A simple model of aggregate pension expenditure

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April 2011

Abstract

This paper develops a simple model that can be used to analyze the long-term sustainability of the contributive pension system and the steady-state response of pension expenditure to changes in some key demographic and economic variables, in the characteristics of the average pensioner and in the parameters that describe how pensions are calculated in Spain as a function of workers' Social Security contribution histories.

Key words: pensions, sustainability, Spain

JEL Classification: H55

* Financial support from BBVA Research and from the Spanish Ministry of Science and Innovation (under project ECO2008-04837/ECON) is gratefully acknowledged. I would like to thank Maite Naranjo for her help with some of the calculations.

1. Introduction

This paper develops a simple model that can be used to analyze the long-term sustainability of the contributive pension system and the steady-state response of pension expenditure to changes in some key demographic and economic variables, in the characteristics of the average pensioner and in the parameters that describe how pensions are calculated in Spain as a function of workers' Social Security contribution histories.

The model achieves tractability at the price of some very strong assumptions, including deterministic life spans and constant rates of growth of total employment and wages, ignores the heterogeneity of agents and the endogeneity of decisions to enter and exit the labor market, and does not take into account some important characteristics of the Spanish pension system, including the existence of caps and floors on contribution bases and pension levels and the possibility of early retirement. Under these assumptions, the model can be used to calculate the average pension and the ratio of this variable to the average salary, the ratio of pensioners to employed workers, the pension system's total current revenues and expenditures and its internal rate of return. It also provides two simple characterizations of the system's long-term financial sustainability: the contributive pension system will be sustainable in the long run if and only if its internal rate of return does not exceed the growth rate of aggregate wage income or, equivalently, if its initial replacement rate (the ratio between the initial pension and the wage at the time of retirement) does not exceed a critical value.

In spite of its simplistic assumptions, the model highlights the main determinants of spending in contributory pensions and the necessary conditions for the system's sustainability. It can also be a useful complement of the standard short-cut procedure for projecting pension expenditure, which is based on a decomposition of this variable, measured as a fraction of GDP, into three factors that capture, respectively, the effects on pension outlays of demographics, labor market performance and the *generosity* of the pension system.[1] In particular, the model introduces a certain amount of discipline when projecting into the future the system's generosity factor (generally defined as the ratio between the average pension and average output per employed worker), which is the component of pension expenditure that is hardest to forecast directly.

The remainder of the paper is divided into six sections and an appendix. Section 2 sets out the model's assumptions regarding demographics and the evolution of wages. Section 3 contains a simplified description of how retirement and widowers' pensions are set in Spain. In sections 4 and 5, wages and pensions are aggregated across individuals to calculate the key magnitudes of the pension system and two alternative characterizations of its long-term sustainability are derived. Section 6 contains a numerical analysis of the comparative statics of the model. Finally, section 7 concludes and the Appendix contains the details of the calculations.

[1] See for instance Jimeno, Rojas and Puente (2008) and Doménech and Melguizo (2008). For an application that combines the decomposition sketched in the text with the model developed in this paper in order to quantify the effects of Spain's most recent pension reform, see de la Fuente and Doménech (2011).

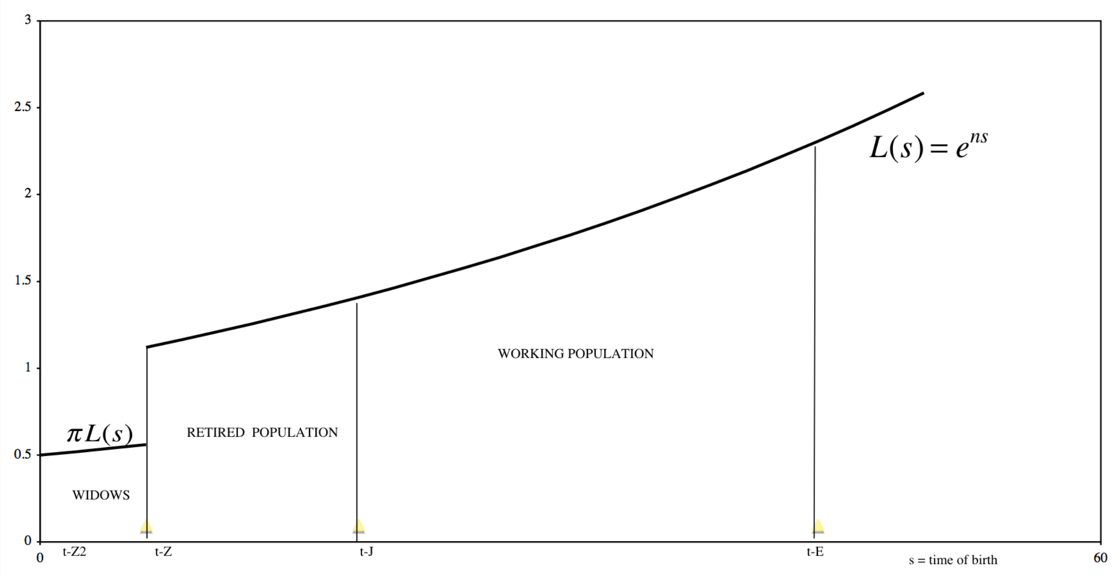
2. Demographic assumptions and the evolution of wages

The model economy is populated by overlapping generations of a continuum of finitely-lived homogeneous agents. The number of births grows over time at a constant exponential rate, n, so that the number of agents born at time s is equal to

$$(1)\quad L(s) = e^{ns}$$

An individual born at time s enters the labor market at $s+E$ and starts to work immediately, retires at $s+J$ and dies at $s+Z$. With probability π, he leaves behind a spouse who survives until $s+Z2$.

Figure 1: Breakdown of the population at time *t* by its economic status



I will treat wages as exogenous. It will be assumed that average real wages increase over time at a constant rate due to technical progress and capital accumulation and that individual wages rise with experience as well. The real wage at time $t \in [s+E, s+J]$ of a worker born at time s will be given by

$$(2)\quad W(s,t) = A_t e^{v(t-(s+E))} = A_o e^{gt} e^{v(t-s-E)} = A_o e^{(g+v)t} e^{-v(s+E)}$$

where $A_t = A_o e^{gt}$ captures the effects of technical progress and capital accumulation on average wages and the term $e^{v(t-s-E)}$ is the experience premium. To simplify the calculations, I have assumed that the experience premium grows at a constant rate, v, and does not therefore display the hump-shaped pattern that is usually found in the data.

3. Pension determination

I will assume that pensions are set using the rules that are currently applied in Spain. The starting pension of an individual born at time s who retires at $s+J$ is given by

$$(3)\quad P(s, s+J, C, N) = \phi(C) B(s, s+J, N)$$

where $\phi()$ is a percentage that depends on the number of years the agent has paid Social Security contributions,[2]

(4) $C = J - E$,

and *B()*, the so-called *regulatory base* of the pension, is an average of the agent's past wages calculated over the last *N* years prior to retirement. I will refer to *N* as the pension's *calculation period*. Assuming that wages are valued in real terms in the calculation (which is approximately true in Spain), the regulatory base is given by

$$(5)\ B(s,s+J,N) = \frac{1}{N}\int_{s+J-N}^{s+J} W(s,t)dt = \frac{\left(1-e^{-(g+\nu)N}\right)}{(g+\nu)N} W(s,s+J) \equiv b(N)W(s,s+J)$$

Hence, the regulatory base can be written as a fraction *b()* of retirement wages, *W(s, s+J)*. It is easy to check (see the Appendix) that this fraction is a decreasing function of $(g+\nu)N$, where *N* is the length of the calculation period and $g+\nu$ the growth rate of individual real wages. Notice that we can write the initial pension in the form

$$(3')\ P(s,s+J,C,N) = \phi(C)b(N)W(s,s+J) \equiv \rho(C,N)W(s,s+J)$$

Hence, $\rho() = \phi()b()$ is the ratio between the wage at retirement and the starting pension. I will refer to this quantity as the *initial replacement rate*.

Once its initial value is set, it will be assumed that an individual's pension *(P)* grows over time at a constant rate ω in real terms. If pensions are indexed to consumer prices, as is the case in Spain, we will have $\omega = 0$ and the real value of individual pensions will remain constant over time. In the general case, the real pension at time *t* of a worker who has retired at *s+J* will be given by

$$(6)\ P(s,t,C,N) = P(s,s+J,C,N)e^{\omega(t-(s+J))} = \rho(C,N)W(s,s+J)e^{\omega t}e^{-\omega(s+J)} = \rho(C,N)A_o e^{\nu C}e^{\omega t}e^{(g-\omega)(s+J)}$$

for $t \in [s+J, s+Z]$. If the pensioner leaves a widower when he or she dies (which happens with probability π), the surviving spouse will enjoy a widower's pension *(PV)* for the rest of his or her life. Assuming widowers' pensions are set at a constant fraction ϕ_v (= 0.52 in Spain) of the deceased spouse's pension at the time of death and grow at the same rate as retirement pensions, the real value of the widower's pension at time *t* will be given by

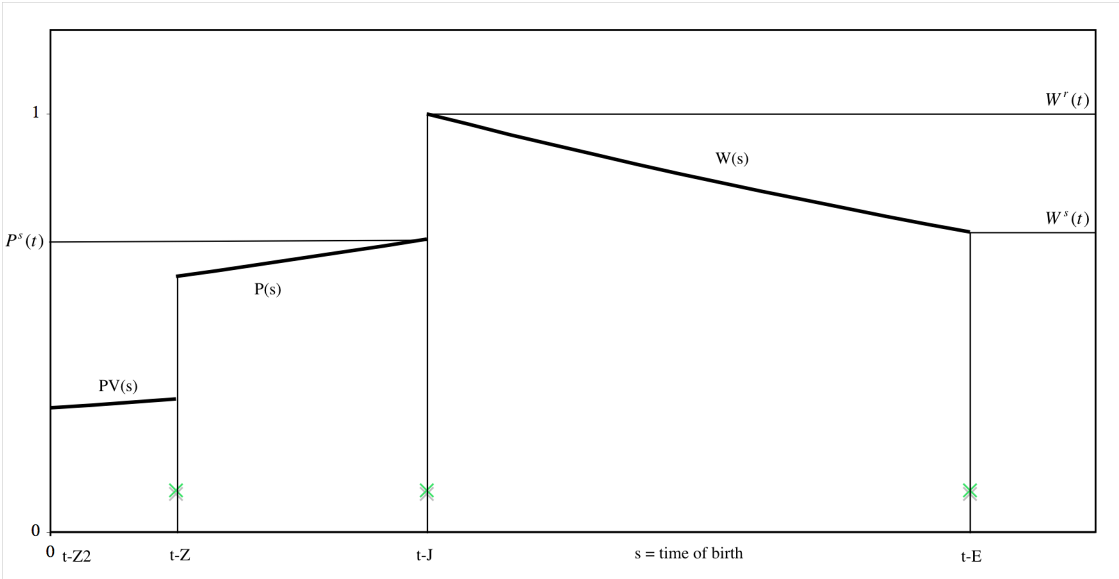
$$(7)\ PV(s,t,C,N) = \phi_v P(s,t,C,N) = \phi_v \rho(C,N)W(s,s+J)e^{\omega t}e^{-\omega(s+J)} \text{ for } t \in [s+Z, s+Z2]$$

Figure 2 shows how wages and pensions change across cohorts, indexed by their time of birth (*s*), at a given point in time (*t*). If there is a positive experience premium $(\nu > 0)$ wages rise with age and are therefore a decreasing function of the time of birth. If productivity growth is faster

[2] Under the current system, $\phi()$ is a piecewise linear function of the number of years of contribution, *C*. A minimum of 15 years is required for access to a contributory pension and entitles the worker to a pension equal to 50% of the regulatory base. This percentage rises by 3 points per year of contribution up to 25 years and by 2 points for each additional year thereafter, reaching 100% after 35 years.

than the real rate of appreciation of pensions ($g > \omega$) then pensions rise as we move to the right to younger cohorts (or decrease with age, as we move to the left).

Figure 2: Pensions and wages at time *t* as a function of the date of birth of each cohort



It will be useful to compute the following three magnitudes as of time t. By (2), the starting salary at time t, i.e. the wage earned by a worker with no experience who has just entered the labor market, will be given by

$$(8)\ W^s(t) = A_o e^{gt}$$

To calculate the retirement wage at t, notice that a worker who retires at that time must have been born at $s = t - J$. Using (2) again, this implies that

$$(9)\ \ W^r(t) \equiv W(t-J,t) = A_t e^{v(t-(t-J+E))} = A_o e^{gt} e^{v(J-E)} = A_o e^{gt} e^{vC} = W^s(t) e^{vC}$$

Finally, the starting pension of this worker will be equal to

$$(10)\ P^s(t) \equiv P(t-J,t) = \phi(C) b(N) W(t-J,t) = \rho(C,N) A_o e^{gt} e^{vC} = \rho(C,N) W^r(t)$$

The internal rate of return of the pension system

From the point of view of a worker, the public system of contributory pensions can be seen as an investment vehicle that allows him to obtain a retirement annuity in return for a flow of contributions during his working life. The internal rate of return (IRR) of this investment can be calculated in the standard way. The expected net present value of the investment, calculated as of time E, is given by

$$(11)\ V(r) = -\int_{s+E}^{s+J} \tau W(t) e^{-r(t-E)} dt + \int_{s+J}^{s+Z} P(t) e^{-r(t-E)} dt + \pi \int_{s+Z}^{s+Z2} PV(t) e^{-r(t-E)} dt$$

where τ is the Social Security contribution rate (defined as the sum of the rates paid by the worker and by his employer), π the probability that the worker is survived by his spouse and r

the discount rate. For convenience, I have suppressed all arguments of the functions *W(), P()* and *PV()* except for *t*. Substituting (2), (6) and (7) into (11), we have

$$V(r) = -\tau A_o e^{-v(s+E)} e^{rE} \int_{s+E}^{s+J} e^{-(r-g-v)t} dt + \rho(C,N) W(s,s+J) e^{-\omega(s+J)} e^{rE} \left(\int_{s+J}^{s+Z} e^{-(r-\omega)t} dt + \pi\phi_v \int_{s+Z}^{s+Z2} e^{-(r-\omega)t} dt \right)$$

The IRR of the pension system from the point of view of the representative worker/pensioner is the value of *r* that makes *V(r)* equal to zero. Setting the previous expression equal to zero and using (2) to write

$$(12)\quad W(s,s+J) = A_o e^{(g+v)(s+J)} e^{-v(s+E)}$$

we have

$$\tau \int_{s+E}^{s+J} e^{-(r-g-v)t} dt = \rho(C,N) e^{(g+v-\omega)(s+J)} \left(\int_{s+J}^{s+Z} e^{-(r-\omega)t} dt + \pi\phi_v \int_{s+Z}^{s+Z2} e^{-(r-\omega)t} dt \right)$$

Solving the integrals that appear in this expression and simplifying the result, we arrive at the following equation, which can be solved numerically for *r*:

$$(13)\quad \tau \frac{\left(e^{(r-g-v)C} - 1\right)}{r-g-v} = \rho(C,N) \frac{1-(1-\pi\phi_v)e^{-(r-\omega)X} - \pi\phi_v e^{-(r-\omega)(X+X2)}}{r-\omega}$$

where

(14) $X = Z - J$, $\quad X2 = Z2 - Z \quad$ and $\quad C = J - E$

are, respectively, the number of years that a retirement and a widower's pension will be collected and the length of the agent's working career (or the number of years he will have contributed to the Social Security system at the time of retirement).

4. Aggregate magnitudes

To calculate total pension expenditure and other economy-wide aggregates, we need to add things up across all agents who are either employed workers or pensioners at a given point in time. To make the exercise tractable, I will assume that nothing changes over time or across individuals. In addition to constant values of *g*, *n* and *v*, this means that the parameters of the system (including the contribution rate, τ, the retirement age, *J*, the pension calculation period, *N*, and the rules for computing the percentage ϕ of the regulatory base that is paid out as pension) remain constant over time and that all pensioners have the same characteristics both within and across generations (and, in particular, the same number of contribution years, *C*, and the same life expectancy, *Z* and *Z2*). Hence, I am essentially solving directly for a steady state of the model with constant life expectancy. As a result, the solution I will obtain will describe the equilibrium point to which the system will converge if we let it run undisturbed during a sufficiently long period under stationary circumstances, but it will tell us nothing about the transition path it will follow to reach this target from given initial conditions.

The average wage

Under these assumptions, it is easy to calculate aggregate magnitudes by integrating over the time of birth, s. Let us start with the working population. At time t, the labor force is composed of all the agents who entered the labor market between t-C and t, and were therefore born between t-C-E and t-E. Hence, the labor force at time t is given by

$$(15)\ LF(t) = \int_{t-C-E}^{t-E} L(s)ds = \int_{t-C-E}^{t-E} e^{ns}ds = \frac{\left(e^{nC}-1\right)}{n} e^{nt} e^{-n(C+E)}$$

The aggregate wage bill *(WB)* is the sum of the earnings of all employed workers, that is,

$$(16)\ WB(t) = \int_{t-C-E}^{t-E} L(s)W(s,t)ds = \int_{t-C-E}^{t-E} e^{ns} A_o e^{-\nu(s+E)} e^{(g+\nu)t} ds = A_o e^{(g+n)t} e^{-nE} \frac{\left(1-e^{-(n-\nu)C}\right)}{n-\nu}$$

Hence, the average salary is

$$(17)\ \bar{W}(t) = \frac{WB(t)}{LF(t)} = \left(A_o e^{gt}\right)\left(\frac{n}{n-\nu}\frac{1-e^{-(n-\nu)C}}{1-e^{-nC}}\right) \equiv W^s(t)D^w(n,\nu,C)$$

where $W^s(t)$ is the starting wage at time t and $D^w(n,\nu,C)$ a correction factor that captures the effect of the age distribution of the population on average wages. In the absence of an experience premium ($\nu = 0$), all workers who are active at time t earn the same wage regardless of their date of birth and the correction factor collapses to 1 independently of the age distribution of the working population. When $\nu > 0$, however, wages rise with age, making the average wage higher than the current starting wage $(D^w > 1)$, and the demographic structure of the population matters.

As the experience premium (ν) rises, the upward sloping wage-age profile becomes steeper and the average wage rises relative to the starting wage. Similarly, when $\nu > 0$, an increase in the length of the working career, C, raises the average wage (relative to the starting wage). Finally, as the rate of population growth (n) increases, the relative weight of younger workers in the labor force increases. If these workers have lower wages than older ones (i.e. if $\nu > 0$), then the average wage falls. Hence, we have[3]

$$(18)\ \frac{\partial D^w}{\partial \nu} > 0\,,\ \frac{\partial D^w}{\partial n} < 0 \text{ if } \nu > 0 \text{ and } \frac{\partial D^w}{\partial C} > 0 \text{ if } \nu > 0$$

The average pension

Next, we consider the population of pensioners *(LP)*. Referring to Figure 1, we see that at time t the population is comprised of those agents born between t-Z and t. Of these, those born between t-Z and t-J are retired. In addition to them, a fraction π of those born between t-$Z2$ = t-Z-$X2$ and t-Z have spouses that are still alive and are drawing a widower's pension. Hence, the

[3] See the Appendix for a proof.

pensioner population at time t, including widows (or rather, the number of pensions, since widowers may be counted twice) is given by

$$(19)\ LP(t)=\int_{t-Z}^{t-J}L(s)ds+\pi\int_{t-Z2}^{t-Z}L(s)ds=\int_{t-Z}^{t-J}e^{ns}ds+\pi\int_{t-Z2}^{t-Z}e^{ns}ds=e^{n(t-J)}\frac{1-(1-\pi)e^{-nX}-\pi e^{-n(X+X2)}}{n}$$

Adding up over living pensioners, including widowers, total pension expenditure *(PE)* at time t is given by

$$(20)\ PE(t)=\int_{t-Z}^{t-J}L(s)P(s)ds+\pi\int_{t-Z2}^{t-Z}L(s)PV(s)ds$$

where I have suppressed all arguments of *P()* and *PV()* except for *s*, the time of birth of the (original) beneficiary. Using (6), (7) and (12), this expression becomes

$$\begin{aligned}PE(t)&=\int_{t-Z}^{t-J}L(s)P(s)ds+\pi\int_{t-Z2}^{t-Z}L(s)PV(s)ds\\&=A_o\rho(C,N)e^{\omega t}e^{(g-\omega)J}e^{\nu C}\left(\int_{t-Z}^{t-J}e^{(n+g-\omega)s}ds+\pi\phi_v\int_{t-Z2}^{t-Z}e^{(n+g-\omega)s}ds\right)\end{aligned}$$

and, solving the integrals inside the parentheses and simplifying (see the Appendix for details),

$$(21)\ PE(t)=A_o\rho(C,N)e^{\nu C}e^{(g+n)t}e^{-nJ}\frac{1-(1-\pi\phi_v)e^{-(n+g-\omega)X}-\pi\phi_v e^{-(n+g-\omega)(X+X2)}}{n+g-\omega}$$

Hence, the average pension is given by

$$(22)\quad\begin{aligned}\overline{P}(t)&=\frac{PE(t)}{LP(t)}=A_o\rho(C,N)e^{\nu C}e^{gt}\left(\frac{1-(1-\pi\phi_v)e^{-(n+g-\omega)X}-\pi\phi_v e^{-(n+g-\omega)(X+X2)}}{1-(1-\pi)e^{-nX}-\pi e^{-n(X+X2)}}\frac{n}{n+g-\omega}\right)\\&\equiv P^s(t)D^p(n,g-\omega)\end{aligned}$$

where $P^s(t)$ is the starting pension at time t and D^p a correction factor that depends on the age distribution of pensioners and on how pensions vary with age at a given point in time.

If productivity does not grow over time, pensions are indexed to consumer prices and widowers inherit their spouse's full pension (i.e. if $g=\omega=0$ and $\phi_v=1$) then all pensions paid out at a given point in time (including widowers' pensions) are equal and the correction factor collapses to 1 for any value of n. Otherwise, real pensions vary with the date of birth of the original beneficiary and the age distribution of the pensioner population matters.

Under normal circumstances, older pensioners will have lower pensions than younger ones due to productivity growth and to the fact that survivors' pensions are only a fraction of the original retirement pension. As a result, the average pension will be below the current starting pension (i.e. $D^p<1$ if $g-\omega>0$ and/or $\phi_v<1$). Under these conditions, moreover, an increase in the rate of population growth *(n)* will increase the average pension by raising the relative weight of younger individuals, who have higher than average pensions, in the stock of live pensioners. As productivity growth (g) rises (or ω declines), the downward sloping pension-age profile becomes steeper and the average pension falls relative to the starting pension. Finally, the average pension rises with the generosity of widowers' pensions, measured by ϕ_v. Hence, we have[4]

$$(23)\quad \frac{\partial D^p}{\partial(g-\omega)} < 0 \quad \text{and} \quad \frac{\partial D^p}{\partial \phi_v} > 0$$

and,

$$(24)\quad \frac{\partial D^p}{\partial n} > 0 \ \text{ if } \ g-\omega > 0 \ \text{ and/or } \ \phi_v < 1$$

The components of pension expenditure as a fraction of the wage bill

Finally, pension expenditure as a fraction of the wage bill can be written

$$(25)\quad EXPW = \frac{PE(t)}{WB(t)} = \frac{LP}{LF} * \frac{\bar{P}}{\bar{W}} \equiv DEMLAB * GENW$$

The first term on the right-hand side of this expression, *DEMLAB*, gives the number of pensions per employed worker, a useful summary of the joint impact of demographics and labor market conditions on the pension system. (Notice that the labor market component of this factor is trivial in the current model since it assumes that all agents are continuously employed during their entire active life). Using (15) and (19), this ratio can be written

$$(26)\quad DEMLAB \equiv \frac{LP}{LF} = \frac{e^{n(t-J)}\dfrac{1-(1-\pi)e^{-nX}-\pi e^{-n(X+X2)}}{n}}{\dfrac{\left(e^{nC}-1\right)}{n}e^{nt}e^{-n(C+E)}} = \frac{1-(1-\pi)e^{-nX}-\pi e^{-n(X+X2)}}{e^{nC}-1}$$

Recalling that $C = J - E$ and $X = Z - J$, it is easy to see that *DEMLAB* increases with life expectancy *(Z)* and decreases with the retirement age *(J)* and with the growth rate of population (n).

The second factor in (25), *GENW*, is the ratio between the average pension and the average wage. I will refer to this term as the *generosity factor* of the pension system (defined in terms of the average wage). Using (17) and (22), *GENW* will be given by

$$(27)\quad GENW(t) = \frac{\bar{P}(t)}{\bar{W}(t)} = \rho(C,N)e^{vC}\frac{D^p(n,g-\omega)}{D^w(n,v,C)}$$

Our earlier results about the comparative statics of the numerator and denominator of this ratio imply that *GENW* will be a decreasing function of the rate of productivity growth and an increasing function of the rate of population growth.

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[4] See the Appendix for a proof.

5. The financial sustainability of the pension system

At time *t*, the total revenue of the pension system from social contributions is given by

$$(28)\ REV(t)=\int_{t-C-E}^{t-E} L(s)\tau W(s,t)ds=\tau WB(t)=\tau A_o e^{(g+n)t}e^{-nE}\frac{\left(1-e^{-(n-v)C}\right)}{n-v}$$

whereas total pension expenditure will be

$$(21)\ PE(t)=A_o\rho(C,N)e^{vC}e^{(g+n)t}e^{-nJ}\frac{1-(1-\pi\phi_v)e^{-(n+g-\omega)X}-\pi\phi_v e^{-(n+g-\omega)(X+X2)}}{n+g-\omega}$$

Hence, the system will be running a current surplus whenever the following condition holds:

$$\tau e^{-nE}\frac{\left(1-e^{-(n-v)C}\right)}{n-v}\geq\rho(C,N)e^{vC}e^{-nJ}\frac{1-(1-\pi\phi_v)e^{-(n+g-\omega)X}-\pi\phi_v e^{-(n+g-\omega)(X+X2)}}{n+g-\omega}$$

which can be simplified to

$$(29)\ \tau\frac{\left(e^{(n-v)C}-1\right)}{n-v}\geq\rho(C,N)\frac{1-(1-\pi\phi_v)e^{-(n+g-\omega)X}-\pi\phi_v e^{-(n+g-\omega)(X+X2)}}{n+g-\omega}$$

Working with (29) written as an equality, we can solve for the *sustainable initial replacement rate* given the Social Security contribution rate, τ,

$$(30)\ \tilde{\rho}=\tau\frac{n+g-\omega}{n-v}\frac{e^{(n-v)C}-1}{1-(1-\pi\phi_v)e^{-(n+g-\omega)X}-\pi\phi_v e^{-(n+g-\omega)(X+X2)}}$$

or for the social contribution rate that is required to keep the system in balance given its other parameters,

$$(31)\ \tilde{\tau}=\rho(N,C)\frac{n-v}{n+g-\omega}\frac{1-(1-\pi\phi_v)e^{-(n+g-\omega)X}-\pi\phi_v e^{-(n+g-\omega)(X+X2)}}{e^{(n-v)C}-1}$$

A useful summary statistic of the long-term sustainability of the system that can be compared across "regimes" defined by different sets of parameter values will be the ratio between its steady-state expenditure and revenues, which turns out to be equal to the ratio between the system's initial replacement rate and the sustainable value of the same variable,[5]

$$(32)\ SUST_\rho\equiv\frac{PE}{REV}=\frac{PE}{\tau WB}=\frac{EXPW}{\tau}=DEMLAB*GENW*\frac{1}{\tau}=\frac{\rho(C,N)}{\tilde{\rho}}$$

I will refer to this variable as the *inverse sustainablity ratio in terms of* ρ because an increase in the ratio signals a deterioration of the system's financial position.

[5] By equation (21), total pension expenditure is directly proportional to the system's observed initial replacement rate. By definition, the system's revenues will be equal to its sustainable expenditure, which can be obtained by replacing $\rho(C,N)$ by $\tilde{\rho}$ in equation (21). Hence, $PE/REV=\rho/\tilde{\rho}$

A second sustainability criterion

In a classical paper, Samuelson (1958) shows in the context of an overlapping generation model that a pay-as-you-go pension system is sustainable in the long run if and only if its internal rate of return (IRR) does not exceed the growth rate of aggregate income. In this section I will show that this result also holds in the present model. This provides a second intuitive way to evaluate the long-term sustainability of the pension system and the effects on it of possible changes in parameter values.[6]

Let us return to the equation that implicitly defines the IRR of the pension system, r, (which under our assumptions is the same for all pensioners)

$$(13)\quad \tau \frac{\left(e^{(r-g-v)C}-1\right)}{r-g-v} = \rho(C,N)\frac{1-(1-\pi\phi_v)e^{-(r-\omega)X}-\pi\phi_v e^{-(r-\omega)(X+X2)}}{r-\omega}$$

Using this expression, we can write the initial replacement rate, ρ, as a function of r:

$$(33)\quad \rho(C,N) = \tau\frac{r-\omega}{r-g-v}\frac{e^{(r-g-v)C}-1}{1-(1-\pi\phi_v)e^{-(r-\omega)X}-\pi\phi_v e^{-(r-\omega)(X+X2)}}$$

Substituting (33) into (29), the no-deficit condition becomes

$$\frac{e^{(n-v)C}-1}{n-v} \geq \frac{r-\omega}{g+n-\omega}\left(\frac{e^{(r-g-v)C}-1}{r-g-v}\right)\left(\frac{1-(1-\pi\phi_v)e^{-(g+n-\omega)X}-\pi\phi_v e^{-(g+n-\omega)(X+X2)}}{1-(1-\pi\phi_v)e^{-(r-\omega)X}-\pi\phi_v e^{-(r-\omega)(X+X2)}}\right)$$

which can be easily shown to be equivalent to

$$(34)\quad \left(\frac{e^{(n-v)C}-1}{e^{(r-g-v)C}-1}\right)\frac{r-g-v}{n-v} \geq \frac{r-\omega}{g+n-\omega}\left(\frac{1-(1-\pi\phi_v)e^{-(g+n-\omega)X}-\pi\phi_v e^{-(g+n-\omega)(X+X2)}}{1-(1-\pi\phi_v)e^{-(r-\omega)X}-\pi\phi_v e^{-(r-\omega)(X+X2)}}\right)$$

Next, we define

$$(35)\ d = n + g - r$$

from where

$$g+n = r+d \quad \text{and} \quad r - g = n - d$$

and rewrite (34) in terms of d:

$$(36)\quad \frac{e^{(n-v)C}-1}{e^{(n-v-d)C}-1}\,\frac{n-v-d}{n-v} \geq \frac{r-\omega}{r-\omega+d}\,\frac{1-(1-\pi\phi_v)e^{-(r-\omega+d)X}-\pi\phi_v e^{-(r-\omega+d)(X+X2)}}{1-(1-\pi\phi_v)e^{-(r-\omega)X}-\pi\phi_v e^{-(r-\omega)(X+X2)}}$$

Notice that if $d = 0$ then (36) holds as an equality. That is, if the IRR of the system is equal to the growth rate of its revenues, $n+g$, then its budget is in balance. Taking derivatives of this expression with respect to d it is straightforward to show that if $r > n+g$ then the system will experience a deficit and will not therefore be sustainable in the long run (see the Appendix). Hence, we have the following result: *the pension system will not be in deficit provided that its IRR does not exceed the growth rate of the wage bill, i.e. that*

[6] Jimeno and Licandro (1999) use this approach to evaluate the sustainability of the Spanish pension system.

$$(37)\ r \leq g+n$$

This result allows us to define a second inverse sustainability ratio as the quotient between the observed IRR of the system, r, and its sustainable IRR, given by $g+n$,

$$(38)\ SUST_r = \frac{r}{g+n}$$

As in the case of $SUST_\rho$, a value of $SUST_r$ equal to 1 means that the system will be running a balanced budget in the steady state and a reduction in this indicator signals an improvement in the system's long-term financial position.

6. Comparative statics

This section numerically explores the comparative statics of the model. To fix the starting point for the required calculations, the values of the model's coefficients will be set taking as a reference the average values of the variables of interest over the period 1981-2007 and the observed values of certain characteristics of the representative pensioner and of the parameters currently used in Spain for pension calculations.

The first column of Table 1 contains the relevant data. The values of g and n are set equal to the average rates of growth of labor productivity and employment during the period 1981-2007, with labor input measured in both cases by full-time equivalent employment according to the Spanish National Accounts (INE, 2011a). Both rates have been calculated by regressing the log of the corresponding variable on a linear trend. The experience premium (v) is set so that the model reproduces the average initial replacement rate (that is, the ratio between the initial pension and the salary at the time of retirement) observed among new retirees who entered the system in 2008, as estimated by Devesa (2009, p. 64) using the panel of work histories put together by the Spanish Ministry of Labor (the so called "muestra continua de vidas laborales"). The Social Security contribution rate linked to the pension system is assumed to be equal to 95% of the contribution rate for ordinary contingencies under the general regime of the Social Security,[7] calculated as the sum of the rates applicable to the company (25,6%) and the worker (4,7%).

The number of years of Social Security contributions paid by the average pensioner (C) is approximated as the product of the average employment rate of the population aged 18-64 during the period of reference and the theoretical maximum duration of the individual's working life, 65 – 18 = 47 years. The average duration of the period during which a pension is drawn (X) is calculated as the difference between the average life expectancy at birth of the population as a whole, Z, (taking its average value during the relevant period) and the retirement age, J, which is set equal to the legal retirement age of 65 years. The period during which a surviving spouse's pension is drawn is approximated as the difference between the life

[7] In Spain, ordinary Social Security contributions cover a series of contingencies apart from retirement, making it impossible to isolate a specific contribution to the pension system. The 95% figure is based on an internal government report cited by Doménech and Melguizo (2008).

expectancy of women and that of the population as a whole, incremented by 2.75 years, which is the average age difference between men and women at the time of marriage according to the Spanish National Statistical Institute's marriage statistics (INE, 2011c). The probability (π) that a retiree is survived by a spouse entitled to a widower's pension is set to ½.

Table 1: Baseline model parameterization based on data for 1980-2007 and model's long-term predictions

	parameters	*predicted ratios*	
Growth of output per worker (g)	1.13%	*DEMLAB*	0.373
Growth of total employment (n)	1.90%	*GENW*	0.705
Experience premium per year (v)	1.28%	*EXPW*	0.263
Social Security contribution rate (τ)	27.89%	$SUST_{\rho}$	0.976
*Avge. employment rate of pop. 18-64**	56.03%	$SUST_{r}$	0.960
Avge. years of contribution (C)	26.34		
Life expectancy			
Entire population	76.66	*Note:*	
men	73.37	*Observed* ρ	0.694
women	79.93	*Sustainable* ρ	0.711
X = avge. duration of retirement pension	11.66	*Observed IRR*	2.91%
X2 = avge. duration widower's pension	6.02	*Sustainable IRR*	3.03%
M = years of contribution required for a full pension	35		

- *Note: (*)* The employment rate is calculated as the ratio between total employment (using full-time equivalent figures taken from the National Accounts) and the population 18-64, taken from INE (2011b). The table reports the average value of this ratio during the period 1980-2007.

The second column of Table 1 shows the model's predictions for some variables and ratios of interest under the parameter values listed in the first column. Since steady-state expenditure as a fraction of the wage bill *(EXPW)* is slightly below the Social Security contribution rate (τ), the system would be expected to enjoy a modest budget surplus if operating under stationary conditions. As a result, both inverse sustainability ratios are below one. In the long run, the initial replacement rate, ρ, would be expected to converge to 0.694, which is slightly below its sustainable value of 0.711, and the system's IRR should approach 2.91%.

Starting from the situation described in Table 1, I have calculated the effects on the variables of interest of a small change in each of the parameters of the model. Table 2 summarizes the results. The table shows the percentage change in the steady-state value of the ratios of interest that would be induced by each of the parameter changes described in the table. The ratios of interest are pension expenditure as a fraction of the wage bill (*EXPW*), the demographic-employment and generosity components of this variable (*DEMLAB* and *GENW*) and the two inverse sustainability ratios, $SUST_{\rho}$ and $SUST_{r}$. Recall that the first of the sustainability indicators is the ratio between *EXPW* and the Social Security contribution rate, τ, which is not included among the columns of the table.

Table 2: Expected long-term % change in the ratios of interest induced by changes in the parameters of the model

	GENW	*DEMLAB*	*EXPW*	$SUST_\rho$	$SUST_r$
$g\uparrow$ *by 0.25 p.p.*	-3.28%		-3.28%	-3.28%	-5.06%
$n\uparrow$ *by 0.25 p.p.*	0.41%	-5.24%	-4.86%	-4.86%	-7.62%
$v\uparrow$ *by 0.25 p.p.*	1.62%		1.62%	1.62%	2.77%
$\tau\uparrow$ *by 1 p.p.*				-3.59%	-6.34%
$N\uparrow$ *by 1 year*	-1.12%		-1.12%	-1.12%	-1.96%
$C\uparrow$ *by 1 year*	3.15%	-4.65%	-1.64%	-1.64%	-2.70%
$Z\uparrow$ *by 1 year**	0.21%	5.89%	6.11%	6.11%	10.13%
$J\uparrow$ *by 1 year**	-0.31%	-6.01%	-6.30%	-6.30%	-11.65%
$X2\uparrow$ *by 1 year*	-1.36%	2.78%	1.38%	1.38%	2.40%
$\omega\uparrow$ *by 0.25 p.p.*	1.60%		1.60%	1.60%	2.75%
$M\uparrow$ *by 1 year*	-1.72%		-1.72%	-1.72%	-3.35%

- (*) $Z\uparrow$ means an increase in average life expectancy, holding constant the difference between men and women (and hence *X2*) and $J\uparrow$ an increase in the age of retirement holding constant the number of contribution years, *C*.

The table shows the effect that different parameter changes would have on the long-term financial health of the system and the channels that would be involved in each case. For instance, an increase in the annual growth rate of (population and) employment of a quarter of a percentage point (p.p), from 1.90% to 2.15%, would slightly raise the average pension to wage ratio ($GENW\uparrow$) by changing the age structure of the working and retired populations[8] but would also induce a much larger reduction in the ratio of pensioners to employed workers ($DEMLAB\downarrow$). The net effect would be a significant reduction in the expenditure and ρ-sustainability ratios, which would fall by 4.86% for each quarter-point increase in *n*. A one-point increase in the Social Security contribution rate (τ) would have no effect on expenditure but would increase revenue, thereby improving (i.e. lowering) the sustainability ratios.

The last row of the table shows the effect of a one-year increase in the period required to attain a "full pension," understood as 100% of the regulatory base of the pension. The impact of this reform has been calculated under the counterfactual assumption that ϕ increases linearly with *C* once the minimum period of 15 years has been completed (see footnote 2). That is, I am assuming that ϕ is given by

$$\phi(C,M) = 0.5 + (C-15) * \frac{0.5}{M-15}$$

and exploring the sensitivity of the model's predictions to a one-year increase in *M*, starting from its current value of 35.

[8] Faster population (and employment) growth will increase the weight of the relatively young both among employed workers and among pensioners. Other things equal, this will reduce the average wage (because young workers have less experience and hence lower wages) and increase the average pension (because recent retirees will have higher pensions than older ones as long as productivity growth is positive).

The results of the exercise indicate that pension expenditure and the system's sustainability ratios are rather sensitive to many of the model's parameters. A one-year increase in life expectancy would increase the expenditure ratio by more than six percent (from 0.263 to 0.279 of the wage bill) and would push the sustainability ratio above the threshold value of 1. To offset the effects of such a change, the retirement age would also have to rise by one year or social contribution rates would have to be raised by 1.7 percentage points.[9] Raising the growth rates of employment and productivity by a quarter of a point would reduce steady-state expenditure by 4.86% and 3.28% respectively. Faster employment growth works mostly through the demographic-labor market component of expenditure by increasing the ratio of employed workers to pensioners. Productivity growth, on the other hand, works through the generosity ratio: faster productivity growth implies steeper wage profiles which in turn translate into lower initial replacement rates through the averaging formula used to calculate the regulatory base of the pension and through the distribution factor discussed in the previous section. Hence, wage gains arising from higher productivity growth do not translate entirely to pensions and, as a result, do help improve sustainability ratios. By contrast, an increase in the experience premium (ν) would have a greater positive effect on the average pension than on average wages, thereby increasing the generosity ratio.

7. Conclusion

Pensions currently constitute one of the biggest public spending items in most advanced nations and one of the potentially most serious threats to the long-term sustainability of their public finances in the face of rapid population aging. The problem is particularly acute in those countries, like Spain, that have a pay-as-you-go system with defined benefits in which pensions are financed by current contributions from active workers and benefit levels are set in advance without reference to actuarial sustainability criteria.

In these circumstances, policy makers and analysts need tools that allow them to project the evolution of pension expenditure under different economic and demographic scenarios and to analyze the effects of possible policy reforms. This paper has developed a simple model that highlights the main forces at work and that may be a useful instrument for performing "quick and dirty" calculations of this sort. An exercise of this type using recent Spanish data suggests that, in the absence of reforms, our public pension system will soon fall below the sustainability threshold. The biggest threats in this regard are the inevitable decrease in the growth rate of employment that will ensue as the working-age population stabilizes and may even decline over the coming decades and the rapid increase in life expectancy that is expected to continue in the foreseeable future. Lagging productivity growth is also a concern, as fast gains in output per worker would partially offset the adverse effects of aging on the financial health of the public pension system.

[9] Raising the age of retirement would have an effect of roughly the same size in the opposite direction. In fact, the effect of raising the retirement age by one year is slightly larger than that of increasing life expectancy in the same amount because the former affects "young pensioners" that have a greater weight in the retired population and on total pension expenditure than the oldest pensioners affected by the latter.

APPENDIX

1. Some useful results

This section collects the results of some simple calculations that will be useful later on.

• In this paper we often have to calculate integrals of a certain type. Introducing a change of variable, it is easy to show that

$$\text{(A.1)}\quad \int_a^b e^{\gamma t}dt = \begin{cases} \dfrac{\left(e^{\gamma(b-a)}-1\right)e^{\gamma a}}{\gamma} = \dfrac{\left(1-e^{-\gamma(b-a)}\right)e^{\gamma b}}{\gamma} & \text{for } \gamma \neq 0 \\ b-a \text{ for } \gamma = 0 \end{cases}$$

Proceeding in the same manner with $\gamma = -r \neq 0$, we have

$$\text{(A.2)}\quad \int_a^b e^{-rt}dt = \frac{\left(1-e^{-r(b-a)}\right)e^{-ra}}{r}$$

• For any x, it is easy to show that

$$\text{(A.3)}\ \varphi(x) \equiv e^{-x}(1+x) \leq 1$$

$$\text{(A.4)}\ \mu(x) \equiv e^{x}(x-1) \geq -1$$

and that both expressions hold as strict inequalities for $x \neq 0$.

• Next, consider the function

$$\text{(A.5)}\quad h(\gamma, D) = \begin{cases} \dfrac{1-e^{-\gamma D}}{\gamma} & \text{for } \gamma \neq 0 \\ D \text{ for } \gamma = 0 \end{cases} \qquad \text{with } D > 0$$

It is easy to show that *h()* is always positive, tends to ∞ as $\gamma \to -\infty$ and to 0 as $\gamma \to \infty$, is continuous at 0 and increases with *D*. Differentiating with respect to each argument and using (A.3) we have

$$\text{(A.6)}\quad \begin{aligned} h_D(\gamma, D) &= \frac{-e^{-\gamma D}(-\gamma)}{\gamma} = e^{-\gamma D} > 0 \\ h_\gamma(\gamma, D) &= \frac{\partial h(\gamma, D)}{\partial \gamma} = \frac{e^{-\gamma D}(1+\gamma D)-1}{\gamma^2} = \frac{\varphi(\gamma D)-1}{\gamma^2} < 0 \end{aligned}$$

for $\gamma \neq 0$. Notice that this expression can also be written

$$\text{(A.7)}\quad h_\gamma(\gamma, D) = \frac{\gamma D e^{-\gamma D} - (1-e^{-\gamma D})}{\gamma^2} = \frac{De^{-\gamma D}}{\gamma} - \frac{(1-e^{-\gamma D})}{\gamma^2} = \frac{De^{-\gamma D}}{\gamma} - \frac{h(\gamma, D)}{\gamma} < 0$$

This implies that

$$\text{(A.8)}\quad h(\gamma, D) > De^{-\gamma D} \text{ for } \gamma > 0 \text{ and } h(\gamma, D) < De^{-\gamma D} \text{ for } \gamma < 0$$

When $\gamma = 0$, moreover, we have

$$\lim_{\gamma\to 0} h_\gamma = \frac{0-1+1}{0} = \lim_{\gamma\to 0}\frac{-\gamma D^2 e^{-\gamma D}}{2\gamma} = \frac{-D^2}{2} < 0$$

by L'Hopital's rule.

• Similarly, the function

$$(A.9)\quad m(\gamma,D) = \begin{cases} \dfrac{e^{\gamma D}-1}{\gamma} \text{ for } \gamma \neq 0 \\ D \text{ for } \gamma = 0 \end{cases} \quad \text{with } D > 0$$

takes on only positive values regardless of the sign of γ, is continuous at zero, tends to 0 as $\gamma \to -\infty$ and to infinity as $\gamma \to \infty$ and increases with *D*. Differentiating with respect to γ and *D* and using (A.4) we have

$$(A.10)\quad \begin{aligned} \frac{\partial m(\gamma,D)}{\partial\gamma} &= \frac{e^{\gamma D}(\gamma D-1)+1}{\gamma^2} = \frac{\mu(\gamma D)+1}{\gamma^2} > 0 \\ \frac{\partial m(\gamma,D)}{\partial D} &= \frac{e^{\gamma D}\gamma}{\gamma} = e^{\gamma D} > 0 \end{aligned}$$

for $\gamma \neq 0$. For $\gamma = 0$ we have

$$\lim_{\gamma\to 0} m_\gamma = \frac{0+1-1}{0} = \lim_{\gamma\to 0}\frac{D^2 e^{-\gamma D}}{2} = \frac{D^2}{2} > 0$$

• Next, consider the function

$$(A.11)\quad f(\gamma,p,X,Y) = \frac{1-(1-p)e^{-\gamma X} - pe^{-\gamma Y}}{\gamma} \quad \text{with} \quad 0 < X \leq Y \quad \text{and} \quad p \in [0,1]$$

and observe that it can be written

$$(A.12)\quad f(\gamma,p,X,Y) = (1-p)h(\gamma,X) + ph(\gamma,Y)$$

Differentiating this expression and using previous results we have

$$(A.13)\quad \begin{aligned} & f_\gamma(\gamma,p,X,Y) = (1-p)h_\gamma(\gamma,X) + ph_\gamma(\gamma,Y) < 0 \\ & f_X(\gamma,p,X,Y) = (1-p)h_X(\gamma,X) = (1-p)e^{-\gamma X} > 0 \\ & f_Y(\gamma,p,X,Y) = ph_Y(\gamma,Y) = pe^{-\gamma Y} > 0 \\ & f_p(\gamma,p,X,Y) = -h(\gamma,X) + h(\gamma,Y) = h(\gamma,Y) - h(\gamma,X) \geq 0 \end{aligned}$$

where the last inequality holds because *h()* is increasing in its second argument and $Y \geq X$.

• *Some useful bounds on the exponential function*

Let $f: R \to R$ be *n+1* times differentiable on some open interval, *I*. It is well known that for any $a, x \in I$, *f* can be written in the form of a Taylor polynomial with remainder,

$$f(x) = f(a) + \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(b)}{(n+1)!}(x-a)^{n+1}$$

where b is some point between a and x and $f^{(k)}(a)$ is the k-th derivative of *f()* evaluated at a.

Letting $f(x) = e^x - 1$, we have $f^{(k)}(x) = e^x$ for all k and, setting a to zero and n to 2, we can write

$$e^x - 1 = \left(e^0 - 1\right) + e^0 x + \frac{e^0 x^2}{2} + \frac{e^b x^3}{3!} = x + \frac{x^2}{2} + \frac{e^b x^3}{6}$$

for some b between 0 and x. If we constrain x to be positive, the remainder (the last term on the right-hand side of this expression) will also be positive and this implies that

$$(A.14)\ \ e^x - 1 > x + \frac{x^2}{2} \quad \text{for all } x > 0$$

If $x < 0$, on the other hand, the remainder will be negative and we will have

$$(A.14')\ \ e^x - 1 < x + \frac{x^2}{2} \quad \text{for all } x < 0$$

Proceeding in a similar way with $f(x) = 1 - e^{-x}$ we have

$$f(0) = 0,\ f'(x) = e^{-x},\ f''(x) = -e^{-x} \text{ and } f'''(x) = e^{-x}$$

and therefore

$$1 - e^{-x} = \left(1 - e^{-0}\right) + e^{-0} x + \frac{-e^{-0} x^2}{2} + \frac{e^b x^3}{3!} = x - \frac{x^2}{2} + \frac{e^b x^3}{6}$$

from where

$$(A.15)\ \ 1 - e^{-x} > x - \frac{x^2}{2} \quad \text{for all } x > 0 \quad \text{and} \quad 1 - e^{-x} < x - \frac{x^2}{2} \quad \text{for } x < 0$$

Using these inequalities, it is easy to establish the following result, which will be useful below.

• **Claim 1**: $\left(e^x - 1\right)^2 \geq x^2 e^x$ with strict inequality whenever $x \neq 0$

Proof:

We want to show that

$$(A.16)\ \ v(x) \equiv \left(e^x - 1\right)^2 \geq x^2 e^x \equiv u(x)$$

with strict inequality whenever $x \neq 0$. Notice that (A.16) holds as an equality for $x = 0$ since

$$v(0) = \left(e^0 - 1\right)^2 = 0 = 0e^0 = u(0)$$

Next, we need to compare the functions *v(x)* and *u(x)* for $x > 0$. Since both functions vanish when $x = 0$, their relative size will depend on that of their derivatives. Formally, since we can write

$$v(x) = v(0) + \int_0^x v'(s)ds = \int_0^x v'(s)ds \quad \text{and} \quad u(x) = \int_0^x u'(s)ds$$

a sufficient condition for (A.16) to hold for all $x > 0$ is that

$$(A.17) \quad v'(x) > u'(x) \quad \text{for all } x > 0$$

Calculating the relevant derivatives,

$$(A.18) \quad \begin{aligned} v'(x) &= 2\left(e^x - 1\right)e^x \\ u'(n) &= x^2 e^x + 2xe^x = \left(x^2 + 2x\right)e^x \end{aligned}$$

this condition can be written

$$\left(e^x - 1\right) > x + \frac{x^2}{2}$$

which is true for all $x > 0$ by (A.14).

Finally, assume that $x < 0$. As before, we need to compare the functions *v(x)* and *u(x)* using their derivatives. Notice, however, that the direction of the inequality between the relevant derivatives will be reversed as we cross the origin. Since $v(x) = u(x) = 0$, *v(x)* will lie above *u(x)* as we move from 0 to the right if *v()* is steeper than *u()*. As we move from 0 to the left, however, we need *v()* to be flatter than *u()* in order to obtain the same result. Hence, in the first case we need $v'(x) > u'(x)$ for $x > 0$ and in the second $v'(x) < u'(x)$ for $x < 0$.

Formally, when $x < 0$ we can write

$$v(x) = \int_0^x v'(s)ds = \int_x^0 -v'(s)ds \quad \text{and} \quad u(x) = \int_0^x u'(s)ds = \int_x^0 -u'(s)ds$$

so a sufficient condition for (A.16) to hold for all $x < 0$ is

$$(A.17') \quad -v'(x) > -u'(x) \Leftrightarrow v'(x) < u'(x) \quad \text{for all } x < 0$$

or, using (A.18),

$$\left(e^x - 1\right) < x + \frac{x^2}{2}$$

which we know to hold for all $x < 0$ by (A.14'). □

Letting $z = -x$, claim 1 can be rewritten

$$(A.19) \quad \left(e^{-z} - 1\right)^2 \geq (-z)^2 e^{-z} \Leftrightarrow \left(1 - e^{-z}\right)^2 \geq z^2 e^{-z}$$

with strict inequality whenever $z \neq 0$. If we define the function *A(z)* by

$$(A.20)\ \ A(z) = \left(1 - e^{-z}\right)^2 - e^{-z} z^2$$

we have by (A.19) that

$$(A.21)\ \ A(0) = 0 \quad \text{and} \quad A(z) > 0 \text{ for } z \neq 0$$

Differentiating *A()*, we have

$$A'(z) = 2e^{-z}\left[\left(1 - e^{-z}\right) - \left(z - \frac{z^2}{2}\right)\right]$$

Using (A.15), this expression implies that

$$(A.22)\ \ A'(z) = \begin{cases} > 0 & \text{for } z > 0 \\ < 0 & \text{for } z < 0 \end{cases} \qquad \text{or} \qquad A'(z)z > 0 \quad \text{for all } z \neq 0$$

Finally, notice that

$$A'(z) + 2A(z) = 2e^{-z}\left[\left(e^{z} - 1\right) - \left(z + \frac{z^2}{2}\right)\right]$$

Using (A.14) and (A.14') we see that

$$(A.23)\ \ A'(z) + 2A(z) = \begin{cases} > 0 & \text{for } z > 0 \\ < 0 & \text{for } z < 0 \end{cases} \quad \Leftrightarrow \quad \left[A'(z) + 2A(z)\right]z > 0 \quad \text{for all } z \neq 0$$

2. Calculation and comparative statics of the regulatory base of the pension

The *regulatory base* of the pension is defined as the average wage of the worker calculated over the N years prior to retirement:

$$\begin{aligned} B(s, s+J, N) &= \frac{1}{N}\int_{s+J-N}^{s+J} W(s,t)dt = \frac{1}{N}\int_{s+J-N}^{s+J} A_o e^{(g+v)t} e^{-v(s+E)} dt = \frac{1}{N} A_o e^{-v(s+E)} \int_{s+J-N}^{s+J} e^{(g+v)t} dt = \\ &= \frac{1}{N} A_o e^{-v(s+E)} \frac{\left(1 - e^{-(g+v)N}\right) e^{(g+v)(s+J)}}{g+v} = \frac{\left(1 - e^{-(g+v)N}\right)}{(g+v)N} A_o e^{-v(s+E)} e^{(g+v)(s+J)} \\ &= \frac{\left(1 - e^{-(g+v)N}\right)}{(g+v)N} W(s, s+J) \equiv b\big((g+v)N\big) W(s, s+J) \end{aligned}$$

or

$$B(s, s+J, N) = b\big((g+v)N\big) W(s, s+J) \quad \text{with} \quad b\big((g+v)N\big) = \frac{1 - e^{-(g+v)N}}{(g+v)N}$$

To abbreviate, define

$$x = (g+v)N$$

and write *b()* in the form

$$b(x) = \frac{1 - e^{-x}}{x}$$

Differentiating this function and using (A.3), we have

$$b'(x) = \frac{xe^{-x} - (1-e^{-x})}{x^2} = \frac{e^{-x}(1+x)-1}{x^2} = \frac{\varphi(x)-1}{x^2} < 0$$

for $x \neq 0$. Hence, the ratio *b()* is a decreasing function of *N* and *g+v*.

3. Calculation of the IRR of the pension system

The IRR of the pension system is the value of *r* that solves the following equation

$$\tau \int_{s+E}^{s+J} e^{-(r-g-v)t}\,dt = \phi(C)b(N)e^{(g+v-\omega)(s+J)} \left(\int_{s+J}^{s+Z} e^{-(r-\omega)t}\,dt + \pi\phi_v \int_{s+Z}^{s+Z2} e^{-(r-\omega)t}\,dt \right)$$

Solving the integrals that appear in this expression and operating, we have

$$\begin{aligned}
\int_{s+J}^{s+Z} e^{-(r-\omega)t}\,dt + \pi\phi_v \int_{s+Z}^{s+Z2} e^{-(r-\omega)t}\,dt &= \frac{\left(1-e^{-(r-\omega)(J-Z)}\right)e^{-(r-\omega)(s+J)}}{r-\omega} + \pi\phi_v \frac{\left(1-e^{-(r-\omega)(Z2-Z)}\right)e^{-(r-\omega)(s+Z)}}{r-\omega} \\
&= e^{-(r-\omega)(s+J)} \frac{\left(1-e^{-(r-\omega)X}\right) + \pi\phi_v\left(1-e^{-(r-\omega)X2}\right)e^{-(r-\omega)[(s+Z)-(s+J)]}}{r-\omega} \\
&= e^{-(r-\omega)(s+J)} \frac{1-(1-\pi\phi_v)e^{-(r-\omega)X} - \pi\phi_v e^{-(r-\omega)(X+X2)}}{r-\omega}
\end{aligned}$$

and

$$\int_{s+E}^{s+J} e^{-(r-g-v)t}\,dt = e^{-(r-g-v)(s+E)} \frac{\left(1-e^{-(r-g-v)C}\right)}{r-g-v}$$

where

$$X = Z - J \qquad X2 = Z2 - Z \qquad \text{and} \qquad C = J - E.$$

Collecting results, the IRR of the system is the value of *r* that solves the following equation:

$$\tau e^{-(r-g-v)(s+E)} \frac{\left(1-e^{-(r-g-v)C}\right)}{r-g-v} = \phi(C)b(N)e^{(g+v-\omega)(s+J)} e^{-(r-\omega)(s+J)} \frac{1-(1-\pi\phi_v)e^{-(r-\omega)X} - \pi\phi_v e^{-(r-\omega)(X+X2)}}{r-\omega}$$

which can be somewhat simplified to

$$\tau \frac{\left(e^{(r-g-v)C}-1\right)}{r-g-v} = \rho(C,N) \frac{1-(1-\pi\phi_v)e^{-(r-\omega)X} - \pi\phi_v e^{-(r-\omega)(X+X2)}}{r-\omega}$$

4. Aggregate magnitudes

The average wage

The labor force at time *t* is given by

$$LF(t) = \int_{t-C-E}^{t-E} L(s)\,ds = \int_{t-C-E}^{t-E} e^{ns}\,ds = \frac{\left(e^{nC}-1\right)e^{n(t-C-E)}}{n} = \frac{\left(e^{nC}-1\right)}{n} e^{nt} e^{-n(C+E)}$$

The aggregate wage bill *(WB)* is the sum of the earnings of all employed workers, that is,

$$WB(t) = \int_{t-C-E}^{t-E} L(s)W(s,t)ds = \int_{t-C-E}^{t-E} e^{ns} A_o e^{-\nu(s+E)} e^{(g+\nu)t} ds = A_o e^{(g+\nu)t} e^{-\nu E} \int_{t-C-E}^{t-E} e^{(n-\nu)s} ds =$$

$$= A_o e^{(g+\nu)t} e^{-\nu E} \frac{\left(1-e^{-(n-\nu)C}\right) e^{(n-\nu)(t-E)}}{n-\nu} = A_o e^{(g+n)t} e^{-nE} \frac{\left(1-e^{-(n-\nu)C}\right)}{n-\nu}$$

Hence, the average salary is

$$\bar{W}(t) = \frac{WB(t)}{LF(t)} = \frac{A_o e^{(g+n)t} e^{-nE} \frac{\left(1-e^{-(n-\nu)C}\right)}{n-\nu}}{\frac{\left(e^{nC}-1\right)}{n} e^{nt} e^{-n(C+E)}} = \left(A_o e^{gt}\right)\left(\frac{n}{n-\nu} \frac{1-e^{-(n-\nu)C}}{1-e^{-nC}}\right) \equiv W^s(t) D^w(n,\nu,C)$$

Notice that

$$D^w(n,\nu,C) = \frac{1-e^{-(n-\nu)C}}{n-\nu} \frac{n}{1-e^{-nC}} = \frac{h(n-\nu,C)}{h(n,C)}$$

where *h()* has been defined above in (A.5). Since *h*() is decreasing in its first argument, it follows that $D^w > 1$ for $\nu > 0$ and

$$\frac{\partial D^w}{\partial \nu} = \frac{1}{h(n,C)} \frac{\partial h(n-\nu,C)}{\partial (n-\nu)}(-1) > 0$$

so the average wage increases with the experience premium. As we increase *n* the weight of the younger workers in the labor force increases. If these workers have lower wages than older ones, i.e. if $\nu > 0$, then the average wage falls with *n*. Hence, we have

- **Claim 2**: Given $\nu > 0$, we have $\frac{\partial D^w(n,\nu,C)}{\partial n} < 0$ for all $n \neq 0$

Proof:

Fix $\nu > 0$ and define the functions *F()* and *g()* by

$$F(n) \equiv \ln D^w(n,\nu,C) = \ln h(n-\nu,C) - \ln h(n,C) \equiv g(n-\nu) - g(n)$$

Since *ln()* is an increasing function, the derivative of *f()* will have the same sign as that of $D^w()$. Hence, the desired result will follow if we can show that

$$F'(n) = g'(n-\nu) - g'(n) < 0 \Leftrightarrow g'(n-\nu) < g'(n) \quad \text{for } \nu > 0$$

i.e. that *g'()* is an increasing function. Hence, it will be sufficient to show that

$$g''(n) > 0$$

To continue, we need to distinguish two cases depending on the sign of *n*.

- *Case i)* Assume *n > 0*. Then we can write

$$(A.24)\;\; g(n) = \ln h(n,C) = \ln\frac{1-e^{-nC}}{n} = \ln\left(1-e^{-nC}\right) - \ln n$$

and differentiating this expression

$$g'(n) = \frac{e^{-nC}C}{1-e^{-nC}} - \frac{1}{n} = \frac{C}{e^{nC}-1} - \frac{1}{n}$$
$$g''(n) = \frac{-C^2e^{nC}}{\left(e^{nC}-1\right)^2} + \frac{1}{n^2}$$

To establish the desired result we need to show that

$$(A.25)\;\; \begin{aligned} g''(n) = \frac{-C^2e^{nC}}{\left(e^{nC}-1\right)^2} + \frac{1}{n^2} > 0 \Leftrightarrow \frac{1}{n^2} > \frac{C^2e^{nC}}{\left(e^{nC}-1\right)^2} \\ \Leftrightarrow v(n) \equiv \left(e^{nC}-1\right)^2 > n^2C^2e^{nC} \equiv u(n) \end{aligned}$$

for all $n > 0$, which holds by claim 1 with $x = nC$.

- Case ii) We now consider the case when $n < 0$. Then, we need to write (A.24) in a slightly different way

$$(A.24')\;\; g(n) = \ln h(n,C) = \ln\frac{e^{-nC}-1}{-n} = \ln\left(e^{-nC}-1\right) - \ln(-n)$$

to ensure that we are taking logs of positive numbers. This does not change the derivative of *g()*, however, so we still need to show that (A.25) holds for all $n < 0$, which is true by claim 1. □

- **Claim 3:** Given $\nu > 0$, we have $\dfrac{\partial D^w(n,\nu,C)}{\partial C} > 0$.

Proof:

Fix $\nu > 0$ and define the functions *F()* and *g()* by

$$F(n,C) \equiv \ln D^w(n,\nu,C) = \ln h(n-\nu,C) - \ln h(n,C) \equiv g(n-\nu,C) - g(n,C)$$

We want to show that

$$F_C(n,C) \equiv g_C(n-\nu,C) - g_C(n,C) > 0 \Leftrightarrow g_C(n-\nu,C) > g_C(n,C) \;\;\text{for } \nu > 0$$

i.e. that $g_C()$ is a decreasing function of its first argument. Hence, it will be sufficient to show that

$$g_{Cn}(n,C) < 0$$

Differentiating

$$g(n,C) = \ln h(n,C) = \ln\frac{1-e^{-nC}}{n}$$

we have

$$g_C(n,C) = \frac{h_C(n,C)}{h(n,C)} = \frac{e^{-nC}}{\frac{1-e^{-nC}}{n}} = \frac{n}{e^{nC}-1}$$

and

$$g_{Cn}(n,C) = \frac{\left(e^{nC}-1\right) - ne^{nC}C}{\left(e^{nC}-1\right)^2} = \frac{-e^{nC}(nC-1)-1}{\left(e^{nC}-1\right)^2} = \frac{-\mu(nC)-1}{\left(e^{nC}-1\right)^2} < 0$$

where the inequality follows by multiplying both sides of (A.4) by -1. □

The average pension

The pensioner population at time t, including widows, is given by

$$LP(t) = \int_{t-Z}^{t-J} L(s)ds + \pi \int_{t-Z2}^{t-Z} L(s)ds = \int_{t-Z}^{t-J} e^{ns}ds + \pi \int_{t-Z2}^{t-Z} e^{ns}ds = \frac{\left(1-e^{-n(Z-J)}\right)e^{n(t-J)}}{n} + \pi \frac{\left(1-e^{-n(Z2-Z)}\right)e^{n(t-Z)}}{n} =$$

$$= e^{n(t-J)} \frac{\left(1-e^{-nX}\right) + \pi\left(1-e^{-nX2}\right)e^{-nX}}{n} = e^{n(t-J)} \frac{1-(1-\pi)e^{-nX} - \pi e^{-n(X+X2)}}{n}$$

Adding up over live pensioners, total pension expenditure *(PE)* at time t is given by

$$PE(t) = \int_{t-Z}^{t-J} L(s)P(s)ds + \pi \int_{t-Z2}^{t-Z} L(s)PV(s)ds$$

Using equations (6), (7) and (12) in the text, this expression can be written

$$PE(t) = \rho(C,N)e^{\omega t} \int_{t-Z}^{t-J} e^{ns}W(s,s+J)e^{-\omega(s+J)}ds + \pi\phi_v \rho(C,N)e^{\omega t} \int_{t-Z2}^{t-Z} e^{ns}W(s,s+J)e^{-\omega(s+J)}ds$$

$$= A_o \rho(C,N)e^{\omega t}e^{(g-\omega)J}e^{vC}\left(\int_{t-Z}^{t-J} e^{(n+g-\omega)s}ds + \pi\phi_v \int_{t-Z2}^{t-Z} e^{(g+n-\omega)s}ds\right)$$

Now, the term in parentheses becomes

$$\int_{t-Z}^{t-J} e^{(n+g-\omega)s}ds + \pi\phi_v \int_{t-Z2}^{t-Z} e^{(g+n-\omega)s}ds = \frac{\left(1-e^{-(g+n-\omega)(Z-J)}\right)e^{(g+n-\omega)(t-J)}}{g+n-\omega} + \pi\phi_v \frac{\left(1-e^{-(g+n-\omega)(Z2-Z)}\right)e^{(g+n-\omega)(t-Z)}}{g+n-\omega}$$

$$= e^{(g+n-\omega)t}e^{-(g+n-\omega)J} \frac{\left(1-e^{-(g+n-\omega)X}\right) + \pi\phi_v\left(1-e^{-(g+n-\omega)X2}\right)e^{-(g+n-\omega)X}}{g+n-\omega}$$

$$= e^{(g+n-\omega)t}e^{-(g+n-\omega)J} \frac{1-(1-\pi\phi_v)e^{-(g+n-\omega)X} - \pi\phi_v e^{-(g+n-\omega)(X+X2)}}{g+n-\omega}$$

Substituting this into the previous expression,

$$PE(t) = A_o \rho(C,N)e^{\omega t}e^{(g-\omega)J}e^{vC}\left(\int_{t-Z}^{t-J} e^{(n+g-\omega)s}ds + \pi\phi_v \int_{t-Z2}^{t-Z} e^{(g+n-\omega)s}ds\right)$$

$$= A_o \rho(C,N)e^{vC}e^{(g+n)t}e^{-nJ} \frac{1-(1-\pi\phi_v)e^{-(g+n-\omega)X} - \pi\phi_v e^{-(g+n-\omega)(X+X2)}}{g+n-\omega}$$

Hence, the average pension is given by

$$\bar{P}(t) = \frac{PE(t)}{LP(t)} = \frac{A_o \rho(C,N) e^{vC} e^{(g+n)t} e^{-nJ} \dfrac{1-(1-\pi\phi_v)e^{-(g+n-\omega)X} - \pi\phi_v e^{-(g+n-\omega)(X+X2)}}{g+n-\omega}}{e^{n(t-J)} \dfrac{1-(1-\pi)e^{-nX} - \pi e^{-n(X+X2)}}{n}}$$

$$= A_o \rho(C,N) e^{vC} e^{gt} \left(\frac{1-(1-\pi\phi_v)e^{-(g+n-\omega)X} - \pi\phi_v e^{-(g+n-\omega)(X+X2)}}{1-(1-\pi)e^{-nX} - \pi e^{-n(X+X2)}} \frac{n}{g+n-\omega} \right) \equiv P^s(t) D^p(n, g-\omega)$$

where $P^s(t)$ is the starting pension at time *t* and D^p a correction factor that depends on the age distribution of pensioners and on how pensions vary with age at a given point in time.

Notice that D^p can be written

$$D^p(n, g-\omega, \pi, \phi_v, X, X+X2) = \frac{f(g+n-\omega, \pi\phi_v, X, X+X2)}{f(n, \pi, X, X+X2)}$$

where

$$f(\gamma, p, X, Y) = \frac{1-(1-p)e^{-\gamma X} - pe^{-\gamma Y}}{\gamma} = (1-p)h(\gamma, X) + ph(\gamma, Y)$$

has been defined above in (A.11). Since *f*() is decreasing in its first argument and increasing in the second (see A.13), it follows that

$$\frac{\partial D^p}{\partial(g-\omega)} = \frac{1}{f(n,\pi)} \frac{\partial f(n+g-\omega, \pi\phi_v)}{\partial(n+g-\omega)} * 1 < 0$$
$$\frac{\partial D^p}{\partial \phi_v} = \frac{1}{f(n,\pi)} \frac{\partial f(n+g-\omega, \pi\phi_v)}{\partial(\pi\phi_v)} \pi > 0$$

i.e. the average pension, written as a fraction of the starting pension, decreases with g, increases with ω and increases with the generosity of widower pensions, ϕ_v.

Notice that $D^p = 1$ when $g - \omega = 0$ and $\phi_v = 1$. Combining this with the signs of the two partial derivatives we have just calculated, we see that $D^p < 1$ for $g - \omega > 0$ and/or $\phi_v < 1$. Finally, it can be shown that D^p is an increasing function of n.

- **Claim 4**: Given $g - \omega > 0$, we have $\dfrac{\partial D^p(n, g-\omega)}{\partial n} > 0$ for all $n \neq 0$

Proof:

To simplify a bit the notation, we can assume $\omega = 0$ without loss of generality and work with g rather than $g - \omega$. We are interested in the function

$$D^p(n, g, \pi, \phi_v, X, Y) = \frac{1-(1-\pi\phi_v)e^{-(g+n)X} - \pi\phi_v e^{-(g+n)Y}}{1-(1-\pi)e^{-nX} - \pi e^{-nY}} \frac{n}{g+n} = \frac{f(g+n, \pi\phi_v, X, Y)}{f(n, \pi, X, Y)}$$

with

$g > 0$, $\pi \in [0,1]$, $0 \le X \le Y$ and $\phi_v \in [0,1]$.

Define the functions *F()* and *q()* by

$$F(n,\pi,\phi_v) \equiv \ln D^p(n,g,\pi,\phi_v) = \ln f(n+g,\pi\phi_v) - \ln f(n,\pi) \equiv q(n+g,\pi\phi_v) - q(n,\pi)$$

Since *ln()* is an increasing function, the derivatives of *F()* will have the same sign as those of D^p. Hence, what we want to show is that

$$F_n(n,\pi,\phi_v) = q_n(n+g,\pi\phi_v) - q_n(n,\pi) > 0 \Leftrightarrow q_n(n+g,\pi\phi_v) > q_n(n,\pi)$$

For this, it will be sufficient to show that $q_n()$ is increasing in n and decreasing in π. In terms of the second partials of *q*, we need to show that

$$q_{nn} > 0 \quad \text{and} \quad q_{n\pi} < 0$$

Differentiating

$$q(n,\pi) = \ln f(n,\pi) = \ln\left(1-(1-\pi)e^{-nX} - \pi e^{-nY}\right) - \ln n$$

the function $q_n()$ is given by

$$(A.26)\quad q_n(n,\pi) = \frac{(1-\pi)e^{-nX}X + \pi e^{-nY}Y}{1-(1-\pi)e^{-nX} - \pi e^{-nY}} - \frac{1}{n}$$

• *Part i:* $q_{nn} > 0$ *:*

Differentiating again with respect to n and operating, $q_{nn}(n,\pi)$ can be written in the form

$$q_{nn}(n,\pi) = \frac{N_{nn}}{n^2\left[(1-\pi)\left(1-e^{-nX}\right) + \pi\left(1-e^{-nY}\right)\right]^2}$$

(A.27) with

$$N_{nn} = \left[(1-\pi)\left(1-e^{-nX}\right) + \pi\left(1-e^{-nY}\right)\right]^2 - (1-\pi)e^{-nX}n^2X^2 - \pi e^{-nY}n^2Y^2 + (1-\pi)\pi e^{-n(X+Y)}n^2(Y-X)^2$$

Since the denominator of $q_{nn}(n,\pi)$ is always positive, we only need to show that $N_{nn} > 0$.

To proceed, notice that the first term of N_{nn} can be written

$$\left[(1-\pi)\left(1-e^{-nX}\right) + \pi\left(1-e^{-nY}\right)\right]^2 = (1-\pi)\left(1-e^{-nX}\right)^2 + \pi\left(1-e^{-nY}\right)^2 - (1-\pi)\pi\left(e^{-nY} - e^{-nX}\right)^2$$

Substituting this expression into (A.27) we have

$$\begin{aligned}
N_{nn} &= (1-\pi)\left[\left(1-e^{-nX}\right)^2 - e^{-nX}n^2X^2\right] + \pi\left[\left(1-e^{-nY}\right)^2 - e^{-nY}n^2Y^2\right] \\
&\quad + (1-\pi)\pi\left[e^{-n(X+Y)}n^2(Y-X)^2 - \left(e^{-nY} - e^{-nX}\right)^2\right] \\
&= (1-\pi)\left[\left(1-e^{-nX}\right)^2 - e^{-nX}n^2X^2\right] + \pi\left[\left(1-e^{-nY}\right)^2 - e^{-nY}n^2Y^2\right] \\
&\quad + (1-\pi)\pi e^{-2nX}\left\{e^{-n(Y-X)}n^2(Y-X)^2 - \left(e^{-n(Y-X)} - 1\right)^2\right\}
\end{aligned}$$

Using the function

$$A(z) = \left(1 - e^{-z}\right)^2 - e^{-z} z^2$$

defined in (A.20) we can write N_{nn} in the form

$$(A.28)\ \ N_{nn}(X,Y) = (1-\pi)A(nX) + \pi A(nY) - (1-\pi)\pi e^{-2nX} A\left(n(Y-X)\right)$$

Next, we observe that

$$N_{nn}(0,Y) = (1-\pi)*0 + \pi A(nY) - (1-\pi)\pi e^{0} A\left(nY\right) = \pi^2 A(nY) > 0$$

and, using (A.22) and (A.23),

$$\frac{\partial N_{nn}}{\partial X} = (1-\pi)nA'(nX) + (1-\pi)\pi e^{-2nX} n\left[A'\left(n(Y-X)\right) + 2A\left(n(Y-X)\right)\right] > 0$$

Hence, N_{nn} is strictly positive for all $X \geq 0$, as was to be shown.

• *Part ii:* $q_{n\pi} < 0$*:*

Differentiating

$$(A.26)\ \ q_n(n,\pi) = \frac{(1-\pi)e^{-nX}X + \pi e^{-nY}Y}{1-(1-\pi)e^{-nX} - \pi e^{-nY}} - \frac{1}{n}$$

with respect to π, we have:

$$q_{n\pi}(n,\pi) = \frac{\left[1-(1-\pi)e^{-nX} - \pi e^{-nY}\right]*\left(e^{-nY}Y - e^{-nX}X\right) - \left[(1-\pi)e^{-nX}X + \pi e^{-nY}Y\right]*\left(e^{-nX} - e^{-nY}\right)}{\left[1-(1-\pi)e^{-nX} - \pi e^{-nY}\right]^2}$$

which can be simplified to

$$q_{n\pi}(n,\pi) = \frac{N_{n\pi}}{\left[1-(1-\pi)e^{-nX} - \pi e^{-nY}\right]^2}$$

with

$$(A.29)\ \ N_{n\pi} = Ye^{-nY}\left(1-e^{-nX}\right) - Xe^{-nX}\left(1-e^{-nY}\right)$$

To show that $N_{n\pi} < 0$, notice that we can write

$$\begin{aligned} N_{n\pi} < 0 &\Leftrightarrow Ye^{-nY}\left(1-e^{-nX}\right) < Xe^{-nX}\left(1-e^{-nY}\right) \\ &\Leftrightarrow B(Y) \equiv \frac{Ye^{-nY}}{1-e^{-nY}} < \frac{Xe^{-nX}}{1-e^{-nX}} \equiv B(X) \end{aligned}$$

both when $n > 0$ and when $n < 0$ since in both cases $\left(1-e^{-nX}\right)$ and $\left(1-e^{-nY}\right)$ have the same sign. Now, since $X \leq Y$ by assumption, to establish the desired result we only need to show that *B()* is a decreasing function. Differentiating

$$B(Z) = \frac{Ze^{-nZ}}{1-e^{-nZ}}$$

we have

$$B'(Z) = \frac{\left(1-e^{-nZ}\right)*\left(Ze^{-nZ}(-n)+e^{-nZ}\right) - Ze^{-nZ}(-e^{-nZ})(-n)}{\left(1-e^{-nZ}\right)^2}$$

$$= e^{-nZ}\frac{1-nZ-e^{-nZ}}{\left(1-e^{-nZ}\right)^2} < 0$$

for all $n \neq 0$ by

$$(A.4)\quad \mu(x) \equiv e^x(x-1) \geq -1 \Leftrightarrow 1-x \leq e^{-x}$$

with $x = nZ$. □

Components of the ratio of expenditure to GDP

Given

$$DEMLAB \equiv \frac{LP}{LF} = \frac{e^{n(t-J)}\dfrac{1-(1-\pi)e^{-nX}-\pi e^{-n(X+X2)}}{n}}{\dfrac{\left(e^{nC}-1\right)}{n}e^{nt}e^{-n(C+E)}} = \frac{f(n,\pi,X)}{m(n,C)} = \frac{f(n,\pi,Z-J)}{m(n,J-E)}$$

and using (A.10) and (A.13) we have

$$\frac{\partial DEMLAB}{\partial Z} = \frac{f_X(n,\pi,X)}{m(n,C)} > 0$$

$$\frac{\partial DEMLAB}{\partial J} = \frac{m(n,C)f_X(n,\pi,X)(-1) - f(n,\pi,X)m_C(n,C)}{m(n,C)^2} = \frac{-m(n,C)f_X(n,\pi,X) - f(n,\pi,X)m_C(n,C)}{m(n,C)^2} < 0$$

$$\frac{\partial DEMLAB}{\partial n} = \frac{m(n,C)f_\gamma(n,\pi,X) - f(n,\pi,X)m_\gamma(n,C)}{m(n,C)^2} = \frac{(-)-(+)}{+} < 0$$

5. Proof of the second sustainability condition

We want to see for what values of d the following condition holds:

$$(A.30)\quad G(d) \equiv \frac{e^{(n-\nu)C}-1}{e^{(n-\nu-d)C}-1}\frac{n-\nu-d}{n-\nu} - \frac{r-\omega}{r-\omega+d}\frac{1-(1-\pi\phi_v)e^{-(r-\omega+d)X} - \pi\phi_v e^{-(r-\omega+d)(X+X2)}}{1-(1-\pi\phi_v)e^{-(r-\omega)X} - \pi\phi_v e^{-(r-\omega)(X+X2)}} \geq 0$$

where $d = n + g - r$ is the difference between the growth rate of the system's revenues and its IRR. Roughly speaking, the first term of *G()* describes the system's revenues, the second term its expenditures and *G()* itself its financial surplus.

Notice that

G(0) = 1 - 1 = 0

Hence, the inequality holds weakly for $d = 0$. Next, we compute the derivative of *G()*. It is helpful to note that *G()* can be written in the form

$$G(d) \equiv \frac{m(n-\nu,C)}{m(n-\nu-d,C)} - \frac{h(r-\omega+d,\pi\phi_v)}{h(r-\omega,\pi\phi_v)}$$

Differentiating this expression and using (A.6) and (A.10) we have

$$G'(d) \equiv m(n-\nu,C)\frac{-m_\gamma(n-\nu-d,C)(-1)}{m(n-\nu-d,C)^2} - \frac{h_\gamma(r-\omega+d,\pi\phi_v)}{h(r-\omega,\pi\phi_v)} = m(n-\nu,C)\frac{m_\gamma(n-\nu-d,C)}{m(n-\nu-d,C)^2} - \frac{h_\gamma(r-\omega+d,\pi\phi_v)}{h(r-\omega,\pi\phi_v)} > 0$$

so the surplus of the system, given by *G()*, is an increasing function of

$$d = n + g - r$$

with $G(0) = 0$ and therefore a decreasing function of its internal rate of return, r. If r increases above $g+n$ we have $G(r) < 0$ and the system is in deficit, which is what we wanted to prove. □

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