(a)$.

For one-to-one matching, each agent's preferences are responsive with quota 1, and R1 and R2 are equivalent. R1 and/or R2 were established for different *substitutable* preference domains:[9]

**One-to-one:** R1≡R2: McVitie and Wilson (1970), Gale and Sotomayor (1985a,b)

**Many-to-one:**[10] { R1: firms w. cardinally monotonic preferences: Alkan (2002) ($\star$)
R2: firms w. separable preferences: Martínez et al. (2000) ($\star\star$)

**Many-to-many:** { R1: cardinally monotonic preferences: Alkan (2002) ($\star$)
R2: reponsive preferences: Alkan (1999)– see also Figure 1

Here, ($\star$) and ($\star\star$) refer to the following two examples which complement the results.
**($\star$)** Martínez et al. (2000, Example 5): a many-to-one market where one firm's (substitutable) preferences violate cardinal monotonicity, the other agents have responsive preferences, and for which R1 does not hold.
**($\star\star$)** Kojima (2011, Example 1):[11] a many-to-one market where one firm has cardinally monotonic (and substitutable) preferences that are not separable, the other agents have responsive preferences, and for which R2 does not hold– see also Figure 1.

For many-to-many matching, R2 was only shown to hold for responsive preferences. Below we show that R2 even holds for separable (and substitutable) preferences, i.e., the most general preference domain for which R2 has been established in the many-to-one framework– see also Figure 1. The proof utilizes the structural result Theorem 1.[12]

**Theorem 2.** *For many-to-many matching, R2 holds for all profiles $P$ of separable (and substitutable) preferences.*

[9] The list is not exhaustive. First, only the papers that established the most general results concerning R1 or R2 are mentioned. Second, the literature on 'generalized two-sided matching' established similar or more general results than R1 in Alkan (2002); see, for instance, Hatfield and Milgrom (2005), Hatfield and Kojima (2010), and Hatfield and Kominers (2011). However, in the context of 'classical matching' these results are implied by or coincide with Alkan's (2002) result. Below we also comment on Kojima (2011).

[11] Kojima (2011) also introduced the domain of separable preferences with so-called affirmative action constraints, which have a natural interpretation in the context of many-to-one matching. This domain is a strict superset of the domain of separable preferences but a strict subset of the domain of cardinally monotonic preferences. Kojima (2011) showed that on his domain an appropriately adjusted version of R2 holds.

[12] In fact, for separable preferences R1 easily follows from Remark 1 in Martínez et al. (2004) which consists of Theorem 1 and two related results from Blair (1988). Details are available upon request from the author but omitted here since Alkan (2002) showed R1 for the strictly larger domain of cardinally monotonic preferences.

*Proof.* Since R1 holds for $P$, it suffices to show that for any $f \in F$ with $|\mu_F(f)| < q_f$, $\mu(f) \subseteq \mu_F(f)$. (Similar arguments can be used to show that for any $w \in W$ with $|\mu_W(w)| < q_w$, $\mu(w) \subseteq \mu_W(w)$.)

Let $f \in F$ with $|\mu_F(f)| < q_f$. Suppose $\mu(f) \nsubseteq \mu_F(f)$. Then, there exists $w \in \mu(f)$ with $w \notin \mu_F(f)$. Suppose $w \, P_f \, \emptyset$. Then, from separability and $|\mu_F(f)| < q_f$ it follows that

$$(\mu_F(f) \cup w) \, P_f \, \mu_F(f) \overset{\text{Th.1(i)}}{=\!=\!=} Ch(\mu_F(f) \cup \mu(f), P_f),$$

which, since $w \in \mu(f)$, contradicts the definition of $Ch$.

Now suppose $\emptyset \, P_f \, w$. Then, by separability and $w \in \mu(f)$,

$$(\mu(f) \backslash w) \, P_f \, \mu(f),$$

which implies that $f$ blocks $\mu$, in contradiction to $\mu \in S(P)$. Hence, $\mu(f) \subseteq \mu_F(f)$. □

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