Scoring Rules on Dichotomous Preferences

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August 9, 2004

Abstract

In this paper, we study individual incentives to report preferences truthfully for the special case when individuals have dichotomous preferences on the set of alternatives and preferences are aggregated in form of scoring rules. In particular, we show that (a) the Borda Count coincides with Approval Voting on the dichotomous preference domain, (b) the Borda Count is the only strategy-proof scoring rule on the dichotomous preference domain, and (c) if at least three individuals participate in the election, then the dichotomous preference domain is the unique maximal rich domain under which the Borda Count is strategy-proof.

Keywords: Approval Voting, Borda Count, Dichotomous Preferences, Social Choice Function.

JEL-Number: D71.

1 Introduction

The objective of this paper is to analyze the aggregation of preferences in form of positional voting methods or scoring rules when individuals have dichotomous preferences on the set of alternatives (there are just two indifference classes, the set of good alternatives and the set of bad alternatives). In particular, we are interested in strategy-proof scoring rules, that is we look for social choice functions belonging to the class of scoring rules that give individuals incentives to report preferences truthfully. Our main results are that the Borda Count

*I thank Jordi Massó for his supervision and his never-ending encouragement. Salvador Barberà, Dolors Berga, Carmen Beviá, Bhaskar Dutta, Lars Ehlers, Alejandro Neme, Shmuel Nitzan and Yves Sprumont helped me a lot with their comments. This research was undertaken with support from the fellowship 2001FI 00451 of the Generalitat de Catalunya and from the research grant BEC2002-02130 of the Ministerio de Ciencia y Tecnología of Spain.
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is equivalent to Approval Voting on the dichotomous preference domains and that the Borda Count is the unique non-manipulable scoring rule.

In a series of papers Saari and van Newenhizen [12] and [13] and Brams et al. [5] discuss the advantages and disadvantages of Approval Voting versus scoring rules in general and versus the Borda Count in particular. Saari and van Newenhizen [12] and [13] argue that Approval Voting is highly indeterminate for a lot of preference profiles (many different alternatives can be selected for a given preference profile) and suggest the Borda Count as an alternative to the widely established Plurality Rule. But according to Brams et al. [5] the indeterminacy of Approval Voting is rather a virtue, because it eliminates the voter's incentives not to vote sincerely whereas scoring rules are highly manipulable.[1]

One way how to contribute to this discussion is to compare scoring rules with Approval Voting on different preference domains.[2] But this task is generally not an easy one, because scoring rules and Approval Voting aggregate preferences in a rather different way. To see this consider at first the following definition of the Borda Count: Given a strict preference relation for some individual and a pair of alternatives, assign one point to the preferred alternative and zero points to the other alternative. Repeat this process for all possible pairs of alternatives in order to finish the point assignment process for this particular individual. Hence, the individual gives 0 points to her/his worst alternative, 1 point to her/his second worst alternative, and so on. The set of Borda Winners is defined to be the set of alternatives that obtain the highest amount of points after aggregating preferences over all individuals participating in the election.

The Borda Count is the most intuitive scoring rule, because, given a strict preference relation, the points assigned to a particular alternative, say x, is simply the number of alternatives that are worse than x according to the considered preference relation. For a generic scoring rule the point assignment process is

[1] Among others Saari [11], Dummett [7] and Smith [14] document the manipulability of scoring rules.

[2] See a recent work of Regenwetter and Tsetlin [10] who take an alternative approach by comparing Approval Voting and scoring rules according to an inference based model. They analyze the data from the Society of Social Choice and Welfare election and find strong similarities between Approval Voting and the Borda Count whenever the data allows for a solid statistical analysis.

more complicate: Given a strict preference relation, the worst alternative gets 0 points. Let $m \geq 1$ and suppose that the points have already been assigned to all of the m worst alternatives according to the considered preference relation. Then, the $(m+1)$'th worst alternative receives at least as many points as the m'th worst alternative (notice that the process is anonymous and neutral, that is the amount of points assigned to the $(m+1)$'th worst alternative is independent of the individual and the alternative). The last requirement to be met is that the top alternative gets strictly more points than the worst alternative, because otherwise one does not distinguish among alternatives and no meaningful decision could be taken. After summing the points up over all individuals society selects again the set of alternatives with the highest amount of points.

While scoring rules are social choice functions and thus take into account the whole preference structure, Approval Voting is a truncated voting rule that endows individuals with the right to vote for as many alternatives as they wish to and selects the alternatives with the largest support. Therefore, the level of information available about individual preferences is generally lower under Approval Voting. Yet, the indeterminacy of Approval Voting disappears when preferences are restricted to be dichotomous, because if we interpret the voting decisions as the set of good alternatives of the individuals, then it is possible to recover the preferences completely from the observed voting decisions. Hence, Approval Voting becomes a social choice function on the domain of dichotomous preferences. Since this is not true any more for larger preference domain, the dichotomous preference domain constitutes an ideal starting point for comparing scoring rules with Approval Voting.

So far scoring rules have only been defined for strict preference domains, and therefore, we still miss to generalize the point assignment process for situations when individuals have weak preferences. Perhaps the most natural way to do this is to assign to all alternatives belonging to the same indifference class the same amount of points. For the Borda Count this implies that, given a weak preference relation and a pair of alternatives, both alternative receive half a point whenever an individual is indifferent between the two alternatives. On the other hand, if the preferences between the two alternative are strict, then, as before, the preferred alternative gets the point (this extension is mentioned by Sarri and van Newenhizen [12]). Our contribution to the former discussion is to show that if the Borda Count is generalized to weak preferences as described above, then it is an affine transformation of Approval Voting on the domain of dichotomous preferences (Proposition 1). Thus, the two social choice functions are equivalent on this preference domain.

Until now we have justified the assumption of dichotomous preferences because of its technical suitability, but we can also identify voting environments where individual preferences on the set of alternatives are reasonably assumed to be dichotomous. Consider a group of individuals that has to select one alternative from a large set of alternatives by voting, for example a recruitment committee has to choose a candidate for a new position in an economic department. Rather than determining the winning alternative immediately, the decision process is very often such that a subset of alternatives is pre-selected and afterwards the winning alternative is chosen from the set of pre-selected alternatives. Since in the first step of the two-step decision process individuals have very few information about the characteristics of the alternatives and it has to be decided whether an alternative should be pre-selected or not, individual preferences on the set of alternatives are naturally assumed to be dichotomous. That is, an individual just determines whether an alternative should be pre-selected or not according to his opinion. Using the example mentioned above one can think of the set of pre-selected alternatives as the group of candidates that is invited to give a seminar. The impression obtained in the seminar and the personal interviews held afterwards help the committee members to refine and revise their preferences in order to identify the candidate, which fits best into the department. Our concern is to identify strategy-proof scoring rules that can be applied in the first step of the two-step decision process whenever individuals believe that all pre-selected alternatives have the same chance of being finally chosen. Thus, we do not model explicitly how individuals refine their dichotomous preferences and how the final decision is taken, rather we make a simplifying assumption on the individual beliefs.

Since scoring rules and Approval Voting are set-valued social choice functions (for some preference profiles more than one alternative is chosen) and individuals have preferences on the set of alternatives and not on the non-empty family of its subsets, one has to make assumptions on how individuals order non-empty subsets of alternatives in order to have a well-defined notion of strategy-proofness. The cohesive preference extension we propose is such that, given a weak preference relation and two non-empty subsets of alternatives, individuals strictly prefer the set of alternatives with the higher proportion of top alternatives. If preferences are dichotomous and individuals attach to all pre-selected alternatives the same chance of being finally chosen, then individuals behave according to the cohesive preference extension as if they were expected utility maximizers. This interpretation is the reason why we do not apply the weaker preference extension of Brams and Fishburn [4] who show in their seminal paper that Approval Voting is strategy-proof on their extended preference domain when the underlying preferences are dichotomous and that Approval Voting is equivalent to the Condorcet Rule on the domain of dichotomous preferences.

The results of Brams and Fishburn [4] together with the first result of our paper (the Borda Count and Approval Voting are equivalent on the domain of dichotomous preferences) open at least two possible lines of research. First, notice that three of the most well known aggregation rules coincide on the dichotomous preference domain, but so far only Fishburn [8] and [9] has characterized this rule by means of normative properties. This line of research is not further followed here, but the interested reader can find additional characterizations of Approval Voting on the domain of dichotomous preferences in an accompanying paper [15]. Second, if we want to exclude the possibility of manipulations in the pre-selection process, then one may wonder whether it is possible to apply other scoring rules as well. Proposition 2 states that a scoring rule is strategy-proof on the cohesive dichotomous domain if and only of it is the Borda Count. That is, the Borda Count is characterized among all scoring rules by means of strategy-proofness whenever preferences on alternatives are dichotomous. Since the whole class of strategy-proof social choice functions on the domain of dichotomous preferences is rather big, we need additional properties in order to characterize the Borda Count (Approval Voting) if we do not restrict the set of social choice functions. To our best knowledge the only two characterizations of Approval Voting on the domain of dichotomous preferences that incorporate the property of strategy-proofness are due to [8] and Vorsatz [15].

Although we have already identified voting environments where individuals have dichotomous preferences, it is reasonable to think that in many other situations preferences are going to be richer. The last result of the paper deals with the question of whether we can enlarge the underlying preference domain without losing strategy-proofness for Borda Count. Barbie et al. [1] study strategy-proof domains for the Borda Count under the assumptions that individual preferences are strict and ties are broken in a non-neutral way. Basically, they find that the Borda Count is non-manipulable on all domains which contain one fixed preference relation and all its cyclic permutation. Since these domains are rather small, their result confirms the common opinion that scoring rules are highly manipulable. Proposition 3 points into the same direction, because the dichotomous preference domain is the largest domain such that the Borda Count is strategy-proof on the cohesive preference extension whenever the number of individuals is greater than two.

The paper is organized as follows. In the next Section we introduce notation and some basic definitions. Afterwards, we present our results.

2 Notation and Definitions

Let N be a group of individuals with preferences on the set K of alternatives. The cardinalities of the two sets are finite and given by $n \geq 2$ and $k \geq 3$. We assume that $k \geq 3$, because otherwise all scoring rules are going to be equal to the Borda Count as it will become clear from the definitions later on. Elements of K are denoted by b, g, x, y and z, and elements of N by i, j and

l. Let R_i be the weak preference relation of individual i on K. The set of all weak preference relations on K is denoted by $\mathcal{R}$. A domain $\bar{\mathcal{R}}$ is a subset of $\mathcal{R}$. Given a domain $\bar{\mathcal{R}} \subseteq \mathcal{R}$, a preference profile $R = (R_1, ..., R_n) \in \bar{\mathcal{R}}^N$ is a vector of individual preference relations. To stress the role of individual i in the preference profile $R \in \bar{\mathcal{R}}^N$, we write $R = (R_i, R_{-i})$. The strict and the indifference preference relations associated with R_i are denoted by P_i and I_i, respectively. The preference relation R_i is dichotomous if it consists of up to two indifference classes, the set of good alternatives and the set of bad alternatives. Given $R_i \in \bar{\mathcal{R}}$, define the set of good alternatives associated with R_i as $G(R_i) = \{g \in K : gR_iy \text{ for all } y \in K\}$. Similarly, let $B(R_i) = \{b \in K : yR_ib \text{ for all } y \in K\}$ be the set of bad alternatives corresponding to R_i. The cardinalities of the two sets are given by $g(R_i)$ and $b(R_i)$. Hence, the preference relation $R_i \in \bar{\mathcal{R}}$ is *dichotomous* if and only if $G(R_i) \cup B(R_i) = K$. The domain of all dichotomous preferences is denoted by $\mathcal{D} \subset \mathcal{R}$. Let $D_i \in \mathcal{D}$ be the generic dichotomous preference relation for individual i. Finally, given a preference profile $D = (D_1, ..., D_n) \in \mathcal{D}^N$ and an alternative $x \in K$, let $N_x(D) = |\{i \in N : x \in G(D_i)\}|$ be the support for x at $D \in \mathcal{D}^N$.

A social choice function $f : \bar{\mathcal{R}}^N \to 2^K \backslash \{\emptyset\}$ selects for all preference profiles $R \in \bar{\mathcal{R}}^N$ a non-empty set of alternatives $f(R) \in 2^K \backslash \{\emptyset\}$.[3] Any social choice function belonging to the class of scoring rules can be represented by a vector $s = (s_0, s_1, \ldots, s_{k-1}) \in \mathbb{R}^k$ satisfying the conditions $s_{j-1} \leq s_j$ for all $j = 1, ..., k-1$ and $s_0 < s_{k-1}$. The range of s is normalized by assuming that $s_0 = 0$ and $s_{k-1} = k-1$. Scoring rules are typically applied to the domain of strict preferences $\mathcal{P}$ on K. In this case, points are assigned to every alternative in such a way that if alternative x is in the j'th position according to P_i, then alternative x receives $p_x^s(P_i) = s_{k-j}$ points from individual i. Given a preference

[3]Notice that scoring rules have multiple winning alternatives for certain preference profiles. One way to deal with this problem is to study non-anonymous or non-neutral tie breaking rules in order to assure a single-valued image. But instead of following this approach we leave the tie breaking rule unspecified and assume that individuals believe that all winning alternatives have the same chance of being finally selected. Thus, we interpret $f(R) \in 2^K \backslash \{\emptyset\}$ as the set of pre-selected alternatives. Finally, we exclude the empty set from the image, because every social choice function corresponding to a scoring rule has for any preference profile at least one winning alternative.

profile $P \in \mathcal{P}^N$ and an alternative $x \in K$, let $p_x^s(P) = \sum_{i=1}^n p_x^s(P_i)$ be the score of x at P when the generic scoring rule s is applied. Finally, society selects for all preference profiles the set of alternatives with the highest score.

However, if individual preferences are not strict, then the point assignment process has to be generalized. One possibility is to give to every alternative of the same indifference class the same amount of points, an extension which has already been mentioned for the Borda Count in [12]. Formally, this is done as follows: Let $C^1(R_i)$ be the set of top alternatives for individual i when her/his preference relation is $R_i \in \bar{\mathcal{R}}$. The cardinality of $C^1(R_i)$ is $c^1(R_i)$. Then every alternative $y \in C^1(R_i)$ receives $p_y^s(R_i) = \frac{1}{c^1(R_i)} \sum_{j=1}^{c^1(R_i)} s_{k-j}$ points from individual i. Let $m \geq 2$ and suppose that the points have already been assigned to all alternatives contained in the first $m-1$ indifference classes of R_i. Moreover, denote the cardinality of the set of all alternatives contained in R_i's first $m-1$ indifference classes by q^{m-1}. Let $C^m(R_i)$ be the set of alternatives belonging to the m'th indifference class of R_i. The cardinality of $C^m(R_i)$ is $c^m(R_i)$. Then every alternative $z \in C^m(R_i)$ gets $p_z^s(R_i) = \frac{1}{c^m(R_i)} \sum_{j=1}^{c^m(R_i)} s_{(k-q^{m-1}-j)}$ points from individual i. Given a preference profile $R \in \bar{\mathcal{R}}^N$ and an alternative $x \in K$, let $p_x^s(R) = \sum_{i=1}^n p_x^s(R_i)$ be the score of alternative x at R when the generic scoring rule s is applied. Now it is straightforward to define the social choice function corresponding to the scoring rule s for all weak preference domains.

Definition 1 The social choice function $f_s : \bar{\mathcal{R}}^N \to 2^K \backslash \{\emptyset\}$ associated to the scoring rule s is such that for all $R \in \bar{\mathcal{R}}^N$, $x \in f(R)$ if and only if $p_x^s(R) \geq p_y^s(R)$ for all $y \in K$.

The most well known scoring rule is the Borda Count. It is given by $s_j = j$ for all $j = 0, ..., k-1$. The social choice function corresponding to the Borda Count is denoted by f_B. With a slight abuse of notation we write $p_x(R_i)$ and $p_x(R)$ whenever the Borda Count is applied. Now, we repeat the intuition of the generalized point assignment process for the Borda Count: Given $R_i \in \bar{\mathcal{R}}$, compare alternative x with every alternative $y \in K \backslash \{x\}$. If xP_iy, then assign one point to x and zero points to y (give the point to y whenever yP_ix). If xI_iy, then split the point equally. The sum of the points alternative x obtains after performing all possible pair-wise comparisons is equal to $p_x(R_i)$.

Among others The Mathematical Association of America, The Econometric Society and The Society of Social Choice and Welfare apply Approval Voting [4] in their elections.[4] Its main novelty with respect to the Plurality Rule is that Approval Voting endows individuals with the right to vote for not just one but for as many alternatives as they wish to. That is, the mapping $M_i : \bar{\mathcal{R}} \to 2^K$ determines for all preference relations $R_i \in \bar{\mathcal{R}}$ the set of alternatives $M_i(R_i) \in 2^K$ individual i votes for and the Approval Voting function $v : M_1(R_1) \times \cdots \times M_n(R_n) \to 2^K \backslash \{\emptyset\}$ aggregates the individual voting decisions by selecting the alternatives with the highest number of votes. Hence, $x \in f(M_1(R_1), \ldots, M_n(R_n))$ if and only if $|\{i \in N : x \in M_i(R_i)\}| \geq |\{i \in N : y \in M_i(R_i)\}|$ for all $y \in K$. There are different probabilistic models making assumptions on how the mappings $(M_i)_{i \in N}$ look like in order to compare Approval Voting in expected terms to other social choice functions such as the Condorcet Rule or the Borda Count (for a discussion of these probabilistic models see [10]). But for the case of dichotomous preferences there is a simpler way how to do this. If the mappings $M_i : \mathcal{D} \to 2^K$ are defined such that for all $i \in N$ and for all $D_i \in \mathcal{D}$, $M_i(D_i) = G(D_i)$, then the voting decision reveals the individual preferences completely. Hence, Approval Voting can be defined as a social choice function on the dichotomous preference domain.

Definition 2 The social choice function $f : \mathcal{D}^N \to 2^K \backslash \{\emptyset\}$ is said to be *Approval Voting* if for all $D \in \mathcal{D}^N$, $x \in f(D)$ if and only if $N_x(D) \geq N_y(D)$ for all $y \in K$.

The social choice function corresponding to Approval Voting is denoted by f_A. If we consider the preference domain $\bar{\mathcal{R}} \supset \mathcal{D}$, then Approval Voting is not a social choice function any more, because, given the voting decision $M_i(R_i)$ for a particular preference relation $R_i \in \bar{\mathcal{R}}$ that consists of at least three indifference classes, we cannot recover the true preferences just by observing $M_i(R_i)$. To

[4] See a recent article of Brams and Fishburn [3] where the success of Approval Voting in different organizations is analyzed.

see this let the preference relation $R_i \in \mathcal{R}$ be such that $x \notin G(R_i) \cup B(R_i)$, $y \in G(R_i)$ and $z \in B(D_i)$. In this case, we cannot infer from $x \in M_i(R_i)$ that yP_ix. Similarly, if $x \notin M_i(R_i)$, then we cannot deduce that xP_iz.

3 Results

We have already seen that the dichotomous preference domain is a natural starting point for a comparison of Approval Voting and scoring rules, because it is the largest domain on which Approval Voting constitutes a well-defined social choice function. Proposition 1 states that the Borda Count is equivalent to Approval Voting on the dichotomous preference domain, because, given a preference profile and an alternative, the score of the alternative under the Borda Count is an affine transformation of the number of individuals who approve that alternative.

Proposition 1 *For all $D \in \mathcal{D}^N$, $f_B(D) = f_A(D)$.*

Proof: Suppose that i's preferences are represented by the dichotomous preference relation D_i and let the scoring rule s be such that for all $j = 0, ..., k-1$, $s_j = j$. We deduce from the equation $\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$ that every alternative $g \in G(D_i)$ receives $p_g(D_i) = \frac{\sum_{j=1}^{g(D_i)} k-j}{g(D_i)} = \frac{g(D_i)k - \sum_{j=1}^{g(D_i)} j}{g(D_i)} = \frac{2g(D_i)k - g(D_i)(g(D_i)+1)}{2g(D_i)} = \frac{2k-g(D_i)-1}{2}$ points from individual i. Similarly, this individual gives to all alternatives $b \in B(D_i)$, $p_b(D_i) = \frac{\sum_{j=1}^{b(D_i)} k-g(D_i)-j}{b(D_i)} = \frac{b(D_i)(k-g(D_i)) - \sum_{j=1}^{b(D_i)} j}{b(D_i)} = \frac{2b(D_i)(k-g(D_i)) - b(D_i)(b(D_i)+1)}{2b(D_i)} = \frac{k-g(D_i)-1}{2}$ points, where the last equation uses that $k - g(D_i) = b(D_i)$.

We complete the proof by showing that, given a preference profile $D \in \mathcal{D}^N$ and an alternative $x \in K$, the score $p_x(D)$ is an increasing function of $N_x(D)$.

This is done as follows,

$$\begin{aligned}
p_x(D) &= \sum_{i\in N:x\in G(D_i)} p_x(D_i) + \sum_{i\in N:x\notin G(D_i)} p_x(D_i) \\
&= \sum_{i\in N:x\in G(D_i)} \tfrac{2k-g(D_i)-1}{2} + \sum_{i\in N:x\notin G(D_i)} \tfrac{k-g(D_i)-1}{2} \\
&= \sum_{i\in N:x\in G(D_i)} \tfrac{k}{2} + \sum_{i\in N} \tfrac{k-g(D_i)-1}{2} \\
&= \tfrac{k}{2}N_x(D) + \tfrac{n}{2}(k-1) - \sum_{i\in N} \tfrac{g(D_i)}{2}.
\end{aligned}$$

Hence, for all $D \in \mathcal{D}^N$, $N_x(D) \geq N_y(D)$ for all $y \in K$ if and only if $p_x(D) \geq p_y(D)$ for all $y \in K$. ■

One aim of the literature on social choice theory is to study normative properties of social choice functions. Well known properties are for example the ones of anonymity (all individuals have the same voting power), neutrality (all alternatives are treated equally) and monotonicity (additional support does not hurt an alternative). Since individuals may try to vote strategically, we are especially interested in scoring rules that provide incentives to represent preferences truthfully. Usually individual incentives are modelled with the help of the notion of strategy-proofness, a property demanding that truth telling is a dominant strategy in the direct revelation game.[5] Yet, whether or not a social choice function is strategy-proof depends crucially on the preference domain on which it operates, because for more restricted preference domains individuals have less strategies, and, as a result, a social choice function is more likely to be strategy-proof the stronger the domain restriction.

So far we cannot define strategy-proofness, because individuals have preferences on the set of alternatives, but they have to compare non-empty subsets of alternatives as the image of a scoring rule is potentially set-valued. To deal with this kind of problem one has to make assumptions on how individuals order non-empty subsets of alternatives given an ordering on the set of alternatives. Brams and Fishburn [4] assume that the weak preference relation $\dot{\succsim}_{R_i}$

[5] In the direct revelation game individuals announce simultaneously and independently from each other their preferences, and therefore, the individual set of strategies is equal to the considered preference domain. Afterwards the announced preferences are aggregated according to a social choice function, which is known to all individuals. Seen from this mechanism design point of view strategy-proofness asks that a social choice function is implementable in dominant strategies.

on $2^K\backslash\{\emptyset\}$ corresponding to the preference relation $R_i \in \bar{\mathcal{R}}$ on K is as follows: (a) $\{x\}\dot{\succ}_{R_i}\{x,y\}\dot{\succ}_{R_i}\{y\}$ if $x \in G(R_i)$ and $y \in B(R_i)$; (b) for all $S,T \in 2^K\backslash\{\emptyset\}$, $S\dot{\succsim}_{R_i}T$ if $T \subseteq B(R_i)$ or $S \subseteq G(R_i)$ or $[S\backslash T \subseteq G(R_i)$ and $T\backslash S \subseteq B(R_i)]$.

The following example illustrates this preference extension when the underlying preferences on alternatives are dichotomous.

Example 1: If the preference relation $D_i \in \mathcal{D}$ is such that $G(D_i) = \{x,y\}$ and $B(D_i) = \{z\}$, then the preference relation $\dot{\succsim}_{D_i}$ satisfies the following set of conditions: First, $\{y\}\dot{\sim}_{D_i}\{x,y\}\dot{\sim}_{D_i}\{x\}\dot{\succ}_{D_i}\{x,z\}\dot{\succ}_{D_i}\{z\}$. Second, it has to be that $\{x,y\}\dot{\succsim}_{D_i}\{x,y,z\}\dot{\succsim}_{D_i}\{x,z\}$ (at least one of the two weak relations has to be strict), and finally, $\{x,y\}\dot{\succsim}_{D_i}\{x,y,z\}\dot{\succsim}_{D_i}\{y,z\}$ (again at least one weak relation has to be strict). Notice that the sets $\{x,z\}$ and $\{y,z\}$ remain unordered. ■

We propose instead a preference extension after which the set $S \in 2^K\backslash\{\emptyset\}$ is evaluated by calculating the percentage of good alternatives restricted to S.

Definition 3 The preference relation $\succsim_{R_i}$ on $2^K\backslash\{\emptyset\}$ is *cohesive with respect to* $R_i \in \bar{\mathcal{R}}$ whenever for all $S,T \in 2^K\backslash\{\emptyset\}$, $S \succsim_{R_i} T$ if and only if $\frac{|G(R_i)\cap S|}{|S|} \geq \frac{|G(R_i)\cap T|}{|T|}$ ($\succsim_{R_i}$ is strict whenever the inequality is strict).

The reason why we suggest the cohesive preference extension is two-fold. First, the cohesive extension can be rationalized in terms of expected utility maximization if preferences on alternatives are dichotomous. To see this suppose that the objective is to determine a unique winning alternative. Then, as it has already been outlined, we can interpret the set $f(D) \in 2^K\backslash\{\emptyset\}$ as the preliminary result of a two-step decision process. If the winning alternative is finally determined by a lottery with support on $f(D)$, then individuals care only about the probability that a good alternative is chosen. That is, the lottery with support on the set of pre-selected alternatives S is weakly preferred to the lottery with support on T if and only if the probability that a good alternative is selected is at least as high in the lottery with support on S as in the lottery with support on T. If, in addition, individuals attach to every pre-selected alternative the same probability of being finally chosen, then it is easy to see that the lottery with support on S is weakly preferred to the lottery with support on T if and only if the percentage of good alternatives is at least as high in S as in T, that is if and only if $S \succsim_{D_i} T$. Second, the cohesive preference extension provides a complete weak ordering of all non-empty subsets of alternatives as long as preferences on alternatives are dichotomous whereas the extension of Brams and Fishburn [4] is incomplete as it has been shown in Example 1. This is important, because otherwise the notion of strategy-proofness can become vague as it will become particularly clear in Example 3 at the end of the paper. The intuition is as follows: Consider a social choice function and suppose that, given the preference relations for all individuals but i, individual i can obtain the set S by telling the truth and the set T by declaring some other preference relation. If the sets S and T are incomparable, then, given this ambiguity, i cannot manipulate the social choice function at this preference profile, and therefore, it is more likely that a social choice function is strategy-proof.

If the preference relation $R_i \in \bar{\mathcal{R}}$ consists of more than two indifference classes, then the cohesive preference extension is not appropriated any more as it can be seen clearly in the next example. Nonetheless, we are going to discuss at the end of the paper that only Proposition 3 changes slightly if we apply the preference extension of Brams and Fishburn [4] instead.

Example 2: Let $K = \{x, y, z\}$ and suppose that the dichotomous preference relation D_i is such that $G(D_i) = \{x, y\}$ and $B(D_i) = \{z\}$. In this case, the cohesive preference relation $\succsim_{D_i}$ is equal to $\{x\} \sim_{D_i} \{y\} \sim_{D_i} \{x, y\} \succ_{D_i} \{x, y, z\} \succ_{D_i} \{x, z\} \sim_{D_i} \{y, z\} \succ_{D_i} \{z\}$. If the preference relation $R_i \in \mathcal{R}$ is prescribed by the sets $G(D_i) = \{x\}$ and $B(D_i) = \{z\}$, then $\{y, z\} \sim_{R_i} \{z\}$ and $\{x\} \succ_{R_i} \{x, y\}$. Hence, alternative y is treated as if it was bad. ∎

Now we are ready to introduce the notion of strategy-proofness according to which individuals do not have incentives to lie about their preferences independently of the reported preferences of the others. Obviously this requirement is rather strong, but its advantage is that strategy-proof social choice functions predict the outcome of the preference aggregation process regardless of the information individuals have about the preferences of others. Formally, the social choice function $f : \bar{\mathcal{R}}^N \to 2^K \backslash \{\emptyset\}$ is said to be *manipulable by i on $\bar{\mathcal{R}}^N$* if for some $R \in \bar{\mathcal{R}}^N$ and $R'_i \in \bar{\mathcal{R}}$, $f(R'_i, R_{-i}) \succ_{R_i} f(R)$.

Definition 4 The social choice function $f : \bar{\mathcal{R}}^N \to 2^K \backslash \{\emptyset\}$ is said to be *strategy-proof on the cohesive $\bar{\mathcal{R}}$ domain* if f is not manipulable by any individual on $\bar{\mathcal{R}}^N$.

Brams and Fishburn [4] show that if preferences on alternatives are dichotomous, then Approval Voting is strategy-proof on their extended preference domain. The incompleteness of their extended preference relation is not crucial for this result, because Approval Voting remains to be strategy-proof for any completion of their preference extension. To see why Approval Voting is strategy-proof on the cohesive dichotomous domain, recall the definition of Approval Voting after which all alternatives with the largest support are selected. If an individual misrepresents her/his preferences by approving some bad alternative and/or disapproving some good alternative, s/he either does not change the image or achieves to remove some good alternative from the image and/or add some bad alternative to the image. Hence, the misrepresentation cannot increase the percentage of good alternative in the set of pre-selected alternatives, and therefore, Approval Voting has to be strategy-proof on the cohesive dichotomous domain. Using this observation together with Proposition 1 we have argued that the Borda Count is strategy-proof on the cohesive dichotomous domain. Proposition 2 states that the Borda Count is the only strategy-proof scoring rule on the cohesive dichotomous domain.

Proposition 2 *The social choice function $f_s : \mathcal{D}^N \to 2^K \backslash \{\emptyset\}$ corresponding to the scoring rule s is strategy-proof on the cohesive dichotomous domain if and only if s is the Borda Count.*

Proof: It has already been argued that the Borda Count is strategy-proof on the cohesive dichotomous domain. To prove the other implication we construct a set of necessary conditions which any social choice function f_s must satisfy in order to be strategy-proof on the cohesive dichotomous domain and show afterwards that only the Borda Count meets these conditions.

Consider the preference profile $D \in \mathcal{D}^N$ which is as follows: If $n = 2$, then $D = (D_i, D_j)$, where D_i is such that $G(D_i) = \{x\}$ and D_j is prescribed by the set of good alternatives $G(D_j) = \{y\}$. If $n > 2$, then the preference relations for individual i and j are as described above and for all $l \neq i, j$, D_l is the dichotomous preference relation associated to the set of good alternatives $G(D_l) = \{x, y\}$. Then, given a scoring rule s, it has to be that $f_s(D) = \{x, y\}$. We analyze under which conditions individual i may not manipulate f_s at D via D'_i, where D'_i satisfies the conditions $g(D'_i) > 1$, $x \in G(D'_i)$ and $y \notin G(D'_i)$.

Let $m = g(D'_i) - 1$ be the difference in the cardinality of the set of good alternatives with respect to the preference relations D'_i and D_i. At $(D'_i, D_{-i}) \in \mathcal{D}^N$, the score of alternative x is equal to $p^s_x(D'_i, D_{-i}) = \frac{\sum_{j=k-m-1}^{k-1} s_j}{m+1} + \frac{\sum_{j=1}^{k-2} s_j}{k-1} + (n-2)\frac{s_{k-1}+s_{k-2}}{2}$, because $p^s_x(D'_i) = \frac{1}{m+1}\sum_{j=1}^{m+1} s_{k-j}$, $p^s_x(D_j) = \frac{1}{k-1}\sum_{j=1}^{k-1} s_{k-1-j}$ and $p^s_x(D_l) = \frac{1}{2}(s_{k-1} + s_{k-2})$ for all $l \neq i, j$. At the same preference profile the score of alternative y is equal to $p^s_y(D'_i, D_{-i}) = \frac{\sum_{j=1}^{k-m-2} s_j}{k-m-1} + s_{k-1} + (n-2)\frac{s_{k-1}+s_{k-2}}{2}$, because $p^s_y(D'_i) = \frac{1}{k-m-1}\sum_{j=1}^{k-m-1} s_{k-(m+1)-j}$, $p^s_y(D_j) = s_{k-1}$ and $p^s_y(D_l) = \frac{1}{2}(s_{k-1} + s_{k-2})$ for all $l \neq i, j$. Since the score of alternative x at $D \in \mathcal{D}^N$ is for sure as high as the score of any alternative $z \neq y$, individual i cannot manipulate f_s at $D \in \mathcal{D}^N$ via $D'_i \in \mathcal{D}$ whenever $p_x(D'_i, D_{-i}) \leq p_y(D'_i, D_{-i})$, or for all $m = 1, ..., k-2$,

$$\frac{\sum_{j=k-m-1}^{k-1} s_j}{m+1} + \frac{\sum_{j=1}^{k-2} s_j}{k-1} \leq s_{k-1} + \frac{\sum_{j=1}^{k-m-2} s_j}{k-m-1}.$$

On the other hand, if the former weak inequality is strict for some m, then i can manipulate f_s at $(D'_i, D_{-i}) \in \mathcal{D}^N$ via $D_i \in \mathcal{D}$. Hence, the set of equations

$$\frac{\sum_{j=k-m-1}^{k-1} s_j}{m+1} + \frac{\sum_{j=1}^{k-2} s_j}{k-1} = s_{k-1} + \frac{\sum_{j=1}^{k-m-2} s_j}{k-m-1}, \tag{1}$$

for all $m = 1, ..., k-2$, defines a set of necessary conditions for strategy-proofness on the cohesive dichotomous domain for the generic social choice function f_s. Since the Borda Count is strategy-proof on the cohesive dichotomous domain, we already know that the Borda Count solves the linear system of $k-2$ equations (the possible deviations $m = 1, ..., k-2$) and $k-2$ unknowns (the scores s_j for all $j = 1, ..., k-2$). Nonetheless, we present the calculus before showing that the system of linear equations (1) has a unique solution.

Suppose that $s_j = j$ for all $j = 1, \ldots, k-2$. We have to verify the equation

$$\frac{\sum_{j=k-m-1}^{k-1} j}{m+1} + \frac{\sum_{j=1}^{k-2} j}{k-1} = k-1+\frac{\sum_{j=1}^{k-m-2} j}{k-m-1}.$$

Rewrite it as

$$\frac{\sum_{j=1}^{k-1} j - \sum_{j=1}^{k-m-2} j}{m+1} = k-1+\frac{\sum_{j=1}^{k-m-2} j}{k-m-1} - \frac{\sum_{j=1}^{k-2} j}{k-1}$$

and apply the equation $\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$ in order to get that the left and the right hand sides of the former equation are equal to $\frac{(k-1)k-(k-m-2)(k-m-1)}{2(m+1)}$ and $\frac{2(k-1)+(k-m-2)-(k-2)}{2}$, respectively. Perform all the necessary multiplications to yield the expression

$$\frac{k^2-k-\left(k^2-2km-3k+m^2+3m+2\right)}{2\left(m+1\right)} = \frac{2\left(k-1\right)-m}{2}$$

which is equivalent to $2km+2k-m^2-3m-2 = \left(m+1\right)\left(2k-m-2\right)$. The result follows from simple algebra.

Finally, we prove that there is no other solution to the system of linear equations. Since s_{k-1} and s_0 are normalized to $k-1$ and 0, respectively, we rewrite equation (1) for the generic parameter m as

$$\frac{\sum_{j=k-m-1}^{k-2} s_j}{m+1} + \frac{\sum_{j=1}^{k-2} s_j}{k-1} - \frac{\sum_{j=1}^{k-m-2} s_j}{k-m-1} = \frac{m(k-1)}{m+1}.$$

Next, consider the matrix representation $\mathbf{As} = \mathbf{b}$ of the former set of equations where the rows of the matrix $\mathbf{A}$ correspond to the different values of $m = 1, \ldots, k-2$. For example, $\mathbf{A} = \left(\frac{1}{(k-1)}\mathbf{E} + \bar{\mathbf{A}}\right)$, where $\mathbf{E}$ is a $(k-2) \times (k-2)$ matrix with a 1 in every entry and

$$\bar{\mathbf{A}} = \begin{pmatrix} -\frac{1}{k-2} & -\frac{1}{k-2} & \cdots & -\frac{1}{k-2} & \frac{1}{2} \\ -\frac{1}{k-3} & -\frac{1}{k-3} & \cdots & \frac{1}{3} & \frac{1}{3} \\ \vdots & \vdots & & \vdots & \vdots \\ -\frac{1}{2} & \frac{1}{k-2} & \cdots & \frac{1}{k-2} & \frac{1}{k-2} \\ \frac{1}{k-1} & \frac{1}{k-1} & \cdots & \frac{1}{k-1} & \frac{1}{k-1} \end{pmatrix}.$$

Moreover, the vector $\mathbf{b}$ can be represented as $\mathbf{b} = \frac{1}{k-1}\mathbf{b}'$, where

$$\mathbf{b}' = \begin{pmatrix} \frac{1}{2} & \frac{2}{3} & \cdots & \frac{k-3}{k-2} & \frac{k-2}{k-1} \end{pmatrix}.$$

The system of $k-2$ linear equations and $k-2$ unknowns has a unique solution if the matrix of coefficients $\mathbf{A}$ with the generic element $a_{m,r}$ has full rank. Multiply the m'th row of $\mathbf{A}$ by $\frac{1}{a_{m,k-2}} = \left(\frac{1}{k-1} + \frac{1}{m+1}\right)^{-1} = \frac{(k-1)(m+1)}{m+k}$. The resulting matrix $\tilde{\mathbf{A}}$ has the same rank as $\mathbf{A}$ and it is equal to

$$\tilde{\mathbf{A}} = \begin{pmatrix} \tilde{a}_1 & \tilde{a}_1 & \tilde{a}_1 & \dots & \tilde{a}_1 & \tilde{a}_1 & 1 \\ \tilde{a}_2 & \tilde{a}_2 & \tilde{a}_2 & \dots & \tilde{a}_2 & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \tilde{a}_{k-3} & 1 & 1 & \dots & 1 & 1 & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 & 1 \end{pmatrix},$$

where $\tilde{a}_m = \frac{(m+1)(k-1)}{k+m}\left(\frac{1}{k-1} - \frac{1}{k-m-1}\right)$. Observe that for all $m = 1, \dots, k-2$, $\frac{(m+1)(k-1)}{k+m} > 0$ and $\frac{1}{k-1} - \frac{1}{k-m-1} < 0$. Therefore, $\tilde{a}_m < 0$ for all $m = 1, \dots, k-2$.

Let $\mathbf{v_r}$ be the vector notation for the r'th column of $\tilde{\mathbf{A}}$. The matrix $\tilde{\mathbf{A}}$ has full rank if there does not exist a vector $\lambda \neq \underline{0}$ such that for all $r = 1, \dots, k-2$ the scalar product $\lambda \cdot \mathbf{v_r} = 0$. Suppose contrary that there exists a $\lambda \neq \underline{0}$ such that for all $r = 1, \dots, k-2$, $\lambda \cdot \mathbf{v_r} = 0$. Consider $\mathbf{v_{k-2}}$ and $\mathbf{v_{k-3}}$. By assumption $\sum_{j=1}^{k-2} \lambda_j = 0$ and $\lambda_1 \tilde{a}_1 + \sum_{j=2}^{k-2} \lambda_j = 0$. Combining the two equations yields $\lambda_1 = \lambda_1 \tilde{a}_1$. Since $\tilde{a}_1 < 0$, it has to be that that $\lambda_1 = 0$. Let $m \geq 2$ and suppose that $\lambda_j = 0$ for all $j < m < k-3$. To see that $\lambda_{m+1} = 0$ consider $\mathbf{v_{k-2-m}}$ and $\mathbf{v_{k-3-m}}$. By the induction hypothesis $\sum_{j=m+1}^{k-2} \lambda_j = 0$ and $\lambda_{m+1}\tilde{a}_{m+1} + \sum_{j=m+2}^{k-2} \lambda_j = 0$. Combining the two equations yields $\lambda_{m+1} = \lambda_{m+1}\tilde{a}_{m+1}$. Since $\tilde{a}_{m+1} < 0$, it has to be that $\lambda_{m+1} = 0$. We conclude that $\lambda_j = 0$ for all $j = 1, ..., k-3$. Finally, since the scalar product $\lambda \cdot \mathbf{v_1} = 0$ by assumption, $\lambda_j = 0$ for all $j = 1, ..., k-3$, and $\tilde{a}_{1,k-2} \neq 0$, we conclude that $\lambda_{k-2} = 0$ as well. Hence, the matrix $\tilde{\mathbf{A}}$ has full rank. ■

Proposition 2 is a characterization of the Borda Count stating that in terms of incentives the Borda Count is the best scoring rule to apply whenever individual preferences are dichotomous. In the Introduction we have already described voting environments where individuals have dichotomous preferences on the set of alternatives. Yet, for the rest of the paper we want to ask whether the dichotomous domain restriction can be weakened, or, to say it in different words, whether there are domains containing the set of dichotomous preferences under which the Borda Count is strategy-proof on the associated cohesive preference domain. Following Ching and Serizawa [6] the domain $\bar{\mathcal{R}} \subseteq \mathcal{R}$ is said to be a *maximal domain for a list of properties for the social choice function* $f : \bar{\mathcal{R}}^N \to 2^K \backslash \{\emptyset\}$ if (a) $f : \bar{\mathcal{R}}^N \to 2^K \backslash \{\emptyset\}$ satisfies the list of properties, and (b) for all $\tilde{\mathcal{R}} \supsetneq \bar{\mathcal{R}}$, $f : \tilde{\mathcal{R}}^N \to 2^K \backslash \{\emptyset\}$ does not satisfy the list of properties. We consider in addition to strategy-proofness on the cohesive $\bar{\mathcal{R}}$ domain a richness condition that eliminates all small domains for which the Borda Count is strategy-proof on the corresponding cohesive domain. The condition we apply is stronger than the one of Berga and Serizawa [2] who propose the following: A domain $\bar{\mathcal{R}} \subseteq \mathcal{R}$ is rich if for all $x \in K$ there exists a preference relation $R_i \in \bar{\mathcal{R}}$ such that xP_iy for all $y \in K\backslash\{x\}$. Here, a domain $\bar{\mathcal{R}} \subseteq \mathcal{R}$ is said to be *rich* if for all $x \in K$ there exists a dichotomous preference relation $D_i \in \bar{\mathcal{R}}$ such that $G(D_i) = \{x\}$. The strengthening of the richness condition is needed, because otherwise, given a preference profile, we cannot calculate the score of all alternatives in order to determine the set of Borda Winners. The last Proposition of the paper states that the dichotomous preference domain is the unique maximal rich domain for strategy-proofness for the Borda Count whenever at least three individuals participate in the election.

Proposition 3 *If* $n \geq 3$*, then the dichotomous preference domain is the unique maximal rich domain for strategy-proofness for the Borda Count.*

Proof: The proof is organized as follows. We show at first that if $n \geq 3$, then the dichotomous preference domain is a maximal rich domain for strategy-proofness for the Borda Count. Afterwards we proof uniqueness.

(1) Since for all $x \in K$ the preference relation D_i which is such that $G(D_i) = \{x\}$ belongs to $\mathcal{D}$, it follows that the dichotomous preference domain is rich. Moreover, it has already been seen that the Borda Count is strategy-proof on the cohesive dichotomous domain. To see that the Borda Count is manipulable on the cohesive preference extension if the we enlarge the underlying preference domain, we add the preference relation $T_i \in \mathcal{R}$ with at least three indifference classes to the domain $\mathcal{D}$. Assume without loss of generality that T_i satisfies

$C_i^1 = \{1, ..., x^1\}$, $C_i^2 = \{x^1+1, ..., x^2\}$, ..., and, $C_i^h = \{x^{h-1}+1, ..., x^h\}$, where $x^h = k$ and $h \geq 3$. The cardinality for the generic set C_i^j is equal to $|C_i^j| = x^j - x^{j-1} > 0$, where x^0 is normalized to 0. Therefore, we consider now the domains $\tilde{\mathcal{R}} \supseteq \mathcal{D} \cup T_i$ and the objective is to show that the Borda Count is manipulable by i on $\tilde{\mathcal{R}}^N$. Consider the preference profile $D_{-i} \in \tilde{\mathcal{R}}^{N \setminus \{i\}}$ which is such that the preference relation D_j for individual j satisfies $G(D_j) = \{x^h\}$ and the preference relation D_l for all $l \neq i, j$ is given by $G(D_l) = \{x^1, x^h\}$. Given the dichotomous preference relation D_i for individual i which is completely prescribed by the set of good alternatives $G(D_i) = \{x^1\}$, we show that if $n \geq 3$, then individual i can manipulate the Borda Count at $D \in \tilde{\mathcal{R}}^N$ via $T_i \in \tilde{\mathcal{R}}$. At the preference profile (T_i, D_{-i}), $p_{x^h}(D_j) = k-1$, $p_{x^h}(D_i) = \frac{1}{x^h - x^{h-1}} \sum\limits_{m=1}^{k-x^{h-1}-1} m$, and for all $l \neq i, j$, $p_{x^h}(D_l) = \frac{k-1+k-2}{2}$. Similar at the preference profile (T_i, D_{-i}), $p_{x^1}(D_j) = \frac{1}{k-1} \sum\limits_{m=1}^{k-2} m$, $p_{x^1}(D_i) = \frac{1}{x^1} \sum\limits_{m=k-x^1}^{k-1} m$, and for all $l \neq i, j$, $p_{x^1}(D_l) = \frac{k-1+k-2}{2}$. Therefore,

$$p_{x^h}(T_i, D_{-i}) = k - 1 + (n-2)\frac{1}{2}(k-1+k-2) + \frac{1}{x^h - x^{h-1}} \sum_{m=1}^{k-x^{h-1}-1} m$$

and

$$p_{x^1}(T_i, D_{-i}) = (n-2)\frac{1}{2}(k-1+k-2) + \frac{1}{x^1} \sum_{m=k-x^1}^{k-1} m + \frac{1}{k-1} \sum_{m=1}^{k-2} m.$$

Since for all $k > j > 0$,

$$\begin{aligned}
\frac{1}{x^{j+1}-x^j} \sum_{m=k-x^{j+1}}^{k-x^j-1} m &= \frac{1}{x^{j+1}-x^j} \left(\sum_{m=1}^{k-x^j-1} m - \sum_{m=1}^{k-x^{j+1}-1} m \right) \\
&= \frac{1}{2} \frac{(k-x^j-1)(k-x^j)}{x^{j+1}-x^j} - \frac{1}{2} \frac{(k-x^{j+1}-1)(k-x^{j+1})}{x^{j+1}-x^j} \\
&= \frac{k^2 - 2kx^j + (x^j)^2 - k + x^j}{2(x^{j+1}-x^j)} - \frac{k^2 - 2kx^{j+1} + (x^{j+1})^2 - k + x^{j+1}}{2(x^{j+1}-x^j)} \\
&= \frac{-2kx^j + (x^j)^2 + x^j + 2kx^{j+1} - (x^{j+1})^2 - x^{j+1}}{2(x^{j+1}-x^j)} \\
&= \frac{(x^{j+1}-x^j)(2k-1) - (x^{j+1}-x^j)(x^{j+1}+x^j)}{2(x^{j+1}-x^j)} = \frac{2k-1-x^{j+1}-x^j}{2}
\end{aligned}$$

it can be concluded that the difference in the score between x^h and x^1 at the preference profile (T_i, D_{-i}), $p_{x^h}(T_i, D_{-i}) - p_{x^1}(T_i, D_{-i})$, is equal to

$$\frac{2(k-1)}{2} + \frac{2k-1-x^h-x^{h-1}}{2} - \frac{2k-1-x^1-x^0}{2} - \frac{k-2}{2} = \frac{-x^{h-1}+x^1}{2} < 0.$$

To see this remember that $x^h = k$ and $x^0 = 0$. Therefore, $f_B(T_i, D_{-i}) = \{x^1\}$ whenever $n \geq 3$ (if $n = 2$, then $f_B(T_i, D_{-i}) = C_i^1$ and i can manipulate the Borda Count only if the cardinality of C_i^1 is equal to one). On the other hand, the score of x^1 and x^h are the same at $D \in \tilde{\mathcal{R}}^N$ which implies that $f_B(D) = \{x^1, x^h\}$. Since the preference relation $\succsim_{D_i}$ relative to $D_i \in \tilde{\mathcal{R}}$ is cohesive, individual i with the preference relation D_i strictly prefers $\{x^1\}$ to $\{x^1, x^h\}$. This is a manipulation, and therefore, the dichotomous preference domain is a maximal rich domain for strategy-proofness for the Borda Count if $n \geq 3$.

(2) To prove uniqueness, suppose otherwise. Then, there exists a rich domain $\bar{\mathcal{R}} \subseteq \mathcal{R}$ that is not a subset of the dichotomous preference domain and the Borda Count is strategy-proof on the cohesive $\bar{\mathcal{R}}$ domain. Since the domain $\bar{\mathcal{R}}$ is rich, given $x \in K$, the dichotomous preference relation D_i which is such that $G(D_i) = \{x\}$ belongs to $\bar{\mathcal{R}}$. Moreover, since $\bar{\mathcal{R}}$ is not a proper subset of the dichotomous preference domain $\mathcal{D}$, the preference relation T_i with at least three indifference classes belongs to $\bar{\mathcal{R}}$ as well. Without loss of generality T_i is as described in the first part of the proof. Construct the preference profile $D_{-i} \in \bar{\mathcal{R}}^{N \setminus \{i\}}$ as follows: If n is even, then let there be $\frac{n}{2} - 1$ individuals with the preference relation D_j which is such that $G(D_j) = \{x^1\}$ and $\frac{n}{2}$ individuals with the preference relation D_l which is such that $G(D_l) = \{x^h\}$. If n is odd, then let there be $\frac{n-1}{2} - 1$ individuals with the preference relation D_j which is such that $G(D_j) = \{x^1\}$, $\frac{n-1}{2}$ individuals with the preference relation D_l which is such that $G(D_l) = \{x^h\}$ and one individual $m \in N$ with the preference relation D_m which is such that $G(D_m) = \{x^2\}$. If the preference relation D_i is prescribed by the set of good alternatives $G(D_i) = \{x^1\}$, then it is easy to see that $f_B(D) = \{x^h, x^1\}$ whenever $n > 3$ and $f_B(D) = \{x^1, x^2, x^h\}$ if $n = 3$. Apply the same calculus as in the first part of the proof of Proposition 3 to see that at (T_i, D_{-i}), $f_B(T_i, D_{-i}) = \{x^1\}$ if $n > 3$. Moreover, it can be shown in a similar way as in the first part of the proof that if $n = 3$, then the difference in the score between alternative x^2 and x^1 at the preference profile (T_i, D_{-i}), $p_{x^2}(T_i, D_{-i}) - p_{x^1}(T_i, D_{-i})$, is equal to $\frac{k - x^2}{2} > 0$. Hence,

in this case $f_B(T_i, D_{-i}) = \{x^2\}$. Since individual i with the preference relation D_i strictly prefers $\{x^1\}$ to $\{x^1, x^h\}$ according to the cohesive preference extension, the Borda Count is manipulable by i at $D \in \bar{\mathcal{R}}^N$ via $T_i \in \bar{\mathcal{R}}$ whenever $n > 3$. On the other hand, individual i with the preference relation T_i strictly prefers $\{x^1, x^2, x^h\}$ to $\{x^2\}$ according to the cohesive preference extension, and therefore, the Borda Count is manipulable by i at $(T_i, D_{-i}) \in \bar{\mathcal{R}}^N$ via $D_i \in \bar{\mathcal{R}}$ whenever $n = 3$. Therefore, the domain of dichotomous preferences is the unique maximal rich domain for strategy-proofness for the Borda Count whenever $n \geq 3$. ■

Finally, we indicate how Proposition 3 changes if we use the preference extension of Brams and Fishburn [4] instead of the cohesive preference extension. Consider the following example to see why in this case there is another maximal rich domain for strategy-proofness for the Borda if the number of individuals is equal to three.

Example 3: Suppose that $n = 3$ and $K = \{x, y, z\}$. Let the preference domain $\bar{\mathcal{R}} = \{D_i, D_j, D_l, T_i\}$ be completely prescribed by the sets $G(D_i) = \{x\}, G(D_j) = \{y\}, G(D_l) = \{z\}, G(T_i) = \{x\}$ and $B(T_i) = \{z\}$. Notice that the domain $\bar{\mathcal{R}}$ is rich. If the preference profile $R \in \bar{\mathcal{R}}$ is such that two individuals have the same preference relation D_m, $m = i, j, l$, or one individual has the preference relation D_i and a second individual has the preference relation T_i, then the Borda Count selects the top alternative according to D_m or the alternative x, respectively. We can see that at these preference profiles the top alternative of two individuals is chosen. Since the third individual cannot change this by misrepresenting her/his preferences, there are only two possible manipulations: Individual i either manipulates the Borda Count at $(D_i, D_j, D_l) \in \bar{\mathcal{R}}^N$ via $T_i \in \bar{\mathcal{R}}$ or s/he manipulates the Borda Count at $(T_i, D_j, D_l) \in \bar{\mathcal{R}}^N$ via $D_i \in \bar{\mathcal{R}}$. Notice that $f_B(D_i, D_j, D_l) = K$ and $f_B(T_i, D_j, D_l) = \{y\}$. Since individual i with the preference relation D_i, or T_i respectively, does not order the sets $\{y\}$ and $\{x, y, z\}$ according to the preference extension of Brams and Fishburn [4], there does not exist a viable manipulation (at this point one sees most clearly why it can be important to have a complete ordering of all non-empty subsets of alternatives). Therefore, the Borda Count is strategy-proof on the $\{D_i, D_j, D_l, T_i\}$ domain according to the preference extension of Brams and Fishburn [4]. ■

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