On Cooperative Solutions of a Generalized Assignment Game: Limit Theorems to the Set of Competitive Equilibria*

Jordi Massó[†] and Alejandro Neme[‡]

February 2010

Abstract: We study two cooperative solutions of a market with indivisible goods modeled as a generalized assignment game: Set-wise stability and Core. We first establish that the Set-wise stable set is contained in the Core and it contains the non-empty set of competitive equilibrium payoffs. We then state and prove three limit results for replicated markets. First, the sequence of Cores of replicated markets converges to the set of competitive equilibrium payoffs when the number of replicas tends to infinity. Second, the Set-wise stable set of a two-fold replicated market already coincides with the set of competitive equilibrium payoffs. Third, for any number of replicas there is a market with a Core payoff that is not a competitive equilibrium payoff.

*Support for the research of J. Massó was received through the prize "ICREA Acadèmia" for excellence in research, funded by the Generalitat de Catalunya. He also acknowledges the support of MOVE, where he is an affiliated researcher, and of the Barcelona Graduate School of Economics (through its Research Recognition Programme), where he is an affiliated professor. His work is also supported by the Spanish Ministry of Science and Innovation, through grants ECO2008-04756 (Grupo Consolidado-C) and CONSOLIDER-INGENIO 2010 (CDS2006-00016), and by the Generalitat de Catalunya, through grant SGR2009-419 and a BE-2008 fellowship to visit the Instituto de Matemática Aplicada de San Luis (Argentina). The work of A. Neme is partially supported by the Universidad Nacional de San Luis, through grant 319502, and by the Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), through grant PIP 112-200801-00655.

[†]Departament d'Economia i d'Història Econòmica and CODE. Universitat Autònoma de Barcelona. 08193, Bellaterra (Barcelona), Spain. E-mail: jordi.masso@uab.es

[‡]Instituto de Matemática Aplicada de San Luis. Universidad Nacional de San Luis and CONICET. Ejército de los Andes 950. 5700, San Luis, Argentina. E-mail: aneme@unsl.edu.ar

Journal of Economic Literature Classification Numbers: C78; D78.

Keywords: Assignment game; Core; Set-wise stability; Competitive equilibrium.

1 Introduction

We study two cooperative solutions for a class of markets with indivisible goods modeled as generalized assignment games. Shapley and Shubik (1972) defined an assignment game as a market where each seller owns one indivisible object and each buyer, who wants to buy at most one object, has valuations over all objects. An assignment is a description of deliveries of objects from sellers to buyers and a price vector is a list of prices, one for each object. A competitive equilibrium of a market is a price vector and a feasible assignment at which each seller maximizes revenues, each buyer maximizes net valuations, and markets clear. Shapley and Shubik (1972) showed that the set of competitive equilibria is non-empty, competitive equilibrium assignments are optimal (the first welfare theorem holds), any optimal assignment is part of a competitive equilibrium with any of the competitive equilibrium price vectors (a strong version of the second welfare theorem holds without requiring any redistribution of the initial endowments), and the set of competitive equilibrium payoffs coincides with the Core of a naturally associated TU$-$game (no enlargement or replica of the market is required for their coincidence).

We consider a generalized assignment game representing a market with a given number of indivisible units of different goods, where sellers may own different units of each of the goods and buyers, who may want to buy several units of different goods up to an exogenous total amount, have constant marginal valuations of each good. Jaume, Massó, and Neme (2009) extend Shapley and Shubik (1972)'s results for this generalized assignment game. In particular, they show that the set of competitive equilibria is non-empty, it is the Cartesian product of the set of competitive equilibrium price vectors and the set of optimal assignments, the set of competitive equilibrium price vectors has a lattice structure with the natural partial order of vectors $\geq$ "to be larger or equal than", and this lattice structure is partly translated in a dual way to the sets of buyers and sellers' utilities that are attainable at competitive equilibria.

In this paper we study two different cooperative solutions for this class of markets and their relationship with the set of competitive equilibrium payoffs. The two solutions differ on how a coalition of buyers and sellers can block a proposed payoff vector. Given an assignment and a coalition of buyers and sellers, some of them may be buying or selling some units of some goods to sellers or buyers outside the coalition. The notion of the Core corresponds to the notion of blocking that requires that all members of the coalition have to break all exchanges performed with all agents outside the coalition and buy or sell only with members within the coalition. In contrast, the concept of Set-wise stability corresponds to the notion of blocking that admits that members of the coalition may completely or partly keep their exchanges performed with non-members. Since Set-wise blocking is easier than Core-wise blocking, the Set-wise stable set is a subset of the Core. We show here that the non-empty set of competitive equilibrium payoffs is contained in the Set-wise stable set. Hence, the Set-wise stable set as well as the Core are non-empty. Moreover, we exhibit two simple markets showing that these inclusions may be strict.

The main contribution of the paper is to answer affirmatively the following question. Do the Core and the Set-wise stable set converge to the set of competitive equilibrium payoffs when the market becomes large? The question is relevant because competitive equilibrium requires price-taking behavior which only makes sense when individual quantity decisions are perceived by each agent as being negligible. To create a setting where price-taking behavior is meaningful we follow the well established tradition in Economics to enlarge the environment by replicating the market. We first show that the Core converges to the set of competitive equilibrium payoffs when the number of replica tends to infinity and hence, the Set-wise stable set converges as well. However, we show that the Set-wise stable set already coincides with the set of competitive equilibrium payoffs for a two-fold replicated market. Finally, we show that for any number of replicas there is a market with a Core payoff that is not a competitive equilibrium payoff. Thus, the notion of Set-wise stability is much closer (not only in terms of set-wise inclusion) to competitive equilibrium than the notion of Core.

There are many other papers that recently have studied the relationship between the set of competitive equilibrium payoffs and alternative cooperative solutions in many-to-one or many-to-many generalizations of Shapley and Shubik (1972)'s assignment game. Sotomayor (1992 and 1999a) study a many-to-many assignment game with two finite and disjoint sets of agents. Each agent from each side can form a maximal number of partnerships with the agents from the other side. Each partnership generates a total payoff that may be shared by its two members. Observe that in this extension partnerships are binary; specifically, if a buyer and a seller form a partnership they can exchange just one indivisible unit of the good held by the seller. Sotomayor (1992) proves that all pair-wise stable assignments are optimal and Sotomayor (1999a) shows that the set of pair-wise stable payoffs has a complete and dual lattice structure. Sotomayor (1999b) proposes the notion of Set-wise stability for the former model and shows that the pair-wise stable set (that may be empty) is a subset of the Core. Camiña (2006) studies a market with one seller, that owns a given number of (potentially) different objects, and several buyers who want to buy at most one object. She shows that the Core and the Set-wise stable set coincide, the set of competitive equilibrium payoffs is non-empty and it is a subset of the Core. Moreover, she shows that the Core has a complete lattice structure with the partial order coming from comparing buyers' payoff vectors with the partial order $\geq$ and this structure is not dual. Sotomayor (2007) studies a generalized assignment game similar to ours but with two important differences: (i) sellers only own units of a unique good and each good is only owned by a particular seller and (ii) buyers may want to buy several units but partnerships are also binary because buyers are not interested in buying more than one unit from each seller. She shows that the set of competitive equilibrium payoffs is a non-empty, complete and dual lattice. Sotomayor (2009) extends Sotomayor (1992 and 1999a) and considers a time-sharing assignment game where both buyers and sellers own a fixed amount of a divisible good (labor time) and to form a partnership a buyer and a seller have to agree to contribute each with the same amount of labor time and to share, in a particular proportion, the amount of money that is proportionally obtained from the jointly contributed amount of labor time. Sotomayor (2009) studies different solution concepts for different kinds of coalitional interactions. In particular, she shows the inclusion relationships that hold among the non-empty sets of competitive equilibrium payoffs, the Core, the Set-wise stable set, the Strong stable set and the set of dual allocations. Moreover, she also shows that some of these sets have a lattice structure. Milgrom (2009) introduces and studies the space of assignment messages to investigate (and solve) the difficulty that agents face when reporting their "types" (or valuations of goods, or sets of goods) in some mechanism design settings. The model is very general and contains as particular cases multi-unit auctions (with substitutable goods), exchange economies, and integer assignment games. Milgrom (2009) focuses on the study of the non-emptyness of the set of competitive equilibrium prices and its lattice structure but he does not analyze any cooperative solution. Jaume, Massó, and Neme (2009) study using linear programming the same model than the present one but they only focus on the study of the Cartesian product and lattice structures of the set of competitive equilibria and the corresponding sets of agents' utilities.

The paper is organized as follows. In Section 2 we define a market. In Section 3 we present the notions of Core and Set-wise stability and show that the Set-wise stable set is a non-empty subset of the Core. In Section 4, and closely following Jaume, Massó, and Neme (2009), we define a competitive equilibrium of a market. We then show that the non-empty set of competitive equilibrium payoffs is contained in the Set-wise stable set. In Section 5 we define, for any positive integer ρ, a $\rho-$fold replica of a market and show in Theorem 1 that the limit of the sequence of the Cores of replicated markets coincides with the set of competitive equilibrium payoffs when the number of replicas tends to infinity. In Theorem 2 we show that the Set-wise stable set of a two-fold replicated market already coincides with the set of competitive equilibrium payoffs. Finally, in Theorem 3 we show that for any number of replicas there is a market with a Core payoff that is not a competitive equilibrium payoff. An appendix at the end of the paper collects the proofs that have been omitted in the main text.

2 Preliminaries

A generalized assignment game (a *market*) consists of seven objects. Three finite and disjoint sets: the set $B = \{b_1, ..., b_m\}$ of *buyers*, the set $G = \{g_1, ..., g_n\}$ of *goods*, and the set $S = \{s_1, ..., s_t\}$ of *sellers*. We identify a generic buyer with b_i or with just i, a generic good with g_j or with just j, and a generic seller with s_k or with just k.

Buyers have a constant marginal valuation of each good. Let $v_{ij} \geq 0$ be the monetary valuation that buyer i assigns to each unit of good j; namely, v_{ij} is the maximum price that buyer i is willing to pay for each unit of good j. Denote by $V = (v_{ij})_{(i,j)\in B\times G}$ the *matrix of valuations*. We assume that buyer $i \in B$ can buy at most $d_i \in \mathbb{Z}_+\backslash\{0\}$ units in total, where $\mathbb{Z}_+$ is the set of non-negative integers. The strictly positive integer d_i should be interpreted as a capacity constraint due to limits on i's ability for storage, transport, etc. Denote by $d = (d_i)_{i\in B}$ the *vector of maximal demands*. Each seller $k \in S$ has $q_{jk} \in \mathbb{Z}_+$ indivisible units of each good $j \in G$. Denote by $Q = (q_{jk})_{(j,k)\in G\times S}$ the *capacity matrix*. Let $r_{jk} \geq 0$ be the monetary valuation that seller k assigns to each unit of good j; that is, r_{jk} is the reservation (or minimum) price that seller k is willing to accept for each unit of good j. Denote by $R = (r_{jk})_{(j,k)\in G\times S}$ the *matrix of reservation prices*. Some sellers may not have any unit of some of the goods. However, we require that the seller's reservation price of a good that he has no units to sell has to be equal to zero; namely, for all $k \in S$ and all $j \in G$,

$$\text{if } q_{jk} = 0 \text{ then } r_{jk} = 0. \tag{1}$$

We also assume that there is a strictly amount of each good; namely,

$$\text{for each } j \in G \text{ there exists } k \in S \text{ such that } q_{jk} > 0. \tag{2}$$

A *market* M is a 7-tuple (B, G, S, V, d, R, Q) satisfying conditions (1) and (2). Shapley and Shubik (1972)'s (one-to-one) assignment game is a special case of a market where each buyer can buy at most one unit, there is only one unit of each good, and each seller only owns one unit of one of the goods; *i.e.*, $d_i = 1$ for all $i \in B$, $n = t$, and for all $(j,k) \in G \times S$, $q_{jk} = 1$ if $j = k$ and $q_{jk} = 0$ if $j \neq k$.

Let M be a market. An *assignment* for market M is a three-dimensional integer matrix $A = (A_{ijk})_{(i,j,k)\in B\times G\times S} \in \mathbb{Z}_+^{B\times G\times S}$ describing a collection of deliveries of units of the goods from buyers to sellers. Each A_{ijk} should be interpreted as "buyer i receives A_{ijk} units of good j from seller k." We often omit the sets to which the subscripts belong to and write, for instance, $\sum_{ijk} A_{ijk}$ and $\sum_i A_{ijk}$ instead of $\sum_{(i,j,k)\in B\times G\times S} A_{ijk}$ and $\sum_{i\in B} A_{ijk}$, respectively.

The assignment A is *feasible* for market M if each buyer i buys at most d_i units and each seller k sells at most q_{jk} units of each good j. We are only interested on feasible assignments. Denote by F the set of all feasible assignments of market M; namely,

$$F = \{A \in \mathbb{Z}_+^{B\times G\times S} \mid \textstyle\sum_{jk} A_{ijk} \leq d_i \text{ for all } i \in B \text{ and } \sum_i A_{ijk} \leq q_{jk} \text{ for all } (j,k) \in G \times S\}.$$

For each $(i,j,k) \in B \times G \times S$, let

$$\tau_{ijk} = \begin{cases} v_{ij} - r_{jk} & \text{if } q_{jk} > 0 \\ 0 & \text{if } q_{jk} = 0 \end{cases} \tag{3}$$

be the *per unit gain* from trade of good j between buyer i and seller k. If seller k does not have any unit of good j the per unit gain from trade of good j with all buyers is equal to zero. The *total gain from trade of market M at assignment A* is

$$T^M(A) = \textstyle\sum_{ijk} \tau_{ijk} \cdot A_{ijk}.$$

Definition 1 *A feasible assignment $\widetilde{A}$ is* optimal *for market M if, for any feasible assignment $A \in F$, $T^M(\widetilde{A}) \geq T^M(A)$.*

Let $\widetilde{F}$ be the set of all optimal assignments for market M. The set $\widetilde{F}$ is always non-empty.[1]

[1]See Milgrom (2009) for a proof of this statement in a more general model. See Jaume, Massó and Neme (2009) for a proof of the statement using only linear programming arguments in the same model as the one studied here.

3 Cooperative Solutions

We present now two alternative cooperative solutions for market M. They differ on how a coalition (a subset) of agents can block a proposal of how to distribute among all agents the total gain from trade obtained at any optimal assignment. The Core assumes that members of a blocking coalition can only form partnerships among themselves and have to break all former partnerships with non-members. Set-wise stability allows members of a blocking coalition to keep or reduce their former exchanges with members outside the blocking coalition. Thus, Set-wise blocking is easier than Core-wise blocking. It seems to us that Set-wise stability is also a more reasonable solution for this class of markets. Our results will indicate from two points of view that Set-wise stability is closer to the set of competitive equilibrium payoffs than the Core is: (i) (set inclusion) closer and (ii) Set-wise stability and the set of competitive equilibrium payoffs already coincide in a two-fold replicated market.

3.1 Core

Let $M = (B, G, S, V, d, R, Q)$ be a market and let $C \subseteq B \cup S$ be a *coalition.* Denote the subsets of buyers and sellers in C by $B^C = C \cap B$ and $S^C = C \cap S$, respectively. The *submarket* M^C is the (natural) restriction of market M to coalition C; namely, M^C is the market $(B^C, G^C, S^C, V^C, d^C, R^C, Q^C)$, where $G^C = \{j \in G \mid \text{there exists } k \in S^C \text{ such that } q_{jk} > 0\}$, $V^C = (v_{ij})_{(ij) \in B^C \times G^C}$, $d^C = (d_i)_{i \in B^C}$, $R^C = (r_{jk})_{(j,k) \in G^C \times S^C}$, and $Q^C = (q_{jk})_{(j,k) \in G^C \times S^C}$.

Definition 2 *A feasible assignment A is* Core$-$compatible *with coalition C if $A_{ijk} \neq 0$ implies $\{i, k\} \subset C$.*

That is, a feasible assignment A is Core$-$compatible with C if all members of C interact only among themselves. Let A be an assignment Core$-$compatible with coalition C and denote by A^C the feasible assignment for submarket M^C, where $A^C = (A_{ijk})_{(i,j,k) \in B^C \times G^C \times S^C}$. When the reference coalition is clear from the context we often omit the superscript C. Denote by F^C the set of feasible assignments for submarket M^C and by $\widetilde{F}^C$ the set of its optimal assignments; *i.e.,*

$$\widetilde{F}^C = \{A^C \in F^C \mid T^{M^C}(A^C) \geq T^{M^C}(\widetilde{A}^C) \text{ for all } \widetilde{A}^C \in F^C\}.$$

Fix a market M. To define a cooperative game v with transferable utility associated to M, let $C \subseteq B \cup S$ be a coalition and set

$$v(C) = T^{M^C}(\widetilde{A}^C),$$

where $\widetilde{A}^C$ is any optimal assignment of submarket M^C. Namely, $v(C)$ is the maximal total utility that members of C can guarantee by exchanging their resources *only* among themselves. Obviously, $v(C) = 0$ for all C such that either $B^C = \varnothing$ or $S^C = \varnothing$, and hence, $v(\varnothing) = 0$. Moreover, $v(\{i\}) = 0$ for all $i \in B$ and $v(\{k\}) = 0$ for all $k \in S$.

Let M be a market. A pair $(u, w) \in \mathbb{R}^B \times \mathbb{R}^S$ is a (feasible) *payoff* of market M if

$$\sum_{i \in B} u_i + \sum_{k \in S} w_k = v(B \cup S).$$

A payoff of market M is a distribution among agents of the total gains from trade at any optimal assignment of market M.

Definition 3 *A payoff* $(u, w) \in \mathbb{R}^B \times \mathbb{R}^S$ *of market* M *is* Core$-$blocked *by coalition* $C \subseteq B \cup S$ *if*

$$\sum_{i \in B^C} u_i + \sum_{k \in S^C} w_k < v(C).$$

Definition 4 *A payoff* $(u, w) \in \mathbb{R}^B \times \mathbb{R}^S$ *of market* M *belongs to the Core if there does not exist a coalition* $C \subset B \cup S$ *such that* (u, w) *is Core$-$blocked by* C*.*

Let $\mathcal{C}$ be the set of payoffs belonging to the Core of market M. When we want to emphasize market M we write $\mathcal{C}^M$. Proposition 1 below states that the Core is always non-empty.

Proposition 1 *Every market has a non-empty Core.*

Proof See the appendix. ■

3.2 Set-wise Stability

The notion of Core$-$blocking requires that all members of the blocking coalition have to give up *all* previous exchange agreements with non-members. However this may be too drastic because, in some circumstances, it is reasonable to let members of the blocking coalition to keep some (or all) previous exchanges with members outside the blocking coalition. This stronger notion of blocking gives rise to the notion of Set-wise stability.[2]

Definition 5 *Let M be a market and C be a coalition. A feasible assignment $\widehat{A}$ for market M is* SW$-$compatible with C *if there exists an optimal assignment $\widetilde{A} \in \widetilde{F}$ such that:*
(i) *For every $i \in B^C$, $\widehat{A}_{ijk} > 0$ implies that either $k \in S^C$ or else $\widehat{A}_{ijk} \leq \widetilde{A}_{ijk}$.*
(ii) *For every $k \in S^C$, $\widehat{A}_{ijk} > 0$ implies that either $i \in B^C$ or else $\widehat{A}_{ijk} \leq \widetilde{A}_{ijk}$.*
(iii) *For every $i \notin B^C$ and $k \notin S^C$, $\widehat{A}_{ijk} = 0$ for every $j \in G$.*

Let M be a market. A three-dimensional matrix $\Gamma = (\Gamma_{ijk})_{(i,j,k)\in B\times G\times S}$ is a *distribution matrix* if for all $(i,j,k) \in B \times G \times S$ such that $v_{ij} \geq r_{jk}$, it holds that $v_{ij} \geq \Gamma_{ijk} \geq r_{jk}$. Let Γ be a distribution matrix and assume $v_{ij} \geq r_{jk}$ for some $(i,j,k) \in B \times G \times S$. Then, Γ_{ijk} describes a way of how buyer i and seller k could split the gain $v_{ij} - r_{jk}$ that they would obtain from trading one unit of good j: buyer i receives $v_{ij} - \Gamma_{ijk}$ and seller k receives $\Gamma_{ijk} - r_{jk}$. If $v_{ij} < r_{jk}$ then the value Γ_{ijk} will be irrelevant because i and k do not trade good j at any optimal assignment. Observe that a distribution matrix is not necessarily anonymous because a buyer can obtain different per unit gains from buying good j from two different sellers, and viceversa.

Definition 6 *A payoff $(u,w) \in \mathbb{R}^B \times \mathbb{R}^S$ for market M is* SW$-$blocked *by coalition $C \subset B \cup S$ if for any distribution matrix $\Gamma = (\Gamma_{ijk})_{(i,j,k)\in B\times G\times S}$ there exists a feasible assignment $\widehat{A}$ that is SW$-$compatible with C and*

$$\sum_{i\in B^C} u_i + \sum_{k\in S^C} w_k \quad < \sum_{(i,j,k)\in B^C\times G\times S^C} \tau_{ijk}\cdot \widehat{A}_{ijk} + \sum_{(i,j,k)\in B^C\times G\times (S^C)^c} (v_{ij} - \Gamma_{ijk})\cdot \widehat{A}_{ijk}$$
$$+ \sum_{(i,j,k)\in (B^C)^c\times G\times S^C} (\Gamma_{ijk} - r_{jk})\cdot \widehat{A}_{ijk}.$$

Namely, members of a coalition $SW-$block a payoff vector if *independently* of the agreements they have with non-members they can jointly obtain a strictly higher payoff by reassigning their exchanges among themselves and by keeping or reducing their exchanges with non-members.

Definition 7 *A payoff $(u,w) \in \mathbb{R}^B \times \mathbb{R}^S$ for market M is* Set-wise stable *if there does not exist a coalition $C \subset B \cup S$ such that (u,w) is SW$-$blocked by coalition C.*

[2] Sotomayor (1999b) defines and studies this concept for a many-to-many generalization of Shapley and Shubik (1972)'s assignment game. See also Sotomayor (2007 and 2009) for an analysis of Set-wise stability in her time-sharing assignment games.

Denote by $\mathcal{SW}$ the set of Set-wise stable payoffs. When we want to emphasize market M we write $\mathcal{SW}^M$. Let $(u,w) \in \mathbb{R}^B \times \mathbb{R}^S$ be a payoff of market M and assume that coalition C Core−blocks (u,w). Let $\widetilde{A}^C \in \widetilde{F}^C$ be arbitrary. Then,

$$\sum_{i\in B^C} u_i + \sum_{k\in S^C} w_k < v(C) = T^{M^C}(\widetilde{A}^C).$$

Let $\widehat{A}$ be the feasible assignment where, for all $(i,j,k) \in B \times G \times S$,

$$\widehat{A}_{ijk} = \begin{cases} \widetilde{A}^C_{ijk} & \text{if } (i,k) \in B^C \times S^C \\ 0 & \text{otherwise.} \end{cases}$$

Then, $\widehat{A}$ is a feasible assignment SW−compatible with C and for any distribution matrix Γ,

$$\begin{aligned} \sum_{i\in B^C} u_i + \sum_{k\in S^C} w_k \quad &< \sum_{(i,j,k)\in B^C\times G\times S^C} \tau_{ijk}\cdot \widehat{A}_{ijk} + \sum_{(i,j,k)\in B^C\times G\times (S^C)^c} (v_{ij} - \Gamma_{ijk})\cdot \widehat{A}_{ijk} \\ &\quad + \sum_{(i,j,k)\in (B^C)^c\times G\times S^C} (\Gamma_{ijk} - r_{jk})\cdot \widehat{A}_{ijk}. \\ &= T^{M^C}(\widetilde{A}^C). \end{aligned}$$

Hence, coalition C SW−blocks (u,w). Thus, the Set-wise stable set is a subset of the Core. For further reference, we state this fact below as Proposition 2.

Proposition 2 *For any market the Set-wise stable set is a subset of the Core.*

4 Competitive Equilibria and Basic Results

4.1 Definitions and Preliminaries

We define a competitive equilibrium of market M by following Jaume, Massó and Neme (2009). Assume buyers and sellers trade through competitive markets. That is, there is a unique market (and its corresponding unique price) for each of the goods and buyers and sellers are price-takers. Given a price vector $p = (p_j)_{j\in G} \in \mathbb{R}^G_+$ sellers supply units of the goods (up to their capacity) in order to maximize revenues at p and buyers demand units of the goods (up to their maximal demands) in order to maximize the total net valuation at p.

Supply of seller k: For each price vector $p = (p_j)_{j\in G} \in \mathbb{R}^G_+$, seller k supplies of every good j any feasible amount that maximizes revenues; namely,

$$S_{jk}(p_j) = \begin{cases} \{q_{jk}\} & \text{if } p_j > r_{jk} \\ \{0,1,...,q_{jk}\} & \text{if } p_j = r_{jk} \\ \{0\} & \text{if } p_j < r_{jk}. \end{cases} \tag{4}$$

To define the demands of buyers we need the following notation. Let $p \in \mathbb{R}_+^G$ be given and consider buyer i. Let

$$\nabla_i^>(p) = \{j \in G \mid v_{ij} - p_j = \max_{j' \in G}\{v_{ij'} - p_{j'}\} > 0\} \tag{5}$$

be the set of goods that give to buyer i the maximum (and strictly positive) net valuation at p. Obviously, for some p, the set $\nabla_i^>(p)$ may be empty. Let

$$\nabla_i^{\geq}(p) = \{j \in G \mid v_{ij} - p_j = \max_{j' \in G}\{v_{ij'} - p_{j'}\} \geq 0\} \tag{6}$$

be the set of goods that give to buyer i the maximum (and non-negative) net valuation at p. Obviously, for some p, the set $\nabla_i^{\geq}(p)$ may also be empty. Obviously, for all $p \in \mathbb{R}_+^n$ and all $i \in B$,

$$\nabla_i^>(p) \subseteq \nabla_i^{\geq}(p). \tag{7}$$

Demand of buyer i: For each price vector $p = (p_j)_{j \in G} \in \mathbb{R}_+^n$, buyer i demands any feasible amounts of the goods that maximize the net valuations at p; namely,

$$D_i(p) = \{\alpha = (\alpha_{jk})_{(j,k) \in G \times S} \in \mathbb{Z}^{G \times S} \mid \begin{array}{l} \text{(D.a) } \alpha_{jk} \geq 0 \text{ for all } (j,k) \in G \times S, \\ \text{(D.b) } \sum_{jk} \alpha_{jk} \leq d_i, \\ \text{(D.c) } \nabla_i^>(p) \neq \emptyset \Longrightarrow \sum_{jk} \alpha_{jk} = d_i, \text{ and} \\ \text{(D.d) } \sum_k \alpha_{jk} > 0 \Longrightarrow j \in \nabla_i^{\geq}(p)\}. \end{array}$$

Thus, $D_i(p)$ describes the set of all trades that maximize the net valuation of buyer i at p. Observe that the set of trades described by each element in the set $D_i(p)$ give the same net valuation to buyer i; *i.e.*, i is indifferent among all trade plans $\alpha \in D_i(p)$.

Let A be an assignment and let i be a buyer. We denote by $A(i) = (A(i)_{jk})_{(j,k) \in G \times S}$ the element in $\mathbb{Z}_+^{G \times S}$ such that, for all $(j,k) \in G \times S$, $A(i)_{jk} = A_{ijk}$.

Definition 8 *A* competitive equilibrium *of market M is a pair $(p, A) \in \mathbb{R}_+^G \times F \subseteq \mathbb{R}_+^G \times \mathbb{Z}_+^{B \times G \times S}$ such that:*

(E.D) *For each buyer $i \in B$, $A(i) \in D_i(p)$.*

(E.S) *For each good $j \in G$ and each seller $k \in S$, $\sum_i A_{ijk} \in S_{jk}(p_j)$.*

We say that a price vector p and a feasible assignment A are *compatible* if (p, A) is a competitive equilibrium of market M. The vector $p \in \mathbb{R}_+^G$ is a *competitive equilibrium price* of market M if there exists $A \in F$ such that (p, A) is a competitive equilibrium of market M.

Let $\widetilde{P}$ be the set of competitive equilibrium prices of market M. The set $\widetilde{P}$ is always non-empty.[3] For further reference, we state this fact without proof as a proposition below.

Proposition 3 *The set of competitive equilibrium prices of any market is non-empty.*

Moreover, by Proposition 4 in Jaume, Massó, and Neme (2009), the set of competitive equilibria has a Cartesian product structure. We also state this fact without proof as Proposition 4.

Proposition 4 *Let M be a market. Then, (p, A) is a competitive equilibrium of M if and only if $p \in \widetilde{P}$ and $A \in \widetilde{F}$.*

4.2 The Set of Competitive Equilibrium Payoffs

Let $p \in \mathbb{R}_+^G$ be a price vector and $A \in F$ a feasible assignment of market M. We define the *utility of buyer* $i \in B$ at the pair (p, A) as the total net gain obtained by i from his exchanges specified by A at price p. We denote it by $u_i(p, A)$; namely,

$$u_i(p, A) = \sum_{jk} (v_{ij} - p_j) \cdot A_{ijk}.$$

We define the *utility of seller* $k \in S$ at the pair (p, A) as the total net gain obtained by k from his exchanges specified by A at price p. We denote it by $w_k(p, A)$; namely,

$$w_k(p, A) = \sum_{ij} (p_j - r_{jk}) \cdot A_{ijk}.$$

Given (p, A), denote by $u(p, A) = (u_i(p, A))_{i \in B}$ and $w(p, A) = (w_k(p, A))_{k \in S}$ the vector of buyers and sellers' utilities at (p, A), respectively. Let

$$\mathcal{CE} = \{(u, w) \in \mathbb{R}^B \times \mathbb{R}^S \mid \text{there exists } (\widetilde{p}, \widetilde{A}) \in \widetilde{P} \times \widetilde{F} \text{ s.t. } (u, w) = (u(\widetilde{p}, \widetilde{A}), w(\widetilde{p}, \widetilde{A}))\}$$

be the set of Competitive Equilibrium payoffs of market M. However, competitive equilibrium payoff vectors are independent of the particular optimal assignment. To see that,

[3] For the proof of this statement in a more general model see Milgrom (2009), and for a proof in our setting using only linear programming see Jaume, Massó and Neme (2009).

define the mappings of per-unit gains $\gamma(\cdot) : \mathbb{R}_+^G \to \mathbb{R}^B$ and $\pi(\cdot) : \mathbb{R}_+^G \to \mathbb{R}^{G\times S}$ as follows. Let $p \in \mathbb{R}_+^G$ be given. For each $i \in B$, define

$$\gamma_i(p) = \begin{cases} v_{ij} - p_j & \text{if there exists } j \in \nabla_i^{>}(p) \\ 0 & \text{otherwise,} \end{cases} \tag{8}$$

and for each $(j,k) \in G \times S$, define

$$\pi_{jk}(p) = \begin{cases} p_j - r_{jk} & \text{if } p_j - r_{jk} > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

The number $\gamma_i(p)$ is the gain obtained by buyer i from each unit that he wants to buy at p (if any) and the number $\pi_{jk}(p)$ is the profit obtained by seller k from each unit of good j that he wants to sell at p (if any).

Let $\widetilde{p} \in \widetilde{P}$ be a competitive equilibrium price of market M and let $(\gamma(\widetilde{p}), \pi(\widetilde{p}))$ be its associated per unit gains. Define $(u(\widetilde{p}), w(\widetilde{p})) \in \mathbb{R}^B \times \mathbb{R}^S$ by

$$\begin{aligned} u_i(\widetilde{p}) &= d_i \cdot \gamma_i(\widetilde{p}) \text{ for all } i \in B \text{ and} \\ w_k(\widetilde{p}) &= \sum_{j\in G} q_{jk} \cdot \pi_{jk}(\widetilde{p}) \text{ for all } k \in S. \end{aligned} \tag{10}$$

By Lemma 6 in Jaume, Massó, and Neme (2009), the set of competitive equilibrium payoffs of market M can also be written as

$$\mathcal{CE} = \{(u,w) \in \mathbb{R}^B \times \mathbb{R}^S \mid \text{there exists } \widetilde{p} \in \widetilde{P} \text{ such that} (u,w) = (u(\widetilde{p}), w(\widetilde{p}))\}; \tag{11}$$

that is, the set of competitive equilibrium payoffs of market M can be described without explicitly referring to any particular optimal assignment because, for all $\widetilde{A} \in \widetilde{F}$, $u_i(\widetilde{p}, \widetilde{A}) = u_i(\widetilde{p})$ for all $i \in B$ and $w_k(\widetilde{p}, \widetilde{A}) = w_k(\widetilde{p})$ for all $k \in S$.

4.3 Basic Results

In this subsection we describe the inclusion relationships among the set of competitive equilibrium payoffs, Set-wise stability and Core. First, the set of competitive equilibrium payoffs is contained in the set of Set-wise stable payoffs.

Proposition 5 *Let $\widetilde{p} \in \widetilde{P}$ be a competitive equilibrium price vector of market M. Then, $(u(\widetilde{p}), w(\widetilde{p})) \in \mathcal{SW}$.*

Proof See the appendix. ∎

Proposition 5 above says that

$$\mathcal{CE} \subset \mathcal{SW}. \tag{12}$$

Thus, by Proposition 3, $\mathcal{SW} \neq \varnothing$. Example 1 below shows that the inclusion in (12) may be strict because there exist markets with a payoff $(u, w) \in \mathbb{R}^B \times \mathbb{R}^S$ with the property that $(u, w) \in \mathcal{SW} \backslash \mathcal{CE}$.

Example 1 Let $M = (B, S, G, V, d, R, Q)$ be a market where $B = \{b_1\}$, $G = \{g_1, g_2\}$, $S = \{s_1\}$, $V = (8, 4)$, $d_1 = 6$, $R' = (4, 2)$, and $Q' = (3, 3)$, where X' is the transposed matrix of X. The unique optimal assignment of market M is $\widetilde{A}' = \left(\widetilde{A}_{111}, \widetilde{A}_{121}\right) = (3, 3)$ and

$$T^M(\widetilde{A}) = (v_{11} - r_{11}) \cdot \widetilde{A}_{111} + (v_{21} - r_{21}) \cdot \widetilde{A}_{121} = 12 + 6 = 18.$$

It is easy to see that the set of equilibrium price vectors of market M is $\widetilde{P} = \{(\widetilde{p}_1, \widetilde{p}_2) \in \mathbb{R}^G_+ \mid 2 \leq \widetilde{p}_2 \leq 4 \text{ and } 8 - \widetilde{p}_1 = 4 - \widetilde{p}_2\}$. For every $\widetilde{p} \in \widetilde{P}$, the per-unit gains are

$$\gamma_1(\widetilde{p}) = v_{11} - \widetilde{p}_1 = v_{12} - \widetilde{p}_2$$

and

$$\pi_{11}(\widetilde{p}) = \widetilde{p}_1 - r_{11} \text{ and } \pi_{21}(\widetilde{p}) = \widetilde{p}_2 - r_{21}.$$

Moreover $u_1(\widetilde{p}) = \gamma_1(\widetilde{p}) \cdot 6$ and $w_1(\widetilde{p}) = \pi_{11}(\widetilde{p}) \cdot 3 + \pi_{21}(\widetilde{p}) \cdot 3$.

Consider the payoff $(u_1, w_1) = (15, 3)$. We first show that $(u_1, w_1) \in \mathcal{SW}$. Let $\Gamma = (\Gamma_{111}, \Gamma_{121}) = (4, 3)$ be a distribution matrix and let C be a coalition. Consider the following three cases:

Case 1: $C = \{b_1, s_1\}$. Since $u_1 + w_1 = 18 = T^M(\widetilde{A})$ coalition C can not $SW-$block the payoff $(u_1, w_1) = (15, 3)$.

Case 2: $C = \{b_1\}$. We have to show that for any $\widehat{A}$ such that $\widehat{A}_{111} \leq \widetilde{A}_{111} = 3$ and $\widehat{A}_{121} \leq \widetilde{A}_{121} = 3$,

$$u_1 \geq (v_{11} - \Gamma_{111}) \cdot \widehat{A}_{111} + (v_{12} - \Gamma_{121}) \cdot \widehat{A}_{121}.$$

But this holds because

$$u_1 = 15 = (8 - 4) \cdot 3 + (4 - 3) \cdot 3 \geq 4 \cdot \widehat{A}_{111} + \widehat{A}_{121}.$$

Case 3: $C = \{s_1\}$. We have to show that for any $\widehat{A}$ such that $\widehat{A}_{111} \leq \widetilde{A}_{111} = 3$ and $\widehat{A}_{121} \leq \widetilde{A}_{121} = 3$,

$$w_1 \geq (\Gamma_{111} - r_{11}) \cdot \widehat{A}_{111} + (\Gamma_{121} - r_{21}) \cdot \widehat{A}_{121}.$$

But this holds because

$$w_1 = 3 = (4-4)\cdot 3 + (3-2)\cdot 3 \geq \widehat{A}_{121}.$$

Finally, we show that there does not exist a competitive equilibrium price vector $\widetilde{p}$ such that $(u(\widetilde{p}), w(\widetilde{p})) = (15, 3)$. Assume otherwise; then, by (10), $15 = \gamma_1(\widetilde{p})\cdot 6$ and hence, $\gamma_1(\widetilde{p}) = \frac{5}{2}$. Thus,

$$\gamma_1(\widetilde{p}) = \frac{5}{2} = v_{12} - \widetilde{p}_2 = 4 - \widetilde{p}_2$$

and $\widetilde{p}_2 = \frac{3}{2}$. But this contradicts that $2 \leq \widetilde{p}_2 \leq 4$. □

Observe that the distribution matrix used in the definition of Set-wise stability is not necessarily anonymous (*i.e.*, Γ_{ijk} could be different to $\Gamma_{i'jk'}$). However, the subset of Set-wise stable payoffs that are obtained from anonymous distribution matrices (*i.e.*, price vectors) is indeed the set of competitive equilibrium payoff vectors. We state this fact as Proposition 6 below.

Proposition 6 *Let $(u, w) \in \mathbb{R}^B \times \mathbb{R}^S$ be a payoff of market M. Then, $(u, w) \in \mathcal{CE}$ if and only if there exists a competitive equilibrium price vector $\widetilde{p}$ such that for every coalition $C \subset B \cup S$ and any feasible assignment $\widehat{A}$ $SW-$compatible with C we have that*

$$\sum_{i\in B^C} u_i + \sum_{k\in S^C} w_k \geq$$

$$\sum_{(i,j,k)\in B^C\times G\times S^C} r_{ijk}\cdot \widehat{A}_{ijk} + \sum_{(i,j,k)\in B^C\times G\times (S^C)^c} (v_{ij} - \widetilde{p}_j)\cdot \widehat{A}_{ijk}. + \sum_{(i,j,k)\in (B^C)^c\times G\times S^C} (\widetilde{p}_j - r_{jk})\cdot \widehat{A}_{ijk}.$$

Proof See the appendix. ■

Example 2 below shows that the Set-wise stable set may be a strict subset of the Core because there exist markets with $(u, w) \in \mathcal{C}\backslash\mathcal{SW}$.

Example 2 Let $M = (B, S, G, V, d, R, Q)$ be a market where $B = \{b_1\}$, $G = \{g_1\}$ $S = \{s_1, s_2\}$, $v_{11} = 2$, $d_1 = 3$, $R' = (1, 1)$, and $Q' = (2, 2)$. Market M only has two optimal assignments: $\widetilde{A}^{1\prime} = (\widetilde{A}^1_{111}, \widetilde{A}^1_{112}) = (2, 1)$ and $\widetilde{A}^{2\prime} = (\widetilde{A}^2_{111}, \widetilde{A}^2_{112}) = (1, 2)$. Observe that

$$T^M(\widetilde{A}^{1\prime}) = (v_{11} - r_{11})\cdot \widetilde{A}^1_{111} + (v_{12} - r_{12})\cdot \widetilde{A}^1_{112} = 1\cdot 2 + 1\cdot 1 = 3 = T^M(\widetilde{A}^{2\prime}).$$

We first show that $\mathcal{SW} = \{(3, 0, 0)\}$. Let $(u_1, w_1, w_2) \in \mathcal{SW}$ be arbitrary. Then, $u_1 + w_1 + w_2 = 3$ and let $\Gamma = (\Gamma_{111}, \Gamma_{112})$ be any distribution matrix. Let C be a singleton coalition. Three cases are possible.

Case 1: $C = \{b_1\}$. Consider the optimal assignment $\widetilde{A}^{1\prime}$. Then,

$$u_1 \geq (v_{11} - \Gamma_{111}) \cdot \widetilde{A}^1_{111} + (v_{11} - \Gamma_{112}) \cdot \widetilde{A}^1_{112} = (2 - \Gamma_{111}) \cdot 2 + (2 - \Gamma_{112}) \cdot 1.$$

Case 2: $C = \{s_1\}$. Consider the optimal assignment $\widetilde{A}^{1\prime}$. Then,

$$w_1 \geq (\Gamma_{111} - r_{11}) \cdot \widetilde{A}^1_{111} = (\Gamma_{111} - 1) \cdot 2.$$

Case 3: $C = \{s_2\}$. Consider the optimal assignment $\widetilde{A}^{2\prime}$. Then,

$$w_2 \geq (\Gamma_{112} - r_{12}) \cdot \widetilde{A}^2_{112} = (\Gamma_{112} - 1) \cdot 2.$$

Hence,

$$3 = u_1 + w_1 + w_2 \geq (2 - \Gamma_{111}) \cdot 2 + (2 - \Gamma_{112}) \cdot 1 + (\Gamma_{111} - 1) \cdot 2 + (\Gamma_{112} - 1) \cdot 2 = 2 + \Gamma_{112}.$$

This implies that $\Gamma_{112} = 1$ because $\Gamma_{112} \geq 1 = r_{12}$. Symmetrically, and by exchanging the roles of $\widetilde{A}^{1\prime}$ and $\widetilde{A}^{2\prime}$, we obtain that $\Gamma_{111} = 1$. Hence, $w_1 = w_2 = 0$ and $u_1 = 3$. Therefore, by just checking singleton coalitions we already know (since $\mathcal{SW} \neq \varnothing$) that $\mathcal{SW} = \{(3, 0, 0)\}$.

We now show that $(2, 1, 0) \in \mathcal{C}$. Since $v(\{b_1, s_1, s_2\}) = 3$, $v(\{b_1, s_1\}) = v(\{b_1, s_2\}) = 2$, and $v(\{s_1, s_2\}) = v(\{b_1\}) = v(\{s_1\}) = v(\{s_2\}) = 0$, we conclude that $\mathcal{C} = \{(3 - \alpha_1 - \alpha_2, \alpha_1, \alpha_2) \mid \alpha_1 \geq 0, \alpha_2 \geq 0 \text{ and } \alpha_1 + \alpha_2 \leq 1\}$. Hence, $(2, 1, 0) \in \mathcal{C}$. □

Thus, we have already showed that the statement of the following corollary holds.[4]

Corollary 1 *For every market M, $\varnothing \neq \mathcal{CE} \subset \mathcal{SW} \subset \mathcal{C}$. Moreover, the two inclusions may be strict.*

5 The $\rho-$fold Replicated Market: Three Limit Results

Competitive equilibrium presupposes that agents are price-takers. This assumption makes sense only when the number of agents is large and individual quantity decisions are insignificant. Thus, and at the light of Corollary 1, it is natural to ask whether the Core and the set of competitive equilibrium payoffs are approximately the same when the number of agents becomes large. By Corollary 1, an affirmative answer to this question would imply that the Set-wise stable set tends to the set of competitive equilibrium payoffs as well. To

[4]The same inclusion relationships hold in the time-sharing assignment game considered by Sotomayor (2009).

enlarge the market, we follow a procedure with a long tradition in Economics which consists of replicating the market.[5] Given a market $M = (B, G, S, V, d, R, Q)$ and a strictly positive integer ρ we will consider the $\rho-$fold replicated market ρM to be composed of ρ agents of each type. For two buyers $i_\alpha \in B_\alpha$ and $i_{\alpha'} \in B_{\alpha'}$ (in replicas α and α', respectively) to be of the same type we require them to have the same valuations of all goods (*i.e.*, $v_{i_\alpha j} = v_{i_{\alpha'} j} = v_{ij}$ for all $j \in G$) and the same maximal demands (*i.e.*, $d_{i_\alpha} = d_{i_{\alpha'}} = d_i$). For two sellers $k_\alpha \in S_\alpha$ and $k_{\alpha'} \in S_{\alpha'}$ (in replicas α and α', respectively) to be of the same type we require them to have the same reservation prices of all goods (*i.e.*, $r_{jk_\alpha} = r_{jk_{\alpha'}} = r_{jk}$ for all $j \in G$) and the same amounts of all goods (*i.e.*, $q_{jk_\alpha} = q_{jk_{\alpha'}} = q_{jk}$ for all $j \in G$).

The following proposition says that the classical result stating that any payoff vector in the Core assigns the same utility to all agents of the same type also holds in this setting.[6]

Proposition 7 *Let M be a market and let $\rho \geq 2$. Then*

$$\mathcal{C}^{\rho M} \subset \{(u^\rho, w^\rho) \equiv (\underbrace{(u, w), ..., (u, w)}_{\rho-\text{times}}) \in (\mathbb{R}^{B_1} \times \mathbb{R}^{S_1}) \times ... \times (\mathbb{R}^{B_\rho} \times \mathbb{R}^{S_\rho}) \mid (u, w) \in \mathcal{C}^M\}.$$

Proof See the appendix. ■

We will say that a payoff vector $(u, w) \in \mathbb{R}^B \times \mathbb{R}^S$ is in the Core of the $\rho-$replicated market if $(u^\rho, w^\rho) \in \mathcal{C}^{\rho M}$. Our first limit result states that, for every market M, the sequence of Cores of the ρM markets converges, when $\rho \to \infty$, to the set of competitive equilibrium payoffs of the replicated market.

Theorem 1 *Let M be a market. If $(u, w) \in \mathbb{R}^B \times \mathbb{R}^S$ is in the Core of the $\rho-$fold replicated market for all $\rho \geq 1$, then (u, w) is a competitive equilibrium payoff of market M.*

Proof See the appendix. ■

[5] It started by Edgeworth (1881) and pursued by Debreu and Scarf (1963) for classical economies with production and by Owen (1975) for linear production games, among others. A linear production game consists of a set of players, each with an endowment (non necessarily integer valued) of m goods that can only be used to produce in a linear way units of p different goods for which there are competitive markets. Owen (1975) shows that the sequence of Cores of replicated linear production games converges to the set of competitive equilibrium payoffs. Moreover, Owen (1975) also shows that if the competitive equilibrium price is unique then the Core of a large but finitely replicated game coincides with the (unique) competitive equilibrium payoff.

[6] See Debreu and Scarff (1963) and Owen (1975) for this equal treatment result in classical economies with production and in linear production games, respectively.

Note that, by Corollary 1 and Proposition 7, we have that for all $\rho \geq 1$,

$$\mathcal{SW}^{\rho M} \subset \{\underbrace{((u,w),...,(u,w))}_{\rho-\text{times}} \in (\mathbb{R}^{B_1} \times \mathbb{R}^{S_1}) \times ... \times (\mathbb{R}^{B_\rho} \times \mathbb{R}^{S_\rho}) \mid (u,w) \in \mathcal{SW}^M\}$$

Theorem 1 only guarantees convergence in the limit. In contrast, our second main result states that the Set-wise stable set of the $2-$fold replicated market already coincides with the set of competitive equilibrium payoffs.

Theorem 2 *Let $(u,w) \in \mathbb{R}^B_+ \times \mathbb{R}^S_+$ be a payoff vector of market M. Then,*

$$((u,w),(u,w)) \in \mathcal{SW}^{2M} \textit{ if and only if } (u,w) \in \mathcal{CE}.$$

Proof See the appendix. ■

Theorem 3 shows that a similar result does not hold for the Core. Namely, for each number ρ of replicas there exists a market M for which the Core of the $\rho-$fold replicated market contains a payoff that is not a competitive equilibrium payoff.

Theorem 3 *Let $\rho \in \mathbb{Z}_+ \backslash \{0\}$. Then, there exist a market M and a payoff vector $(u,w) \notin \mathcal{CE}$ such that $(u^\rho, w^\rho) \in \mathcal{C}^{\rho M}$.*

Proof See the appendix. ■

References

[1] Camiña, E. "A generalized assignment game," *Mathematical Social Sciences* 52, 152-161 (2006).

[2] Dantzig, G. *Linear Programming and Extensions.* Princeton University Press, Princeton (1963).

[3] Debreu, G. and Scarf, H. "A limit theorem on the Core of an economy," *International Economic Review* 4, 235-246 (1963).

[4] Edgeworth, F.Y. *Mathematical Psychics.* London: Kegan Paul (1881).

[5] Jaume, D., Massó, J., and Neme, A. "The multi-partners assignment game with heterogeneous sales and multi-unit demands: competitive equilibria," mimeo UAB (2009).

[6] Milgrom, P. "Assignment messages and exchanges," *American Economic Review: Micro* 1, 95-113 (2009).

[7] Owen, G. "On the Core of linear production games," *Mathematical Programming* 9, 358-370 (1975).

[8] Shapley, L. S., and Shubik, M. "The assignment game I: the core," *International Journal of Game Theory* 1, 111-130 (1972).

[9] Sotomayor, M. "The multiple partners game," in Majumdar, M. (editor) *Equilibrium and dynamics: Essays in honor to David Gale.* Macmillian, 322-336 (1992).

[10] Sotomayor, M. "The lattice structure of the set of stable outcomes of the multiple partners assignment game," *International Journal of Game Theory* 28, 567-583 (1999a).

[11] Sotomayor, M. "Three remarks on the many-to-many stable matching problem," *Mathematical Social Sciences* 38, 55-70 (1999b).

[12] Sotomayor, M. "Connecting the cooperative and competitive structures of the multiple-partners assignment game," *Journal of Economic Theory* 134, 155-174 (2007).

[13] Sotomayor, M. "Correlating new cooperative and competitive concepts in the time-sharing assignment game," mimeo (2009).

Appendix: Preliminaries and Omitted Proofs

We start with some preliminaries. Let M be a market and C be a coalition. Consider the primal linear problem to which any optimal assignment $\widetilde{A}^C \in \widetilde{F}^C$ is a solution.

$$(\mathrm{PLP})^C: \quad \max_{(A_{ijk})_{(i,j,k)\in B^C\times G^C\times S^C}\in\mathbb{R}^{B^C\times G^C\times S^C}} \quad \sum_{(i,j,k)\in B^C\times G^C\times S^C} \tau_{ijk}\cdot A_{ijk}$$

$$\begin{array}{llrll}
\text{s. t.} & (\mathrm{P.1}) & \sum\limits_{(j,k)\in G^C\times S^C} A_{ijk} & \le d_i & \text{for all } i\in B^C, \\
 & (\mathrm{P.2}) & \sum\limits_{i\in B^C} A_{ijk} & \le q_{jk} & \text{for all } (j,k)\in G^C\times S^C, \\
 & (\mathrm{P.3}) & A_{ijk} & \ge 0 & \text{for all } (i,j,k)\in B^C\times G^C\times S^C.
\end{array}$$

The dual linear problem associated to $(\mathrm{PLP})^C$ is the following.

$$(\mathrm{DLP})^C: \quad \min_{(\gamma^C,\pi^C)\in\mathbb{R}^{B^C}\times\mathbb{R}^{G^C\times S^C}} \quad \textstyle\sum_{i\in B^C} d_i\cdot\gamma_i^C + \sum_{(j,k)\in G^C\times S^C} q_{jk}\cdot\pi_{jk}^C$$

$$\begin{array}{llrll}
\text{s. t.} & (\mathrm{D.1}) & \gamma_i^C+\pi_{jk}^C & \ge \tau_{ijk} & \text{for all } (i,j,k)\in B^C\times G^C\times S^C, \\
 & (\mathrm{D.2}) & \gamma_i^C & \ge 0 & \text{for all } i\in B^C, \\
 & (\mathrm{D.3}) & \pi_{jk}^C & \ge 0 & \text{for all } (j,k)\in G^C\times S^C.
\end{array}$$

Let D^C be the set of pairs (γ^C,π^C) for which (D.1), (D.2) and (D.3) hold and let $\widetilde{D}^C$ be the set of all solutions of the $(\mathrm{DLP})^C$. It is well-known that $\widetilde{D}^C$ is non-empty. We will denote the sets $\widetilde{D}^{B\cup S}$ and $D^{B\cup S}$ by $\widetilde{D}$ and D respectively, and $(\gamma^{B\cup S},\pi^{B\cup S})$ by $(\gamma,\pi)\in D$. Let M be a market and let C be a coalition. Then, it is immediate to check that the following two implications hold.

$$\text{If } (\gamma,\pi)\in D \text{ then } ((\gamma_i)_{i\in B^C},(\pi_{jk})_{(j,k)\in G^C\times S^C})\in D^C \tag{13}$$

and

$$\text{if } (\gamma,\pi)\in \widetilde{D} \text{ then } ((\gamma_i)_{i\in B^C},(\pi_{jk})_{(j,k)\in G^C\times S^C})\in \widetilde{D}^C. \tag{14}$$

Let M be a market and $(\gamma, \pi) \in D$ be a dual feasible solution. We write $TD^M(\gamma, \pi)$ to denote the value of the objective function of the $(\text{DLP})^{B \cup S}$ at (γ, π); that is,

$$TD^M(\gamma, \pi) = \sum_i d_i \cdot \gamma_i + \sum_{jk} q_{jk} \cdot \pi_{jk}.$$

The Strong Duality Theorem (SDT) of Linear Programming applied to our setting says the following (see Dantzig, 1963).

Strong Duality Theorem *Let M be a market and assume $A \in F$ and $(\gamma, \pi) \in D$. Then,*

$$A \in \widetilde{F} \text{ and } (\gamma, \pi) \in \widetilde{D} \text{ if and only if } T^M(A) = TD^M(\gamma, \pi). \tag{15}$$

Given $(\gamma, \pi) \in D$, define its associated payoff $(u^{(\gamma,\pi)}, w^{(\gamma.\pi)}) \in \mathbb{R}^B \times \mathbb{R}^S$ as follows:

$$\begin{aligned} u_i^{(\gamma,\pi)} &= \gamma_i \cdot d_i \text{ for all } i \in B \\ w_k^{(\gamma,\pi)} &= \sum_{j \in G} \pi_{jk} \cdot q_{jk} \text{ for all } k \in S. \end{aligned}$$

Proposition 1 *Every market has a non-empty Core.*

Proof of Proposition 1 Let M be a market and let $(\widetilde{\gamma}, \widetilde{\pi}) \in \widetilde{D}$ be a solution of the $(\text{DLP})^{B \cup S}$. We will show that the payoff vector $(u^{(\widetilde{\gamma},\widetilde{\pi})}, w^{(\widetilde{\gamma},\widetilde{\pi})}) \in \mathbb{R}^B \times \mathbb{R}^S$ belongs to the Core of M. We first show that

$$\sum_{i \in B} u_i^{(\widetilde{\gamma},\widetilde{\pi})} + \sum_{k \in S} w_k^{(\widetilde{\gamma},\widetilde{\pi})} = v(B \cup S). \tag{16}$$

By the Strong Duality Theorem, $TD^M(\widetilde{\gamma}, \widetilde{\pi}) = T^M(\widetilde{A})$ for all $\widetilde{A} \in \widetilde{F}$. Thus,

$$\sum_{i \in B} d_i \cdot \widetilde{\gamma}_i + \sum_{(j,k) \in G \times S} q_{jk} \cdot \widetilde{\pi}_{jk} = TD^M(\widetilde{\gamma}, \widetilde{\pi})$$

and $T^M(\widetilde{A}) = v(B \cup S)$. Hence, (16) holds.

Let $C \subseteq B \cup S$ be an arbitrary coalition. We shall show that

$$\sum_{i \in B^C} u_i^{(\widetilde{\gamma}^C,\widetilde{\pi}^C)} + \sum_{k \in S^C} w_k^{(\widetilde{\gamma}^C,\widetilde{\pi}^C)} \geq v(C). \tag{17}$$

Observe first that, by (14), $(\widetilde{\gamma}^C, \widetilde{\pi}^C) \in \widetilde{D}^C$. Therefore, for every $(\gamma^C, \pi^C) \in D^C$, we have that

$$\sum_{i \in B^C} d_i \cdot \gamma_i^C + \sum_{(j,k) \in G^C \times S^C} q_{jk} \cdot \pi_{jk}^C \geq \sum_{i \in B^C} d_i \cdot \widetilde{\gamma}_i^C + \sum_{(j,k) \in G^C \times S^C} q_{jk} \cdot \widetilde{\pi}_{jk}^C = v(C). \tag{18}$$

The last equality follows from the Strong Duality Theorem. By the definition of the payoff vector $(u^{(\widetilde{\gamma}^C,\widetilde{\pi}^C)}, w^{(\widetilde{\gamma}^C,\widetilde{\pi}^C)}) \in \mathbb{R}^{B^C} \times \mathbb{R}^{S^C}$,

$$\sum_{i\in B^C} u_i^{(\widetilde{\gamma}^C,\widetilde{\pi}^C)} + \sum_{k\in S^C} w_k^{(\widetilde{\gamma}^C,\widetilde{\pi}^C)} = \sum_{i\in B^C} d_i \cdot \widetilde{\gamma}_i^C + \sum_{(j,k)\in G^C\times S^C} q_{jk} \cdot \widetilde{\pi}_{jk}^C.$$

Hence, by (18), (17) holds. Since C was an arbitrary coalition, $(u^{(\widetilde{\gamma},\widetilde{\pi})}, w^{(\widetilde{\gamma},\widetilde{\pi})}) \in \mathbb{R}^B \times \mathbb{R}^S$ belongs to the Core of M. ■

Proposition 5 *Let $\widetilde{p} \in \widetilde{P}$ be a competitive equilibrium price vector of market M. Then, $(u(\widetilde{p}), w(\widetilde{p})) \in \mathcal{SW}$.*

Proof of Proposition 5 Let $\widetilde{p} \in \widetilde{P}$. We first show that

$$\sum_{i\in B} u_i(\widetilde{p}) + \sum_{k\in S} w_k(\widetilde{p}) = v(B\cup S). \tag{19}$$

By Theorem 2 in Jaume, Massó, and Neme (2009), $(\gamma(\widetilde{p}), \pi(\widetilde{p})) \in \widetilde{D}$. Hence,

$$TD^M(\gamma(\widetilde{p}), \pi(\widetilde{p})) = \sum_{i\in B} d_i \cdot \gamma_i(\widetilde{p}) + \sum_{(j,k)\in G\times S} q_{jk} \cdot \pi_{jk}(\widetilde{p}). \tag{20}$$

By the definition of $(u(\widetilde{p}), w(\widetilde{p})) \in \mathbb{R}^m \times \mathbb{R}^t$,

$$\sum_{i\in B} d_i \cdot \gamma_i(\widetilde{p}) + \sum_{(j,k)\in G\times S} q_{jk} \cdot \pi_{jk}(\widetilde{p}) = \sum_{i\in B} u_i(\widetilde{p}) + \sum_{k\in S} w_k(\widetilde{p}). \tag{21}$$

By (20) and (21),

$$TD^M(\gamma(\widetilde{p}), \pi(\widetilde{p})) = \sum_{i\in B} u_i(\widetilde{p}) + \sum_{k\in S} w_k(\widetilde{p}).$$

Hence, by the Strong Duality Theorem, (19) holds.

Assume $(u(\widetilde{p}), w(\widetilde{p})) \notin \mathcal{SW}$. Then, there exists a coalition $C \subset B\cup S$ that SW$-$blocks it. Hence, for every distribution matrix $\Gamma = (\Gamma_{ijk})_{(i,j,k)\in B\times G\times S}$ there exists a feasible assignment $\widehat{A}$ that is SW$-$compatible with C such that

$$\begin{aligned}\sum_{i\in B^C} u_i(\widetilde{p}) + \sum_{k\in S^C} w_k(\widetilde{p}) \quad &< \sum_{(i,j,k)\in B^C\times G\times S^C} \tau_{ijk} \cdot \widehat{A}_{ijk} + \sum_{(i,j,k)\in B^C\times G\times (S^C)^c} (v_{ij} - \Gamma_{ijk}) \cdot \widehat{A}_{ijk} \\ &+ \sum_{(i,j,k)\in (B^C)^c\times G\times S^C} (\Gamma_{ijk} - r_{jk}) \cdot \widehat{A}_{ijk}.\end{aligned}$$

Consider the distribution matrix $\Gamma = (\Gamma_{ijk})_{(i,j,k)\in B\times G\times S}$ where for each $(i,j,k) \in B\times G\times S$, $\Gamma_{ijk} = \widetilde{p}_j$. Then, there must exist a feasible assignment $\widehat{A}$ that is SW$-$compatible with C

such that

$$\begin{aligned}\sum_{i\in B^C} u_i(\widetilde{p})+\sum_{k\in S^C} w_k(\widetilde{p}) &< \sum_{(i,j,k)\in B^C\times G\times S^C} \tau_{ijk}\cdot \widehat{A}_{ijk}+\sum_{(i,j,k)\in B^C\times G\times (S^C)^c} (v_{ij}-\widetilde{p}_j)\cdot \widehat{A}_{ijk}\\ &+\sum_{(i,j,k)\in (B^C)^c\times G\times S^C} (\widetilde{p}_j-r_{jk})\cdot \widehat{A}_{ijk}.\end{aligned} \tag{22}$$

Now, define the feasible assignment $\overline{A}$ as follows: for each $(i,j,k)\in B\times G\times S$,

$$\overline{A}_{ijk}=\begin{cases} 0 & \text{if either } \{i,k\}\subset C \text{ or } \{i,k\}\subset C^c\\ \widehat{A}_{ijk} & \text{otherwise,}\end{cases}$$

where C^c is the complementary set of C. Define a new market $\overline{M}=\left(B,S,G,V,\overline{d},R,\overline{Q}\right)$, where the new vector of maximal demands $\overline{d}$ is defined by setting

$$\overline{d}_i=d_i-\sum_{(j,k)\in G\times S}\overline{A}_{ijk}$$

for all $i\in B$, and the new matrix of capacities $\overline{Q}$ is defined by setting

$$\overline{q}_{jk}=q_{jk}-\sum_{i\in B}\overline{A}_{ijk}$$

for all $(j,k)\in G\times S$. Note that if $i\in C$ then, $\overline{d}_i=d_i-\sum_{(j,k)\in G\times (S^C)^c}\widehat{A}_{ijk}$ and if $k\in C$ then, $\overline{q}_{jk}=q_{jk}-\sum_{i\in (B^C)^c}\widehat{A}_{ijk}$ for all $j\in G$.

By (10),

$$\sum_{i\in B^C} u_i(\widetilde{p})+\sum_{k\in S^C} w_k(\widetilde{p})=\sum_{i\in B^C} d_i\cdot\gamma_i(\widetilde{p})+\sum_{(j,k)\in G\times S^C} q_{jk}\cdot\pi_{jk}(\widetilde{p}),$$

and note that

$$\begin{aligned}\sum_{i\in B^C} d_i\cdot\gamma_i(\widetilde{p}) &= \sum_{i\in B^C}\overline{d}_i\cdot\gamma_i(\widetilde{p})+\sum_{i\in B^C}\Big(\sum_{(j,k)\in G\times S}\overline{A}_{ijk}\Big)\cdot\gamma_i(\widetilde{p})\\ &= \sum_{i\in B^C}\overline{d}_i\cdot\gamma_i(\widetilde{p})+\sum_{i\in B^C}\Big(\sum_{j\in G}\sum_{k\in (S^C)^c}\widehat{A}_{ijk}\Big)\cdot\gamma_i(\widetilde{p})\end{aligned}$$

and for each $j\in G$,

$$\begin{aligned}\sum_{k\in S^C} q_{jk}\cdot\pi_{jk}(\widetilde{p}) &= \sum_{k\in S^C}\overline{q}_{jk}\cdot\pi_{jk}(\widetilde{p})+\sum_{k\in S^C}\Big(\sum_{i\in B}\overline{A}_{ijk}\Big)\cdot\pi_{jk}(\widetilde{p})\\ &= \sum_{k\in S^C}\overline{q}_{jk}\cdot\pi_{jk}(\widetilde{p})+\sum_{k\in S^C}\Big(\sum_{i\in (B^C)^c}\widehat{A}_{ijk}\Big)\cdot\pi_{jk}(\widetilde{p}).\end{aligned}$$

Hence,

$$\sum_{i\in B^C} u_i(\widetilde{p}) + \sum_{k\in S^C} w_k(\widetilde{p}) = \sum_{i\in B^C} \overline{d}_i \cdot \gamma_i(\widetilde{p}) + \sum_{i\in B^C} (\sum_{j\in G} \sum_{k\in (S^C)^c} \widehat{A}_{ijk}) \cdot \gamma_i(\widetilde{p}) + \sum_{k\in S^C} \sum_{j\in G} \overline{q}_{jk} \cdot \pi_{jk}(\widetilde{p}) + \sum_{k\in S^C} \sum_{j\in G} (\sum_{i\in (B^C)^c} \widehat{A}_{ijk}) \cdot \pi_{jk}(\widetilde{p}). \quad (23)$$

By (8), for every $i \in B$ and $j \in G$,

$$\gamma_i(\widetilde{p}) \geq v_{ij} - \widetilde{p}_j. \quad (24)$$

Moreover, by (9), for every $(j,k) \in G \times S$,

$$\pi_{jk}(\widetilde{p}) \geq \widetilde{p}_j - r_{jk}. \quad (25)$$

By (22) and (23)

$$\sum_{(i,j,k)\in B^C\times G\times S^C} \tau_{ijk} \cdot \widehat{A}_{ijk} + \sum_{(i,j,k)\in B^C\times G\times (S^C)^c} (v_{ij} - \widetilde{p}_j) \cdot \widehat{A}_{ijk} + \sum_{(i,j,k)\in (B^C)^c\times G\times S^C} (\widetilde{p}_j - r_{jk}) \cdot \widehat{A}_{ijk} > \sum_{i\in B^C} \overline{d}_i \cdot \gamma_i(\widetilde{p}) + \sum_{i\in B^C} (\sum_{j\in G} \sum_{k\in (S^C)^c} \widehat{A}_{ijk}) \cdot \gamma_i(\widetilde{p}) + \sum_{k\in S^C} \sum_{j\in G} \overline{q}_{jk} \cdot \pi_{jk}(\widetilde{p}) + \sum_{k\in S^C} \sum_{j\in G} (\sum_{i\in (B^C)^c} \widehat{A}_{ijk}) \cdot \pi_{jk}(\widetilde{p}). \quad (26)$$

By (24),

$$\sum_{i\in B^C} (\sum_{j\in G} \sum_{k\in (S^C)^c} \widehat{A}_{ijk}) \cdot \gamma_i(\widetilde{p}) \geq \sum_{(i,j,k)\in B^C\times G\times (S^C)^c} (v_{ij} - \widetilde{p}_j) \cdot \widehat{A}_{ijk}.$$

Hence, by (26),

$$\sum_{(i,j,k)\in B^C\times G\times S^C} \tau_{ijk} \cdot \widehat{A}_{ijk} + \sum_{(i,j,k)\in (B^C)^c\times G\times S^C} (\widetilde{p}_j - r_{jk}) \cdot \widehat{A}_{ijk} > \sum_{i\in B^C} \overline{d}_i \cdot \gamma_i(\widetilde{p}) + \sum_{k\in S^C} \sum_{j\in G} \overline{q}_{jk} \cdot \pi_{jk}(\widetilde{p}) + \sum_{k\in S^C} \sum_{j\in G} (\sum_{i\in (B^C)^c} \widehat{A}_{ijk}) \cdot \pi_{jk}(\widetilde{p}). \quad (27)$$

By (25),

$$\sum_{k\in S^C} \sum_{j\in G} (\sum_{i\in (B^C)^c} \widehat{A}_{ijk}) \cdot \pi_{jk}(\widetilde{p}) \geq \sum_{(i,j,k)\in (B^C)^c\times G\times S^C} (\widetilde{p}_j - r_{jk}) \cdot \widehat{A}_{ijk}.$$

Hence, by (27),

$$\sum_{(i,j,k)\in B^C\times G\times S^C} \tau_{ijk} \cdot \widehat{A}_{ijk} > \sum_{i\in B^C} \overline{d}_i \cdot \gamma_i(\widetilde{p}) + \sum_{k\in S^C} \sum_{j\in G} \overline{q}_{jk} \cdot \pi_{jk}(\widetilde{p}). \quad (28)$$

Observe that since $\widehat{A}$ is a feasible assignment,

$$v^{\overline{M}}(C) \geq \sum_{(i,j,k)\in B^C\times G\times S^C} \tau_{ijk}\cdot \widehat{A}_{ijk},$$

where $v^{\overline{M}}(C) = T^{\overline{M}}(\widetilde{A}^C)$ for any optimal assignment $\widetilde{A}^C$ of market $\overline{M}$. Since $(\gamma(\widetilde{p}), \pi(\widetilde{p})) \in \widetilde{D}$ then by (14), $(\gamma^C(\widetilde{p}), \pi^C(\widetilde{p})) \in \widetilde{D}^C$ for market $\overline{M}$. Hence, by the Strong Duality Theorem,

$$\sum_{i\in B^C} \overline{d}_i\cdot\gamma_i(\widetilde{p}) + \sum_{k\in S^C}\sum_{j\in G} \overline{q}_{jk}\cdot\pi_{jk}(\widetilde{p}) = v^{\overline{M}}(C),$$

contradicting (28). ■

Proposition 6 *Let $(u, w) \in \mathbb{R}^B \times \mathbb{R}^S$ be a payoff of market M. Then, $(u, w) \in \mathcal{CE}$ if and only if there exists a competitive equilibrium price vector $\widetilde{p}$ such that for every coalition $C \subset B \cup S$ and any feasible assignment $\widehat{A}$ $SW-$compatible with C we have that*

$$\sum_{i\in B^C} u_i + \sum_{k\in S^C} w_k \geq$$

$$\sum_{(i,j,k)\in B^C\times G\times S^C} \tau_{ijk}\cdot\widehat{A}_{ijk} + \sum_{(i,j,k)\in B^C\times G\times (S^C)^c} (v_{ij}-\widetilde{p}_j)\cdot\widehat{A}_{ijk}. + \sum_{(i,j,k)\in (B^C)^c\times G\times S^C} (\widetilde{p}_j - r_{jk})\cdot\widehat{A}_{ijk}.$$

Proof of Proposition 6

$\Rightarrow$) It follows from Proposition 5.

$\Leftarrow$) Let $(u, w) \in \mathbb{R}^B \times \mathbb{R}^S$ be a payoff of market M. Consider first any coalition $C = \{i\}$, where $i \in B$. Then, by assumption,

$$u_i \geq \sum_{(j,k)\in G\times S} (v_{ij} - \widetilde{p}_j)\cdot\widehat{A}_{ijk}, \tag{29}$$

where $\widehat{A}$ is any feasible assignment.

Consider now any coalition $C = \{k\}$, where $k \in S$. Then, by assumption,

$$w_k \geq \sum_{(i,j)\in B\times G} (\widetilde{p}_j - r_{jk})\cdot\widehat{A}_{ijk}, \tag{30}$$

where $\widehat{A}$ is any feasible assignment.

Finally, assume that $\widetilde{A}$ is an optimal assignment. Then,

$$\sum_{i\in B} u_i + \sum_{k\in S} w_k = v(B\cup S) = \sum_{(i,j,k)\in B\times G\times S} \tau_{ijk}\cdot\widetilde{A}_{ijk}.$$

By definition of the per unit gains τ_{ijk},

$$\sum_{(i,j,k)\in B\times G\times S} \tau_{ijk}\cdot \widetilde{A}_{ijk} = \sum_{i\in B}\sum_{(j,k)\in G\times S}(v_{ij}-\widetilde{p}_j)\cdot \widetilde{A}_{ijk} + \sum_{k\in S}\sum_{(i,j)\in B\times G}(\widetilde{p}_j - r_{jk})\cdot \widetilde{A}_{ijk}.$$

Hence, (29) and (30) imply that for every $i \in B$ and $k \in S$,

$$\begin{aligned} u_i &= \sum_{(j,k)\in G\times S}(v_{ij}-\widetilde{p}_j)\cdot \widetilde{A}_{ijk} \\ w_k &= \sum_{(i,j)\in B\times G}(\widetilde{p}_j - r_{jk})\cdot \widetilde{A}_{ijk}, \end{aligned}$$

and consequently, by Lemma 6 in Jaume, Massó, and Neme (2009), for every $i \in B$ and $k \in S$,

$$\begin{aligned} u_i &= d_i \cdot \gamma_i(\widetilde{p}) \geq 0 \\ w_k &= \sum_{j\in G} q_{jk}\cdot \pi_{jk}(\widetilde{p}) \geq 0. \end{aligned}$$

Thus, $(u,w) = (u(\widetilde{p}), w(\widetilde{p}))$ and by (11), $(u,w)\in \mathcal{CE}$. ■

Proposition 7 *Let M be a market and let $\rho \geq 2$. Then*

$$\mathcal{C}^{\rho M} \subset \{(u^\rho, w^\rho) \equiv (\underbrace{(u,w),...,(u,w)}_{\rho-\text{times}}) \in (\mathbb{R}^{B_1}\times\mathbb{R}^{S_1})\times ...\times(\mathbb{R}^{B_\rho}\times\mathbb{R}^{S_\rho}) \mid (u,w)\in \mathcal{C}^M\}.$$

Proof of Proposition 7 Let $((\widehat{u}_{i_1},...,\widehat{u}_{i_\rho})_{i\in B}, (\widehat{w}_{k_1},...,\widehat{w}_{k_\rho})_{k\in S}) \in \mathcal{C}^{\rho M}$. For every $\alpha = 1,...,\rho$,

$$\sum_{i_\alpha \in B_\alpha} \widehat{u}_{i_\alpha} + \sum_{k_\alpha\in S_\alpha}\widehat{w}_{k_\alpha} \geq v(B_\alpha \cup S_\alpha) = v(B\cup S)$$

must hold; otherwise, any coalition $C = B_\alpha \cup S_\alpha$ would Core−block $(\widehat{u},\widehat{w})$. Hence, for every $\alpha = 1,...,\rho$,

$$\sum_{i_\alpha \in B_\alpha} \widehat{u}_{i_\alpha} + \sum_{k_\alpha\in S_\alpha}\widehat{w}_{k_\alpha} = v(B\cup S).$$

Assume that there exists a buyer type $\widehat{i}\in B$ and two replicas α and α' such that

$$\widehat{u}_{\widehat{i}_\alpha} > \widehat{u}_{\widehat{i}_{\alpha'}}.$$

Then, the coalition $C = [(B_\alpha \cup S_\alpha)\backslash\{\widehat{i}_\alpha\}]\cup\{\widehat{i}_{\alpha'}\}$ Core−blocks $(\widehat{u},\widehat{w})$ because

$$v(C) = v(B\cup S) > \sum_{i_\alpha\in B_\alpha\backslash\{\widehat{i}_\alpha\}} \widehat{u}_{i_\alpha} + \widehat{u}_{\widehat{i}_{\alpha'}} + \sum_{k_\alpha\in S_\alpha}\widehat{w}_{k_\alpha}.$$

Similarly for any seller type $\widehat{k} \in S$. Thus, $((\widehat{u}_{i_1}, ..., \widehat{u}_{i_\rho})_{i\in B}, (\widehat{w}_{k_1}, ..., \widehat{w}_{k_\rho})_{k\in S}) = (u^\rho, w^\rho)$ for some payoff vector $(u, w) \in \mathbb{R}^B \times \mathbb{R}^S$ of market M.

To obtain a contradiction, assume that $(u, w) \notin \mathcal{C}^M$. Then, there exists a coalition C that Core$-$blocks (u, w). But then, C also Core$-$blocks $(\widehat{u}, \widehat{w})$, a contradiction with $(\widehat{u}, \widehat{w}) \in \mathcal{C}^{\rho M}$. ■

Theorem 1 *Let M be a market. If $(u, w) \in \mathbb{R}^B \times \mathbb{R}^S$ is in the Core of the $\rho-$fold replicated market for all $\rho \geq 1$, then (u, w) is a competitive equilibrium payoff of market M.*

Proof of Theorem 1 It follows from Lemma 1 below.[7] ■

Lemma 1 *Let M be a market and assume that $(u, w) \in \mathbb{R}^B \times \mathbb{R}^S$ is such that $(u^\rho, w^\rho) \in \mathcal{C}^{\rho M}$ for all $\rho \geq 1$. Then, there exists a solution $(\gamma, \pi) \in \widetilde{D}$ of* (DLP)$^{B\cup S}$ *such that $u_i = d_i \cdot \gamma_i$ for every $i \in B$, and $w_k = \sum_j \pi_{jk} \cdot q_{jk}$ for every $k \in S$; namely, $(u, w) \in \mathcal{CE}$.*

Proof of Lemma 1 Let $(u, w) \in \mathbb{R}^B \times \mathbb{R}^S$ be a payoff vector of market M. Consider the following system of inequalities

$$\begin{array}{rll} (-d_i) \cdot \gamma_i & \geq -u_i & \text{for all } i \in B \\ \sum_j (-q_{jk}) \cdot \pi_{jk} & \geq -w_k & \text{for all } k \in S \\ \gamma_i + \pi_{jk} & \geq \tau_{ijk} & \text{for all } (i,j,k) \in B \times G \times S \\ \gamma_i & \geq 0 & \text{for all } i \in B \\ \pi_{jk} & \geq 0 & \text{for all } (j,k) \in G \times S. \end{array} \tag{31}$$

Claim $(u, w) \in \mathcal{CE}$ if and only if the system in (31) has at least one solution (γ, π).

Proof of Claim Necessity follows from (10). To show sufficiency suppose (γ^*, π^*) is a solution of (31). Then,

$$\begin{array}{rll} \sum_i d_i \cdot \gamma_i^* + \sum_{jk} q_{jk} \cdot \pi_{jk}^* & \leq v(B \cup S) & \\ \gamma_i^* + \pi_{jk}^* & \geq \tau_{ijk} & \text{for all } (i,j,k) \in B \times G \times S \\ \gamma_i^* & \geq 0 & \text{for all } i \in B \\ \pi_{jk}^* & \geq 0 & \text{for all } (j,k) \in G \times S. \end{array}$$

[7] Our proof adapts Owen (1975)'s proof of the convergence of the Core to the set of competitive equilibrium payoffs for linear programming games.

Thus, by the Strong Duality Theorem, (γ^*, π^*) is a solution of the $(\text{DLP})^{B \cup S}$. Hence, $(u, w) \in \mathcal{CE}$. This proves the claim. $\Diamond$

Consider the following Primal Linear Problem

$$\begin{array}{llrl}
\min\limits_{(\gamma,\pi)\in\mathbb{R}^B\times\mathbb{R}^{G\times S}} & \sum\limits_{i\in B}(\gamma_i\cdot 0)+\sum\limits_{(j,k)\in G\times S}(\pi_{jk}\cdot 0) & & \\
\text{s.t.} & (-d_i)\cdot\gamma_i & \geq -u_i & \text{for all } i\in B\\
& \sum_j(-q_{jk})\cdot\pi_{jk} & \geq -w_k & \text{for all } k\in S\\
& \gamma_i+\pi_{jk} & \geq \tau_{ijk} & \text{for all } (i,j,k)\in B\times G\times S\\
& \gamma_i & \geq 0 & \text{for all } i\in B\\
& \pi_{jk} & \geq 0 & \text{for all } (j,k)\in G\times S,
\end{array}$$

and its associated Dual Linear Problem

$$\begin{array}{llrl}
\max\limits_{\substack{y^1\in\mathbb{R}^B\\ y^2\in\mathbb{R}^S\\ y^3\in\mathbb{R}^{B\times G\times S}}} & \sum_i y_i^1\cdot(-u_i)+\sum_k y_k^2\cdot(-w_k)+\sum_{ijk}y_{ijk}^3\cdot\tau_{ijk} & & \\
\text{s.t.} & (-d_i)\cdot y_i^1+\sum_{jk}y_{ijk}^3 & \leq 0 & \text{for all } i\in B\\
& (-q_{jk})\cdot y_k^2+\sum_i y_{ijk}^3 & \leq 0 & \text{for all } (j,k)\in G\times S\\
& y^1 & \geq 0 & \\
& y^2 & \geq 0 & \\
& y^3 & \geq 0. &
\end{array}$$

Assume the system in (31) has no solution. Then, the Primal Linear Problem has no solution and the Dual Linear Problem has no solution either, and since $\overrightarrow{0}$ is a feasible vector of the Dual Linear Problem, the linear function $\sum_i y_i^1\cdot(-u_i)+\sum_k y_k^2\cdot(-w_k)+\sum_{ijk}y_{ijk}^3\cdot\tau_{ijk}$ is unbounded. But this implies that there exists (y^1, y^2, y^3) such that

$$\begin{array}{rll}
\sum_i y_i^1\cdot u_i+\sum_k y_k^2\cdot w_k & < y_{ijk}^3\cdot\tau_{ijk} & \\
\sum_{jk}y_{ijk}^3 & \leq d_i\cdot y_i^1 & \text{for all } i\in B\\
\sum_i y_{ijk}^3 & \leq q_{jk}\cdot y_k^2 & \text{for all } (j,k)\in G\times S\\
y^1 & \geq 0 & \\
y^2 & \geq 0 & \\
y^3 & \geq 0. &
\end{array}$$

Since the first restriction holds with strict inequality, y^1, y^2 and y^3 can be vectors with rational components. Multiplying them by the lowest common denominator we can assume, without loss of generality, that $y_i^1 \in \mathbb{N}$ for all $i \in B$ and $y_k^2 \in \mathbb{N}$ for all $k \in S$.

Define $\rho = \max\{y_1^1, ..., y_m^1, y_1^2, ..., y_t^2\}$ and let C be a coalition containing y_i^1 buyers of type i, and y_k^2 sellers of type k. Observe that $v(C) \geq \sum_{ijk} y_{ijk}^3 \cdot \tau_{ijk}$ and hence, $v(C) > \sum_i y_i^1 \cdot u_i + \sum_k y_k^2 \cdot w_k$. Thus, (u^ρ, w^ρ) is Core-blocked by C, a contradiction with the general assumption of Lemma 1. Hence, the Primal Linear Problem has a solution. By the Claim, $(u, w) \in \mathcal{CE}$. ■

Theorem 2 *Let $(u, w) \in \mathbb{R}^B \times \mathbb{R}^S$ be a payoff vector of market M. Then,*

$$((u, w,), (u, w)) \in \mathcal{SW}^{2M} \textit{ if and only if } (u, w) \in \mathcal{CE}.$$

Proof of Theorem 2 Assume first that $(u, w) \in \mathcal{CE}$. Then, by Corollary 1 and Proposition 7, $((u, w), (u, w)) \in \mathcal{SW}^{2M}$.

Assume now that $((u, w,), (u, w)) \in \mathcal{SW}^{2M}$. Then, there exists a distribution matrix $\Gamma = (\Gamma_{ijk})_{(i,j,k) \in 2B \times G \times 2S}$ such that for every coalition $C \subset 2B \cup 2S$ and every feasible assignment $\widehat{A}$ that is SW-compatible with C we have that:

$$\begin{aligned}\sum_{i \in (2B)^C} u_i + \sum_{k \in (2S)^C} w_k \quad &\geq \sum_{(i,j,k) \in (2B)^C \times G \times (2S)^C} \tau_{ijk} \cdot \widehat{A}_{ijk} + \sum_{(i,j,k) \in (2B)^C \times G \times ((2S)^C)^c} (v_{ij} - \Gamma_{ijk}) \cdot \widehat{A}_{ijk} \\ &+ \sum_{(i,j,k) \in ((2B)^C)^c \times G \times (2S)^C} (\Gamma_{ijk} - r_{jk}) \cdot \widehat{A}_{ijk}.\end{aligned} \tag{32}$$

Fix Γ and let $\widetilde{A} \in \widetilde{F}$ be any optimal assignment of market M.

Claim 1 *For every $i \in B$ and every $k \in S$,*

$$u_i = \sum_{(j,k) \in G \times S} (v_{ij} - \Gamma_{ijk}) \cdot \widetilde{A}_{ijk}$$

and

$$w_k = \sum_{(i,j) \in B \times G} (\Gamma_{ijk} - r_{jk}) \cdot \widetilde{A}_{ijk}.$$

hold.

Proof of Claim 1 By considering either $C = \{i\}$ or $C = \{k\}$, we have that, by (32),

$$u_i \geq \sum_{(j,k) \in G \times S} (v_{ij} - \Gamma_{ijk}) \cdot \widetilde{A}_{ijk} \text{ for every } i \in B$$

and

$$w_k \geq \sum_{(i,j) \in B \times G} (\Gamma_{ijk} - r_{jk}) \cdot \widetilde{A}_{ijk} \text{ for every } k \in S.$$

Since

$$\sum_{i\in B}\sum_{(j,k)\in G\times S}(v_{ij}-\Gamma_{ijk})\cdot\widetilde{A}_{ijk}+\sum_{k\in S}\sum_{(i,j)\in B\times G}(\Gamma_{ijk}-r_{jk})\cdot\widetilde{A}_{ijk}=\sum_{(i,j,k)\in B\times G\times S}\tau_{ijk}\cdot\widetilde{A}_{ijk}=T^M(\widetilde{A})$$

and

$$T^M(\widetilde{A})=\sum_{i\in B}u_i+\sum_{k\in S}w_k,$$

the statement of Claim 1 follows. $\square$

Claim 2 *Let i_1,i_2,j',k_1,k_2 be such that $\widetilde{A}_{i_1j'k_1}\neq 0\neq\widetilde{A}_{i_2j'k_2}$. Then, $\Gamma_{i_1j'k_1}=\Gamma_{i_2j'k_2}$.*

Proof of Claim 2 Assume otherwise; for instance, $\Gamma_{i_1j'k_1}>\Gamma_{i_2j'k_2}$. If $\Gamma_{i_2j'k_2}>\Gamma_{i_1j'k_1}$, then replace in the argument that follows the roles of i_1 by i_2 and k_2 by k_1. Consider the coalition $C=\{i_1,k_2\}$. From $\widetilde{A}$ we define the assignment $\widehat{A}$ SW$-$compatible with C by decreasing in 1 unit the exchanges between i_1 and k_1 and between i_2 and k_2 and by simultaneously increasing in 1 unit the exchange between i_1 and k_2. Namely, for every $(i,j,k)\in B\times G\times S$, define

$$\widehat{A}_{ijk}=\begin{cases}\widetilde{A}_{ijk}-1 & \text{if } i=i_1', j=j' \text{ and } k=k_1'\\ \widetilde{A}_{ijk}-1 & \text{if } i=i_2', j=j' \text{ and } k=k_2'\\ \widetilde{A}_{ijk}+1 & \text{if } i=i_1', j=j' \text{ and } k=k_2'\\ \widetilde{A}_{ijk} & \text{otherwise.}\end{cases}$$

Observe that since by assumption $\widetilde{A}_{i_1j'k_1}\neq 0\neq\widetilde{A}_{i_2j'k_2}$, $\widehat{A}$ is a feasible assignment. Moreover, $\widehat{A}$ is SW$-$compatible with $C=\{i_1,k_2\}$. Define

$$\widehat{u}_{i_1}=\sum_{(j,k)\in G\times S}(v_{ij}-\Gamma_{ijk})\cdot\widehat{A}_{ijk}$$

and

$$\widehat{w}_{k_2}=\sum_{(i,j)\in B\times G}(\Gamma_{ijk}-r_{jk})\cdot\widehat{A}_{ijk}.$$

By Claim 1 and the definition of $\widehat{A}$,

$$\begin{aligned}u_{1_1}+w_{k_2}-(\widehat{u}_{1_1}+\widehat{w}_{k_2}) &= (v_{i_1j'}-\Gamma_{i_1j'k_1})\cdot\widetilde{A}_{i_1j'k_1}+(v_{i_1j'}-\Gamma_{i_1j'k_2})\cdot\widetilde{A}_{i_1j'k_2}\\ &\quad+(\Gamma_{i_1j'k_2}-r_{j'k_2})\cdot\widetilde{A}_{i_1j'k_2}+(\Gamma_{i_2j'k_2}-r_{j'k_2})\cdot\widetilde{A}_{i_2j'k_2}\\ &\quad-(v_{i_1j'}-\Gamma_{i_1j'k_1})\cdot\widehat{A}_{i_1j'k_1}-(v_{i_1j'}-\Gamma_{i_1j'k_2})\cdot\widehat{A}_{i_1j'k_2}\\ &\quad-(\Gamma_{i_1j'k_2}-r_{j'k_2})\cdot\widehat{A}_{i_1j'k_2}-(\Gamma_{i_2j'k_2}-r_{j'k_2})\cdot\widehat{A}_{i_2j'k_2}\\ &= (v_{i_1j'}-\Gamma_{i_1j'k_1})-(v_{i_1j'}-\Gamma_{i_1j'k_2})\\ &\quad-(\Gamma_{i_1j'k_2}-r_{j'k_2})+(\Gamma_{i_2j'k_2}-r_{j'k_2})\\ &= -\Gamma_{i_1j'k_1}+\Gamma_{i_2j'k_2}.\end{aligned}$$

Since by assumption $\Gamma_{i_1 j' k_1} > \Gamma_{i_2 j' k_2}$, we have that $u_{1_1} + w_{k_2} < \widehat{u}_{1_1} + \widehat{w}_{k_2}$, a contradiction with (32). $\square$

To proceed with the proof of Theorem 2 we define a price vector $p = (p_j)_{j \in G} \in \mathbb{R}^G$ as follows. Consider first any $j \in G$ for which there exist $i \in B$ and $k \in S$ such that $A^*_{ijk} \neq 0$ for some optimal assignment A^*. Then, define $p_j = \Gamma_{ijk}$. By Claim 2, p_j is well defined. Suppose now that $j \in G$ is such that for all optimal assignment A^* and all $i \in B$ and $k \in S$, $A^*_{ijk} = 0$. Then, define $p_j = \min\{r_{jk} \mid k \text{ is such that } q_{jk} > 0\}$; by (2), p_j is well-defined. Let $A \in \widetilde{F}$ be arbitrary. We shall show that (p, A) is a competitive equilibrium of M by showing that the equilibrium conditions (E.D) and (E.S) are satisfied.

(E.D) For each buyer $i \in B$, $A(i) \in D_i(p)$.
Since $A \in \widetilde{F}$, (D.a) and (D.b) hold.
(D.c)**:** $\nabla_i^{>}(p) \neq \emptyset \Longrightarrow \sum_{jk} A(i)_{jk} = d_i$.

Assume $\nabla_{i'}^{>}(p) \neq \emptyset$; *i.e.*, there exists $j' \in G$ such that $v_{i'j'} - p_{j'} = \max_{j \in G}\{v_{i'j} - p_j\} > 0$. Assume that

$$\sum_{jk} A_{i'jk} < d_{i'}. \tag{33}$$

Without loss of generality suppose that i' belongs to the first replica; *i.e.*, $i' = i_1$. Consider first the case where there are $i_2 \in B_2$ and $k_2 \in S_2$ with the property that $A_{i_2 j' k_2} \neq 0$. Consider the coalition $C = \{i_1, k_2\}$ and its SW$-$compatible assignment $\widehat{A}$ where, for all $(i, j, k) \in B \times G \times S$,

$$\widehat{A}_{ijk} = \begin{cases} A_{ijk} + 1 & \text{if } i = i_1, j = j' \text{ and } k = k_2 \\ A_{ijk} - 1 & \text{if } i = i_2, j = j' \text{ and } k = k_2 \\ A_{ijk} & \text{otherwise.} \end{cases}$$

By (33) and $A_{i_2 j' k_2} \neq 0$, $\widehat{A}$ is a feasible assignment and SW$-$compatible with coalition $\{i_1, k_2\}$. Then, proceeding in a similar way as we did in the proof of Claim 2, define $\widehat{u}_{i_1}$ and $\widehat{w}_{k_2}$ as the payoffs of buyer i_1 and seller k_2 at assignment $\widehat{A}$, respectively. Then, By Claim 1 and the definition of $\widehat{A}$,

$$\begin{aligned} u_{i_1} + w_{k_2} - (\widehat{u}_{i_1} + \widehat{w}_{k_2}) &= (v_{i_1 j'} - \Gamma_{i_1 j' k_2}) \cdot A_{i_1 j' k_2} \\ &\quad + (\Gamma_{i_1 j' k_2} - r_{j' k_2}) \cdot A_{i_1 j' k_2} + (\Gamma_{i_2 j' k_2} - r_{j' k_2}) \cdot A_{i_2 j' k_2} \\ &\quad - (v_{i_1 j'} - \Gamma_{i_1 j' k_2}) \cdot \widehat{A}_{i_1 j' k_2} \\ &\quad - (\Gamma_{i_1 j' k_2} - r_{j' k_2}) \cdot \widehat{A}_{i_1 j' k_2} - (\Gamma_{i_2 j' k_2} - r_{j' k_2}) \cdot \widehat{A}_{i_2 j' k_2} \\ &= -(v_{i_1 j'} - \Gamma_{i_1 j' k_2}) - (\Gamma_{i_1 j' k_2} - r_{j' k_2}) + (\Gamma_{i_2 j' k_2} - r_{j' k_2}) \\ &= -v_{i_1 j'} + \Gamma_{i_2 j' k_2}. \end{aligned}$$

By (32), $-v_{i_1j'} + \Gamma_{i_2j'k_2} \geq 0$. Since $A_{i_2j'k_2} \neq 0$, by definition of $p_{j'}$, $0 \geq v_{i_1j'} - p_{j'}$, a contradiction with $j' \in \nabla_{i_1}^{>}(p)$.

Assume now that for all $i'' \in 2B$ and all $k'' \in 2S$, $A_{i''j'k''} = 0$. By definition, $p_{j'} = \min\{r_{j'k} \mid k \text{ is such that } q_{j'k} > 0\}$. Let $k^* \in 2S$ be such that $q_{j'k^*} > 0$ and $p_{j'k^*} = r_{j'k^*}$. By (2), such k^* does exist. Consider the coalition $C = \{i_1, k^*\}$ and its SW$-$compatible assignment $\widehat{A}$ where, for all $(i,j,k) \in 2B \times G \times 2S$

$$\widehat{A}_{ijk} = \begin{cases} A_{ijk} + 1 & \text{if } i = i_1, j = j' \text{ and } k = k^* \\ A_{ijk} & \text{otherwise.} \end{cases}$$

By (33) and $q_{j'k^*} > 0$, $\widehat{A}$ is a feasible assignment. Then, as before,

$$\begin{aligned} u_{i_1} + w_{k^*} - (\widehat{u}_{i_1} + \widehat{w}_{k^*}) &= (v_{i_1j'} - \Gamma_{i_1j'k^*}) \cdot A_{i_1j'k^*} + (\Gamma_{i_1j'k^*} - r_{j'k^*}) \cdot A_{i_1j'k^*} \\ &\quad -(v_{i_1j'} - \Gamma_{i_1j'k^*}) \cdot \widehat{A}_{i_1j'k^*} - (\Gamma_{i_1j'k^*} - r_{j'k^*}) \cdot \widehat{A}_{i_1j'k^*} \\ &= -(v_{i_1j'} - \Gamma_{i_1j'k^*}) - (\Gamma_{i_1j'k^*} - r_{j'k^*}) \\ &= -v_{i_1j'} + r_{i_1j'k^*}. \end{aligned}$$

By (32), $-v_{i_1j'} + r_{i_1j'k^*} \geq 0$. Since $p_{j'} = r_{j'k^*}$, $v_{i_1j'} - p_{j'} \leq 0$, contradicting that $j' \in \nabla_{i_1}^{>}(p)$.

(D.d): $\sum_k A(i)_{jk} > 0 \Longrightarrow j \in \nabla_i^{\geq}(p)$.

Assume otherwise; *i.e.*, there exist i', j', k' such that $A_{i'j'k'} \neq 0$ and $j' \notin \nabla_{i'}^{\geq}(p)$. We distinguish between the following two cases.

<u>Case 1</u>: $v_{i'j'} - p_{j'} < 0$. Consider the coalition $C = \{i'\}$ and its compatible assignment $\widehat{A}$ where

$$\widehat{A}_{ijk} = \begin{cases} 0 & \text{if } i = i', j = j' \text{ and } k = k' \\ A_{ijk} & \text{otherwise.} \end{cases}$$

Define $\widehat{u}_{i'}$ as the utility of buyer i' at assignment $\widehat{A}$. Then, it is immediate to see that $u_{i'} < \widehat{u}_{i'}$, contradicting (32).

<u>Case 2</u>: There exists $j'' \in \nabla_{i'}^{\geq}(p)$ such that

$$(v_{i'j''} - p_{j''}) > (v_{i'j'} - p_{j'}) \geq 0. \tag{34}$$

Note that $A_{i'j'k'} \neq 0$. By definition of p, $p_{j'} = \Gamma_{i'j'k'}$. Assume first that there exist $i'' \in 2B$ and $k'' \in 2S$ such that $A_{i''j''k''} \neq 0$. Again, by definition p, $p_{j''} = \Gamma_{i''j''k''}$. Consider the coalition $C = \{i', k''\}$ and its SW$-$compatible assignment $\widehat{A}$ where, for all $(i,j,k) \in B \times$

$G \times S$,

$$\widehat{A}_{ijk} = \begin{cases} A_{ijk} - 1 & \text{if } i = i', j = j' \text{ and } k = k' \\ A_{ijk} - 1 & \text{if } i = i'', j = j'' \text{ and } k = k'' \\ A_{ijk} + 1 & \text{if } i = i', j = j'' \text{ and } k = k'' \\ A_{ijk} & \text{otherwise.} \end{cases}$$

Then, proceeding in a similar way as we did in the proof of Claim 2, define $\widehat{u}_{i'}$ and $\widehat{w}_{k''}$ as the payoffs of buyer i' and seller k'' at assignment $\widehat{A}$, respectively. Then, By Claim 1 and the definition of $\widehat{A}$,

$$\begin{aligned} u_{i'} + w_{k''} - (\widehat{u}_{i'} + \widehat{w}_{k''}) &= (v_{i'j'} - \Gamma_{i'j'k'}) \cdot A_{i'j'k'} + (v_{i'j''} - \Gamma_{i'j''k''}) \cdot A_{i'j''k''} \\ &\quad + (\Gamma_{i'j''k''} - r_{j''k''}) \cdot A_{i'j''k''} + (\Gamma_{i''j''k''} - r_{j''k''}) \cdot A_{i''j''k''} \\ &\quad - (v_{i'j'} - \Gamma_{i'j'k'}) \cdot \widehat{A}_{i'j'k'} - (v_{i'j''} - \Gamma_{i'j''k''}) \cdot \widehat{A}_{i'j''k''} \\ &\quad - (\Gamma_{i'j''k''} - r_{j''k''}) \cdot \widehat{A}_{i'j''k''} - (\Gamma_{i''j''k''} - r_{j''k''}) \cdot \widehat{A}_{i''j''k''} \\ &= (v_{i'j'} - \Gamma_{i'j'k'}) - (v_{i'j''} - \Gamma_{i'j''k''}) - (\Gamma_{i'j''k''} - r_{j''k''}) + (\Gamma_{i''j''k''} - r_{j''k''}) \\ &= v_{i'j'} - p_{j'} - v_{i'j''} + p_{j''} - p_{j''} + r_{j''k''} + p_{j''} - r_{j''k''} \\ &= v_{i'j'} - p_{j'} - (v_{i'j''} - p_{j''}). \end{aligned}$$

By (34), $u_{i'} + w_{k''} - (\widehat{u}_{i'} + \widehat{w}_{k''}) < 0$, a contradiction with (32). Assume now that for all $i'' \in 2B$ and all $k'' \in 2S$, $A_{i''j''k''} = 0$. By definition, $p_{j''} = \min\{r_{j''k} \mid k \text{ is such that } q_{j''k} > 0\}$. Let $k^* \in 2S$ be such that $q_{j''k^*} > 0$ and $p_{j''} = r_{j''k^*}$. By (2), such k^* does exist. Consider the coalition $C = \{i', k^*\}$ and its SW$-$compatible assignment $\widehat{A}$ where, for all $(i, j, k) \in B \times G \times S$,

$$\widehat{A}_{ijk} = \begin{cases} A_{ijk} - 1 & \text{if } i = i', j = j' \text{ and } k = k' \\ 1 & \text{if } i = i', j = j'' \text{ and } k = k^* \\ A_{ijk} & \text{otherwise.} \end{cases}$$

Then, proceeding as before, define $\widehat{u}_{i'}$ and $\widehat{w}_{k^*}$ as the payoffs of buyer i' and seller k^* at assignment $\widehat{A}$, respectively. Then, By Claim 1 and the definition of $\widehat{A}$,

$$\begin{aligned} u_{i'} + w_{k^*} - (\widehat{u}_{i'} + \widehat{w}_{k^*}) &= (v_{i'j'} - \Gamma_{i'j'k'}) \cdot A_{i'j'k'} - (v_{i'j'} - \Gamma_{i'j'k'}) \cdot \widehat{A}_{i'j'k'} \\ &\quad - (v_{i'j''} - \Gamma_{i'j''k^*}) \cdot \widehat{A}_{i'j''k^*} - (\Gamma_{i'j''k^*} - r_{j''k^*}) \cdot \widehat{A}_{i'j''k^*} \\ &= (v_{i'j'} - \Gamma_{i'j'k'}) - (v_{i'j''} - \Gamma_{i'j''k^*}) - (\Gamma_{i'j''k^*} - r_{j''k^*}) \\ &= v_{i'j'} - p_{j'} - (v_{i'j''} - r_{j''k^*}). \end{aligned}$$

By (32), $v_{i'j'} - p_{j'} \geq v_{i'j''} - r_{j''k^*}$. Since by its definition, $p_{j''} = r_{j''k^*}$, $v_{i'j'} - p_{j'} \geq v_{i'j''} - p_{j''}$, a contradiction with (34).

(E.S) For each good $j \in G$ and each seller $k \in S$, $\sum_i A_{ijk} \in S_{jk}(p_j)$.

Fix $j' \in G$ and $k' \in S$. Assume first that $p_{j'} < r_{j'k'}$. We want to show that $\sum_i A_{ij'k'} = 0$. Suppose that $A_{i'j'k'} \neq 0$. Consider the coalition $C = \{k'\}$ and its SW−compatible assignment $\widehat{A}$ where, for every $(i, j, k) \in B \times G \times S$,

$$\widehat{A}_{ijk} = \begin{cases} 0 & \text{if } i = i', j = j' \text{ and } k = k' \\ A_{ijk} & \text{otherwise.} \end{cases}$$

Define $\widehat{w}_{k'}$ as the utility of seller k' at assignment $\widehat{A}$. Then, it is immediate to see that $w_{k'} < \widehat{w}_{k'}$, contradicting (32).

Assume now that $p_{j'} = r_{j'k'}$. We want to show that $0 \leq \sum_i A_{ij'k'} \leq q_{j'k'}$. But this holds because A is a feasible assignment.

Finally, assume that $p_{j'} > r_{j'k'}$. We want to show that $\sum_i A_{ij'k'} = q_{j'k'}$. Assume $\sum_i A_{ij'k'} < q_{j'k'}$. Hence,

$$q_{j'k'} > 0. \tag{35}$$

Consider first the case where there exist $i' \in 2B$ and $k'' \in 2S$ such that $A_{i'j'k''} \neq 0$. Then, by definition of p and Claim 2, $p_{j'} = \Gamma_{i'j'k''} = \Gamma_{i'j'k'}$. Consider the coalition $C = \{i', k'\}$ and its SW−compatible assignment $\widehat{A}$ where, for all $(i, j, k) \in B \times G \times S$,

$$\widehat{A}_{ijk} = \begin{cases} A_{ijk} - 1 & \text{if } i = i', j = j' \text{ and } k = k'' \\ A_{ijk} + 1 & \text{if } i = i', j = j' \text{ and } k = k' \\ A_{ijk} & \text{otherwise.} \end{cases}$$

Then, proceeding in a similar way as we did in the proof of Claim 2, define $\widehat{u}_{i'}$ and $\widehat{w}_{k'}$ as the payoffs of buyer i' and seller k' at assignment $\widehat{A}$, respectively. Then, by Claim 1 and the definition of $\widehat{A}$,

$$\begin{aligned} u_{i'} + w_{k'} - (\widehat{u}_{i'} + \widehat{w}_{k'}) &= (v_{i'j'} - \Gamma_{i'j'k''}) \cdot A_{i'j'k''} + (v_{i'j'} - \Gamma_{i'j'k'}) \cdot A_{i'j'k'} \\ &\quad + (\Gamma_{i'j'k'} - r_{j'k'}) \cdot A_{i'j'k'} \\ &\quad - (v_{i'j'} - \Gamma_{i'j'k''}) \cdot \widehat{A}_{i'j'k''} - (v_{i'j'} - \Gamma_{i'j'k'}) \cdot \widehat{A}_{i'j'k'} \\ &\quad - (\Gamma_{i'j'k'} - r_{j'k'}) \cdot \widehat{A}_{i'j'k'} \\ &= (v_{i'j'} - \Gamma_{i'j'k''}) - (v_{i'j'} - \Gamma_{i'j'k'}) \\ &\quad - (\Gamma_{i'j'k'} - r_{j'k'}) \\ &= -p_{j'} + r_{j'k'}. \end{aligned}$$

Since by assumption $p_{j'} > r_{j'k'}$, $u_{i'} + w_{k'} - (\widehat{u}_{i'} + \widehat{w}_{k'}) < 0$, a contradiction with (32). Assume now that for all $i' \in 2B$ and all $k'' \in 2S$, $A_{i'j'k''} = 0$. By definition, $p_{j'} = \min\{r_{j'k} \mid k$ is such that $q_{j'k} > 0\}$. Let $k^* \in 2S$ be such that $q_{j'k^*} > 0$ and $p_{j'} = r_{j'k^*}$. By (2), such k^* does exist. By (35) and the definition of $p_{j'}$, $p_{j'} \leq r_{j'k'}$, a contradiction with the initial assumption that $p_{j'} > r_{j'k'}$. ■

Theorem 3 *Let $\rho \in \mathbb{Z}_+ \backslash \{0\}$. Then, there exist a market M and a payoff vector $(u, w) \notin \mathcal{CE}$ such that $(u^\rho, w^\rho) \in \mathcal{C}^{\rho M}$.*

Proof of Theorem 3 Fix $\rho \in \mathbb{Z}_+ \backslash \{0\}$. Define M as follows: $B = \{b_1\}$, $S = \{s_1, s_2\}$, $G = \{g_1\}$, $v_{11} = 1$, $r_{11} = r_{12} = 0$, $d_1 = 4\rho - 1$ and $q_{11} = q_{12} = 2\rho$. It is easy to see that since the short side of the market is the demand, the unique competitive equilibrium price is $\widetilde{p}_1 = 0$ and $\mathcal{CE} = \{(4\rho - 1, 0, 0)\}$. Consider the payoff vector $(4\rho - 3, 1, 1) \notin \mathcal{CE}$. We show that $((4\rho - 3)^\rho, (1, 1)^\rho) \in \mathcal{C}^{\rho M}$. Let C be a coalition in market ρM with $\#(\bigcup_{\alpha=1}^{\rho} B_\alpha^C) = \beta$ and $\#(\bigcup_{\alpha=1}^{\rho} S_\alpha^C) = \sigma$. Thus,

$$\beta \leq \rho \text{ and } \sigma \leq \rho/2. \tag{36}$$

The value of coalition C is

$$v(C) = \begin{cases} \beta(4\rho - 1) & \text{if } 2\beta \leq \sigma \\ 2\rho\sigma & \text{if } 2\beta > \sigma \end{cases} \tag{37}$$

and

$$\sum_{i \in B^C} u_i + \sum_{k \in S^C} w_k = \beta(4\rho - 3) + \sigma. \tag{38}$$

We want to show that for all β and σ satisfying (36),

$$\sum_{i \in B^C} u_i + \sum_{k \in S^C} w_k \geq v(C). \tag{39}$$

Assume first that C is such that $2\beta \leq \sigma$. Then, by (37) and (38), (39) holds if and only if $\beta(4\rho - 3) + \sigma \geq \beta(4\rho - 1)$ holds, which follows from $2\beta \leq \sigma$.

Assume now that C is such that $2\beta > \sigma$. Then, by (37) and (38), (39) holds if and only if $\beta(4\rho - 3) + \sigma \geq 2\rho\sigma$ holds. Thus, to show that (39) holds is equivalent to show that

$$\beta(4\rho - 3) \geq \sigma(2\rho - 1) \tag{40}$$

holds. By (36), the most disfavourable case for which (40) holds is when σ is larger; *i.e.*, $\sigma = 2\gamma - 1$. Hence, (40) follows if $\beta(4\rho - 3) \geq (2\beta - 1)(2\rho - 1)$. But this last inequality can be written as

$$4\beta\rho - 3\beta \geq 4\beta\rho - 2\beta - 2\rho + 1,$$

which holds because $\beta \leq \rho$ and $\rho \geq 1$ imply $2\rho - 1 \geq \beta$. ■