Invoking a Cartesian Product Structure on Social States: New Resolutions of Sen's and Gibbard's Impossibility Theorems

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Abstract

The purpose of this article is to introduce a Cartesian product structure into the social choice theoretical framework and to examine if new possibility results to Gibbard's and Sen's paradoxes can be developed thanks to it. We believe that a Cartesian product structure is a pertinent way to describe individual rights in the social choice theory since it discriminates the personal features comprised in each social state. First we define some conceptual and formal tools related to the Cartesian product structure. We then apply these notions to Gibbard's paradox and to Sen's impossibility of a Paretian liberal. Finally we compare the advantages of our approach to other solutions proposed in the literature for both impossibility theorems.

1 Introduction

Could society forbid me to read a book considered as perverse if I wanted to? Or, on the contrary, could society compel me to read it if I did not want to? What values should society invoke to force me to paint the walls of my room in the same color as my neighbor's?

These questions refer to some impossibility theorems developed in the social choice theoretical framework devised by Arrow [1], such as the liberal paradox (Sen [24], [25]) or Gibbard's result [8]. Since Sen's seminal article,

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there has been an extensive literature concerning the treatment of individual rights in economics. According to Mill [16] and Hayek [14], the existence of a personal protected sphere in which individuals are free to do what they want seems to be acknowledged. In 1970, Sen introduced this concept into the social choice theoretical framework with a condition of liberalism based on the notion of decisiveness: individuals must be decisive – their preferences must be acknowledged by society – over some pairs of social states, which belong to their private sphere. Sen shows that this condition of liberalism and a weak Pareto principle lead to an impossibility of social choice: it is the impossibility of a Paretian liberal. But Sen's formal analysis does not need to distinguish between decisive pairs that enable an individual to take decisions that are "personal" to her and those that are not. Formally, Sen makes no distinction between a man deciding whether to sleep in a prone or supine position, and a religious leader dictating whether he does so. Gibbard [8] investigates this issue: he uses a Cartesian product structure to describe individual rights and points out the internal inconsistency caused by an extended condition of liberalism. This result is called Gibbard's paradox or Gibbard's First Libertarian Claim. Besides, Gibbard shows that his paradox arises only if individuals express *conditional* preferences. In other words, an individual expresses conditional preferences if her preferences depend on those of another individual. For example, Connie is said to have conditional preferences if her desire is to wear a dress of the same color as Anita's. On the contrary, if Anita's desire is to differentiate from Connie, it leads to Gibbard's paradox. Gibbard stresses that his paradox does not arise if *unconditional* preferences only are acknowledged by society. It is Gibbard's Second Libertarian Claim, which solves the problem caused by the internal inconsistency of individual rights. However the liberal paradox still occurs with unconditional preferences.

This topic gave rise to many debates and attempts to develop new tools to take individual rights into account and to solve Gibbard's and Sen's paradoxes. The purpose of this article is to introduce a Cartesian product structure on social states and to examine if new possibility results can be developed. We believe that the Cartesian product structure is a pertinent way to describe individual rights in the social choice theory since it discriminates personal features comprised in each social state. Consequently, the concept of personal protected sphere is clarified. Although some authors already introduced a Cartesian product structure on social states (see especially Gibbard [8], Hammond [10], [11], [12], Coughlin [4], Pétron-Brunel and Salles [18]), this has never been applied in a thorough way to Gibbard's result and to Sen's paradox.

But a Cartesian product structure is inadequate in itself in order to deal with both impossibility results. It is necessary to determine a relevant way to take into account the implementation of these individual rights thus clarified. Indeed a formal right granted by society to an individual does not imply a real right since other individuals are able to prevent her from exercising it, i.e., other individuals can express “invasive” preferences towards her protected sphere. We propose two definitions so that individual rights can be guaranteed. The first definition aims to distinguish individuals’ “strong” and “light” preferences between two options: if an individual prefers red to blue and blue to white, her preference between red and white is said strong. Hence, if she prefers her neighbor’s living-room to be painted in red whereas her neighbor wishes to paint it in white, she has an invasive preference into her neighbor’s protected sphere. In other words, an individual expresses an invasive preference if she has a strong preference over two social states, which belong to the private sphere of another individual, and if that individual has the inverse preference. This statement of invasive preferences corresponds to our second definition. Finally, some possibility results for Gibbard’s and Sen’s theorems are designed by preventing one or several individuals from having invasive preferences. Even if possibility results, which follow the same line of reasoning have been already proposed (see Sen [26] or Pétron-Brunel [17]), ours extend the domains devised by these authors and require weaker constraints on individual preferences thanks to the Cartesian product structure.

The organisation of the rest of the paper is as follows. In the second section we define some conceptual and formal tools related to the Cartesian product structure. In the third section we apply these notions to Gibbard’s paradox. The fourth section is devoted to Sen’s impossibility of a Paretian liberal. The fifth section provides a discussion of the advantages of our approach compared to other solutions proposed in the literature for both impossibility theorems. The sixth section concludes.

2 Some conceptual and formal tools related to the Cartesian product structure

Let $N = \{1, 2, ..., n\}$ be the finite set of individuals[1], which constitutes society ($n \geq 2$). With a Cartesian product structure on social states, each individual is concerned about a set X of personal features, this set being the same for all individuals. X is a finite set, where $|X| \geq 2$. A social state is a n-list $(x_1, x_2, ..., x_n)$ of personal features of the world, where $x_i \in X, \forall\, i \in N$. The

[1]The mathematical concept of n-list can then be called upon. See Salles [23].

set of all social states X^n is given by

$$X^n = \underbrace{X \times X \times ... \times X}_{n-fold}.$$

Each individual $i \in N$ has a binary relation $\succeq_i$ on X^n, which is a linear ordering. A $n-$list of individual preferences $(\succeq_1, \succeq_2, ..., \succeq_n)$ is designed by d. A collective choice rule f specifies a social preference relation for each d: $\succeq = f(d)$. If $\succeq$ is a complete pre-ordering for all d in the domain, f is a "social welfare function" (SWF) in the sense of Arrow [1]. A weaker requirement is that $\succeq$ must be complete and acyclic for all d in the domain[2]. In this case, f is called a "social decision function" or SDF (see Sen [25], p. 52).

Thanks to the Cartesian product structure on social states, we can now describe individual rights. Let us just define a few more notations: for any $i \in N$ and any $x = (x_1, ..., x_{i-1}, x_i, x_{i+1}, ..., x_n) \in X^n$, $x_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$, where $x_{-i} \in X^n_{-i}$. If $x_i \in X$ and $a_{-i} = (a_1, ..., a_{i-1}, a_{i+1}, ..., a_n) \in X^n_{-i}$, then $(x_i; a_{-i}) = (a_1, ..., a_{i-1}, x_i, a_{i+1}, ..., a_n)$. The personal sphere of individual i is the family of sets $\{D_i(a_{-i})\}_{a_{-i} \in X^n_{-i}}$ where $D_i(a_{-i})$ is defined as follows:

$$D_i(a_{-i}) = \{x \in X^n \mid x_{-i} = a_{-i}\}.$$

Two social states belong to a set of individual i's personal sphere when they differ only in the personal feature of this individual.

As stated in our introduction, the difficulty we face with the problem of individual rights in the social choice theory is less analytical than conceptual. Consequently, it is crucial to find out first which values could be wished by the members of society and how they can be guaranteed.

According to Mill ([16], pp. 4-5, emphasis added): "protection (...) against the tyranny of the magistrate is not enough: there needs protection also against the *tyranny of the prevailing opinion and feeling*; *against the tendency of society to impose, by other means than civil penalties, its own ideas and practices as rules of conduct on those who dissent from them* (...). *There is a limit to the legitimate interference of collective opinion with individual independence*: and to find that limit, and maintain it against encroachment, is as indispensable to a good condition of human affairs, as protection against political despotism".

The "limit to the legitimate interference of collective opinion" Mill describes is the personal or private sphere of an individual. Inside it, an individual is free to decide. Sugden ([28], p. 229) adopts the same point of view: "it is perfectly consistent to claim, as Mill did, that every individual

[2] A binary relation $\succeq$ on X^n is acyclic if $x \succeq z$ follows $x \succ y$ and $y \succ z$ for any $x, y, z \in X^n$.

is entitled to an area of life to be controlled by himself or herself alone – however meddlesome or fashion-conscious or jealous or otherwise unworthy people's preferences may be". Society should not give a formal right to an individual for this does not match a real right of action: "if a person's religious meditation is made impossible through loud and disturbing noises made by others (...), his or her liberty is violated, even though this violation does not take the form of *prohibiting* the mediater from choosing his or her own acts or strategies" (Sen [27], p. 142). Hence it is essential to determine a way to achieve effective rights. As stated by Hausman and McPherson ([13], p. 125): "rights typically involve both 'privileges' for the right-holder and correlative duties for others" or Gaertner, Pattanaik and Suzumura ([7], p. 173): "every active right of [individual] i implies the freedom of i (...) from a certain set of options or actions; and i's choice of one of these options, in its turn, implies obligations of certain other agents to do or not to do something". Obviously, effective rights can be achieved by constraining individual preferences.

To summarize: each individual has a private sphere in which she is free to act. However some invasive preferences of other individuals should be excluded so that her formal rights can be real. All authors acknowledge the fundamental necessity of protecting individual rights. Therefore our resolution of Sen's and Gibbard's impossibility theorems takes this basic requirement into account.

Nevertheless, one issue is still unsolved: how to define invasive preferences? In order to do so, two definitions are developed below.

The first one deals with the status of an individual preference between two social states: it could be strong or light. Some authors such as Blau [3], Saari ([19], [21]), Pétron-Brunel [17], and Saari and Pétron-Brunel [22] already inquired about this topic, i.e., how to introduce more information into individual preferences. The idea is the following: when listing a binary ranking coming from a transitive ranking, also specify if other alternatives separate two social states. If there is any, this individual preference between these social states is called a strong preference. If there is none, the preference is said light. Actually, our definition establishes how to evaluate the status of an individual preference between two social states.

Definition 1 ***Strong and light preferences*** *For any $x, y \in X^n$, for any $j \in N$, if $x \succ_j y$ and if there exists at least one $z \in X^n$ such that $x \succ_j z$ and $z \succ_j y$, then individual j strongly prefers x to y: it will be denoted by $T[x \succ_j y] = S$. If $x \succ_j y$ but if such a social state z does not exist, individual j lightly prefers x to y: it is denoted by $T[x \succ_j y] = L$.*

For example, let us consider the following individual linear ordering:

$X^n = \{x, y, z, w\}$ and $x \succ_j w \succ_j z \succ_j y$. By transitivity, $x \succ_j y$. We then obtain $T[x \succ_j y] = S$. However if $x \succ_j y \succ_j w \succ_j z$, $T[x \succ_j y] = L$.

Our second definition aims to characterize individual invasive preferences. In fact, we now have at hand all conceptual and formal tools to deal with this issue. An invasive preference arises if an individual j goes against a preference of another individual i in her personal sphere; in other words, if two options, x and y, belong to the personal sphere of i and if j strongly prefers x to y whereas i prefers y to x. Let us characterize the set of options coming from j's invasive preferences. It is denoted by Y_j and called "set of invasive options". Let individual j be fixed. Consider two options x and y and suppose that $x \succ_j y$ with $T[x \succ_j y] = S$. Suppose moreover that x and y belong to another individual i's personal sphere, i.e., they belong to some $D_i(a_{-i})$, and that $y \succ_i x$. Then $y \in Y_j(a_{-i})$ where $Y_j(a_{-i})$ is a subset of Y_j which is composed of all options y coming from an invasive preference of individual j on a given $D_i(a_{-i})$. Hence, for an individual j, the set Y_j is the union of all subsets $Y_j(a_{-i})$ over every $a_{-i} \in X^n_{-i}$ and over every $i \neq j$[3].

Definition 2 ***Set of invasive options*** *For a given d, the set Y_j is composed of all social states for which the individual j has a preference which goes against a preference of another individual $i \neq j$ in her personal sphere:*

$$Y_j(a_{-i}) = \left\{ \begin{array}{c} y \in D_i(a_{-i}) \mid T[x \succ_j y] = S \textit{ for at least one } x \in D_i(a_{-i}) \\ \textit{such that } y \succ_i x \end{array} \right\}$$

$$\textit{and } Y_j = \bigcup_{i \neq j} \bigcup_{a_{-i} \in X^n_{-i}} Y_j(a_{-i}).$$

For example, consider two individuals 1 and 2 and $X = \{r, w\}$. Thus, $X^n = \{(r, w), (w, r), (r, r), (w, w)\}$. Suppose that individual 1 has the following linear ordering: $(r, r) \succ_1 (w, r) \succ_1 (w, w) \succ_1 (r, w)$. Suppose moreover that $(r, w) \succ_2 (r, r)$ and $(w, w) \succ_2 (w, r)$. Hence, since $(r, w), (r, r) \in D_2(r)$ and $T[(r, r) \succ_1 (r, w)] = S$, $Y_1(r) = \{(r, w)\}$. And since $(w, w), (w, r) \in D_2(w)$ and $T[(w, r) \succ_1 (w, w)] = L$, $Y_1(w) = \emptyset$. Finally, $Y_1 = \{(r, w)\}$. But, if $(r, r) \succ_1 (w, r) \succ_1 (r, w) \succ_1 (w, w)$, all other things remaining equal, then $Y_1 = \{(r, w), (w, w)\}$ since $T[(w, r) \succ_1 (w, w)] = S$.

Sen's and Gibbard's theorems can now be presented and some possibility results be proposed thanks to the exclusion of invasive preferences.

[3] Note that it would be possible as well to gather in a set all pairs of social states (x, y) coming from such a preference. But, in both cases, our results would be the same.

3 A solution to Gibbard's paradox

First of all, let us describe briefly Gibbard's result [8]. It is based on an extended interpretation of the concept of personal sphere: every individual should be decisive over all pairs of social states, which differ only in her personal feature. Gibbard suggests the following condition:

Condition 1 (GL) ***First Libertarian Claim*** *For any* $x, y \in X^n$*, for any* $i \in N$*, for any* $a_{-i} \in X^n_{-i}$*, if* $x, y \in D_i(a_{-i})$ *and* $x \succ_i y$*, then* $x \succ y$*.*

Moreover, the collective choice rule f should respect the condition of unrestricted domain:

Condition 2 (U) ***Unrestricted domain*** *The domain of* f *includes all logically possible* n*-lists of individual linear orderings.*

The extended condition of liberalism proposed by Gibbard [8] and the condition of unrestricted domain bring about an internal inconsistency of individual rights: there is no SDF satisfying conditions U and GL.

Thanks to the conceptual and formal tools elaborated above, a possibility result for Gibbard's theorem can be developed. More precisely, a condition constraining the number of individuals with an empty set of invasive options is defined:

Condition 3 (PM1) ***Preference Modification 1*** $Y_k = \emptyset, \forall\, k \in K$ *where* $K \subseteq N$ *and* $|K| \geq n - 1$*.*

It should be understood that condition PM1 is a restriction of condition U: it requires that at least $(n-1)$ individuals of society should have an empty set of invasive options. The following proposition can then be stated:

Proposition 1 *There exists a SDF satisfying conditions PM1 and GL.*

Proof The theorem is proved by constructing a SDF, which gives each person i an appropriate special voice on her feature. Let Q be the relation between x and y,

$$(\exists i)[x, y \in D_i(a_{-i}) \text{ and } x \succ_i y].$$

Let $\succeq = f(d)$ be generated from Q in the following manner:

$$\forall\, x, y \in X^n : x \succeq y \Longleftrightarrow \neg(yQx).$$

Firstly, we prove that whenever yQx, $y \succ x$. Suppose that $\neg(y \succ x)$. Hence $x \succeq y$ and from the construction of f, we have $\neg(yQx)$. Then, from $\neg(y \succ x)$, it followed that $\neg(yQx)$; therefore, if yQx, we obtain $y \succ x$ as asserted. Secondly, we show that f satisfies GL. By the construction of Q, if $x, y \in D_i(a_{-i})$ and $x \succ_i y$, then xQy. Therefore, $x \succ y$, and hence f satisfies GL.

Next, we check that f is really a SDF, i.e., f is complete and acyclic. Since Q is an asymmetric relation, f is necessarily complete. It remains to be demonstrated that f is acyclic – that a cycle of the social preference relation never occurs. Suppose there is a cycle $\mathcal{O}$,

$$x^1 Q x^2, ..., x^{\tau-1} Q x^{\tau}, x^{\tau} Q x^1$$

where $x^1, ..., x^{\tau}$ belong to X^n (for the subscripts, we shall use mod τ arithmetic, so that $1 - 1 = \tau$ and $\tau + 1 = 1$). Since individual preferences are transitive, a single individual cannot bring about a cycle through condition GL. Then at least two individuals lead to cycle $\mathcal{O}$. Moreover, they are both responsible for at least two steps of the social preference relation so that cycle $\mathcal{O}$ can exist.

Now, consider the individual, which is responsible for the step $x^1 Q x^2$ and call her individual j. Then, $x^1 \succ_j x^2$. Individual j is necessarily responsible for another step of the cycle $x^{\iota-1} Q x^{\iota}$, with $\iota = 4, ..., \tau$ so that cycle $\mathcal{O}$ can exist. Hence, $x^{\iota-1} \succ_j x^{\iota}$.

Suppose that $x^1 \succ_j x^{\iota-1}$. From x^{ι} to x^{τ}, steps originating from individuals $i \neq j$ or from individual j follow each other. In every case, we cannot obtain $x^{\tau} \succ_j x^{\iota-1}$ so that Y_j can be empty. We necessarily have $x^{\iota-1} \succ_j x^{\tau}$. Therefore, the step $x^{\tau} Q x^1$ necessarily comes from an individual $i \neq j$. Hence, $T[x^1 \succ_j x^{\tau}] = S$ and Y_j is nonempty. If $x^{\iota-1} \succ_j x^1$, we can prove that Y_j is nonempty according to the same line of reasoning.

Finally, we showed that if individual j is involved in cycle $\mathcal{O}$, her set of invasive options is nonempty. But the same conclusion remains for any individual involved in such a cycle. This violates the stipulation that at least $(n - 1)$ individuals of society should have an empty set of invasive options and hence, cycle $\mathcal{O}$ cannot occur. The theorem is proved. ■

We discuss this result in section 5.

4 A solution to Sen's paradox

The liberal paradox results from the incompatibility of three conditions that constrain the collective choice rule: the unrestricted domain condition, the weak Pareto condition, and a condition of liberalism. Since Gibbard's paradox [8], the Second Libertarian Claim or a self-consistent rights system is used as the third condition of Sen's theorem (see, for example, Farrell [5], Sen [26], Suzumura [30], or Pétron-Brunel [17]). This claim requires individuals to express unconditional preferences in order to take their rights into account within the social preference relation. Before defining this condition, a few more notations have to be stated: let $x_i p_i y_i$ mean that the feature x_i is unconditionally preferred to y_i by individual i. It is defined as follows: $(x_i; a_{-i}) \succ_i (y_i; a_{-i})$ for all $a_{-i} \in X^n_{-i}$.

Condition 4 (GL') ***Second Libertarian Claim*** *For any $x, y \in X^n$, for any $i \in N$, if $x, y \in D_i(a_{-i})$, $x \succ_i y$ and $x_i p_i y_i$, then $x \succ y$.*

The weak Pareto condition used by Arrow [1] in his impossibility thereom is as follows:

Condition 5 (P) ***Weak Pareto*** *For any $x, y \in X^n$, if $x \succ_i y$ for all $i \in N$, then $x \succ y$.*

The three conditions give rise to the liberal paradox: there is no SDF satisfying conditions U, P, and GL'.

A resolution of the liberal paradox is proposed thanks to the set of invasive options. Indeed, as for Gibbard's result, a condition establishing how many individuals have to exhibit an empty set of invasive options is devised:

Condition 6 (PM2) ***Preference Modification 2*** *$\exists\, j \in N$ such that $Y_j = \emptyset$.*

Like condition PM1, condition PM2 is a restriction of condition U and states that at least one individual should have an empty set of invasive options.

Proposition 2 *There exists a SDF satisfying conditions PM2, P, and GL'.*

Proof Let Q be the relation between x and y,

$$(\exists j)\ x, y \in D_j(a_{-j}), x \succ_j y \text{ and } x_j p_j y_j \text{ or } (\forall\, i)\ x \succ_i y.$$

As in the proof of proposition 1, let $\succeq = f(d)$ be generated by Q in the following manner:

$$\forall\, x, y \in X^n : x \succeq y \Longleftrightarrow \neg(yQx).$$

From the way Q is defined, it is obvious that f satisfies conditions P and GL'. Next, we check that f is really a SDF, i.e., f is complete and acyclic. Since Q is an asymmetric relation, f is necessarily complete. It remains to be shown that f is acyclic – that a cycle of the social preference relation never occurs. Suppose there is a cycle $\mathcal{O}$,

$$x^1 Q x^2, ..., x^{\tau-1} Q x^{\tau}, x^{\tau} Q x^1$$

where $x^1, ..., x^\tau$ belong to X^n (for the subscripts, we shall use mod τ arithmetic, so that $1-1=\tau$ and $\tau+1=1$. Variables ι and κ will range from 1 to τ). First, let us consider how cycle $\mathcal{O}$ arises. Each step of this cycle originates either from condition P or from condition GL'. It should be realized that the cycle cannot be provoked either by condition P alone since individual preferences are transitive, or by condition GL' alone because it implies a self-consistent rights system: therefore steps proceeding from both conditions are necessary so that the cycle can exist. In addition, at least two steps of the cycle originate from condition GL' for two different individuals. In fact, if only one step of the cycle comes from condition GL', the individual that is responsible for it expresses cyclic individual preferences, since all other steps of the cycle come from condition P. Hence at least two steps originating from condition GL' for two distinct individuals and one step originating from condition P lead to such a cycle. Now, consider $x^{\iota-1}Qx^{\iota}$. We get either $(\forall\, i)$ $x^{\iota-1} \succ_i x^{\iota}$ or $\neg(\forall\, i)$ $x^{\iota-1} \succ_i x^{\iota}$ and $(\exists\, j)$ $x^{\iota-1}, x^{\iota} \in D_j(a_{-j})$, $x^{\iota-1} \succ_j x^{\iota}$ and $x^{\iota-1}p_j x^{\iota}$.

The next part of the proof comprises two stages: first we show that if an individual is responsible for at least one step of the cycle because of condition GL', she has a nonempty set of invasive options (1). We then show that if an individual is involved only in steps of the cycle originating from condition P, she has a nonempty set of invasive options as well (2).

(1) We consider two individuals j and l: each of them is responsible for a step of cycle $\mathcal{O}$ proceeding from condition GL'. Then there is a ι such that $x^{\iota-1}, x^{\iota} \in D_j(a_{-j})$, $x^{\iota-1} \succ_j x^{\iota}$ and $x^{\iota-1}p_j x^{\iota}$ and such that $(\forall\, i)$ $x^{\iota} \succ_i x^{\iota+1}$. Hence we get: $x^{\iota-1} \succ_j x^{\iota}$ and $x^{\iota} \succ_j x^{\iota+1}$. In addition, individual l is involved in the cycle as well. There is a κ such that $x^{\kappa}, x^{\kappa+1} \in D_l(a_{-l})$, $x^{\kappa} \succ_l x^{\kappa+1.}$ and $x^{\kappa}p_l x^{\kappa+1}$. Suppose that $\kappa+1=\iota-1$, in other words, the step $x^{\kappa}Qx^{\iota-1}$ originates from condition GL'. It should be noted that this step can proceed from condition P, but this does not modify our proof: there is somewhere in the cycle a step originating from condition GL' and from an individual different from j. For individual j, we have either $x^{\kappa} \succ_j x^{\iota-1}$ or $x^{\iota-1} \succ_j x^{\kappa}$ with $T[x^{\iota-1} \succ_j x^{\kappa}] = L$ so that Y_j can be empty. In both cases, $T[x^{\kappa} \succ_j x^{\iota+1}] = S$. Then, in cycle $\mathcal{O}$, from $x^{\iota+1}$ to x^{κ}, steps necessarily come from condition GL'. Hence, the set Y_j is nonempty since j necessarily expresses at least one strong preference against a preference of another individual's protected sphere in the following subpart of cycle $\mathcal{O}$: $x^{\iota+1}Qx^{\iota+2}, ..., x^{\kappa}Qx^{\iota-1}$.

(2) The second stage of the proof requires to rely on the Cartesian product structure. Suppose that an individual m is involved in cycle $\mathcal{O}$ only in steps originating from condition P. According to (1), we get: $x^{\iota} \succ_m x^{\iota+1}$. For individual m, we could have $x^{\iota} \succ_m x^{\iota-1}$ and $T[x^{\iota} \succ_m x^{\iota-1}] = L$, $x^{\iota-1} \succ_m x^{\kappa}$ and $T[x^{\iota-1} \succ_m x^{\kappa}] = L$, et cetera, if all steps from $x^{\iota+1}$ to x^{κ} proceed from

condition GL'. In every other cases, Y_m is nonempty. In order to complete this proof, we show that m's above preferences necessarily imply a non-empty set Y_m. For individual j, recall that $x^{\iota-1} \succ_j x^{\iota}$, $x^{\iota-1}, x^{\iota} \in D_j(a_{-j})$ and $x^{\iota-1} p_j x^{\iota}$. For individual l, $x^{\kappa} \succ_l x^{\iota-1}$, $x^{\kappa}, x^{\iota-1} \in D_l(a_{-l})$ and $x^{\kappa} p_l x^{\iota-1}$. Let $x^{\iota-1} = (x^1, ..., x^j, ..., x^l, ..., x^n)$, $x^{\iota} = (x^1, ..., x^{j^*}, ..., x^l, ..., x^n)$ and $x^{\kappa} = (x^1, ..., x^j, ..., x^{l^*}, ..., x^n)$. Since individuals j and l have to express unconditional preferences, we obtain $(x^1, ..., x^j, ..., x^{l^*}, ..., x^n) \succ_j (x^1, ..., x^{j^*}, ..., x^{l^*}, ..., x^n)$ and $(x^1, ..., x^{j^*}, ..., x^{l^*}, ..., x^n) \succ_l (x^1, ..., x^{j^*}, ..., x^l, ..., x^n)$. But $(x^1, ..., x^j, ..., x^{l^*}, ..., x^n) = x^{\kappa}$ and $(x^1, ..., x^{j^*}, ..., x^l, ..., z_n) = x^{\iota}$. Let x^* be the social state $(x^1, ..., x^{j^*}, ..., x^{l^*}, ..., x^n)$. Then individual m has to respect the preferences of individual j in her personal sphere, $x^{\iota-1} \succ_j x^{\iota}$ and $x^{\kappa} \succ_j x^*$, and the preferences of individual l in her personal sphere, $x^{\kappa} \succ_l x^{\iota-1}$ and $x^* \succ_l x^{\iota}$. According to m's above preferences, Y_m is nonempty since individual m necessarily expresses at least one strong preference against a preference of individuals j or l in their protected sphere.

Hence statements (1) and (2) violate condition PM2: cycles cannot occur. The theorem is proved. ■

5 Why does the Cartesian product structure matter?

In the preceding sections, some possibility results for Gibbard's and Sen's theorems have been established. It is then important to determine if these results make it possible to alleviate some drawbacks brought about by other resolutions proposed in the literature.

A chief advantage of conditions PM1 and PM2 is that these conditions allow to solve Gibbard's and Sen's results in an almost identical way. In fact, both conditions are based on the same definitions and differ only in the number of individuals that should have an empty set of invasive options. Hence our possibility results emphasize the similar causes that give rise to these theorems.

Furthermore, conditions PM1 and PM2 involve in themselves some advantages as well. As stated in the introduction, the standard solution used to solve Gibbard's theorem is to forbid conditional preferences (condition GL'). According to Gibbard's [8], there exists a SDF satisfying conditions U and GL'. The ideas included in conditions PM1 and GL' are different: condition GL' prevents individuals from expressing conditional preferences in their protected sphere, whereas condition PM1 allows individuals to decide freely in their personal sphere, but compels them to respect individual rights to decide freely too. Hence condition PM1 implies a reciprocity extended to all members of society: *I have the right to act freely in my private sphere and I admit that others have the right to do the same.* In fact, the treatment of individual rights requires to consider interactional issues. Thus we believe that condition PM1 is more suitable than condition GL' to solve Gibbard's theorem.

Moreover, condition PM1 examines how to apply a liberal collective choice rule in a new light. We can define a liberal collective choice rule as that which gives each person i an appropriate special voice on her feature and implies a SDF. For instance, the proofs of our proposition 1 and Gibbard's theorem 3 (Gibbard [8], pp. 395-397) use this rule. Thus condition PM1 determines a domain different from that of unconditional preferences in which the liberal rule can be applied. Therefore, the set of individual preferences according to which all individuals express unconditional preferences is not the maximal domain of the liberal collective choice rule. But it should be stressed that the union of both domains – unconditional preferences' and PM1's – is not the maximal domain of this rule either. We propose a simple example to illustrate this point: consider two individuals 1 and 2 and $X = \{r, w\}$. Thus, $X^n = \{(r, w), (w, r), (r, r), (w, w)\}$. Suppose that individuals 1 and 2 have the following linear orderings: $(r, r) \succ_1 (w, w) \succ_1 (r, w) \succ_1 (w, r)$ and $(w, r) \succ_2 (w, w) \succ_2 (r, r) \succ_2 (r, w)$. Hence, it is easy to check that this configuration of individual preferences is neither in unconditional preferences' domain since individual 1 does not express unconditional preferences, nor in PM1's domain since both individuals have a strong preference which goes against a preference of the other in her personal sphere. And yet no cycle occurs. Consequently, condition PM1 shows that the liberal collective choice rule can be applied outside Gibbard's domain and thus it can be considered as an extension of Gibbard's result.

We consider next condition PM2. Most studies in the literature propose to solve the liberal paradox by weakening either the weak Pareto condition (Farrell [5], Sen [26], Suzumura [29], Austin-Smith [2], Hammond [10], Coughlin [4], Saari [20], Pétron-Brunel [17]...), or the condition of liberalism (Gibbard [8], Blau [3], Gaertner and Krüger [6], Krüger and Gaertner [15], Wriglesworth [31]...). Obviously, the results we developed conflict with Gibbard's [8] and Blau's [3], which suggest to weaken the condition of liberalism. Our view is entirely different: we support the respect of individual rights. It is therefore crucial to emphasize whether our results are more pertinent than the former studies, which propose to weaken the weak Pareto condition, especially Sen's [26] and Pétron-Brunel's [17].

In order to achieve a comparison between these propositions, the analytical concepts developed by Sen [26] and Pétron-Brunel [17] should be restated in our framework, i.e., with a Cartesian product structure and with invasive preferences expressed via a specific set.

Definition 3 ***Set of invasive options à la Sen*** *For a given d, the set Y_j^{Sen} is composed of all social states for which the individual j has a preference which is different from a preference of another individual $i \neq j$ in her personal sphere:*

$$Y_j^{Sen}(a_{-i}) = \left\{ \begin{array}{c} y \in D_i(a_{-i}) \mid x \succ_j y \text{ for at least one } x \in D_i(a_{-i}) \\ \text{such that } y \succ_i x \end{array} \right\}$$

$$\text{and } Y_j^{Sen} = \bigcup_{i \neq j} \bigcup_{a_{-i} \in X_{-i}^n} Y_j^{Sen}(a_{-i}).$$

Condition 7 (PM^{Sen}) ***Preference Modification à la Sen*** *$\exists\ j \in N$ such that $Y_j^{Sen} = \emptyset$.*

We then obtain:

Proposition 3 (Restatement of Sen [26]) *There exists a SDF satisfying conditions PM^{Sen}, P, and GL'.*

According to this restatement, the difference between our proposition 2 and Sen's proposition is clarified: whereas both conditions PM2 and PM^{Sen} state that at least one individual has to present an empty set of invasive options, the ways these sets are defined by the two approaches are rather distinct. On the one hand, the set Y_j^{Sen} is composed of all social states for which an individual j has a preference which *is different* from a preference of another individual in her personal sphere. On the other hand, our set of invasive options is composed of all social states for which the individual j has a preference which *goes against* a preference of another individual in her personal sphere. There are many more social states in the set Y_j^{Sen} than in our set since we only consider inverse strong preferences. Consequently, condition PM^{Sen} is much more demanding than condition PM2: PM$^{Sen} \Longrightarrow$ PM2.

We then consider Pétron-Brunel's [17]. The restatement of the definition of the rank-difference between two social states proposed by this author fits roughly our definition of strong and light preferences. And the condition according to which the author constrains the domain of the collective choice rule states that all individuals in society should have an empty set of invasive options. We obtain:

Condition 8 (PM^{PB}) ***Preference Modification à la Pétron-Brunel*** *$Y_j^{PB} = \emptyset$ for all $j \in N$.*

Proposition 4 (Restatement of Pétron-Brunel [17]) *There exists a SDF satisfying conditions PM^{PB}, P, and GL'.*

It is obvious that PM^{PB} $\Longrightarrow$ PM2. Therefore condition PM2 seems to be a relevant alternative to the results proposed by Sen [26] and Pétron-Brunel [17] since it weakens the constraints they impose on individuals.

However one question remains: does this significant outcome come from the Cartesian product structure? To answer it briefly, let us consider the proof of proposition 2: in point (2), the Cartesian product structure is necessary in order to guarantee the existence of a SDF with condition PM2. Thus the Cartesian product structure enables us to go beyond the results developed by Sen [26] and Pétron-Brunel [17] since on the one hand it allows us to weaken the constraints imposed on individuals and on the other hand it offers a better representation of individual rights.

6 Concluding remarks

The aim of the article is to devise a reliable way of overcoming two impossibility results developed into a social choice theoretical framework thanks to a Cartesian product structure, which makes it possible to take individual rights into account properly. Some conceptual and formal tools are developed so that the private sphere can be protected from invasive preferences. According to definitions 1 and 2, two new conditions for modifying preferences, conditions PM1 and PM2, are designed. Consequently, original resolutions of Gibbard's and Sen's theorems are deduced.

The Cartesian product structure matters since it provides improved solutions for Gibbard's and Sen's results. First of all, the possibility results for both theorems require the same conceptual and formal tools; thus the similar causes of these theorems are stressed. Secondly, the solution proposed to Gibbard's paradox thanks to condition PM1 emphasizes the reciprocity issue present in this paradox in a more convincing way than the Second Libertarian Claim does. Moreover, it extends Gibbard's result by proposing another domain in which the liberal collective choice rule can be applied. Thirdly, the resolution of the liberal paradox, which uses condition PM2, improves significantly Sen's [26] and Pétron-Brunel's [17] solutions.

References

[1] Arrow KJ (1950) A difficulty in the concept of welfare. J Polit Econ 58: 328-346

[2] Austen-Smith D (1982) Restricted Pareto and Rights. J Econ Theory 26: 89-99

[3] Blau JH (1975) Liberal Values and Independence. R Econ Stud 42: 395-401

[4] Coughlin PJ (1986) Rights and the Private Pareto Principle. Economica 55: 303-320

[5] Farrell MJ (1976) Liberalism in the Theory of Social Choice. R Econ Stud 43: 3-10

[6] Gaertner W and Krüger L (1981) Self-Supporting Preferences and Individual Rights: The Possibility of Paretian Libertarianism. Economica 48: 17-28

[7] Gaertner W, Pattanaik PK and Suzumura K (1992) Individual Rights Revisited. Economica 59: 161-177

[8] Gibbard A (1974) A Pareto-Consistent Libertarian Claim. J Econ Theory 7: 388-410

[9] Gibbard A (1982) Rights and the Theory of Social Choice. In: Cohen LJ, Los J, Pfeiffer H and Podewski KP (eds) Logic, Methodology and Philosophy of Science, vol. 6, North-Holland, Amsterdam, pp 595-605

[10] Hammond PJ (1982) Liberalism, Independent Rights and the Pareto Principle. In: Cohen LJ, Los J, Pfeiffer H and Podewski KP (eds) Logic, Methodology and Philosophy of Science, vol. 6, North-Holland, Amsterdam, pp 607-620

[11] Hammond PJ (1995) Social choice of individual and group rights. In: Barnett WA, Moulin H, Salles M and Schofield NJ (eds), Social choice, welfare, and ethics, Cambridge University Press, Cambridge, pp 55-77

[12] Hammond PJ (1998) Rigths, free exchange and widespread externalities. In: Laslier JF, Fleurbaey M, Gravel N and Trannoy A (eds), Freedom in Economics: New Perspectives in Normative Analysis, Routledge, London, pp 139-157

[13] Hausman DM and McPherson MS (1996) Economic analysis and moral philosophy. Cambridge University Press, Cambridge

[14] Hayek FA (1960) The Constitution of Liberty. Routledge, London

[15] Krüger L and Gaertner W (1983) Alternative Libertarian Claims and Sen's Paradox. Theory Decision 15: 211-229

[16] Mill JS (1859) On Liberty. Appleton-Century-Crofts, New York, 1947

[17] Pétron-Brunel A (1998) Contribution à l'analyse des droits en théorie du choix social. PhD Dissertation, Université de Caen – Basse-Normandie

[18] Pétron-Brunel A and Salles M (1998) Interpretative, Semantic and Formal Difficulties of the Social Choice Rule Approach to Rights. In: Laslier JF, Fleurbaey M, Gravel N and Trannoy A (eds), Freedom in Economics: New Perspectives in Normative Analysis, Routledge, London, pp 101-111

[19] Saari DG (1995) Inner Consistency or not inner Consistency: A reformulation is the answer. In: Barnett WA, Moulin H, Salles M and Schofield NJ (eds), Social choice, welfare, and ethics, Cambridge University Press, Cambridge, pp 187-212

[20] Saari DG (1998) Connecting and resolving Sen's and Arrow's theorems. Soc Choice and Welfare 15: 239-261

[21] Saari DG (2001) Decisions and elections. Explaining the unexpected. Cambridge University Press, Cambridge

[22] Saari DG and Pétron-Brunel A (2006) Negative externalities and Sen's liberalism theorem. Econ Theory 28: 265-281

[23] Salles M (2001) Agrégation des Préférences. In: Jessua C, Labrousse C and Vitry D (eds), Dictionnaire des Sciences Economiques, Presses Universitaires de France, Paris

[24] Sen AK (1970a) The Impossibility of a Paretian Liberal. J Polit Econ 78: 152-157

[25] Sen AK (1970b) Collective Choice and Social Welfare. Holden-Day, San Francisco

[26] Sen AK (1976) Liberty, Unanimity and Rights. Economica 43: 217-245

[27] Sen AK (1992) Minimal Liberty. Economica 59: 139-159

[28] Sugden R (1985) Liberty, Preference, and Choice. Econ Philos 1: 213-229

[29] Suzumura K (1978) On the Consistency of Libertarian Claims. R Econ Stud 45: 329-342

[30] Suzumura K (1980) Liberal Paradox and the Voluntary Exchange of Rights-Exercising. J Econ Theory 22: 407-420

[31] Wriglesworth DL (1985) Respecting Individual Rights in Social Choice. Oxford Econ Papers 37: 100-117