Corrigendum to "On Randomized Matching Mechanisms" [Economic Theory 8(1996)377-381]*

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Summary: Ma (1996) studied the random order mechanism, a matching mechanism suggested by Roth and Vande Vate (1990) for marriage markets. By means of an example he showed that the random order mechanism does not always reach all stable matchings. Although Ma's (1996) result is true, we show that the probability distribution he presented – and therefore the proof of his Claim 2 – is not correct. The mistake in the calculations by Ma (1996) is due to the fact that even though the example looks very symmetric, some of the calculations are not as "symmetric." *JEL classification:* C78

For a description of the marriage model we refer to Roth and Vande Vate (1990). A marriage market is denoted by (M, W, P) where $M = \{m_1, \ldots, m_a\}$ is a set of "men," $W = \{w_1, \ldots, w_a\}$ is a set of "women," and P is a preference profile. The set of stable matchings for (M, W, P) is denoted by $S(P)$. We now recall the random order mechanism.

Random Order (RO) Mechanism

Input: A marriage market (M, W, P).
Set $R_0 := \emptyset$, μ_0 such that for all $i \in N$, $\mu_0(i) = i$, and $t := 1$.

Step *t*: Choose an agent i_t from $N \backslash R_{t-1}$ at random. Set $R_t := R_{t-1} \cup \{i_t\}$.
Suppose $i_t = w \in W$. (Otherwise replace w by m in Step t.)

Stable Room Procedure

Case (i) There exists no blocking pair (m, w) for μ_{t-1} with $m \in R_t$:
Stop if $t = n$ and define $RO(P) := \mu_{t-1}$. Otherwise set $\mu_t = \mu_{t-1}$ and go to Step $t := t + 1$.

Case (ii) There exists a blocking pair (m, w) for μ_{t-1} with $m \in R_t$:
Choose the blocking pair (m^*, w) for μ_{t-1} with $m^* \in R_t$ that w prefers most.
If $\mu_{t-1}(m^*) = m^*$, then define μ_t such that $\mu_t(w) := m^*$, $\mu_t(m^*) := w$, and for all $i \in N \backslash \{w, m^*\}$, $\mu_t(i) := \mu_{t-1}(i)$. Stop if $t = n$ and define $RO(P) := \mu_t$. Otherwise go to Step $t := t + 1$.
If $\mu_{t-1}(m^*) = w' \in W$, then redefine $\mu_{t-1}(w) := m^*$, $\mu_{t-1}(m^*) := w$, $\mu_{t-1}(w') := w'$, and for all $i \in N \backslash \{w, m^*, w'\}$, $\mu_{t-1}(i) := \mu_{t-1}(i)$. Set $w := w'$, and repeat the Stable Room Procedure.

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The algorithm ends in exactly $n := 2a$ steps and its outcome is a random stable matching $RO(P)$, generated by a sequence of agents $(i_1, \ldots, i_n)$. The set of possible sequences of agents equals the set of permutations of all agents denoted by Q. Hence, $|Q| = n!$. Moreover, for any $\mu \in S(P)$, let $Q_\mu \subseteq Q$ be the (possibly empty) set of sequences that lead to μ. Denote $q_\mu = |Q_\mu|$. The random order mechanism induces in a natural way a probability distribution $\mathcal{P}$ over the set of stable matchings: for any $\mu \in S(P)$, the probability that $RO(P) = \mu$ equals $p_\mu = \frac{q_\mu}{n!}$.

By using the following example, Ma (1996) showed that the random order mechanism may not reach all stable matchings. Although Ma's (1996) theorem is true, we show that the probability distribution he presented – and therefore the proof of his Claim 2 – is not correct.

Knuth's (1976) Example. Let (M, W, P) with $a = 4$ and P given below.

Preferences						Stable Matchings				
$P(m_1) =$	w_1	w_2	w_3	w_4	m_1	$\mu_1 =$	m_1	m_2	m_3	m_4
$P(m_2) =$	w_2	w_1	w_4	w_3	m_2	$\mu_2 =$	m_2	m_1	m_3	m_4
$P(m_3) =$	w_3	w_4	w_1	w_2	m_3	$\mu_3 =$	m_1	m_2	m_4	m_3
$P(m_4) =$	w_4	w_3	w_2	w_1	m_4	$\mu_4 =$	m_2	m_1	m_4	m_3
$P(w_1) =$	m_4	m_3	m_2	m_1	w_1	$\mu_5 =$	m_3	m_1	m_4	m_2
$P(w_2) =$	m_3	m_4	m_1	m_2	w_2	$\mu_6 =$	m_2	m_4	m_1	m_3
$P(w_3) =$	m_2	m_1	m_4	m_3	w_3	$\mu_7 =$	m_3	m_4	m_1	m_2
$P(w_4) =$	m_1	m_2	m_3	m_4	w_4	$\mu_8 =$	m_4	m_3	m_1	m_2
						$\mu_9 =$	m_3	m_4	m_2	m_1
						$\mu_{10} =$	m_4	m_3	m_2	m_1

Ma (1996) claimed that $(p_{\mu_1}, p_{\mu_2}, p_{\mu_3}, p_{\mu_4}, p_{\mu_5}, p_{\mu_6}, p_{\mu_7}, p_{\mu_8}, p_{\mu_9}, p_{\mu_{10}}) = (\frac{1}{4}, \frac{1}{8}, \frac{1}{8}, 0, 0, 0, 0, \frac{1}{8}, \frac{1}{8}, \frac{1}{4})$. Below we show that this is not true by calculating the correct probability distribution: $(\frac{9600}{40320}, \frac{5280}{40320}, \frac{5280}{40320}, 0, 0, 0, 0, \frac{5280}{40320}, \frac{5280}{40320}, \frac{9600}{40320})$. The mistake in the calculations is due to the fact that even though the example looks very symmetric, some of the calculations are not as "symmetric," in other words Ma's (1996) statement on page 380 that "the proofs for the remaining cases are similar." is not correct. In spite of the mistake, our computation is still based on Ma's (1996) idea of analyzing sequences of agents backwards, *i.e.*, considering the last agent that enters, subsequently the last but one agent that enters, *etc.*. The difference is that we provide more detailed discussions and justify the (restricted) use of "symmetry." $\diamond$

Proof: We show that $(p_{\mu_1}, p_{\mu_2}, p_{\mu_3}, p_{\mu_4}, p_{\mu_5}, p_{\mu_6}, p_{\mu_7}, p_{\mu_8}, p_{\mu_9}, p_{\mu_{10}}) = (\frac{9600}{40320}, \frac{5280}{40320}, \frac{5280}{40320}, 0, 0, 0, 0, \frac{5280}{40320}, \frac{5280}{40320}, \frac{9600}{40320})$ by checking which stable matchings the random order mechanism induces for various sequences $(i_1, \ldots, i_8)$. Whenever we refer to a unique stable matching obtained for a marriage market not containing all agents, we calculated the man-optimal and the woman-optimal matching for the "submarket" using the deferred acceptance algorithm and detected that they coincide (this calculation is not included in the proof). Furthermore, whenever we "satisfy" a blocking pair, the (unique) proposing agent does not propose to agents that are better than his/her previous match (all these proposals would be rejected).

Case a: m_1 enters last; *i.e.*, the sequence of agents is $(i_1, \ldots, m_1)$. There are only two stable matchings μ' and μ'' when the set of agents consists of all women W and the remaining three men $\{m_2, m_3, m_4\}$:

$$\mu' : \begin{array}{cccc} w_1 & w_2 & w_3 & w_4 \\ | & | & | & | \\ w_1 & m_2 & m_3 & m_4 \end{array} \qquad \mu'' : \begin{array}{cccc} w_1 & w_2 & w_3 & w_4 \\ | & | & | & | \\ w_1 & m_2 & m_4 & m_3 \end{array}$$

When m_1 enters last, he proposes to the single woman w_1, who accepts. So, matching μ' implies matching μ_1 and matching μ'' implies μ_3.

Case a.1: m_2 enters before m_1; *i.e.*, the sequence is $(i_1, \ldots, m_2, m_1)$.

Case a.1.1: m_3 enters before m_2 and m_1; *i.e.*, the sequence is $(i_1, \ldots, m_3, m_2, m_1)$. The unique stable matching before m_3, m_2, and m_1 enter matches m_4 to w_4 and everybody else to themselves. Next, when m_3 enters he proposes to w_3, who accepts. Similarly, when m_2 enters he proposes to w_2, who accepts. Thus, w_1 is single and the resulting matching is μ'. Hence, all 5! sequences induce μ_1.

Case a.1.2: m_4 enters before m_2 and m_1; *i.e.*, the sequence is $(i_1, \ldots, m_4, m_2, m_1)$. Similarly as in *Case a.1.1*, all 5! sequences induce μ_1.

Case a.1.3: w_1 enters before m_2 and m_1; *i.e.*, the sequence is $(i_1, \ldots, w_1, m_2, m_1)$. There are only two stable matchings $\tilde{\mu}'$ and $\tilde{\mu}''$ before w_1, m_2, and m_1 enter:

$$\tilde{\mu}' : \begin{array}{ccc} w_2 & w_3 & w_4 \\ | & | & | \\ w_2 & m_3 & m_4 \end{array} \qquad\qquad \tilde{\mu}'' : \begin{array}{ccc} w_2 & w_3 & w_4 \\ | & | & | \\ w_2 & m_4 & m_3 \end{array}$$

It is easy to check that half of the partial sequences $(i_1, \ldots, i_5)$ with $\{i_1, \ldots, i_5\} \cap \{w_1, m_2, m_1\} = \emptyset$ result in $\tilde{\mu}'$, the other half in $\tilde{\mu}''$: if $[i_5 \in \{m_3, m_4\}]$ then $(i_1, \ldots, i_5)$ results in $\tilde{\mu}'$, if $[i_5 \in \{w_3, w_4\}]$ then $(i_1, \ldots, i_5)$ results in $\tilde{\mu}''$, if $[i_4 \in \{m_3, m_4\}$ and $i_5 = w_2]$ then $(i_1, \ldots, i_5)$ results in $\tilde{\mu}'$, and if $[i_4 \in \{w_3, w_4\}$ and $i_5 = w_2]$ then $(i_1, \ldots, i_5)$ results in $\tilde{\mu}''$. After agents w_1, m_2, and m_1 enter, $\tilde{\mu}'$ induces μ'. Similarly, $\tilde{\mu}''$ induces μ''. Hence, $\frac{5!}{2}$ sequences induce μ_1 and $\frac{5!}{2}$ sequences induce μ_3.

Case a.1.4: w_2 enters before m_2 and m_1; *i.e.*, the sequence is $(i_1, \ldots, w_2, m_2, m_1)$. Similarly as in *Case a.1.3*, $\frac{5!}{2}$ sequences induce μ_3 and $\frac{5!}{2}$ sequences induce μ_1.

Case a.1.5: w_3 enters before m_2 and m_1; *i.e.*, the sequence is $(i_1, \ldots, w_3, m_2, m_1)$. The unique matching before agents w_3, m_2, and m_1 enter matches m_3 to w_4, m_4 to w_2, and w_1 to herself. When w_3 enters she proposes to m_4, who accepts. Now w_2 is single. Next, when m_2 enters he proposes to w_2, who accepts. Thus, w_1 is single and the resulting matching is μ''. Hence, all 5! sequences induce μ_3.

Case a.1.6: w_4 enters before m_2 and m_1; *i.e.*, the sequence is $(i_1, \ldots, w_4, m_2, m_1)$. Similarly as in *Case a.1.5*, all 5! sequences induce μ_3.

Summary Case a.1: 360 sequences $(i_1, \ldots, m_2, m_1)$ induce μ_1 and 360 sequences $(i_1, \ldots, m_2, m_1)$ induce μ_3.

Summary Cases a.2-a.7: In a similar way as in Case a.1 we can calculate the number of sequences that induce μ_1 and μ_3, respectively, in case the last but one position is occupied by an agent different from m_2. We summarize the results in the table below.

Summary Case a: By summing up the boldface numbers in the table below we see that 2400 sequences $(i_1, \ldots, m_1)$ induce μ_1 and 2640 sequences $(i_1, \ldots, m_1)$ induce μ_3.

Case	Sequences	Inducing μ_1	Inducing μ_3
a.1	$(\boldsymbol{i_1}, \ldots, \boldsymbol{m_2}, \boldsymbol{m_1})$	**360**	**360**
a.2	$(\boldsymbol{i_1}, \ldots, \boldsymbol{m_3}, \boldsymbol{m_1})$	**720**	**–**
a.3	$(\boldsymbol{i_1}, \ldots, \boldsymbol{m_4}, \boldsymbol{m_1})$	**720**	**–**
a.4.1	$(i_1, \ldots, m_2, w_1, m_1)$	60	60
a.4.2	$(i_1, \ldots, m_3, w_1, m_1)$	120	–
a.4.3	$(i_1, \ldots, m_4, w_1, m_1)$	120	–
a.4.4	$(i_1, \ldots, w_2, w_1, m_1)$	–	120
a.4.5	$(i_1, \ldots, w_3, w_1, m_1)$	–	120
a.4.6	$(i_1, \ldots, w_4, w_1, m_1)$	–	120
a.4	$(\boldsymbol{i_1}, \ldots, \boldsymbol{w_1}, \boldsymbol{m_1})$	**300**	**420**
a.5.1	$(i_1, \ldots, m_2, w_2, m_1)$	60	60
a.5.2	$(i_1, \ldots, m_3, w_2, m_1)$	120	–
a.5.3	$(i_1, \ldots, m_4, w_2, m_1)$	120	–
a.5.4	$(i_1, \ldots, w_1, w_2, m_1)$	–	120
a.5.5	$(i_1, \ldots, w_3, w_2, m_1)$	–	120
a.5.6	$(i_1, \ldots, w_4, w_2, m_1)$	–	120
a.5	$(\boldsymbol{i_1}, \ldots, \boldsymbol{w_2}, \boldsymbol{m_1})$	**300**	**420**
a.6	$(\boldsymbol{i_1}, \ldots, \boldsymbol{w_3}, \boldsymbol{m_1})$	**–**	**720**
a.7	$(\boldsymbol{i_1}, \ldots, \boldsymbol{w_4}, \boldsymbol{m_1})$	**–**	**720**

Case b: m_2 enters last; *i.e.*, the sequence is $(i_1, \ldots, m_2)$.

Because of the symmetry of the preferences, by changing the roles of agents [m_1 and m_2], [w_1 and w_2], [m_3 and m_4], and [w_3 and w_4] in the proof of *Case a* we can show that in *Case b* 2400 sequences $(i_1, \ldots, m_2)$ induce μ_1 and 2640 sequences $(i_1, \ldots, m_2)$ induce μ_3.

Case c: m_3 enters last; *i.e.*, the sequence is $(i_1, \ldots, m_3)$.

There are only two stable matchings $\hat{\mu}'$ and $\hat{\mu}''$ when the set of agents consists of all women W and the remaining three men $\{m_1, m_2, m_4\}$:

$$\hat{\mu}' : \begin{array}{cccc} w_1 & w_2 & w_3 & w_4 \\ | & | & | & | \\ m_1 & m_2 & w_3 & m_4 \end{array} \qquad \hat{\mu}'' : \begin{array}{cccc} w_1 & w_2 & w_3 & w_4 \\ | & | & | & | \\ m_2 & m_1 & w_3 & m_4 \end{array}$$

When m_3 enters last, he proposes to the single woman w_3, who accepts. So, matching $\hat{\mu}'$ implies matching μ_1 and matching $\hat{\mu}''$ implies μ_2.

In order to determine which sequences induce matchings μ_1 and μ_2, we change the roles of agents [m_1 and m_3], [w_1 and w_3], [m_2 and m_4], and [w_2 and w_4] in the proof of *Case a*. Note that after this change, matching $\hat{\mu}'$ corresponds to μ' in the proof of *Case a* and matching $\hat{\mu}''$ corresponds to μ'' in the proof of *Case a*. Similarly, matching μ_1 corresponds to μ_1 in the proof of *Case a* and μ_2 corresponds to μ_3 in the proof of *Case a*.

Thus, changing the roles of the agents as specified above in the proof of *Case a* implies that in *Case c* 2400 sequences $(i_1, \ldots, m_3)$ induce μ_1 and 2640 sequences $(i_1, \ldots, m_3)$ induce μ_2.

Case d: m_4 enters last; *i.e.*, the sequence is $(i_1, \ldots, m_4)$.

Because of the symmetry of the preferences, by changing the roles of agents [m_3 and m_4], [w_3 and w_4], [m_1 and m_2], and [w_1 and w_2] in the proof of *Case c* we can show that in *Case d* 2400 sequences $(i_1, \ldots, m_4)$ induce μ_1 and 2640 sequences $(i_1, \ldots, m_4)$ induce μ_2.

Summary Cases a to d: Let $m \in M$. Then, 9600 sequences $(i_1, \ldots, m)$ induce μ_1, 5280 sequences $(i_1, \ldots, m)$ induce μ_2, and 5280 sequences $(i_1, \ldots, m)$ induce μ_3.

Let $w \in W$. Similarly to *Cases a* to *d*, 9600 sequences $(i_1, \ldots, w)$ induce μ_{10}, 5280 sequences $(i_1, \ldots, w)$ induce μ_9, and 5280 sequences $(i_1, \ldots, w)$ induce μ_8.

Finally, the probability distribution induced by the random order mechanism equals $(p_{\mu_1}, p_{\mu_2}, p_{\mu_3}, p_{\mu_4}, p_{\mu_5}, p_{\mu_6}, p_{\mu_7}, p_{\mu_8}, p_{\mu_9}, p_{\mu_{10}}) = (\frac{9600}{40320}, \frac{5280}{40320}, \frac{5280}{40320}, 0, 0, 0, 0, \frac{5280}{40320}, \frac{5280}{40320}, \frac{9600}{40320}) \neq (\frac{1}{4}, \frac{1}{8}, \frac{1}{8}, 0, 0, 0, 0, \frac{1}{8}, \frac{1}{8}, \frac{1}{4})$.[1] $\square$

References

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[1]For a version of the proof discussing all cases in detail please contact any of the authors.