A Note on the Separability Principle in Economies with Single-Peaked Preferences*

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Abstract: We consider the problem of allocating an infinitely divisible commodity among a group of agents with single-peaked preferences. A rule that has played a central role in the analysis of the problem is the so-called uniform rule. Chun (2001) proves that the uniform rule is the only rule satisfying *Pareto optimality*, *no-envy*, *separability*, and *Ω-continuity*. We obtain an alternative characterization by using a weak *replication-invariance* condition, called *duplication-invariance*, instead of *Ω-continuity*. Furthermore, we prove that *Pareto optimality*, *equal division lower bound*, and *separability* imply *no-envy*. Using this result, we strengthen one of Chun's (2001) characterizations of the uniform rule by showing that the uniform rule is the only rule satisfying *Pareto optimality*, *equal division lower bound*, *separability*, and *either Ω-continuity or duplication-invariance*.

Keywords: fair division with single-peaked preferences, separability, duplication-invariance, uniform rule.

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1 Introduction

We consider the division of some perfectly divisible commodity among a group of agents with single-peaked preferences. This means that each agent has a most preferred amount, his peak amount, below which and above which his welfare is decreasing. A typical example is rationing in a two-good exchange economy when prices are in disequilibrium (*e.g.,* Benassy, 1982): if the preferences of the agents over the two-dimensional space of bundles are strictly convex, then the restrictions of these preferences to the budget lines are single-peaked. In this context Benassy (1982) considered the uniform rationing scheme. For the more general class of division problems with single-peaked preferences, this solution is known as the uniform rule.

Sprumont (1991) initiated the axiomatic analysis of this class of problems. Since then a wide literature has been concerned with the search for rules with appealing properties. Central axioms in this analysis are axioms of fairness, non-manipulability, monotonicity, and consistency; see for instance Ching (1992,1994), Dagan (1995), Sönmez (1994), and Thomson (1994a,b,1995,1997).[1] An important conclusion of this research is that the uniform rule is now accepted as the most important rule for allocation problems with single-peaked preferences.[2]

In a recent article Chun (2001) considers *separability* for economies with single-peaked preferences: When comparing two economies that are defined over the same set of agents, *separability* requires the following. If each agent in a subgroup has the same preference relation in both economies and the total amount assigned to this subgroup is the same in both economies, then the amounts assigned to each agent in the subgroup should be the same in both economies. *Separability* was introduced by Moulin (1987) in the context of surplus sharing and Chun (1999,2000) studies it in the contexts of bankruptcy and quasi-linear choice.

For economies with single-peaked preferences Chun (2001, Theorem 1) proves that the uniform rule is the only rule satisfying *Pareto optimality*, *no-envy*, *separability*, and Ω-*continuity*. We obtain an alternative characterization by using a weak *replication-invariance* condition, called *duplication-invariance*, instead of Ω-*continuity* (Theorem 2). Furthermore, we prove (Lemma 2) that *Pareto optimality*, *equal division lower bound*, and *separability* imply *no-envy*. Chun (2001, Theorem 1), Theorem 2, and Lemma 2 together imply that the uniform rule is the only rule satisfying *Pareto op-*

[1] This list of references is not exhaustive.
[2] See Klaus (1998) and Thomson (forthcoming) for surveys.

timality, *equal division lower bound*, *separability*, and either Ω-*continuity* (Corollary 2) or *duplication-invariance* (Corollary 3).

2 Economies with Single-Peaked Preferences and Properties of Rules

The purpose of this note is to add a variation of Chun's (2001, Theorem 1) characterization of the uniform rule and strengthen Chun's (2001, Theorem 2) characterization of the uniform rule. Hence, for further motivation and references with respect to economies with single-peaked preferences and their solutions we refer to Chun (2001). In this section we introduce the problem of fair division when preferences are single-peaked and properties for rules (see also Chun (2001), Sections 2 and 3).

Let $I \equiv \{1, 2, \ldots\}$ be an (infinite) universe of "potential" agents and $\mathcal{N}$ be the collection of nonempty and finite subsets of I containing at least three agents.[3] Given $N \in \mathcal{N}$, each agent $i \in N$ has continuous and single-peaked preferences R_i defined over the non-negative real numbers $\mathbb{R}_+$. Single-peakedness of R_i means that there exists a point $p(R_i) \in \mathbb{R}_+$, called *agent i's peak amount*, with the following property: for all $z_i, z_i' \in \mathbb{R}_+$ such that either$z_i < z_i' \leq p(R_i)$ or $z_i > z_i' \geq p(R_i)$, $z_i' \, P_i \, z_i$.[4] Let $\mathcal{R}$ be the class of all continuous, single-peaked preferences over $\mathbb{R}_+$ and $\mathcal{R}^N$ be the set of *preference profiles* $R = (R_i)_{i \in N}$ such that for all $i \in N$, $R_i \in \mathcal{R}$.

Now, an economy can be formalized as follows. Let $\Omega \in \mathbb{R}_+$ be the amount of an infinitely divisible commodity, the *(social) endowment,* that has to be distributed among the agents in $N \in \mathcal{N}$ with preference profile $R \in \mathcal{R}^N$.[5] Thus, $e = (R, \Omega) \in \mathcal{R}^N \times \mathbb{R}_+$ denotes an *economy.* Let $\mathcal{E}^N = \mathcal{R}^N \times \mathbb{R}_+$ be the *class of all economies for* $N \in \mathcal{N}$ and $\mathcal{E} = \bigcup_{\mathcal{N}} \mathcal{E}^N$ be the *class of all economies.* A *feasible allocation for* $e = (R, \Omega) \in \mathcal{E}^N$ is a vector $x \in \mathbb{R}_+^N$ such that $\sum_N x_i = \Omega$. An *(allocation) rule* is a function φ that assigns to every $e \in \mathcal{E}$ a feasible allocation, denoted $\varphi(e)$. Given $e \in \mathcal{E}^N$ and $i \in N$, $\varphi_i(e)$ is the *allotment of agent i.*

We impose the following properties of rules.

Pareto optimality: For all $N \in \mathcal{N}$ and $e \in \mathcal{E}^N$, there is no feasible allocation x such that for all $i \in N$, $x_i \, R_i \, \varphi_i(e)$ and for some $j \in N$, $x_j \, P_j \, \varphi_j(e)$.

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[3] Since for any $N \in \mathcal{N}$ such that $|N| = 2$, our main property of *separability* is vacuously satisfied, we require $|N| \geq 3$ for all $N \in \mathcal{N}$.

[4] P_i denotes the strict preference relation associated with R_i.

[5] Note that free disposal of the commodity is not allowed.

Pareto optimality is equivalent to *same-sidedness* (Sprumont, 1991): for all $N \in \mathcal{N}$ and $e \in \mathcal{E}^N$, either [for all $i \in N$, $\varphi_i(e) \leq p(R_i)$] or [for all $i \in N$, $\varphi_i(e) \geq p(R_i)$]).

By *no-envy* (Foley, 1967) no agent strictly prefers the allotment of another agent to his own allotment.

No-envy: For all $N \in \mathcal{N}$, $e \in \mathcal{E}^N$, and $i, j \in N$, $\varphi_i(e)\, R_i\, \varphi_j(e)$.

By *equal division lower bound* each agent (weakly) prefers his allotment to the "equal division share" of the economy.

Equal Division Lower Bound: For all $N \in \mathcal{N}$, $e = (R, \Omega) \in \mathcal{E}^N$, and $i \in N$, $\varphi_i(e)\, R_i\, \frac{\Omega}{|N|}$.

Our next property, called *separability*, requires that if for two economies with the same set of agents each agent in a subgroup has the same preference relation in both economies and the total amount assigned to this subgroup is the same in both economies, then the allotments of each agent in the subgroup should be the same in both economies.

Let $M \subseteq N$. Then, for all $R \in \mathcal{R}^N$, $R_M \equiv (R_i)_{i \in M}$.

Separability: For all $N \in \mathcal{N}$, $e = (R, \Omega)$, $e' = (R', \Omega') \in \mathcal{E}^N$, and $M \subseteq N$, if $R_M = R'_M$ and $\Sigma_M \varphi_i(e) = \Sigma_M \varphi_i(e')$, then for all $i \in M$, $\varphi_i(e) = \varphi_i(e')$.

By Ω-*continuity* small changes in the social endowment cause small changes in the allocation chosen by the rule.

Ω-Continuity: For all $N \in \mathcal{N}$, $e = (R, \Omega) \in \mathcal{E}^N$, and all sequences $\{\Omega^k\}_{k \in \mathbb{N}}$ in $\mathbb{R}_+$, if $\lim_{k \to \infty} \Omega^k = \Omega$, then $\lim_{k \to \infty} \varphi(R, \Omega^k) = \varphi(R, \Omega)$.

Replication-invariance states that if an economy is replicated, *i.e.,* the amount to divide and the preference profile are replicated, then the replica of the allocation assigned by the rule for the initial economy equals the allocation assigned by the rule for the replicated economy. Since *replication-invariance* is a well-known property and its formal description is somewhat cumbersome, for a formal statement we refer to Thomson (1995a,1997) or Chun (2001).

Note that *replication-invariance* applies to all possible "replica economies," that is, the original economy may have been replicated any number of times. Next we introduce *duplication-invariance*, a weaker form of *replication-invariance* since it is only applied to the duplication of economies.

Let $N, N' \in \mathcal{N}$ such that $|N'| = |N|$ and $N' \cap N = \emptyset$ and $e = (R, \Omega) \in \mathcal{E}^N$. We call $2 * e = ((R, R'), 2\,\Omega) \in R^{N \cup N'} \times \mathbb{R}_+$ a *duplication* or a *duplicate economy of* e, if for each agent $i \in N$ there exists exactly one "*clone*" $i' \in N'$ such that $R_i = R'_{i'}$. Given a feasible allocation x for e, $2 * x$ denotes the *duplication of* x for the duplicate economy $2 * e$, that is, any agent

$i \in N$ and his clone receive the same allotment at $2 * e$, namely agent i's original allotment at e: if $i' \in N'$ is the clone of agent $i \in N$ in $2 * e$, then $(2 * x)_{i'} = (2 * x)_i = x_i$.

Duplication-Invariance: For all $N \in \mathcal{N}$, all $e = (R, \Omega) \in \mathcal{E}^N$, and all duplicate economies $2 * e$ of e, $\varphi(2 * e) = 2 * \varphi(e)$.

3 Separability and the Uniform Rule

A rule satisfying all properties introduced in Section 2 is the *uniform rule*. It has played a central role in the literature of fair division when preferences are single-peaked.

Uniform Rule *U*: For all $N \in \mathcal{N}$, $e = (R, \Omega) \in \mathcal{E}^N$, and all $j \in N$,

$$U_j(e) \equiv \begin{cases} \min\{p(R_j), \lambda\} & \text{if } \sum_N p(R_i) \geq \Omega, \\ \max\{p(R_j), \lambda\} & \text{if } \sum_N p(R_i) \leq \Omega, \end{cases}$$

where λ solves $\sum_N U_i(e) = \Omega$.

The uniform rule satisfies many axioms of fairness, non-manipulability, monotonicity, and consistency. Furthermore, various combinations of these axioms together with Pareto optimality yield characterizations of the uniform rule; see for instance Ching (1992,1994), Dagan (1995), Sönmez (1994), and Thomson (1994a,b,1995,1997). We state the following lemma without proof (see Chun, 2001).

Lemma 1. *The uniform rule satisfies Pareto optimality, no-envy, equal division lower bound, separability, Ω-continuity, and duplication-invariance.*

In this section we present three characterizations of the uniform rule that involve *separability*. First, we state one of Chun's (2001, Theorem 1) characterizations of the uniform rule.

Theorem 1. [Chun (2001), Theorem 1] *The uniform rule is the only rule satisfying Pareto optimality, no-envy, separability, and Ω-continuity.*

We obtain an alternative characterization of the uniform rule by using *duplication-invariance* instead of *Ω-continuity*

Theorem 2. *The uniform rule is the only rule satisfying Pareto optimality, no-envy, separability, and duplication-invariance.*

Proof. By Lemma 1, the uniform rule satisfies *Pareto optimality*, *no-envy*, *separability*, and *duplication-invariance*. Let φ be a solution satisfying *Pareto optimality*, *no-envy*, *separability*, and *duplication-invariance*. Let $e = (R, \Omega) \in \mathcal{E}^N$ and $\Omega = \sum_N p(R_i)$. By *Pareto optimality*, for all $i \in N$, $\varphi_i(e) = p(R_i) = U_i(e)$.

Case 1: Let $e = (R, \Omega) \in \mathcal{E}^N$, $\Omega > \Sigma_N p(R_i)$, and $x = \varphi(e)$. Suppose, by contradiction, that $x \neq U(e)$. By *Pareto optimality* and *no-envy*, there are two agents, without loss of generality $1, 2 \in N$, such that

$$x_1 > p(R_1) > x_2 \geq p(R_2) \text{ and } x_1 \, R_1 \, x_2.$$

Let $N' \in \mathcal{N}$ be such that $|N'| = |N|$ and $N' \cap N = \emptyset$. Consider a duplicate economy $2{*}e = ((R, R'), 2\,\Omega) \in R^{N \cup N'} \times \mathbb{R}_+$ of e. By *duplication-invariance*, $\varphi(2 * e) = 2 * \varphi(e) = 2 * x$. Let $y = \varphi(2 * e)(= 2 * x)$. Hence, since y is a duplication of x, $y_1 = x_1$, $y_2 = x_2$, and

$$y_1 > p(R_1) > y_2 \geq p(R_2) \text{ and } y_1 \, R_1 \, y_2.[6]$$

Note that $|(N \cup N')\backslash\{1, 2\}|$ is even. Next, we partition the set of agents $(N \cup N')\backslash\{1, 2\}$ into two sets of equal size: let $N^1, N^2 \in \mathcal{N}$ be such that $N^1 \cup N^2 = (N \cup N')\backslash\{1, 2\}$, $N^1 \cap N^2 = \emptyset$, and $|N^1| = |N^2|$.
Let $\tilde{e} = (\tilde{R}, \tilde{\Omega}) \in R^{N \cup N'} \times \mathbb{R}_+$ be the economy obtained from $2{*}e$ by defining

- **(a)** $\tilde{R}_1 = R_1$, $\tilde{R}_2 = R_2$, for all $i \in N^1$, $p(\tilde{R}_i) = p(\tilde{R}_1)$ and $y_2 \, \tilde{P}_i \, y_1$, for all $i \in N^2$, $p(\tilde{R}_i) = p(\tilde{R}_2)$, and
- **(b)** $\tilde{\Omega} = |N|(y_1 + y_2)$.

Let $\varphi(\tilde{e}) = \tilde{y}$. By *no-envy* and *same-sidedness*, for all $i, j \in N^1 \cup \{1\}$, $\tilde{y}_i = \tilde{y}_j$. By *no-envy* and *same-sidedness*, for all $i, j \in N^2 \cup \{2\}$, $\tilde{y}_i = \tilde{y}_j$. Note that $|N| = |N^1 \cup \{1\}| = |N^2 \cup \{2\}|$. Hence, $\tilde{\Omega} = |N|(\tilde{y}_1 + \tilde{y}_2)$. Thus, since $\tilde{\Omega} = |N|(y_1 + y_2)$, $\tilde{y}_1 + \tilde{y}_2 = y_1 + y_2$. Hence, by *separability*, $\tilde{y}_1 = y_1$ and $\tilde{y}_2 = y_2$. Then, for any agent $i \in N^1$, $\tilde{y}_i = y_1$ and by the definition of $\tilde{R}_i$, $\tilde{y}_2 \, \tilde{P}_i \, \tilde{y}_i$. Hence, any agent $i \in N^1$ envies agent 2, a contradiction.

Case 2: Let $e = (R, \Omega) \in \mathcal{E}^N$, $\Omega < \Sigma_N p(R_i)$, and $x = \varphi(e)$. Note that we can apply the same proof technique as in Case 1. However, for economies with at least four agents, we give an alternative proof that does not require *duplication-invariance*. Thus, let $|N| \geq 4$.

Suppose, by contradiction, that $x \neq U(e)$. By *Pareto optimality* and *no-envy*, there are two agents, without loss of generality $1, 2 \in N$, such that

[6] Note that whenever the number of agents in the original economy e is at least four and even, we can skip the "duplication step."

$x_1 < p(R_1) < x_2 \le p(R_2)$ and $x_1\, R_1\, x_2$. Since $|N| \ge 4$, there are two agents in $N \backslash \{1,2\}$, without loss of generality $3,4 \in N \backslash \{1,2\}$.
Let $\tilde{e} = (\tilde{R}, \tilde{\Omega}) \in R^N \times \mathbb{R}_+$ be the economy obtained from e by defining

(a) $\tilde{R}_1 = R_1$, $\tilde{R}_2 = R_2$, $p(\tilde{R}_3) = p(\tilde{R}_1)$ and $x_2\, \tilde{P}_3\, x_1$, $p(\tilde{R}_4) = p(\tilde{R}_2)$, for all $i \in N \backslash \{1,2,3,4\}$, $p(\tilde{R}_i) = 0$, and

(b) $\tilde{\Omega} = 2\,(x_1 + x_2)$.

Let $\varphi(\tilde{e}) = \tilde{x}$. By *same-sidedness*, for all $i \in N \backslash \{1,2,3,4\}$, $\tilde{x}_i = p(\tilde{R}_i) = 0$. By *no-envy* and *same-sidedness*, $\tilde{x}_1 = \tilde{x}_3$ and $\tilde{x}_2 = \tilde{x}_4$. Hence, $\tilde{\Omega} = 2\,(\tilde{x}_1 + \tilde{x}_2)$. Thus, since $\tilde{\Omega} = 2\,(x_1 + x_2)$, $\tilde{x}_1 + \tilde{x}_2 = x_1 + x_2$. Hence, by *separability*, $\tilde{x}_1 = x_1$ and $\tilde{x}_2 = x_2$. Then, $\tilde{x}_3 = x_1$ and by the definition of $\tilde{R}_3$, $\tilde{x}_2\, \tilde{P}_3\, \tilde{x}_3$. Hence, agent 3 envies agent 2, a contradiction. □

Remark 1. Note that in Case 1, that is for economies with excess supply, whenever the number of agents in the economy is at least four and even, we can easily adjust the proof without using *duplication-invariance*. Furthermore, in Case 2, that is for economies with excess demand, we do not need *duplication-invariance* for economies with at least four agents. It is an open problem if *duplication-invariance* is independent of the other properties for economies with excess supply with an odd number of agents and for economies with excess demand with three agents.

The proof of Theorem 2 (see also Remark 1) implies the following corollary.

Corollary 1. *Let φ be a rule satisfying Pareto optimality, no-envy, and separability.*
(i) *For all $N \in \mathcal{N}$ and $e = (R, \Omega) \in \mathcal{E}^N$ such that $|N| \ge 4$, $|N|$ even, and $\Sigma_N p(R_i) > \Omega$, $\varphi(e) = U(e)$.*
(ii) *For all $N \in \mathcal{N}$ and $e = (R, \Omega) \in \mathcal{E}^N$ such that $|N| \ge 4$ and $\Sigma_N p(R_i) > \Omega$, $\varphi(e) = U(e)$.*

Next, we state the second of Chun's (2001, Theorem 2) characterization of the uniform rule.

Theorem 3. [Chun (2001), Theorem 2] *The uniform rule is the only rule satisfying Pareto optimality, equal division lower bound, separability, Ω-continuity, and replication-invariance.*

The next observation will allow us to strengthen Theorem 3.

Lemma 2. *Pareto optimality, equal division lower bound, and separability imply no-envy.*

Proof. Let φ be a solution satisfying *Pareto optimality*, *equal division lower bound*, and *separability*. Let $e = (R, \Omega) \in \mathcal{E}^N$ and $x = \varphi(e)$. If $\Omega = \sum_N p(R_i)$, by *Pareto optimality*, for all $i \in N$, $\varphi_i(e) = p(R_i)$ and *no-envy* is satisfied. Without loss of generality, we assume that $\Omega < \Sigma_N p(R_i)$. Suppose, by contradiction, that *no-envy* is violated. Then, there are two agents, without loss of generality $1, 2 \in N$, such that $x_2 \, P_1 \, x_1$. By *same-sidedness*, $x_1 < x_2$.
Let $\bar{x} \equiv \frac{x_1+x_2}{2}$ and $\tilde{e} = (\tilde{R}, \tilde{\Omega})$ be the economy obtained from e by defining

- **(a)** $\tilde{R}_1 = R_1$, $\tilde{R}_2 = R_2$, for all $i \in N\backslash\{1, 2\}$, $p(\tilde{R}_i) = \bar{x}$, and
- **(b)** $\tilde{\Omega} = |N| \, \bar{x}$.

Let $\tilde{x} = \varphi(\tilde{e})$. Note that by (b), $\frac{\tilde{\Omega}}{|N|} = \bar{x}$. Hence, by (a) and *equal division lower bound*, for all $i \in N\backslash\{1, 2\}$, $\tilde{x}_i = \bar{x}$. Thus, $\sum_{N\backslash\{1,2\}} \tilde{x}_i = (|N| - 2) \, \bar{x}$ and $\tilde{x}_1 + \tilde{x}_2 = 2 \, \bar{x} = x_1 + x_2$. Hence, by *separability*, $\tilde{x}_1 = x_1$ and $\tilde{x}_2 = x_2$. This implies $\tilde{x}_2 \, \tilde{P}_1 \, \tilde{x}_1$.

If $p(\tilde{R}_1) \geq \bar{x}$, then $\tilde{x}_1 < \bar{x} \leq p(\tilde{R}_1)$. Thus, by single-peakedness, $\bar{x} \, \tilde{P}_1 \, \tilde{x}_1$, a contradiction to *equal division lower bound*.

If $p(\tilde{R}_1) < \bar{x}$, then $p(\tilde{R}_1) < \bar{x} < \tilde{x}_2$. Thus, by single-peakedness, $\bar{x} \, \tilde{P}_1 \, \tilde{x}_2$. Hence, by $\tilde{x}_2 \, \tilde{P}_1 \, \tilde{x}_1$ and transitivity, $\bar{x} \, \tilde{P}_1 \, \tilde{x}_1$, a contradiction to *equal division lower bound*. □

We can now state two more characterizations of the uniform rule. First, we can drop *replication-invariance* from Theorem 3.

Corollary 2. *The uniform rule is the only solution satisfying* Pareto optimality, equal division lower bound, separability, *and* Ω-continuity.

Proof. By Lemma 1, the uniform rule satisfies *Pareto optimality*, *equal division lower bound*, *separability*, and Ω-*continuity*.

Let φ be a solution satisfying *Pareto optimality*, *equal division lower bound*, *separability*, and Ω-*continuity*. Then, by Lemma 2, φ satisfies *no-envy*. Hence, by Theorem 1, $\varphi \equiv U$. □

Second, we can drop Ω-*continuity* from Theorem 3 and furthermore replace *replication-invariance* by the weaker *duplication-invariance*.

Corollary 3. *The uniform rule is the only solution satisfying* Pareto optimality, equal division lower bound, separability, *and* duplication-invariance.

Proof. By Lemma 1, the uniform rule satisfies *Pareto optimality*, *equal division lower bound*, *separability*, and *duplication-invariance*.

Let φ be a solution satisfying *Pareto optimality*, *equal division lower bound*, *separability*, and *duplication-invariance*. Then, by Lemma 2, φ satisfies *no-envy*. Hence, by Theorem 2, $\varphi \equiv U$. □

Remark 2. Since *consistency* (*e.g.*, Thomson, 1994a) implies *separability*, all our results still hold if we replace *separability* by *consistency*.

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