On Group Strategy-proof Mechanisms for a Many-to-one Matching Model*

by

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Abstract: For the many-to-one matching model in which firms have substitutable and quota $q-$separable preferences over subsets of workers we show that the workers-optimal stable mechanism is group strategy-proof for the workers. In order to prove this result, we also show that under this domain of preferences (which contains the domain of responsive preferences of the college admissions problem) the workers-optimal stable matching is weakly Pareto optimal for the workers and the Blocking Lemma holds as well. We exhibit an example showing that none of these three results remain true if the preferences of firms are substitutable but not quota $q-$separable.

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1 Introduction

Two-sided, many-to-one matching models have been used to study assignment problems where agents can be divided, from the very beginning, into two disjoint subsets: the set of institutions and the set of individuals. The fundamental question of these assignment problems consists of matching each firm, on one side, with a group of workers, on the other side (we will follow the convention of generically referring to institutions as firms and to individuals as workers). A matching is called *stable* if all agents have acceptable partners and there is no unmatched worker-firm pair who both would prefer to be matched to each other rather than staying with their current partners.

The "college admissions model with substitutable preferences" is the name given by Roth and Sotomayor (1990) to the most general many-to-one model with ordinal preferences in which stable matchings exist. Firms are restricted to have substitutable preferences over subsets of workers; namely, all firms continue to want to employ a worker even if other workers become unavailable (Kelso and Crawford (1982) were the first to use this property in a more general model with money). Under this hypothesis the deferred-acceptance algorithms produce either the firms-optimal stable matching or the workers-optimal stable matching, depending on whether the firms or the workers make the offers. The firms (workers)-optimal stable matching is unanimously considered by all firms (respectively, workers) to be the best among all stable matchings.

A more specific many-to-one model, called the "college admissions problem" by Gale and Shapley (1962), supposes that firms have a maximum number of positions to be filled (their quota), and that each firm, given its ranking of individual workers, orders subsets of workers in a responsive manner; namely, for any two subsets that differ in only one student a college prefers the subset containing the most-preferred student. In this model the set of stable matchings satisfies many desirable properties.[1] The first type of properties are more theoretical in nature and are related with its lattice structure. The second type of properties have more practical implications and are related with the strategic incentives of agents participating in markets where centralized mechanisms are used to propose to participants their corresponding partners of a particular stable matching (stable mechanisms).[2]

[1] Observe that the marriage model (i.e., the one-to-one matching model) is a particular instance of the "college admissions problem" when all firms have quota one.

[2] Roth (1984, 1986, 1990, and 1991), Mongell and Roth (1991), Roth and Xing (1994), and Romero-Medina (1998) are examples of papers studying these incentives in particular matching problems like entry-level professional labor markets, student admissions at colleges, american sororities, etc.

However, whether or not a matching is stable depends on the preferences of agents and, since they constitute private information, agents have to be asked for them; hence, untruthful reports might arise. This is the reason why the matching literature has intensively studied the strategic properties of stable mechanisms. In particular, Dubins and Freedman (1981) shows that in the "college admissions problem" the deferred-acceptance algorithm in which workers make offers is group strategy-proof for the workers.[3] This is important because it means that if the mechanism selects for each preference profile its corresponding workers-optimal stable matching then, no group of workers can never benefit by reporting untruthfully their preference relations.

It is well-known that this group strategy-proofness property is *not* necessarily true when the preferences of firms are substitutable. The purpose of this paper is to consider a weaker condition than responsiveness, called quota $q-$separability, that together with substitutability implies that the property that the workers-optimal stable mechanism is group strategy-proof for the workers holds for this more general many-to-one matching model.[4] A firm is said to have separable preferences over all subsets of workers if its partition between acceptable and unacceptable workers has the property that only adding acceptable workers makes any given subset of workers a better one. However, in many applications such as the entry-level professional labor markets, separability alone does not seem very reasonable because firms usually have fewer openings (their quota) than the number of "good" workers looking for a job. In these cases it seems reasonable to restrict the preferences of firms in such a way that the separability condition operates only up to their quota, considering unacceptable all subsets with higher cardinality. Moreover, while responsiveness seems the relevant property for extending an ordered list of individual students to preferences on all subsets of students, it is too restrictive, though, to capture some degree of complementarity among workers, which can be very natural in other settings. The quota $q-$separability con-

[3]To be precise, they show it for the marriage model, but their result can be extended to the college admissions problem. Some results concerning stability in the college admissions problem are immediate consequences of the fact that they hold for the marriage model. Each college is split into as many pieces as positions it has, so transforming the original many-to-one model into a one-to-one model. Responsiveness allows then the translation of stability from one model to another. See Roth and Sotomayor (1990) for a complete description of this procedure as well as for its applications.

[4]We have already showed that if firms have substitutable and quota $q-$separable preferences then, (a) the set of unmatched agents is the same in all stable matchings (Martínez, Massó, Neme, and Oviedo, 2000) and (b) the set of stable matchings has a lattice structure with two natural binary operations (Martínez, Massó, Neme, and Oviedo, 2001).

dition permits greater flexibility in going from orders on individuals to orders on subsets. For instance, candidates for a job can be grouped together by areas of specialization. A firm with quota two may consider as the best subset of workers not the set consisting of the first two candidates on the individual ranking (which may have both the same specialization) but rather the subset composed of the first and fourth candidates in the individual ranking (i.e.; the first in each area of specialization).

As it is the case for the college admissions problem, the property that (in this more general many-to-one matching model) the workers-optimal stable mechanism is group strategy-proof for the workers is an immediate consequence of the following result (known in the literature as the Blocking Lemma): Suppose that the set of workers that strictly prefer an individually rational matching to the workers-optimal stable matching is nonempty. Then, we can always find a firm and a worker (a blocking pair of the individually rational matching) with the following properties: (a) the firm was hiring another worker who strictly prefers the individually rational matching to the workers-optimal stable matching and (b) the worker (member of the blocking pair) considers the workers-optimal stable matching to be at least as good as the individually rational matching. Furthermore, and in order to prove the Blocking Lemma, we also show that the workers-optimal stable matching is weakly Pareto optimal for the workers; namely, there is no individually rational matching that all workers strictly prefer to the workers-optimal stable matching.

Since our many-to-one matching model includes (as a particular subclass) the college admissions problem, all negative results concerning strategic incentives of agents of the latter model carry over to the former one. In particular, (1) the workers-optimal stable mechanism is not group strategy-proof for the firms (it is not even strategy-proof for them) and (2) there is no stable and strategy-proof mechanism.

The paper is organized as follows. In Section 2 we present the preliminary notation and definitions. In Section 3 we prove that if firms have substitutable and quota $q-$separable preferences then, the workers-optimal stable matching is weakly Pareto optimal for the workers (Theorem 1). Furthermore, we exhibit an example (Example 2) showing that this result is not true if the preferences of firms are substitutable but not quota $q-$separable. In Section 4 we formally define a mechanism and state the main result of the paper: if firms have substitutable and quota $q-$separable preferences then, the workers-optimal stable mechanism is group strategy-proof for the workers (Theorem 2). Moreover, in Theorem 3 we state that the Blocking Lemma holds for our many-to-one model. Finally, the Appendix in Section 5 contains the proof of Theorem 3, the key result used to prove Theorem 2.

2 Preliminaries

There are two disjoint sets of *agents*, the set of n *firms* $F = \{f_1, ..., f_n\}$ and the set of m *workers* $W = \{w_1, ..., w_m\}$. Generic elements of both sets will be denoted, respectively, by f, $\overline{f}$, and $\widetilde{f}$, and by w, $\overline{w}$, and $\widetilde{w}$. Each worker $w \in W$ has a strict, transitive, and complete preference relation $P(w)$ over $F \cup \{\emptyset\}$, and each firm $f \in F$ has a strict, transitive, and complete preference relation $P(f)$ over 2^W. *Preference profiles* are $(n+m)$-tuples of preference relations and they are represented by $P = (P(f_1), ..., P(f_n); P(w_1), ..., P(w_m))$. Given a preference relation of a firm $P(f)$ the subsets of workers preferred to the empty set by f are called *acceptable*. Similarly, given a preference relation of a worker $P(w)$ the firms preferred by w to the empty set are called *acceptable*. Therefore, we are allowing for the possibility that firm f may prefer not to hire any worker rather than to hire unacceptable subsets of workers and that worker w may prefer to remain unemployed rather than to work for an unacceptable firm. To express preference relations in a concise manner, and since only acceptable partners will matter, we will represent preference relations as lists of acceptable partners. For instance,

$$\begin{aligned} P(f_i) &: w_1 w_3, w_2, w_1 \\ P(w_j) &: f_1, f_3 \end{aligned}$$

indicate that $\{w_1, w_3\} P(f_i) \{w_2\} P(f_i) \{w_1\} P(f_i) \emptyset$ and $f_1 P(w_j) f_3 P(w_j) \emptyset$.

A *matching market* is a triple (F, W, P), where F is a set of firms, W is a set of workers, and P is a preference profile. Given a matching market (F, W, P) the assignment problem consists of matching workers with firms, keeping the bilateral nature of their relationship and allowing for the possibility that both, firms and workers, may remain unmatched. Formally,

Definition 1 *A* ***matching*** μ *is a mapping from the set* $F \cup W$ *into the set of all subsets of* $F \cup W$ *such that for all* $w \in W$ *and* $f \in F$*:*

1. *Either* $|\mu(w)| = 1$ *and* $\mu(w) \subseteq F$ *or else* $\mu(w) = \emptyset$.

2. $\mu(f) \subseteq 2^W$.

3. $\mu(w) = \{f\}$ *if and only if* $w \in \mu(f)$.

Condition *1* says that a worker can either be matched to at most one firm or remain unmatched. Condition *2* says that a firm can either hire a subset of workers or be unmatched. Finally, condition *3* states the bilateral nature of a matching in the sense that firm f hires worker w if and only if worker w works for firm f. We say that w and f are *unmatched* in a matching μ if $\mu(w) = \emptyset$ and $\mu(f) = \emptyset$. Otherwise, they are matched. A matching μ is said to be *one-to-one* if firms can hire at most one worker; namely, Condition *2* is replaced by: Either $|\mu(f)| = 1$ and $\mu(f) \subseteq W$ or else $\mu(f) = \emptyset$. The model in which all matchings are one-to-one is also known in the literature as the *marriage model*. The model in which all matchings are many-to-one (i.e., they satisfy Definition 1), and firms have responsive preferences,[5] is also known in the literature as the *college admissions model* (Gale and Shapley, 1962). To represent matchings concisely we will follow the widespread notation where, for instance, given $F = \{f_1, f_2, f_3\}$ and $W = \{w_1, w_2, w_3, w_4\}$,

$$\begin{array}{ccccc} & f_1 & f_2 & f_3 & \emptyset \\ \mu & w_3w_4 & w_1 & \emptyset & w_2 \end{array}$$

represents the matching where firm f_1 is matched to workers w_3 and w_4, firm f_2 is matched to worker w_1, and firm f_3 and worker w_2 are unmatched. Given a matching μ and two subsets $F' \subseteq F$ and $W' \subseteq W$ we denote by $\mu(F')$ and $\mu(W')$ the sets $\{w \in W \mid \mu(w) \in F'\}$ and $\{f \in F \mid \exists w \in W' \text{ such that } w \in \mu(f)\}$, respectively.

Let P be a preference profile. Given a set of workers $S \subseteq W$, let $Ch(S, P(f))$ denote firm f's *most-preferred* subset of S according to its preference ordering $P(f)$. Generically we will refer to this set as the *choice set*. Blair (1988) shows that the choice set satisfies the following property.

Remark 1 *Let $P(f)$ be any preference and assume A and B are two subsets of workers such that $Ch(A, P(f)) \subseteq B \subseteq A$. Then, $Ch(A, P(f)) = Ch(B, P(f))$.*

A matching μ is *blocked by worker* w if $\emptyset P(w) \mu(w)$. A matching μ is *blocked by firm* f if $\mu(f) \neq Ch(\mu(f), P(f))$. We say that a matching is *individually rational* if it is not blocked by any individual agent. We will denote by $IR(P)$ the set of individually rational matchings. A matching μ is *blocked by a firm-worker pair* (f, w) if $w \notin \mu(f)$, $w \in Ch(\mu(f) \cup \{w\}, P(f))$, and $fP(w)\mu(w)$.

[5] Roughly, for any two subsets of workers that differ in only one worker a firm prefers the subset containing the most-preferred worker. See Roth and Sotomayor (1990) for a precise and formal definition of responsive preferences as well as for a masterful and illuminating analysis of these models and an exhaustive bibliography.

Definition 2 *A matching μ is* ***stable*** *if it is not blocked by any individual agent or any firm-worker pair.*

Given a preference profile P, denote the set of stable matchings by $S(P)$. It is easy to construct examples of preference profiles with the property that the set of stable matchings is empty. These examples share the feature that at least one firm regards a subset of workers as being complements. This is the reason why the literature has focused on the restriction where workers are regarded as substitutes.

Definition 3 *A firm f's preference relation $P(f)$ satisfies* ***substitutability*** *if for any set S containing workers w and w' $(w \neq w')$, if $w \in Ch(S, P(f))$ then $w \in Ch(S \setminus \{w'\}, P(f))$.*

A preference profile P is *substitutable* if for each firm f, the preference relation $P(f)$ satisfies substitutability.

Blair (1988) shows that the choice set of substitutable preference relations have the following property.

Remark 2 *Let $P(f)$ be a substitutable preference relation and assume A and B are two subsets of workers. Then, $Ch(A \cup B, P(f)) = Ch(Ch(A, P(f)) \cup B, P(f))$.*

Kelso and Crawford (1982) shows that (in a more general model with money) if all firms have substitutable preferences then: (1) the set of stable matchings is non-empty, and (2) firms unanimously agree that a stable matching μ_F is the best stable matching. Roth (1984) extends these results and shows that if all firms have substitutable preferences then: (3) workers unanimously agree that a stable matching μ_W is the best stable matching,[6] and (4) the optimal stable matching for one side is the worst stable matching for the other side. That is, $S(P) \neq \emptyset$ and for all $\mu \in S(P)$ we have that $\mu_F R(f) \mu R(f) \mu_W$ for all $f \in F$ and $\mu_W R(w) \mu R(w) \mu_F$ for all $w \in W$.

The *deferred-acceptance algorithm*, originally defined by Gale and Shapley (1962) for the marriage model, produces either μ_F or μ_W depending on who makes the offers. At any

[6] The matchings μ_F and μ_W are called, respectively, the *firms-optimal stable matching* and the *workers-optimal stable matching*. We are following the convention of extending preferences from the original sets (2^W and $F \cup \{\emptyset\}$) to the set of matchings. However, we now have to consider weak orderings since the matchings μ and μ' may associate to an agent the same partner. These orderings will be denoted by $R(f)$ and $R(w)$. For instance, to say that all firms prefer μ_F to any stable μ means that for every $f \in F$ we have that $\mu_F R(f) \mu$ for all stable μ (that is, either $\mu_F(f) = \mu(f)$ or else $\mu_F(f) P(f) \mu(f)$).

step of the algorithm in which firms make offers, a firm proposes itself to the choice set of the set of workers that have not already rejected it during the previous steps, while a worker accepts the offer of the best firm among the set of current offers plus the one made by the firm provisionally matched in the previous step (if any). The algorithm stops at the step at which all offers are accepted; the (provisional) matching then becomes definite and it is the firms-optimal stable matching μ_F. Symmetrically, at any step of the algorithm in which workers make offers, a worker proposes himself to the best firm among the set of firms that have not already rejected him during the previous steps, while a firm accepts the choice set of the set of current offers plus that of the workers provisionally matched in the previous step (if any). The algorithm stops at the step at which all offers are accepted; the (provisional) matching then becomes definite and it is the workers-optimal stable matching μ_W.

A firm f has separable preferences if the division between *good* workers ($\{w\}P(f)\emptyset$) and *bad* workers ($\emptyset P(f)\{w\}$) guides the ordering of subsets in the sense that adding a good worker leads to a *better* set, while adding a bad worker leads to a *worse* set.[7] Formally,

Definition 4 *A firm f's preference relation $P(f)$ satisfies* ***separability*** *if for all $S \subseteq W$ and $w \notin S$ we have that $(S \cup \{w\})\, P(f) S$ if and only if $\{w\}P(f)\emptyset$.*

A preference profile P is *separable* if for each firm f, the preference relation $P(f)$ satisfies separability.

Remark 3 *All separable preference relations are substitutable. To see this, just note that if $P(f)$ is separable then, for every $S \subseteq W$, $Ch(S, P(f)) = \{w \in S \mid \{w\}P(f)\emptyset\}$. Moreover, the preference relation*

$$P(f) : w_1, w_1w_2, w_2$$

shows that not all substitutable preference relations are separable.

Sönmez (1996) shows that if firms have separable preferences then there exists a unique stable matching. A simple way to construct this unique stable matching μ is as follows:

[7] This condition has been extensively used in social choice; see, for instance, Barberà, Sonnenschein, and Zhou (1991). In the matching literature Sönmez (1996), Dutta and Massó (1997), Martínez, Massó, Neme, and Oviedo (2000) and (2001) have made use of it to study, respectively, strategy-proof implementation, the stability of matchings when workers also care about their colleagues, the set of unmatched agents in different stable matchings, and the lattice structure of the set of stable matchings.

for each $w \in W$, let $\mu(w)$ be the maximal element, according to $P(w)$, on the set of firms for which w is an acceptable worker, i.e., $\{f \in F \mid \{w\} P(f) \emptyset\}$. The stability of μ follows directly from separability of firms' preferences.

Here, we will assume that each firm f has, in addition to substitutable and separable preferences, a maximum number of positions to be filled: its quota q_f. This limitation may arise from, for example, technological, legal, or budgetary reasons. Since we are interested in stable matchings we introduce this restriction by incorporating it into the preference relation of the firm. The college admissions model with responsive preferences (Gale and Shapley, 1962) incorporates the quota restriction of each college by imposing a limit on the number of students that a college may admit . However, from the point of view of stability, this is equivalent to supposing that all sets of students with cardinality larger than the quota are unacceptable for the college. Therefore, even if the number of good workers for firm f is larger than its quota q_f, all sets of workers with cardinality strictly larger than q_f will be unacceptable. Formally,

Definition 5 *A firm f's preference relation $P(f)$ over sets of workers is q_f-**separable** if: (a) for all $S \subsetneq W$ such that $|S| < q_f$ and $w \notin S$ we have that $(S \cup \{w\})P(f)S$ if and only if $\{w\}P(f)\emptyset$, and (b) $\emptyset P(f) S$ for all S such that $|S| > q_f$.*[8]

We will denote by $q = (q_f)_{f \in F}$ the list of quotas and we will say that a preference profile P is quota $q-$separable if each $P(f)$ is quota q_f-separable. In principle we may have firms with different quotas. The case where all firms have quota $1-$separable preferences is equivalent, from the point of view of the set of stable matchings, to the marriage model. Hence, our set-up includes the marriage model as a particular case.

The two preference relations over $2^{\{w_1,w_2,w_3\}}$

$$\begin{aligned} P(f) &: w_1w_2, w_1w_3, w_2w_3, w_1, w_2, w_3 \\ P(\overline{f}) &: w_1w_2w_3, w_1w_2, w_1w_3, w_2w_3, w_1, w_2, w_3 \end{aligned}$$

illustrate the fact that, in general and given a list of quotas q, the sets of separable and quota $q-$separable preferences are unrelated. Firm f's preference relation is $2-$separable but not

[8]For the purpose of studying the set of stable matchings, condition (b) in this definition could be replaced by the following condition: $|Ch(S, P(f))| \leq q_f$ for all S such that $|S| > q_f$. We choose condition (b) since it is simpler. Sönmez (1996) uses an alternative approach which consists of deleting condition (b) in the definition but then requiring in the definition of a matching that $|\mu(f)| \leq q_f$ for all $f \in F$. Notice that in his approach the set of separable preferences is quota q_f-separable for all q_f.

separable, since $\emptyset P(f)\{w_1, w_2, w_3\}$ and all workers are good, while firm $\overline{f}$'s preference relation is separable but not quota 2−separable.

Moreover, the preference relation over $2^{\{w_1,w_2,w_3,w_4\}}$

$$P(f) : w_1w_2, w_3w_4, w_1w_3, w_1w_4, w_2w_3, w_2w_4, w_1, w_2, w_3, w_4$$

illustrates the fact that quota $q-$separability does not imply substitutability. To see this, notice that the preference relation $P(f)$ is quota 2−separable but it is not substitutable since $w_1 \in Ch(\{w_1, w_2, w_3, w_4\}, P(f)) = \{w_1, w_2\}$, but $w_1 \notin Ch(\{w_1, w_3, w_4\}, P(f)) = \{w_3, w_4\}$. However, it is easy to see that all quota $(m-1)-$separable preferences are substitutable.

The preference relation $P(f)$ over $2^{\{w_1,w_2,w_3\}}$

$$P(f) : w_1w_2w_3, w_1w_3, w_1w_2, w_2w_3, w_1, w_2, w_3$$

illustrates the fact that the set of responsive preferences is a strict subset of the set of quota q_f-separable and substitutable preferences, since $P(f)$ is quota 3−separable and substitutable but it is not responsive because $\{w_1, w_3\}P(f)\{w_1, w_2\}$ but $\{w_2\}P(f)\{w_3\}$.

The following example shows that even if all firms have quota $q-$separable preferences the set of stable matchings may be empty.

Example 1 Let $F = \{f_1, f_2\}$ and $W = \{w_1, w_2, w_3, w_4\}$ be the two sets of agents with the preference profile P, where

$$\begin{aligned}
P(f_1) &: w_3w_4, w_2w_4, w_1w_2, w_1w_3, w_2w_3, w_1w_4, w_1, w_2, w_3, w_4, \\
P(f_2) &: w_3, w_4, \\
P(w_1) &: f_1, \\
P(w_2) &: f_1, \\
P(w_3) &: f_1, f_2, \\
P(w_4) &: f_2, f_1.
\end{aligned}$$

Notice that P is quota $(2,1)-$separable. However, $P(f_1)$ is not substitutable since $w_3 \in Ch(W, P(f_1))$ but $w_3 \notin Ch(W\setminus\{w_4\}, P(f_1))$. It can be verified that $S(P) = \emptyset$.

We close this section with a remark about the set of unmatched agents in different stable matchings.

Remark 4 *If firms have substitutable and quota $q-$separable preferences then the set of unmatched agents is the same under every stable matching.*[9]

3 Weak Pareto Optimality

For the marriage model Roth (1982) shows that optimal stable matchings have an even stronger optimality property: there is no *individually rational* matching that all agents of one side of the market strictly prefer to their corresponding optimal stable matching. Roth (1985) partly extends this result to the college admissions problem. He shows (using the result of the marriage model) that this weak Pareto optimality property holds for the students (the "one side" of the market); moreover, he also shows that the property is in general false for the colleges (the "many side" of the market). In this section we exhibit an example (Example 2) where firms have substitutable preferences and the weak Pareto optimality property does not even hold for the workers; that is, there is an individually rational matching strictly preferred by all workers (our "one side" of the market) to the workers-optimal stable matching. Additionally, we show in Theorem 1 that if firms preferences are also quota $q-$separable the weak Pareto optimality for the workers holds again. In our case, however, the proof is genuinely many-to-one since it can not be based on the fact that the result holds for the marriage model.

Theorem 1 *Assume P is substitutable and quota $q-$separable. Then, there is no individually rational matching μ such that $\mu(w)P(w)\mu_W(w)$ for all $w \in W$.*

Proof. Assume that $\mu \in IR(P)$ and

$$\mu(w)P(w)\mu_W(w) \text{ for all } w \in W. \tag{1}$$

Since μ_W is also individually rational, $\mu(w) \neq \emptyset$ for all $w \in W$. Therefore, $\mu(W)$ is nonempty.

CLAIM A $\mu(W) = \mu_W(W)$.

[9] See Martínez, Massó, Neme, and Oviedo (2000) for a proof of Remark 4 and an example illustrating that if firms have substitutable preferences (but not quota $q-$separable) the set of unmatched agents under two stable matchings may be different. Gale and Sotomayor (1985) and Roth (1984) proved independently that for the college admissions problem the set of unmatched agents is the same under every stable matching.

Proof of Claim A. Let $f \in \mu(W)$. Then $f = \mu(w)$ for some w. Observe that $\mu_W(f) \in 2^W \setminus \{\emptyset\}$. Otherwise ($\mu_W(f) = \emptyset$), by the individual rationality of μ and the quota q_f−separability of $P(f)$, $\{w\} P(f) \emptyset$, implying that (f, w) would block μ_W, contradicting its stability. Therefore,

$$\mu(W) \subseteq \mu_W(W). \tag{2}$$

Since all workers are matched at μ we already know that $\sum_{f \in \mu(W)} |\mu(f)| = |W|$. To see that the same property holds for matching μ_W assume $\sum_{f \in \mu(W)} |\mu_W(f)| < |W|$. Therefore, there exists $\overline{f} \in F$ such that $\left|\mu_W\left(\overline{f}\right)\right| < \left|\mu\left(\overline{f}\right)\right| \leq q_{\overline{f}}$, implying that we can find $\overline{w} \in \mu\left(\overline{f}\right) \setminus \mu_W\left(\overline{f}\right)$. Since $\mu \in IR(P)$, $\overline{w} \in Ch\left(\mu\left(\overline{f}\right), P\left(\overline{f}\right)\right)$, yielding by the quota $q_{\overline{f}}$−separability of $P\left(\overline{f}\right)$, $\{\overline{w}\} P\left(\overline{f}\right) \emptyset$; therefore, because $\left|\mu_W\left(\overline{f}\right)\right| < q_{\overline{f}}$,

$$\overline{w} \in Ch\left(\mu_W\left(\overline{f}\right) \cup \{\overline{w}\}, P\left(\overline{f}\right)\right). \tag{3}$$

The fact that $\overline{f} = \mu(\overline{w}) P(\overline{w}) \mu_W(\overline{w})$ and condition (3) imply that $\left(\overline{f}, \overline{w}\right)$ blocks μ_W, contradicting its stability. Therefore,

$$\sum\nolimits_{f \in \mu(W)} |\mu_W(f)| = |W|. \tag{4}$$

Hence, $|W| \geq \sum_{f \in F} |\mu_W(f)| = \sum_{f \in \mu(W)} |\mu_W(f)| + \sum_{f \in F \setminus \mu(W)} |\mu_W(f)|$, implying, by condition (4) that $\sum_{f \in F \setminus \mu(W)} |\mu_W(f)| = 0$. Therefore, if $f \in F \setminus \mu(W)$ then $\mu_W(f) = \emptyset$, which means that $f \notin \mu_W(W)$, implying that $F \setminus \mu(W) \subseteq F \setminus \mu_W(W)$. Hence, $\mu_W(W) \subseteq \mu(W)$ and, by condition (2),

$$\mu(W) = \mu_W(W). \tag{5}$$

Claim A is proved. ■

Claim B $\mu_W(\mu(W)) = W$.

Proof of Claim B. Assume otherwise; that is, there exists $w' \in W$ such that $w' \notin \mu_W(\mu(W))$. Since $\mu_W(W) = \mu(W)$, $\mu_W(w') = \emptyset$; therefore,

$$\sum\nolimits_{f \in \mu(W)} |\mu_W(f)| < |W| = \sum\nolimits_{f \in \mu(W)} |\mu(f)|,$$

where the equality follows from condition (1). Therefore, there exists $\overline{f} \in \mu(W)$ such that $\left|\mu_W\left(\overline{f}\right)\right| < \left|\mu\left(\overline{f}\right)\right| \leq q_{\overline{f}}$, implying that we can find $\overline{w} \in \mu\left(\overline{f}\right) \setminus \mu_W\left(\overline{f}\right)$. By condition (1), $\overline{f} = \mu(\overline{w}) P(\overline{w}) \mu_W(\overline{w})$ and by the quota $q_{\overline{f}}$−separability of $P\left(\overline{f}\right)$ and the individual rationality of μ, the pair $\left(\overline{f}, \overline{w}\right)$ blocks μ_W in contradiction with its stability. Therefore,

$$\mu_W(\mu(W)) = W. \tag{6}$$

Claim B is proved. ■

Consider now the last step of the deferred acceptance algorithm where workers make offers (and which yields, as its outcome, matching μ_W). Let $\overline{w}$ be a worker who makes an offer to an acceptable firm f in the last step of the algorithm. If f rejects some worker w', then this worker is unmatched in μ_W since we are considering the last step of the algorithm; this contradicts Claim B. Therefore, f does not reject any worker. By the substitutability of $P(f)$ and condition (1)

$$\mu_W(f) = Ch(S_{f,\overline{w}} \cup \{\overline{w}\}, P(f)) = Ch(S_{f,\overline{w}}, P(f)) \cup \{\overline{w}\} \supseteq \mu(f) \cup \{\overline{w}\},$$

where $S_{f,\overline{w}} = \{w \in W \backslash \{\overline{w}\} \mid w \text{ makes an offer to } f \text{ during the algorithm}\}$. The last inclusion holds because $\mu(f) \subseteq S_{f,\overline{w}}$ (condition (1) implies that all $w \in \mu(f)$ make an offer to f during the algorithm). Therefore,

$$\mu_W(f) \supseteq \mu(f) \cup \{\overline{w}\}. \tag{7}$$

Then, since $\overline{w} \notin \mu(f)$, we have that $q_f \geq |\mu_W(f)| \geq |\mu(f)| + 1$ and thus, $|\mu(f)| < q_f$. If $|\mu(f)| > 0$ then we can find $w \in \mu(f) \cap \mu_W(f)$ contradicting condition (1). Therefore, $|\mu(f)| = 0$ which implies that $\mu(f) = \emptyset$. Condition (7) says that $\overline{w} \in \mu_W(f)$ holds. Hence, we obtain a contradiction, since $f \in \mu_W(W)$ and $f \notin \mu(W)$ imply that Claim A does not hold. ■

Example 2 below shows that the statement of Theorem 1 is false without the assumption that P is quota $q-$separable.

Example 2 Let $F = \{f_1, f_2, f_3\}$ and $W = \{w_1, w_2, w_3, w_4\}$ be the two sets of agents with the substitutable preference profile P, where

$$\begin{aligned}
P(f_1) &: w_1w_2, w_2, w_1, w_4,\\
P(f_2) &: w_3, w_2w_4, w_1w_2, w_4, w_1, w_2,\\
P(f_3) &: w_4, w_1, w_3,\\
P(w_1) &: f_2, f_3, f_1,\\
P(w_2) &: f_2, f_1,\\
P(w_3) &: f_3, f_2,\\
P(w_4) &: f_2, f_1, f_3.
\end{aligned}$$

The workers-optimal stable matching is

	f_1	f_2	f_3
μ_W	$w_1 w_2$	w_3	w_4.

The individually rational matching

	f_1	f_2	f_3
μ	w_4	$w_1 w_2$	w_3

has the property that $\mu(w) P(w) \mu_W(w)$ for all $w \in W$.

4 Group strategy-proofness

The following notation and definitions are needed to state the main result of the paper. Let $\mathcal{S}$ be the set of substitutable and quota $q-$separable preference relations of firms on 2^W and let $\mathcal{T}$ be the set of all preference relations of workers on $F \cup \{\emptyset\}$. The set of all substitutable and quota $q-$separable preference profiles can be written as the set $\mathcal{P} = \mathcal{S}^n \times \mathcal{T}^m$, where n and m are the number of firms and workers, respectively. Denote by $\mathcal{M}$ the set of all matchings.

A *mechanism* $h : \mathcal{P} \to \mathcal{M}$ maps each preference profile $P \in \mathcal{P}$ to a matching $h(P) \in \mathcal{M}$. Therefore, $h(P)(f)$ is the set of workers assigned to f and $h(P)(w)$ is the firm assigned to w (if any) at preference profile P by mechanism h. To emphasize the role of a subset of workers $\widehat{W}$ we will write the preference profile P as $\left(P_{\widehat{W}}, P_{-\widehat{W}}\right)$. Therefore, given $\widehat{W} \subseteq W$, $P \in \mathcal{P}$ and $\widehat{P}_{\widehat{W}} \in \mathcal{T}^{|\widehat{W}|}$ we write $\left(\widehat{P}_{\widehat{W}}, P_{-\widehat{W}}\right)$ to denote the preference profile P where the preference relations $P_{\widehat{W}} \in \mathcal{T}^{|\widehat{W}|}$ have been replaced by $\widehat{P}_{\widehat{W}} \in \mathcal{T}^{|\widehat{W}|}$. Mechanisms require each agent to report some preference relation. A mechanism is group strategy-proof for the workers if it is always in the best interest of all subsets of workers to reveal their preferences truthfully. Formally,

Definition 6 *A mechanism $h : \mathcal{P} \to \mathcal{M}$ is* ***group strategy-proof for the workers*** *if for all preference profiles $P \in \mathcal{P}$, all subsets of workers $\widehat{W} \subseteq W$, and all reports $\widehat{P}_{\widehat{W}} \in \mathcal{T}^{|\widehat{W}|}$,*

$$h(P)(w)\, R(w)\, h\left(\widehat{P}_{\widehat{W}}, P_{-\widehat{W}}\right)(w)$$

for all $w \in \widehat{W}$.

We say that $h : \mathcal{P} \to \mathcal{M}$ is the *workers-optimal stable mechanism* if it always selects the workers-optimal stable matching; that is, for all $P \in \mathcal{P}$, $h(P)$ is the workers-optimal stable matching relative to P. We are now ready to state the main result of the paper.

Theorem 2 *The workers-optimal stable mechanism $h : \mathcal{P} \to \mathcal{M}$ is group strategy-proof for the workers.*

Theorem 2 (as it is the case for the marriage model) is an immediate consequence of the Blocking Lemma, which states that if the set of workers that strictly prefer an individually rational matching μ to μ_W is nonempty then, we can always find a blocking pair (f, w) of μ with the property that f was hiring at μ a worker strictly preferring μ to μ_W and w considers μ_W being at least as good as μ. Gale and Sotomayor (1985) proved the Blocking Lemma for the marriage model. Here, in Theorem 3 below, we state that the Blocking Lemma also holds for the more general many-to-one model with substitutable and quota $q-$separable preferences, and hence Theorem 2 holds.[10]

Theorem 3 *Let P be a substitutable and quota $q-$separable preference profile and let $\mu \in IR(P)$. Denote by $W' = \{w \in W \mid \mu(w)\, P(w)\, \mu_W(w)\}$ the set of workers who strictly prefer μ to μ_W. If W' is nonempty, then there exist $f \in \mu(W')$ and $w \in W \backslash W'$ such that the pair (f, w) blocks μ.*

Example 2 at the end of Section 3 shows that Theorems 2 and 3 are false without the quota $q-$separability condition. Theorem 3 is false because μ is not stable since the (unique) pair (f_2, w_4) blocks μ but $w_4 \in W'$ because $\mu(w_4) = f_1 P(w_4)\, f_3 = \mu_W(w_4)$. Moreover, consider the preference relations $\widehat{P}_W \in \mathcal{T}^4$, where

$$\begin{aligned} \widehat{P}(w_1) &: f_2, \\ \widehat{P}(w_2) &: f_2, \\ \widehat{P}(w_3) &: f_3, \\ \widehat{P}(w_4) &: f_1. \end{aligned}$$

Let h be the workers-optimal stable mechanism. Then, $h\left(\widehat{P}_W, P_{-W}\right) = \mu P(w)\, \mu_W = h(P)$ for all $w \in W$, implying that h is not group strategy-proof for the workers.

[10] In the marriage model the Blocking Lemma also plays a fundamental role in the proof of the Strong Stability Theorem of Demange, Gale, and Sotomayor (1987).

The proof of Theorem 3 is in the Appendix. We warn the reader that it is involved and long. However, we will try to guide the reader through its main building blocks as well as to point out why the proof of the Blocking Lemma in the marriage model can not be immediately translated to the many-to-one case.

Proof of Theorem 2. The statement of Theorem 2 follows immediately from the following Claim.

Claim Let P be a substitutable and quota $q-$separable profile of preferences and let $\widehat{P}$ differ from P in that some nonempty subset $\widehat{W}$ of workers have different preferences. Then, there is no matching $\mu \in S(\widehat{P})$ such that $\mu P\left(w\right)\mu_W$ by all $w \in \widehat{W}$.

Proof of Claim. Assume otherwise; that is, there exists a nonempty subset of workers $\widehat{W}$ and a matching $\mu \in S(\widehat{P})$ such that for all $w \in \widehat{W}$, $\mu(w)P(w)\mu_W(w)$. We first show that $\mu \in IR\left(P\right)$. Since $\mu \in S(\widehat{P})$ and $\widehat{P}(i) = P(i)$ for all $i \in (F \cup W) \backslash \widehat{W}$, then μ is individually rational for all $i \notin \widehat{W}$. Moreover, if $w \in \widehat{W}$ then $\mu(w)P(w)\mu_W(w)R(w)\emptyset$. Hence, $\mu \in IR\left(P\right)$. Since all matchings $\mu' \in S\left(P\right)$ have the property that $\mu_W R\left(w\right)\mu'$ for all $w \in W$ and there exists at least one $w \in \widehat{W}$ with $\mu(w)P(w)\mu_W(w)$, we conclude that $\mu \notin S\left(P\right)$. Since $\widehat{W} \neq \emptyset$, we can apply the Blocking Lemma (Theorem 3) because $\emptyset \neq \widehat{W} \subseteq W' = \{w \in W \mid \mu(w)P(w)\mu_W(w)\}$. Thus, there is a pair $(\overline{f}, \overline{w})$, where $\overline{f} \in \mu(W')$ and $\overline{w} \in W \backslash W'$, that blocks μ in P, but $\overline{w} \in W \backslash W'$ implies that $\widehat{P}(\overline{w}) = P(\overline{w})$; therefore, $(\overline{f}, \overline{w})$ blocks μ in $\widehat{P}$, contradicting that $\mu \in S(\widehat{P})$. ■

5 Appendix: The Proof of the Blocking Lemma

Through out all this Appendix we will assume that P is a substitutable and quota $q-$separable profile of preferences and that $\mu \in IR\left(P\right)$. The set of workers who strictly prefer μ to μ_W will be denoted by $W' = \{w \in W \mid \mu\left(w\right)P\left(w\right)\mu_W\left(w\right)\}$ and we will assume that W' is nonempty.

The proof of the Blocking Lemma will be decomposed, as in the marriage model, into two propositions depending on whether or not $\mu(W')$ is equal to $\mu_W(W')$ (Proposition 2, in Subsection 5.2, for the more simple case where they are different and Proposition 3, in Subsection 5.3, for the more involved case where they are equal). However, before proving separately Propositions 2 and 3, we give the proof of Proposition 1, in Subsection 5.1, which says that the Blocking Lemma holds for the particular case where, regardless of whether or not $\mu\left(W'\right)$ is equal to $\mu_W\left(W'\right)$, there exists a firm in $\mu\left(W'\right)$ which does not fill its quota at

μ. But before moving to these three subsections, we also prove a series of four claims and Lemma 1 which will be used in the proof of all three propositions since they hold regardless of whether or not $\mu(W')$ is equal to $\mu_W(W')$ and whether or not all firms in $\mu(W')$ fill their quota at μ.

Claim 1.1 For each $f \in \mu(W')$, $|\mu_W(f)| = q_f$.

Proof of Claim 1.1. Assume otherwise and let $f' \in \mu(W')$ be such that $|\mu_W(f')| < q_{f'}$. Since W' is nonempty and $f' \in \mu(W')$ there exists $w' \in W'$ such that $f' = \mu(w') P(w') \mu_W(w')$, which implies that $w' \notin \mu_W(f')$. Moreover, $\mu \in IR(P)$, the quota $q_{f'}$−separability of $P(f')$, and $|\mu_W(f')| < q_{f'}$ imply that $w' \in Ch(\mu_W(f') \cup \{w'\}, P(f'))$. Thus, the pair (f', w') blocks μ_W, which is a contradiction. ■

Claim 1.2 Assume there exist $f \in \mu(W')$ and $w \in Ch(\mu(f) \cup \mu_W(f), P(f)) \backslash \{[\mu_W(f) \cap W'] \cup [\mu(f) \cap (W \backslash W')]\}$. Then, $w \in W \backslash W'$ and the pair (f, w) blocks μ.

Proof of Claim 1.2. Since $w \in Ch(\mu(f) \cup \mu_W(f), P(f))$ and $w \notin [\mu_W(f) \cap W'] \cup [\mu(f) \cap (W \backslash W')]$, we have that either $w \in W'$ and $w \in \mu(f) \backslash \mu_W(f)$ or $w \in W \backslash W'$ and $w \in \mu_W(f) \backslash \mu(f)$. Assume $w \in W'$ and $w \in \mu(f) \backslash \mu_W(f)$. Then,

$$f = \mu(w) P(w) \mu_W(f). \tag{8}$$

Moreover, $w \in Ch(\mu(f) \cup \mu_W(f), P(f))$ implies, by the substitutability of $P(f)$, that $w \in Ch(\mu_W(f) \cup \{w\}, P(f))$, which together with condition (8) imply that the pair (f, w) blocks μ_W. Therefore, we can assume that $w \in W \backslash W'$ and $w \in \mu_W(f) \backslash \mu(f)$. Then,

$$f = \mu_W(w) P(w) \mu(w). \tag{9}$$

Moreover, $w \in Ch(\mu(f) \cup \mu_W(f), P(f))$ implies, by the substitutability of $P(f)$, that $w \in Ch(\mu(f) \cup \{w\}, P(f))$, which together with condition (9) imply that the pair (f, w) blocks μ. ■

Claim 1.3 Assume there exists $f \in \mu(W')$ such that $|\mu(f) \cap W'| > |\mu_W(f) \cap W'|$. Then, there exists $w \in W \backslash W'$ such that the pair (f, w) blocks μ.

Proof of Claim 1.3. Assume $f \in \mu(W')$. We will first show that $|\mu(f) \cap W'| > |\mu_W(f) \cap W'|$ implies that there exists $w \in Ch(\mu(f) \cup \mu_W(f), P(f)) \backslash \{[\mu_W(f) \cap W'] \cup [\mu(f) \cap (W \backslash W')]\}$. To see this, first observe that, by Claim 1.1, $|\mu_W(f)| = q_f$. Moreover,

$$|\mu_W(f)| = |\mu_W(f) \cap W'| + |\mu_W(f) \cap (W \backslash W')| = q_f$$

and

$$|\mu(f)| = |\mu(f) \cap W'| + |\mu(f) \cap) W \backslash W')| \leq q_f.$$

By hypothesis, $|\mu_W(f) \cap W'| + |\mu(f) \cap (W \backslash W')| < q_f$, and $|\mu_W(f)| = q_f$ implies that $|Ch(\mu(f) \cup \mu_W(f), P(f))| = q_f$. Hence, there exists $w \in Ch(\mu(f) \cup \mu_W(f), P(f)) \backslash \{[\mu_W(f) \cap W'] \cup [\mu(f) \cap (W \backslash W')]\}$. By Claim 1.2, the pair (f, w) blocks μ. ■

Claim 1.4 Assume there exists $f \in \mu(W')$ such that $|\mu(f) \cap W'| < |\mu_W(f) \cap W'|$. Then, there exist $\widetilde{f} \in \mu(W')$ and $w \in W \backslash W'$ such that the pair $(\widetilde{f}, w)$ blocks μ.

Proof of Claim 1.4. First observe that

$$|W'| = \sum_{\overline{f} \in \mu(W')} \left|\mu(\overline{f}) \cap W'\right| \geq \sum_{\overline{f} \in \mu(W')} \left|\mu_W(\overline{f}) \cap W'\right|,$$

which implies, by the hypothesis that there exists $f \in \mu(W')$ such that $|\mu(f) \cap W'| < |\mu_W(f) \cap W'|$, that there exists $\widetilde{f} \in \mu(W')$ with the property that $\left|\mu(\widetilde{f}) \cap W'\right| > \left|\mu_W(\widetilde{f}) \cap W'\right|$. This $\widetilde{f}$ satisfies the hypothesis of Claim 1.3, and hence, there exists $w \in W \backslash W'$ such that the pair $(\widetilde{f}, w)$ blocks μ. ■

Lemma 1 *Assume there exists $f \in \mu(W')$ such that $|\mu(f) \cap W'| \neq |\mu_W(f) \cap W'|$. Then, there exist $\widetilde{f} \in \mu(W')$ and $w \in W \backslash W'$ such that the pair $(\widetilde{f}, w)$ blocks μ.*

Proof. It follows from Claims 1.3 and 1.4. ■

5.1 There exists $f \in \mu(W')$ such that $|\mu(f)| < q_f$

We prove in Proposition 1 below that the Blocking Lemma holds for the case where (regardless of whether or not $\mu(W')$ is equal to $\mu_W(W')$) there exists $f \in \mu(W')$ such that $|\mu(f)| < q_f$.[11]

Proposition 1 *Assume $f \in \mu(W')$ is such that $|\mu(f)| < q_f$. Then, there exist $\widetilde{f} \in \mu(W')$ and $w \in W \backslash W'$ such that the pair $(\widetilde{f}, w)$ blocks μ.*

Proof. Let $f \in \mu(W')$ be such that $|\mu(f)| < q_f$. By Claim 1.1, $|\mu_W(f)| = q_f$. We consider two cases:

[11] Notice that $f \in \mu(W')$ and $|\mu(f)| < 1$ are incompatible in the marriage model.

Case 1. $|\mu(f) \cap W'| \neq |\mu_W(f) \cap W'|$. By Lemma 1, there exist $\widetilde{f} \in \mu(W')$ and $w \in W \backslash W'$ such that $(\widetilde{f}, w)$ blocks μ.

Case 2. $|\mu(f) \cap W'| = |\mu_W(f) \cap W'|$. Then, $|\mu(f) \cap (W \backslash W')| < |\mu_W(f) \cap (W \backslash W')|$ since $|\mu(f)| < q_f = |\mu_W(f)|$. Hence, there exists $w \in (\mu_W(f) \cap (W \backslash W')) \setminus (\mu(f) \cap (W \backslash W'))$. In particular, $w \in (\mu_W(f) \cap (W \backslash W')) \setminus \mu(f)$. Therefore, since $w \notin W'$, $w \notin \mu(f)$, and $w = \mu_W(f)$ we have that

$$fP(w)\mu(w). \tag{10}$$

Moreover, by the quota q_f-separability of $P(f)$ and the individual rationality of μ_W, w is a good worker for f. Hence, $|\mu(f)| < q_f$ implies

$$w \in Ch(\mu(f) \cup \{w\}, P(f)). \tag{11}$$

Conditions (10) and (11) say that $f \in \mu(W')$ and $w \in W \backslash W'$ are such that the pair (f, w) blocks μ. ■

Therefore, from now on, and without loss of generality, we will assume that $|\mu(f)| = q_f$ for all $f \in \mu(W')$.

5.2 The case $\mu(W') \neq \mu_W(W')$

Before stating and proving the Blocking Lemma for this case (Proposition 2 below), we find it useful to illustrate the difficulties of extending the proof used in the marriage model to the many-to-one model. In the marriage model the proof proceeds as follows. Since $\mu(W') \neq \mu_W(W')$ consider a firm $f \in \mu(W') \backslash \mu_W(W')$. Look at (*the unique*) $w' = \mu(f)$ who also belongs to W'. Hence,

$$f = \mu(w')P(w')\mu_W(w'). \tag{12}$$

Since $\mu_W \in S(P)$, there must exist $w \in W$ such that

$$w = \mu_W(f)P(f)\mu(f), \tag{13}$$

otherwise (f, w') would block μ_W. But $f \neq \mu_W(w')$ implies that $w \notin W'$, since $f \notin \mu_W(W')$ and $\mu_W(f) = w$. Hence,

$$f = \mu_W(w)P(w)\mu(w). \tag{14}$$

Conditions (13) and (14), $f \in \mu(W')$, and $w \in W \backslash W'$ imply that the conclusion of the Blocking Lemma holds.

Now, we come back to our many-to-one model. Consider a firm $f \in \mu(W')\backslash\mu_W(W')$. Look at one (among potentially many) $w' \in \mu(f)$. By construction, $w' \in W'$ and therefore condition (12) still holds. But now, and as a consequence of the stability of μ_W, we have

$$w' \notin Ch(\mu_W(f) \cup \{w'\}, P(f)). \tag{15}$$

However, condition (15) does not give us immediately the existence of $w \notin W'$ such that $w \in Ch(\mu(f) \cup \{w\}, P(f))$.[12] To find a worker with this property, we have to look at the set $Ch(\mu_W(f) \cup \mu(f), P(f))$, because then, by substitutability of $P(f)$, we will have $w \in Ch(\mu(f) \cup \{w\}, P(f))$. Moreover, condition $\mu_W(f) \subseteq W\backslash W'$ now implies that condition (14) only holds in the weak form; that is, $f = \mu_W(w)R(w)\mu(w)$ because it could well be the case that $w \in \mu(f)$ (and hence $\mu_W(w) = \mu(w)$). Claims 1.2, 1.3, and 1.4 solve this double difficulty of identifying such a worker w in $Ch(\mu(f) \cup \mu_W(f), P(f))$ with the property that $\mu_W(w)P(w)\mu(w)$.[13]

Proposition 2 *Assume $\mu(W') \neq \mu_W(W')$. Then, there exist $f \in \mu(W')$ and $w \in W\backslash W'$ such that the pair (f, w) blocks μ.*

Proof. Consider the following two cases:

Case 1. There exists $f \in \mu(W')\backslash\mu_W(W')$. This means that we can find $w' \in W'$ such that $w' \in \mu(f)$, implying $|\mu(f) \cap W'| \geq 1$. Moreover, $f \notin \mu_W(W')$ implies that $|\mu_W(f) \cap W'| = 0$. Therefore, $|\mu(f) \cap W'| > |\mu_W(f) \cap W'|$. Hence, by Claim 1.3 the statement of Proposition 2 holds.

Case 2. There exists $f \in \mu_W(W')\backslash\mu(W')$. This means that we can find $w' \in W'$ such that $w' \in \mu_W(f)$, implying $|\mu_W(f) \cap W'| \geq 1$. Moreover, $f \notin \mu(W')$ implies that $|\mu(f) \cap W'| = 0$. Therefore, $|\mu(f) \cap W'| < |\mu_W(f) \cap W'|$. Hence, by Claim 1.4 the statement of Proposition 2 holds. ■

5.3 The case $\mu(W') = \mu_W(W')$

Proposition 3 *Assume $\mu(W') = \mu_W(W')$. Then, there exist $f \in \mu(W')$ and $w \in W\backslash W'$ such that the pair (f, w) blocks μ.*

[12] Condition (15) will imply that, since $P(f)$ is quota q_f-separable, $|\mu_W(f)| = q_f$ whenever $f \in \mu(W')$. However, Claim 1.1 is more general because it also holds for the case $\mu(W') = \mu_W(W')$.

[13] Although they do it in the more general case without assuming $\mu(W') \neq \mu_W(W')$ and $|\mu(f)| = q_f$.

Before stating and proving several Claims and Lemmas needed to prove Proposition 3 we describe our strategy of proof as well as discuss why the natural extension of the proof for the marriage model does not work in this many-to-one setup.

There are two alternative proofs of the Blocking Lemma for the marriage model in the case where $\mu(W') = \mu_W(W')$ (see Roth and Sotomayor, 1990). The first one is based on the deferred-acceptance algorithm in which workers make offers. The second one derives the existence of the desired blocking pair of μ directly from the stability and optimality properties of μ_W. Our proof will mix both arguments.

The proof based on the deferred-acceptance algorithm in which workers make offers identifies the blocking pair $(\overline{f}, \overline{w})$ of μ by looking at one of the last firms receiving an offer from a worker $w' \in W'$. It is argued then that $\overline{f}$ will necessarily reject, as the consequence of having received this offer, a worker $\overline{w} \notin W'$. Therefore,

$$\overline{f} P(\overline{w}) \mu_W(\overline{w}) R(\overline{w}) \mu(\overline{w}) \tag{16}$$

and

$$\mu_W(\overline{f}) P(\overline{f}) \overline{w} P(\overline{f}) \mu(\overline{f}) \in W'. \tag{17}$$

The first strict preference of condition (17) is a consequence of $\overline{w}$ being rejected by $\overline{f}$ while the second one follows from the fact that, before receiving $\overline{w}$'s offer (which was accepted by $\overline{f}$), worker $\mu(\overline{f})$ had already made an offer to $\overline{f}$.

In the many-to-one setup, condition (16) is still valid but the second strict preference in condition (17) should be written as $\overline{w} \in Ch(\mu(\overline{f}) \cup \{\overline{w}\}, P(\overline{f}))$. But now we can not guarantee (as we did to justify the second strict preference in condition (17)) that all workers in $\mu(\overline{f})$ had already made an offer to $\overline{f}$ before it receives $\overline{w}$'s offer (for instance, workers in $W \backslash W'$ who strictly prefer μ_W to μ may not have made such offers). Therefore, our proof will be based on the stability and optimality properties of μ_W although, at the end, we will have to comeback to the proof using the deferred-acceptance algorithm applied to a matching market with an adequately modified preference profile P'. Before proceeding with the formal proof of Proposition 3 we want to single out one of its general features as well as some difficulties.

Most of our proof is carried out for the particular case where $\mu(w) R(w) \mu_W(w)$ for all $w \in W$. Once the Blocking Lemma is established for this case, we extend it to the general case.

The first difficulty arises because in the marriage model $\mu(W') = \mu_W(W') = F'$ holds if and only if $\mu(F') = \mu_W(F') = W'$ holds as well. Claim 3 below shows that $\mu(F') =$

$\mu_W(F')$ whenever $\mu(w)R(w)\mu_W(w)$ for all $w \in W$. Moreover, in the many-to-one setup the set $\mu(F') = \mu_W(F') \equiv W''$ may contain workers outside W'. The proof in the marriage model (based on the stability and optimality properties of μ_W) proceeds by defining a new matching market (F', W', P') and its corresponding workers-optimal stable matching μ'_W, where P' coincides with P except that, for all $f \in F'$, $P'(f)$ is the truncation of $P(f)$ below $\mu(f)$ (i.e., $\mu(f)$ is the last acceptable mate for f in $P'(f)$).

The second difficulty we face now is that the truncation $P'(f)$ may generate a non substitutable preference profile, in which case the new matching market (F', W'', P') may not have a workers-optimal stable matching. We overcome this difficulty by undertaking a non-trivial and drastic modification of $P(f)$ which guarantees substitutability as well as other properties that will be required later. The proof in the marriage model continues by showing that $\mu'_W \neq \mu_W$ holds because otherwise, we would have $\mu(w)P'(w)\mu'_W(w)$ for all $w \in W'$ contradicting the weak Pareto-optimality of μ'_W in (F', W', P').

Note that in our artificial many-to-one matching market (F', W'', P'), we can only guarantee that $\mu(w)R'(w)\mu'_W(w)$ for all $w \in W \backslash W'$, which is not sufficient to contradict our weak Pareto-optimality result of Theorem 1.

We solve this third difficulty by defining a new matching market (F', W'', P'') where P'' coincides with P' except that for all $w \in W'' \backslash W'$, $P''(w)$ has $\mu(w)$ as the unique acceptable firm. Since P' was substitutable, P'' remains substitutable. Therefore, we can look at μ''_W and show, using the deferred-acceptance algorithm in which workers make offers, that if $\mu''_W = \mu'_W$ then $\mu''_W \neq \mu_W$ (observe that the Blocking Lemma follows directly when $\mu''_W \neq \mu_W$).

The proof finishes (as in the marriage model) by showing that the matching $\overline{\mu}_W$ in the original matching market (F, W, P) (obtained from μ'_W and μ_W by letting $\overline{\mu}_W(f) = \mu'_W(f)$ if $f \in F'$ and $\overline{\mu}_W(f) = \mu_W(f)$ if $f \notin F'$) is not stable and that the pair (f, w) that blocks $\overline{\mu}_W$ also blocks μ, and $f \in \mu(W')$ and $w \in W \backslash W'$.

We now go back to the formal proof. As we have already said, the proof of Proposition 3 will also be decomposed into two different parts. First, we will show that its conclusion holds for the particular case where $\mu(w)R(w)\mu_W(w)$ for all $w \in W$. Then, using this fact, we will show that Proposition 3 also holds for the general case.

Assume $\mu(W') = \mu_W(W')$ and denote this set of firms by F'; that is, $F' = \mu(W')$.

Claim 3 Assume $\mu R(w)\mu_W$ for all $w \in W$. Then, $\mu(F') = \mu_W(F')$.

Proof of Claim 3. First note that

$$\mu(w) = \mu_W(w) \text{ for all } w \in W \backslash W'. \tag{18}$$

Now,

$$\begin{aligned} \mu(F') &= \bigcup_{f\in F'} \{[\mu(f) \cap W'] \cup [\mu(f) \cap W \backslash W']\} \\ &= [\bigcup_{f\in F'} \mu(f) \cap W'] \cup [\bigcup_{f\in F'} \mu_W(f) \cap W \backslash W'] \\ &= [\mu(F') \cap W'] \cup [\mu_W(F') \cap W \backslash W'] \\ &= [\mu_W(F') \cap W'] \cup [\mu_W(F') \cap W \backslash W'] \\ &= \mu_W(F'), \end{aligned}$$

where the second equality follows from condition (18) and the fourth equality follows from the assumption that $\mu(W') = \mu_W(W')$. ■

Assume $\mu R(w)\mu_W$ holds for all $w \in W$. Define the set of workers $W'' = \mu(F') = \mu_W(F')$ and the new matching market (F', W'', P'), where the preference profile P' is defined from P as follows: For every $w \in W''$, $P'(w)$ coincides with $P(w)$ on $F' \cup \{w\}$. For every $f \in F'$, let $P'(f)$ be any preference relation on the subsets of W'' compatible with the following two properties:

(INT) $SP'(f)\emptyset$ if and only if $S = S \cap Ch\,(S \cup \mu\,(f)\,, P(f))\,.$

(MAX) $S_1P'(f)S_2P'(f)\emptyset$ if and only if $\widehat{S}_i = \max_{P(f)}\{S \subseteq W'' \mid S \cap Ch(S \cup \mu(f), P(f)) = S_i\}$, $i = 1, 2$, and $\widehat{S}_1 P(f) \widehat{S}_2$.

Lemma 2 below states that any $P'(f)$ satisfying properties (INT) and (MAX) above is substitutable. But before stating it, we find useful to describe an algorithmic way of obtaining such a preference relation. Moreover, the algorithm shows that such preference relation on $2^{W''}$ always exists.

Assume $f \in F'$ and let $P(f)$ be its preference relation on 2^W. Given W'', define a new preference relation $P'(f)$ on $2^{W''}$ by the following algorithm:

Step -1 (Cleaning): Eliminate from the ordering $P(f)$ all sets containing workers in $W \backslash W''$. Denote this preference relation on $2^{W''}$ by $\widetilde{P}(f)$.[14]

Step 0 (Renaming): Rename the acceptable sets of workers according to $\widetilde{P}(f)$ in such a way that

$$\widetilde{P}(f) : \widetilde{S}_1, \widetilde{S}_2, ..., \widetilde{S}_r.$$

[14] Obviously, preferences $P(f)$ and $\widetilde{P}(f)$ coincide on $2^{W''}$. Martínez, Massó, Neme, and Oviedo (2001) shows that $\widetilde{P}(f)$ remains substitutable.

Step 1 (Constructing): Let $\widetilde{S}_1$ be the best set according to $P'(f)$. Set $T^1 = \{\widetilde{S}_1\}$.

Let T^k be the output of Step k, with $k < r$.

Step $k+1$ (Constructing): Define $Q_{k+1} = \widetilde{S}_{k+1} \cap Ch(\widetilde{S}_{k+1} \cup \mu(f), P(f))$. If $Q_{k+1} \in T^k$, set $T^{k+1} = T^k$. If $Q_{k+1} \notin T^k$ locate Q_{k+1} in the $(\left|T^k\right| + 1)$-th place in the ordering of $P'(f)$ and set $T^{k+1} = T^k \cup \{Q_{k+1}\}$. The set T^{k+1} is the output of Step $k+1$.

The algorithm stops at Step r with $|T^r|$ acceptable and ordered subsets of W'' (notice that $|T^r| \leq r$). Unacceptable subsets are unordered. Let $P'(f)$ be any preference relation on $2^{W''}$ that coincides with the ordering of these acceptable subsets. Example 3 below illustrates the algorithm.

Example 3 Let $W = \{w_1, w_2, w_3, w_4\}$ be the set of workers and assume that $W'' = \{w_1, w_2, w_4\}$. Consider firm f with $\mu(f) = \{w_2, w_4\}$ and the substitutable and quota $2-$separable preference relation

$$P(f) : w_1w_2, w_1w_3, w_2w_4, w_3w_4, w_1w_4, w_2w_3, w_4, w_3, w_1, w_2.$$

Step -1 (Cleaning): $\widetilde{P}(f) : w_1w_2, w_2w_4, w_1w_4, w_4, w_1, w_2$.

Step 0 (Renaming): Obvious with $r = 6$.

Step 1 (Constructing): $\{w_1, w_2\}$ is the best subset according to $P'(f)$. Set $T^1 = \{\{w_1, w_2\}\}$.

Step 2 (Constructing): Since $Q_2 = \{w_2, w_4\} \cap Ch(\{w_2, w_4\} \cup \{w_2, w_4\}, P(f)) = \{w_2, w_4\} \notin T^1$, the set $\{w_2, w_4\}$ is located in the second place in the ordering $P'(f)$ and let $T^2 = \{\{w_1, w_2\}, \{w_2, w_4\}\}$.

Step 3 (Constructing): Since $Q_3 = \{w_1, w_4\} \cap Ch(\{w_1, w_4\} \cup \{w_2, w_4\}, P(f)) = \{w_1, w_4\} \cap \{w_1, w_2\} = \{w_1\} \notin T^2$, the set $\{w_1\}$ is located in the third place in the ordering $P'(f)$ and let $T^3 = \{\{w_1, w_2\}, \{w_2, w_4\}, \{w_1\}\}$.

Step 4 (Constructing): Since $Q_4 = \{w_4\} \cap Ch(\{w_4\} \cup \{w_2, w_4\}, P(f)) = \{w_4\} \cap \{w_2, w_4\} = \{w_4\} \notin T^3$, the set $\{w_4\}$ is located in the fourth place in the ordering $P'(f)$ and let $T^4 = \{\{w_1, w_2\}, \{w_2, w_4\}, \{w_1\}, \{w_4\}\}$.

Step 5 (Constructing): Since $Q_5 = \{w_1\} \cap Ch(\{w_1\} \cup \{w_2, w_4\}, P(f)) = \{w_1\} \cap \{w_1, w_2\} = \{w_1\} \in T^4$ set $T^5 = T^4$.

Step 6 (Constructing): Since $Q_6 = \{w_2\} \cap Ch(\{w_2\} \cup \{w_2, w_4\}, P(f)) = \{w_2\} \cap \{w_2, w_4\} = \{w_2\} \notin T^5$, the set $\{w_2\}$ is located in the fifth place in the ordering $P'(f)$ and let $T^6 = \{\{w_1, w_2\}, \{w_2, w_4\}, \{w_1\}, \{w_4\}, \{w_2\}\}$.

The algorithm stops at Step 6 with five acceptable and strictly ordered subsets of W''. Therefore, let $P'(f)$ be any preference relation on subsets of $W'' = \{w_1, w_2, w_4\}$ such that

$$P'(f) : w_1w_2, w_2w_4, w_1, w_4, w_2.$$

To finish the example, it is easy to check that $P'(f)$ satisfies properties (INT) and (MAX) of its general definition. To illustrate property (INT) consider $\{w_1\}$ and note that $\{w_1\}P'(f)\emptyset$ and $\{w_1\} = \{w_1\} \cap Ch(\{w_1\} \cup \{w_2, w_4\}, P(f))$. To illustrate property (MAX), consider $\{w_1\}$ and $\{w_4\}$; note that $\{w_1\}P'(f)\{w_4\}$ and

$$\{w_1, w_4\} = \max_{P(f)}\{S \subseteq W'' \mid S \cap Ch(S \cup \{w_2, w_4\}, P(f)) = \{w_1\}\},$$

$$\{w_4\} = \max_{P(f)}\{S \subseteq W'' \mid S \cap Ch(S \cup \{w_2, w_4\}, P(f)) = \{w_4\}\},$$

and $\{w_1, w_4\}P(f)\{w_4\}$.

Lemma 2 *$P'(f)$ is substitutable.*

Proof. See Subsection 5.4 at the end of the paper. ■

Lemma 3 *Assume $\mu(w)R(w)\mu_W(w)$ for all $w \in W$ and $\mu(W') = \mu_W(W')$. Then, there exist $f \in \mu(W')$ and $w \in W \backslash W'$ such that the pair (f, w) blocks μ.*

Proof. The proof follows from Claims 3.1 to 3.6 below.

Claim 3.1 For each $f \in F'$, $\mu_W(f) = Ch\left(\mu_W(f) \cup \mu(f), P(f)\right)$.

Proof of Claim 3.1. Assume $\mu_W(\overline{f}) \neq Ch\left(\mu_W(\overline{f}) \cup \mu(\overline{f}), P(\overline{f})\right)$ for some $\overline{f} \in F'$. By Claim 1.1, $\left|\mu_W(\overline{f})\right| = q_{\overline{f}}$. By the quota $q_{\overline{f}}$−separability of $P(\overline{f})$, $\left|Ch\left(\mu_W(\overline{f}) \cup \mu(\overline{f}), P(\overline{f})\right)\right| = q_{\overline{f}}$. Therefore, there exists $\overline{w} \in \mu(\overline{f})$ such that $\overline{w} \in Ch\left(\mu_W(\overline{f}) \cup \mu(\overline{f}), P(\overline{f})\right) \backslash \mu_W(\overline{f})$. By the substitutability of $P(\overline{f})$,

$$\overline{w} \in Ch\left(\mu_W(\overline{f}) \cup \{\overline{w}\}, P(\overline{f})\right). \tag{19}$$

Since $\overline{w} \in \mu(\overline{f}) \backslash \mu_W(\overline{f})$ and $\overline{f} \in F'$ we have that $\overline{w} \in W'$, which means

$$\overline{f} = \mu(\overline{w})P(\overline{w})\mu_W(\overline{w}). \tag{20}$$

Conditions (19) and (20) imply that the pair $(\overline{f}, \overline{w})$ blocks μ_W, which is a contradiction. ■

Define $\mu_W|_{F'\cup W''}$ as the restriction of μ_W to the sets F' and W''.

Claim 3.2 $\mu_W|_{F'\cup W''} \in S(P')$.

Proof of Claim 3.2. Assume there exists a pair $(\overline{f}, \overline{w})$ which blocks $\mu_W|_{F'\cup W''}$ in the matching market (F', W'', P') . That is,

$$\overline{f} P'(\overline{w}) \mu_W(\overline{w}) \tag{21}$$

and

$$\overline{w} \in Ch\left(\mu_W(\overline{f}) \cup \{\overline{w}\}, P'(\overline{f})\right). \tag{22}$$

Condition (21) implies that

$$\overline{f} P(\overline{w}) \mu_W(\overline{w}). \tag{23}$$

Since $\mu_W \in S(P)$, condition (23) implies

$$\overline{w} \notin Ch\left(\mu_W(\overline{f}) \cup \{\overline{w}\}, P(\overline{f})\right). \tag{24}$$

Now,

$$Ch\left(\mu_W(\overline{f}) \cup \{\overline{w}\}, P(\overline{f})\right) = Ch\left(\mu_W(\overline{f}) \cup \{\overline{w}\}, P'(\overline{f})\right), \tag{25}$$

since

$$\begin{aligned} Ch\left(\mu_W(\overline{f}) \cup \{\overline{w}\}, P(\overline{f})\right) &\overset{(1)}{=} Ch\left(Ch\left(\mu_W(\overline{f}) \cup \mu(\overline{f}), P(\overline{f})\right) \cup \{\overline{w}\}, P(\overline{f})\right) \\ &\overset{(2)}{=} Ch\left(\mu_W(\overline{f}) \cup \mu(\overline{f}) \cup \{\overline{w}\}, P(\overline{f})\right) \end{aligned}$$

Equality (1) follows from Claim 3.1 and equality (2) follows from Remark 2. Then, Claim 2.3 used in the proof of Lemma 2 (see Subsection 5.4 at the end of the paper) implies that condition (25) holds. Therefore, conditions (22) and (24) contradict the equality of condition (25). ■

Let μ'_W be the workers-optimal stable matching corresponding to the matching market (F', W'', P'). This matching exists by Lemma 2.

Claim 3.3. For each $f \in F'$, $\mu'_W(f) = Ch(\mu'_W(f) \cup \mu(f), P(f))$.

Proof of Claim 3.3. We will first show that $|\mu'_W(f)| = q_f$ for all $f \in F'$. By Claim 3.2 and the optimality of μ'_W we know that for all $w \in W''$,

$$\mu'_W(w) R'(w) \mu_W \mid_{F'\cup W''} (w) = \mu_W(w) \neq w.$$

Therefore, $\mu'_W(w) \neq w$, which implies

$$\mu'_W(w) \in F' \text{ for all } w \in W''. \tag{26}$$

Assume there exists $\overline{f} \in F'$ such that $\left|\mu'_W(\overline{f})\right| < q_{\overline{f}}$. Thus,

$$\begin{aligned} |W''| &= \sum_{f\in F'} |\mu_W(f)| && \text{since } W'' = \mu_W(F') \\ &= \sum_{f\in F'} q_f && \text{by Claim 1.1} \\ &> \sum_{f\in F'} \mu\,|'_W(f)| && \text{by the contradiction of the hypothesis } \left|\mu'_W(\overline{f})\right| < q_{\overline{f}}. \end{aligned}$$

Therefore, there must exists $\widetilde{w} \in W''\backslash\mu'_W(F')$; but this contradicts condition (26). Hence,

$$|\mu'_W(f)| = q_f \text{ for all } f \in F'. \tag{27}$$

Moreover, $\mu'_W(f)P'(f)\emptyset$ for all $f \in F'$. By property (INT) in the definition of $P'(f)$,

$$\mu'_W(f) = \mu'_W(f) \cap Ch(\mu'_W(f) \cup \mu(f), P(f)).$$

Therefore, by condition (27), $\mu'_W(f) = Ch(\mu'_W(f) \cup \mu(f), P(f))$. ■

Claim 3.4 $\mu'_W \neq \mu_W|_{F'\cup W''}$.

Proof of Claim 3.4. Consider the deferred-acceptance algorithm where workers make offers in the matching market (F', W'', P'). Let k denote the last step of the algorithm; that is, the provisional matching produced at the end of this step is μ'_W. Let $f \in F'$ and denote by S^j_f the set of workers that made an offer to f at step j of the algorithm. Now, let $\widehat{W}_f$ be the set of workers outside W' that made an offer to f during the algorithm which was provisionally accepted. That is,

$$\widehat{W}_f = \left\{ w \in W''\backslash W' \mid fP'(w)\mu(w) = \mu_W(w) \text{ and } \exists t_w < k \text{ s.t. } w \in Ch(\bigcup_{j=1}^{t_w} S^j_f, P'(f)) \right\}.$$

The proof of Claim 3.4 is by contradiction. However, before proceeding with it, we prove, in Claim 3.4.1 below, that if $\widehat{W}_f \neq \emptyset$ the conclusion of the Blocking Lemma follows, and hence, Lemma 3. Therefore, once Claim 3.4.1 is proven we will be able to assume in the remaining part of the proof of Claim 3.4 that $\widehat{W}_f = \emptyset$ for all $f \in F'$.

Claim 3.4.1 Assume $f \in F'$ and $\widehat{W}_f \neq \emptyset$. Then, there is $w \in W''\backslash W'$ such that the pair (f, w) blocks μ.

Proof of Claim 3.4.1. Let $w \in \widehat{W}_f$. By definition of $\widehat{W}_f$, $fP'(w)\mu(w)$ and, by definition of $P'(w)$,

$$fP(w)\mu(w). \tag{28}$$

Moreover, there exists $t_w < k$ such that $w \in Ch(\bigcup_{j=1}^{t_w} S_f^j, P'(f))$. By property (INT) in the definition of $P'(f)$,

$$w \in Ch(\bigcup_{j=1}^{t_w} S_f^j, P'(f)) \subseteq Ch\left(Ch(\bigcup_{j=1}^{t_w} S_f^j, P'(f)) \cup \mu(f), P(f)\right).$$

By the substitutability of $P(f)$,

$$w \in Ch\left(\mu(f) \cup \{w\}, P(f)\right). \tag{29}$$

Conditions (28) and (29) say that the pair (f, w) blocks μ. ■

To get either a contradiction, or else that we are already able to identify the blocking pair (f, w) of μ whose existence would establish immediately that Lemma 3 holds, we would like to look at the offers made in the deferred acceptance algorithm at the step previous to the last one. However, there may be workers in $W''\backslash W'$ making their offers precisely in the last step and therefore, these workers have not been provisionally assigned to an $f \in F'$ yet. In order to follow our strategy of proof we have to be sure that this is not the case. Therefore, we will modify the preferences of workers in $W''\backslash W'$ in such a way they will make an offer to $\mu(w) \in F'$ just in the first step of the algorithm.

Assume $\mu'_W = \mu_W \mid_{F' \cup W''}$. Let (F', W'', P'') be the matching market where P'' is the substitutable profile of preferences defined from P' as follows:

$$\begin{array}{ll} P''(f) = P'(f) & \text{for all } f \in F', \\ P''(w) = P'(w) & \text{for all } w \in W', \text{ and} \\ P''(w) : \mu(w) & \text{for all } w \in W''\backslash W'. \end{array}$$

Remember that $\mu(w) = \mu_W(w) = \mu'_W(w)$ for all $w \in W''\backslash W'$ and observe that P'' is substitutable.

Let μ''_W be the workers-optimal stable matching for the matching market (F', W'', P''). The next three claims refer to properties of this matching.

Claim 3.4.2 $\mu'_W \in S(P'')$ and $\mu''_W(w) = \mu'_W(w)$ for all $w \in W''\backslash W'$.

Proof of Claim 3.4.2. Assume that there exists a pair $(\overline{f}, \overline{w})$ that blocks μ'_W in the matching market (F', W'', P''). Therefore $\overline{w} \notin \mu'_W(\overline{f})$,

$$\overline{f} P''(\overline{w}) \mu'_W(\overline{w}), \tag{30}$$

and,

$$\overline{w} \in Ch\left(\mu'_W(\overline{f}) \cup \{\overline{w}\}, P''(\overline{f})\right).$$

By the definition of P'',

$$\overline{w} \in Ch\left(\mu'_W(\overline{f}) \cup \{\overline{w}\}, P'(\overline{f})\right). \tag{31}$$

Since $\mu'_W \in S(P')$, condition (31) implies

$$\mu'_W(\overline{w}) P'(\overline{w}) \overline{f}. \tag{32}$$

Conditions (30) and (32) imply that $\overline{w} \in W'' \backslash W'$ and

$$\mu'_W(\overline{w}) = \emptyset. \tag{33}$$

By the optimality of μ'_W in the matching market (F', W'', P'), the definition of $\mu_W \mid_{F' \cup W''}$, and Claim 3.2 we have that, for all $w \in W''$,

$$\mu'_W(w) R'(w) \mu_W \mid_{F' \cup W''}(w) = \mu_W(w).$$

But for all $w \in W''$, $\mu_W(w) \in F'$. This contradicts condition (33). Therefore, $\mu'_W \in S(P'')$. Hence, by the definition of P'', $\mu''_W(w) = \mu'_W(w)$ for all $w \in W'' \backslash W'$. ■

To proceed with the proof of Claim 3.4 we need to assume that $\mu''_W \in S(P')$. Claim 3.4.3 below says that we can do it without loss of generality, because otherwise the conclusion of Lemma 3 follows directly.

Claim 3.4.3 Assume $\mu''_W \notin S(P')$. Then, there exist $f \in F'$ and $w \in W'' \backslash W'$ such that the pair (f, w) blocks μ.

Proof of Claim 3.4.3. Assume $\mu''_W \notin S(P')$. Then, there exists a pair (f, w) such that $w \notin \mu''_W(f)$,

$$f P'(w) \mu''_W(w), \tag{34}$$

and

$$w \in Ch\left(\mu''_W(f) \cup \{w\}, P'(f)\right). \tag{35}$$

By the definition of P'',

$$w \in Ch\left(\mu''_W(f) \cup \{w\}, P''(f)\right).$$

Since $\mu''_W \in S(P'')$, $\mu''_W(w)P''(w)f$, which implies together with condition (34), that $P''(w) \neq P'(w)$, and thus $w \in W''\backslash W'$. By Claim 3.4.2, $\mu''_W(w) = \mu'(w) = \mu(w)$ and condition (34) becomes $fP'(w)\mu(w)$. By definition of P', $P'(w) = P(w)$; thus,

$$fP(w)\mu(w). \tag{36}$$

By property (INT) in the definition of P',

$$Ch\left(\mu''_W(f) \cup \{w\}, P'(f)\right) \subseteq Ch(Ch\left(\mu''_W(f) \cup \{w\}, P'(f)\right) \cup \mu(f), P(f)),$$

which together with condition (35) imply $w \in Ch(Ch\left(\mu''_W(f) \cup \{w\}, P'(f)\right) \cup \mu(f), P(f))$. By the substitutability of $P(f)$,

$$w \in Ch(\{w\} \cup \mu(f), P(f)). \tag{37}$$

Conditions (36) and (37) show that the pair (f, w) blocks μ. Moreover, $f \in F'$ and $w \in W''\backslash W'$. ■

Claim 3.4.4 Assume $\mu''_W \in S(P')$. Then, $\mu''_W = \mu'_W$.

Proof of Claim 3.4.4. Notice that $\mu''_W R''(w)\mu'_W R'(w)\mu''_W$ for all $w \in W''$. The first weak preference follows from the optimality of μ''_W in (F', W'', P'') and Claim 3.4.2. The second follows from the hypothesis. If $w \in W''\backslash W'$, Claim 3.4.2 implies that $\mu'_W(w) = \mu''_W(w)$. If $w \in W'$, by $P''(w) = P'(w)$, $\mu''_W R'(w)\mu'_W$ implies, $\mu''_W(w) = \mu'_W(w)$. ■

Before proceeding with the proof of Claim 3.4 it is useful to recall that, without loss of generality, we may assume now that $\widehat{W}_f = \emptyset$ and $\mu''_W \in S(P')$ because otherwise, Claims 3.4.1 and 3.4.3 imply, respectively, Lemma 3.

Consider the deferred acceptance algorithm where workers make offers in the matching market (F', W'', P''). Notice that the algorithm has at least two steps, since workers in $W' \neq \emptyset$ make offers to their corresponding firms matched through μ, and these offers will be rejected. Moreover, each worker $w \in W''\backslash W'$ makes an offer to $\mu(w)$ in the first step of the algorithm, which is immediately accepted; furthermore, w is never rejected later on during the algorithm, since $\mu(w) = \mu''_W(w)$.

Let $\overline{f} \in F'$ be one of the last firms to receive an offer from a worker $\overline{w} \in W'$. We will distinguish between the following two cases, depending on whether or not $\overline{f}$ rejects some workers.

Case 1. There exists $w' \in W'$ such that $\overline{f}$ rejects w' as a consequence of receiving the offer from $\overline{w}$. Note that w' may be $\overline{w}$. Then, $\mu''_W(w') = \emptyset$. Since $\mu''_W \in S(P')$, Claim 3.4.4 implies $\mu''_W(w') = \mu'_W(w') = \emptyset$. Moreover, $w' \in W' \subseteq W''$ implies

$$\mu_W \mid_{F' \cup W''} (w') = \mu_W(w') \in F'. \tag{38}$$

By Claim 3.2,

$$\mu_W \mid_{F' \cup W''} \in S(P'). \tag{39}$$

Therefore conditions (38) and (39) can not hold simultaneously because the optimality of μ'_W in the matching market (F', W'', P') implies $\mu'_W R(w') \mu_W \mid_{F' \cup W''}$.

Case 2. There does not exist $w' \in W'$ rejected by $\overline{f}$ (as a consequence of receiving $\overline{w}$'s offer). Therefore, by substitutability of $P''(\overline{f})$ and iterated application of Remark 2, $0 < \left| Ch(\bigcup_{j=1}^{k-1} S^j_{\overline{f}}, P''(\overline{f})) \right| < q_{\overline{f}}$. Since $P''(\overline{f}) = P'(\overline{f})$,

$$0 < \left| Ch(\bigcup_{j=1}^{k-1} S^j_{\overline{f}}, P'(\overline{f})) \right| < q_{\overline{f}}. \tag{40}$$

Moreover, we claim that

$$\mu(\overline{f}) \subseteq \bigcup_{j=1}^{k-1} S^j_{\overline{f}}, \tag{41}$$

otherwise, if there exists $\widetilde{w} \in \mu(\overline{f}) \setminus \bigcup_{j=1}^{k-1} S^j_{\overline{f}}$ then, either $\widetilde{w} \in W'$ or $\widetilde{w} \in W'' \setminus W'$. The latter is not possible because, by definition of $P''(\widetilde{w})$, worker $\widetilde{w}$ had already made an offer to $\overline{f}$ in the first step of the algorithm. Assume $\widetilde{w} \in W'$. By the contradiction hypothesis ($\mu'_W = \mu_W \mid_{F' \cup W''}$) and our assumption that $\mu''_W \in S(P')$, $\widetilde{w}$ makes an offer to $\overline{f}$ in the last step of the algorithm, which implies that either $\widetilde{w} \in \mu''_W(\overline{f})$ or $\mu''_W(\widetilde{w}) = \widetilde{w}$. But, both conditions are false.

Finally, we obtain a contradiction as a consequence of the following four relationships.

$$\begin{aligned}
Ch(\bigcup_{j=1}^{k-1} S^j_{\overline{f}}, P'(\overline{f})) &\overset{(1)}{\subsetneq} Ch(Ch(\bigcup_{j=1}^{k-1} S^j_{\overline{f}}, P'(\overline{f})) \cup \mu(\overline{f}), P(\overline{f})) \\
&\overset{(2)}{=} Ch(Ch(\bigcup_{j=1}^{k-1} S^j_{\overline{f}}, P'(\overline{f})) \cup \mu(\overline{f}), P'(\overline{f})) \\
&\overset{(3)}{=} Ch(\bigcup_{j=1}^{k-1} S^j_{\overline{f}} \cup \mu(\overline{f}), P'(\overline{f})) \\
&\overset{(4)}{=} Ch(\bigcup_{j=1}^{k-1} S^j_{\overline{f}}, P'(\overline{f})).
\end{aligned}$$

The inclusion in (1) follows from Property (INT) in the definition of $P'(\overline{f})$ and the strict inclusion follows from (40), $\left|\mu(\overline{f})\right| = q_{\overline{f}}$, and the quota $q_{\overline{f}}$−separability of $P(\overline{f})$. Equality (2) follows from Remark 6, part 2, (see Subsection 5.4 at the end of the paper). The equality (3) is a consequence of the substitutability of $P'(\overline{f})$ and Remark 2. Condition (41) implies equality (4). ■

Now, we extend μ'_W to the original matching market (F, W, P) as follows:

$$\overline{\mu}_W(f) = \begin{cases} \mu'_W(f) & f \in F' \\ \\ \mu_W(f) & f \notin F' \end{cases} \quad \text{and} \quad \overline{\mu}_W(w) = \begin{cases} \mu'_W(w) & w \in W'' \\ \\ \mu_W(w) & w \notin W'' \end{cases}.$$

It is straightforward to verify that $\overline{\mu}_W$ is a matching.

Claim 3.5 For all $w \in W$, $\overline{\mu}_W R(w) \mu_W$. Moreover, $\overline{\mu}_W \notin S(P)$.

Proof of Claim 3.5. By Claim 3.4 and the definition of $\overline{\mu}_W$, $\overline{\mu}_W \neq \mu_W$. If $w \notin W''$, $\overline{\mu}_W(w) = \mu_W(w)$. If $w \in W''$, Claim 3.2 and the optimality of μ'_W say that $\overline{\mu}_W R'(w) \mu_W$. Therefore, $\overline{\mu}_W R(w) \mu_W$ for all $w \in W$. Hence $\overline{\mu}_W \notin S(P)$. ■

Claim 3.6 Assume the pair (f, w) blocks $\overline{\mu}_W$. Then, the following three conditions hold:

(3.6.1) $f \in F'$,

(3.6.2) $w \in W \backslash W''$, and

(3.6.3) (f, w) blocks μ.

Proof of Claim 3.6. Assume (f, w) blocks $\overline{\mu}_W$. Therefore,

$$f P(w) \overline{\mu}_W(w) \tag{42}$$

and

$$w \in Ch\left(\overline{\mu}_W(f) \cup \{w\}, P(f)\right). \tag{43}$$

To prove condition (3.6.1) assume the contrary; that is, $f \notin F'$. By the definition of $\overline{\mu}_W$, $\overline{\mu}_W(f) = \mu_W(f)$. Condition (43) then implies

$$w \in Ch\left(\mu_W(f) \cup \{w\}, P(f)\right). \tag{44}$$

By Claim 3.5, $\overline{\mu}_W(w)R(w)\mu_W(w)$ for all $w \in W$. Hence, either $fP(w)\overline{\mu}_W(w)P(w)\mu_W(w)$ or $fP(w)\overline{\mu}_W(w) = \mu_W(w)$. Therefore,

$$fP(w)\mu_W(w). \tag{45}$$

Conditions (44) and (45) say that the pair (f, w) blocks μ_W, which contradicts its stability.

To prove condition (3.6.2) assume $w \in W''$. Then, by definition of $\overline{\mu}_W$, $\overline{\mu}_W(w) = \mu'_W(w)$. Therefore, by condition (42),

$$fP(w)\mu'_W(w). \tag{46}$$

By condition (3.6.1) of this Claim, $f \in F'$ and by the definition of $\overline{\mu}_W$, $\overline{\mu}_W(f) = \mu'_W(f)$. Therefore, by condition (43),

$$w \in Ch\left(\mu'_W(f) \cup \{w\}, P(f)\right). \tag{47}$$

Conditions (46) and (47) imply $\mu'_W \notin S(P)$. Since $f \in F'$, condition (46) implies that $fP'(w)\mu'_W(w)$. But $\mu'_W \in S(P')$ and $w \notin \mu'_W(f)$ imply

$$w \notin Ch\left(\mu'_W(f) \cup \{w\}, P'(f)\right). \tag{48}$$

Now,

$$Ch\left(\mu'_W(f) \cup \{w\}, P(f)\right) = Ch\left(\mu'_W(f) \cup \{w\}, P'(f)\right), \tag{49}$$

since

$$\begin{aligned} Ch\left(\mu'_W(f) \cup \{w\}, P(f)\right) &= Ch\left(Ch\left(\mu'_W(f) \cup \mu(f), P(f)\right) \cup \{w\}, P(f)\right) && \text{by Claim 3.3} \\ &= Ch\left(\mu'_W(f) \cup \mu(f) \cup \{w\}, P(f)\right) && \text{by Remark 2} \end{aligned}$$

and Claim 2.3 used in the proof of Lemma 2 (see Subsection 5.4 at the end of the paper). Conditions (47) and (48) contradict equality (49). Thus, $w \in W \backslash W''$.

To prove condition (3.6.3) notice that by condition (3.6.1) of this Claim and the definition of $\overline{\mu}_W$,

$$\overline{\mu}_W(f) = \mu'_W(f). \tag{50}$$

Conditions (43) and (50) imply that $w \in Ch\left(\mu'_W(f) \cup \{w\}, P(f)\right)$. By Claim 3.3, $w \in Ch\left(Ch(\mu'_W(f) \cup \mu\left(f\right), P(f)\right) \cup \{w\}, P(f))$. By Remark 2,

$$w \in Ch\left(\mu'_W(f) \cup \mu\left(f\right) \cup \{w\}, P(f)\right).$$

Finally, by the substitutability of $P(f)$,

$$w \in Ch\left(\mu(f) \cup \{w\}, P(f)\right). \tag{51}$$

Since $w \in W \backslash W''$ we have that $\overline{\mu}_W(w) = \mu_W(w) = \mu(w)$. By condition (42),

$$fP(w)\mu(w). \tag{52}$$

Conditions (51) and (52) imply that the pair (f, w) blocks μ. ■

Once Lemma 3 is proved, our objective in order to prove Proposition 3 is still to identify a particular blocking pair (f, w) of the matching μ. Remember that Proposition 3 is more general than Lemma 3 because the former does not assume $\mu(w) R(w) \mu_W(w)$ for all $w \in W$. To look for this blocking pair, we will first identify a blocking pair of the matching that assigns to each worker w his most preferred firm amongst $\mu(w) \cup \mu_W(w)$. Then, we will show that this pair blocks μ as well.

Define $\mu \vee_W \mu_W$ as follows: for every $w \in W$,

$$\mu \vee_W \mu_W(w) = \begin{cases} \mu(w) & \text{if } w \in W' \\ \\ \mu_W(w) & \text{if } w \notin W' \end{cases}$$

and for every $f \in F$,

$$\mu \vee_W \mu_W(f) = \begin{cases} [\mu(f) \cap W'] \cup [\mu_W(f) \cap [W \backslash W']] & \text{if } f \in \mu(W') \\ \\ \mu_W(f) & \text{if } f \notin \mu(W'). \end{cases}$$

Martínez, Massó, Neme, and Oviedo (2001) shows that if μ were a stable matching then $\mu \vee_W \mu_W$ would be stable as well. However, it is immediate to see that under the hypothesis of next lemma, $\mu \vee_W \mu_W$ is an individually rational matching. We state this fact as Remark 5 below.

Remark 5 *Assume* $|\mu(f)| = q_f$ *and* $|\mu(f) \cap W'| = |\mu_W(f) \cap W'|$ *for all* $f \in \mu(W')$. *Then,* $\mu \vee_W \mu_W \in IR(P)$.

Lemma 4 *Assume* $|\mu(f)| = q_f$ *and* $|\mu(f) \cap W'| = |\mu_W(f) \cap W'|$ *for all* $f \in \mu(W')$. *Then,*

(4.1) $\mu_W(f) = Ch\left(\mu_W(f) \cup \mu \vee_W \mu_W(f), P(f)\right)$ *for all* $f \in \mu(W')$.

(4.2) If there exists $\overline{f} \in \mu(W')$ *such that* $\mu(\overline{f}) \neq Ch\left(\mu(\overline{f}) \cup \mu \vee_W \mu_W(\overline{f}), P\left(\overline{f}\right)\right)$ *then, there exists* $\overline{w} \in W \backslash W'$ *such that the pair* $(\overline{f}, \overline{w})$ *blocks* μ.

Proof. To prove (4.1), assume $\mu_W(\overline{f}) \neq Ch\left(\mu_W(\overline{f}) \cup \mu \vee_W \mu_W(\overline{f}), P\left(\overline{f}\right)\right)$ for at least one $\overline{f} \in \mu(W')$. Then, there exists $\overline{w} \in Ch\left(\mu_W(\overline{f}) \cup \mu \vee_W \mu_W(\overline{f}), P\left(\overline{f}\right)\right) \backslash \mu_W(\overline{f})$. By the substitutability of $P(\overline{f})$,

$$\overline{w} \in Ch\left(\mu_W(\overline{f}) \cup \{\overline{w}\}, P\left(\overline{f}\right)\right). \tag{53}$$

By the definition of $\mu \vee_W \mu_W(\overline{f})$, $\overline{w} \in \mu(\overline{f}) \cap W'$. Then,

$$\overline{f} = \mu(\overline{w}) P(\overline{w}) \mu_W(\overline{w}). \tag{54}$$

Conditions (53) and (54) imply that the pair $(\overline{f}, \overline{w})$ blocks μ_W, contradicting $\mu_W \in S(P)$.

To prove condition (4.2), assume there exists $\overline{f} \in \mu(W')$ with the property that $\mu(\overline{f}) \neq Ch\left(\mu(\overline{f}) \cup \mu \vee_W \mu_W(\overline{f}), P\left(\overline{f}\right)\right)$. Then, there exits $\overline{w} \in Ch\left(\mu(\overline{f}) \cup \mu \vee_W \mu_W(\overline{f}), P\left(\overline{f}\right)\right) \backslash \mu(\overline{f})$. By the substitutability of $P(\overline{f})$,

$$\overline{w} \in Ch\left(\mu(\overline{f}) \cup \{\overline{w}\}, P\left(\overline{f}\right)\right). \tag{55}$$

By the definition of $\mu \vee_W \mu_W(\overline{f})$, $\overline{w} \in \mu_W(\overline{f}) \cap W \backslash W'$. Then,

$$\overline{f} = \mu_W(\overline{w}) P(\overline{w}) \mu(\overline{w}). \tag{56}$$

Conditions (55) and (56) imply that the pair $(\overline{f}, \overline{w})$ blocks μ. ■

We are now ready to prove Proposition 3.

Proof of Proposition 3. Remember that we may assume that $|\mu(f)| = q_f$ for all $f \in F'$; otherwise, the conclusion of Proposition 3 follows from Proposition 1 in Subsection 5.1. Furthermore, by Lemma 1, we can also assume that $|\mu(f) \cap W'| = |\mu_W(f) \cap W'|$ for all $f \in F'$ as well. Thus, the hypotheses of Remark 5 and Lemma 4 hold. Therefore, $\mu \vee_W \mu_W \in IR(P)$. Moreover,

$$|\mu \vee_W \mu_W(f)| = q_f \text{ for all } f \in F'. \tag{57}$$

To identify a blocking pair of $\mu \vee_W \mu_W$ we will apply Lemma 3 to the matching $\mu \vee_W \mu_W$. We next check that all the hypotheses of Lemma 3 are satisfied:

- The fact that $\mu \vee_W \mu_W R(w) \mu_W$ for all $w \in W$ follows from its definition.

- The set $\{w \in W \mid \mu \vee_W \mu_W(w) P(w) \mu_W(w)\}$ coincides with the set W' already defined from μ.

- The equality $\mu \vee_W \mu_W(W') = \mu_W(W')$ follows from the fact that

$$\begin{aligned} \mu \vee_W \mu_W(W') &= \bigcup_{w \in W'} \mu \vee_W \mu_W(w) \\ &= \bigcup_{w \in W'} \mu(w) && \text{by the definition of } \mu \vee_W \mu_W \\ &= \mu(W') \\ &= \mu_W(W') && \text{by the general assumption of Subsection 5.2.} \end{aligned}$$

Therefore, by Lemma 3, there exist $f \in \mu(W')$ and $w \in W \backslash W'$ such that (f, w) blocks $\mu \vee_W \mu_W$. That is,

$$w \notin \mu \vee_W \mu_W(f), \tag{58}$$

$$f P(w) \mu \vee_W \mu_W(w), \tag{59}$$

and

$$w \in Ch\left(\mu \vee_W \mu_W(f) \cup \{w\}, P(f)\right). \tag{60}$$

As a consequence of the matching $\mu \vee_W \mu_W$ being blocked by the pair (f, w), we will establish that the following claim holds for this pair (f, w).

Claim P.3 There exists $\overline{w} \in \mu(f) \cap W'$ such that $\overline{w} \notin Ch\left(\mu \vee_W \mu_W(f) \cup \{w\}, P(f)\right)$.

Proof of Claim P.3. Assume $\widetilde{w} \in Ch\left(\mu \vee_W \mu_W(f) \cup \{w\}, P(f)\right)$ for all $\widetilde{w} \in \mu(f) \cap W'$. By conditions (57) and (58), $|\mu \vee_W \mu_W(f) \cup \{w\}| = q_f + 1$. By condition (60), there exists $\widehat{w} \notin Ch\left(\mu \vee_W \mu_W(f) \cup \{w\}, P(f)\right)$. By hypothesis, $\widehat{w} \notin \mu(f) \cap W'$; thus, by the definition of $\mu \vee_W \mu_W(f)$, $\widehat{w} \in \mu_W(f) \cap (W \backslash W')$. By the substitutability of $P(f)$,

$$\widehat{w} \notin Ch\left(\mu \vee_W \mu_W(f) \cup \{w\} \cup \mu_W(f), P(f)\right). \tag{61}$$

But

$$\begin{aligned} & Ch\left(\mu \vee_W \mu_W(f) \cup \{w\} \cup \mu_W(f), P(f)\right) \\ &= Ch\left(Ch\left(\mu \vee_W \mu_W(f) \cup \mu_W(f), P(f)\right) \cup \{w\}, P(f)\right) && \text{by Remark 2} \\ &= Ch\left(\mu_W(f) \cup \{w\}, P(f)\right) && \text{by Lemma 4, part (4.1),} \end{aligned}$$

together with condition (61), the quota q_f-separability of $P(f)$, and the fact that w is a good worker for firm f imply

$$w \in Ch\left(\mu_W(f) \cup \{w\}, P(f)\right). \tag{62}$$

Condition (59) and the definition of $\mu \vee_W \mu_W$ imply that

$$fP(w)\mu \vee_W \mu_W(w)R(w)\mu_W(w). \tag{63}$$

Conditions (62) and (63) contradict $\mu_W \in S(P)$. ■

Consider $\overline{w} \in \mu(f) \cap W'$ with the property that $\overline{w} \notin Ch\left(\mu \vee_W \mu_W(f) \cup \{w\}, P(f)\right)$, whose existence follows from Claim P.3. By the substitutability of the preference relation $P(f)$, $\overline{w} \notin Ch\left(\mu \vee_W \mu_W(f) \cup \{w\} \cup \mu(f), P(f)\right)$. Again by the substitutability of $P(f)$ and Remark 2,

$$\overline{w} \notin Ch\left(Ch\left(\mu \vee_W \mu_W(f) \cup \mu(f), P(f)\right) \cup \{w\}, P(f)\right). \tag{64}$$

We distinguish between the following two cases:

CASE 1. $\mu(f) \neq Ch\left(\mu \vee_W \mu_W(f) \cup \mu(f), P(f)\right)$. Lemma 4, part (4.2), implies that the conclusion of Proposition 3 follows.

CASE 2. $\mu(f) = Ch\left(\mu \vee_W \mu_W(f) \cup \mu(f), P(f)\right)$. Condition (64) implies that $\overline{w} \notin Ch\left(\mu(f) \cup \{w\}, P(f)\right)$, which together with the quota q_f-separability of $P(f)$ implies

$$w \in Ch\left(\mu(f) \cup \{w\}, P(f)\right). \tag{65}$$

Condition (59) and the definition of $\mu \vee_W \mu_W$ yield

$$fP(w)\mu \vee_W \mu_W(w)R(w)\mu(w). \tag{66}$$

Conditions (65) and (66) say that the pair (f, w) blocks μ, where $f \in F'$ and $w \in W \backslash W'$. ■

5.4 $P'(f)$ is substitutable

Since f will be fixed through out all this Subsection 5.4, we will write P, P', q, and μ instead of $P(f)$, $P'(f)$, q_f, and $\mu(f)$, respectively. Moreover, we will use w to denote the set $\{w\}$.

CLAIM 2.1 Assume $S_1 P' \emptyset$. Then, $Ch(\widehat{S}_1, P) = \widehat{S}_1$, where $\widehat{S}_1 = \max_P \{S \subseteq W'' \mid S \cap Ch(S \cup \mu, P) = S_1\}$.

PROOF OF CLAIM 2.1. First, by definition of the choice set, $Ch(\widehat{S}_1, P) \subseteq \widehat{S}_1$. Assume there exists $w \in \widehat{S}_1 \backslash Ch(\widehat{S}_1, P)$. By the substitutability of P, $w \notin Ch(\widehat{S}_1, P)$ implies $w \notin$

$Ch(\widehat{S}_1 \cup \mu, P)$. Therefore, $w \notin S_1$. Hence, $S_1 \subseteq Ch(\widehat{S}_1, P)$. Moreover, S_1 is also a subset of $Ch(\widehat{S}_1 \cup \mu, P)$. Therefore,

$$\begin{aligned} S_1 &\subseteq Ch(\widehat{S}_1, P) \cap Ch(\widehat{S}_1 \cup \mu, P) & \\ &= Ch(\widehat{S}_1, P) \cap Ch(Ch(\widehat{S}_1, P) \cup \mu, P) & \text{by Remark 2} \\ &\subseteq \widehat{S}_1 \cap Ch(\widehat{S}_1 \cup \mu, P) & \text{since } Ch(A,P) \subseteq A \\ &= S_1 & \text{by the definition of } \widehat{S}_1. \end{aligned}$$

Therefore, $Ch(\widehat{S}_1, P) \cap Ch(Ch(\widehat{S}_1, P) \cup \mu, P) = S_1$, contradicting the definition of $\widehat{S}_1$. ■

CLAIM 2.2 Let S and B be such that $S \cap Ch(S \cup \mu, P) = Ch(B, P')$. Then, for all A,

$$Ch(S \cup \mu \cup A, P) = Ch(Ch(B, P') \cup \mu \cup A, P).$$

PROOF OF CLAIM 2.2. Note that S can be written, by hypothesis, as

$$S = Ch(B, P') \mathring{\cup} \overline{S}, \tag{67}$$

where $\overline{S} = \{w \in S \mid w \notin Ch(B, P')\}$ and the symbol $\mathring{\cup}$ means the disjoint union of sets. Thus, $\overline{S} \cap Ch(S \cup \mu, P) = \emptyset$. Therefore, if $w \in \overline{S}$, we have that $w \notin Ch(S \cup \mu, P)$. Then, by the substitutability of P,

$$\text{if } w \in \overline{S} \text{ then } w \notin Ch(S \cup \mu \cup A, P) \text{ for all } A. \tag{68}$$

Thus,

$$\begin{aligned} Ch(S \cup \mu \cup A, P) &= Ch(Ch(B, P') \cup \overline{S} \cup \mu \cup A, P) & \text{by (67)} \\ &= Ch(Ch(B, P') \cup \mu \cup A, P) & \text{by (68).} \end{aligned}$$

■

CLAIM 2.3 Let T be such that $Ch(T, P) = Ch(T \cup \mu, P)$. Then, $Ch(T, P) = Ch(T, P')$.

PROOF OF CLAIM 2.3. By Remark 2, $Ch(T \cup \mu, P) = Ch(Ch(T, P) \cup \mu, P)$. The hypothesis and property (INT) in the definition of P' imply that $Ch(T, P)P'\emptyset$. Assume that $Ch(T, P) \neq Ch(T, P')$, then

$$Ch(T, P')P'Ch(T, P)P'\emptyset. \tag{69}$$

By property (MAX) in the definition of P', let $\widehat{S}$ be the maximal set (according to the preference relation P) such that

$$\widehat{S} \cap Ch(\widehat{S} \cup \mu, P) = Ch(T, P'). \tag{70}$$

Moreover, since

$$Ch(T,P) = Ch(T,P) \cap Ch\left(Ch(T,P) \cup \mu, P\right),$$

property (MAX) in the definition of P' and conditions (69) and (70) imply

$$\widehat{S} P Ch(T,P). \tag{71}$$

Claim 2.2 says that condition (70) implies, for $B = T$, $S = \widehat{S}$, and $A = \emptyset$,

$$Ch(\widehat{S} \cup \mu, P) = Ch(Ch(T,P') \cup \mu, P).$$

Furthermore,

$$\begin{aligned}
Ch(\widehat{S} \cup Ch(T,P), P) &= Ch(\widehat{S} \cup Ch(T \cup \mu, P), P) && \text{by hypothesis} \\
&= Ch(\widehat{S} \cup T \cup \mu, P) && \text{by Remark 2} \\
&= Ch(Ch(T,P') \cup T \cup \mu, P) && \text{by Claim 2.2 (for } A = T\text{)} \\
&= Ch(T \cup \mu, P) && \text{since } Ch(T,P') \subseteq T \\
&= Ch(T,P) && \text{by hypothesis.}
\end{aligned}$$

This equality implies $Ch(T,P) R(f) \widehat{S}$, which contradicts (71). ■

Claim 2.4 Assume T is such that $|Ch\,(T,P')| = q$. Then, $Ch(T,P) = Ch(T \cup \mu, P)$.

Proof of Claim 2.4. Let T be such that $|Ch\,(T,P')| = q$. Then,

$$\begin{aligned}
Ch(T,P) &= Ch(T \cup Ch(T,P'), P) && \text{since } Ch(T,P') \subseteq T \\
&= Ch(Ch(T,P) \cup Ch(T,P'), P) && \text{by Remark 2} \\
&= Ch(Ch(T,P) \cup Ch(T,P') \cup \mu, P) && \\
&= Ch(T \cup Ch(T,P') \cup \mu, P) && \text{by Remark 2 again} \\
&= Ch(T \cup \mu, P) && \text{since } Ch(T,P') \subseteq T,
\end{aligned}$$

where the unjustified equality follows from the fact that, $|Ch(T,P')| = q$ implies, by the quota $q-$separability of P and property (INT) in the definition of P', that $Ch(T,P') = Ch(Ch(T,P') \cup \mu, P)$. ■

As a consequence of Claims 2.3 and 2.4 we can state the following Remark.

Remark 6 *Let T be such that $|Ch(T,P')| = q$. Then,*
(1) $Ch(T,P') = Ch(T,P) = Ch(T \cup \mu, P)$.
(2) $Ch(T \cup \mu, P) = Ch(T \cup \mu, P')$.

Claim 2.5 Assume w and B are such that $w \in B$, $Ch(B,P')P'\emptyset$, and $w \notin Ch(B,P')$. Then, $w \notin Ch(Ch(B,P') \cup w \cup \mu, P)$.

Proof of Claim 2.5. Assume otherwise; that is,

$$w \in Ch\left(Ch(B,P') \cup w \cup \mu, P\right). \tag{72}$$

Since $Ch(B,P')P'\emptyset$, by property (INT) in the definition of P', $Ch(B,P') = Ch(B,P') \cap Ch(Ch(B,P') \cup \mu, P)$. This guarantees the existence of $\widehat{S}$ (the best subset according to the preference relation P) such that

$$Ch(B,P') = \widehat{S} \cap Ch(\widehat{S} \cup \mu, P). \tag{73}$$

Define $V = Ch\left(\widehat{S} \cup w, P\right)$. Then,

$$\begin{aligned} Ch(V \cup \mu, P) &= Ch(Ch(\widehat{S} \cup w, P) \cup \mu, P) && \text{by the definition of } V \\ &= Ch(\widehat{S} \cup w \cup \mu, P) && \text{by Remark 2} \\ &= Ch(Ch(B,P') \cup \mu \cup w, P) && \text{by Claim 2.2.} \end{aligned} \tag{74}$$

Claim 2.5.1 $V \neq \widehat{S}$.

Proof of Claim 2.5.1. Assume otherwise. Then,

$$Ch(V \cup \mu, P) = Ch(\widehat{S} \cup \mu, P). \tag{75}$$

By condition (72) (the contradiction hypothesis of Claim 2.5), $w \in Ch(Ch(B,P') \cup w \cup \mu, P)$. Hence, by equalities in (74) and (75), $w \in Ch(V \cup \mu, P) = Ch\left(\widehat{S} \cup \mu, P\right)$. By the substitutability of P, $w \in Ch\left(\widehat{S} \cup w, P\right) = V$, implying that $w \in \widehat{S}$. Then, $w \in \widehat{S} \cap Ch\left(\widehat{S} \cup \mu, P\right) = Ch(B,P')$, contradicting $w \notin Ch(B,P')$, an hypothesis of Claim 2.5. This proves Claim 2.5.1. ■

By the definition of V and Claim 2.5.1,

$$w \in Ch\left(\widehat{S} \cup w, P\right) = V. \tag{76}$$

Now, by conditions (72) and (74),

$$w \in Ch(V \cup \mu, P). \tag{77}$$

Define

$$L = V \cap Ch(V \cup \mu, P). \tag{78}$$

Observe that, by (76) and (77),

$$w \in L. \tag{79}$$

Now, condition (78), $L \subseteq Ch(V \cup \mu, P)$, substitutability of P, and $L \subseteq V$, imply

$$L \subseteq Ch(L \cup \mu, P),$$

which in turn implies,

$$L = L \cap Ch(L \cup \mu, P).$$

Therefore, by property (INT) of the definition of P', $LP'\emptyset$. Moreover, $L \subseteq B$ because

$$\begin{array}{lll} L & = V \cap Ch(V \cup \mu, P) & \text{by the definition of } L \\ & = Ch(\widehat{S} \cup w, P) \cap Ch(Ch(\widehat{S} \cup w, P) \cup \mu, P) & \text{by the definition of } V \\ & = Ch(\widehat{S} \cup w, P) \cap Ch(\widehat{S} \cup w \cup \mu, P) & \text{by Remark 2} \\ & \subseteq w \cup [Ch(\widehat{S}, P) \cap Ch(\widehat{S} \cup \mu, P)] & \text{since } w \text{ belongs to both sets} \\ & = w \cup [\widehat{S} \cap Ch(\widehat{S} \cup \mu, P)] & \text{by Claim 2.1} \\ & = w \cup Ch(B, P') & \text{by (73) (the definition of } \widehat{S}) \\ & \subseteq B & \text{because } w \in B. \end{array}$$

Since $w \in L$ (condition (79)) and $w \notin Ch(B, P')$ (hypotheses of Claim 2.5), then $L \neq Ch(B, P')$. Therefore, $Ch(B, P')P'LP'\emptyset$, because $L \subseteq B$.

Hence, by property (MAX) in the definition of P',

$$\widehat{S}PV. \tag{80}$$

On the other hand, by definition, $V = Ch(\widehat{S} \cup \{w\}, P)$ and, by Claim 2.5.1, $V \neq \widehat{S}$. Therefore, $VP\widehat{S}$ which contradicts (80). ■

Claim 2.6 Let $w_1, w_2 \in T$ be such that $w_1 \neq w_2$, $w_1 \in Ch(T, P')$, and $w_1 \notin Ch(T \backslash w_2, P')$. Then,

(1) $Ch(Ch(T, P') \cup Ch(T \backslash w_2, P'), P) \neq Ch(T, P')$.

(2) $Ch(Ch(T, P') \cup Ch(T \backslash w_2, P'), P) \neq Ch(Ch(T, P') \cup Ch(T \backslash w_2, P') \cup \mu, P)$.

Proof of Claim 2.6 (1) Assume otherwise. We will distinguish between the following two cases:

Case 1. $|Ch(T, P')| < q$. Then, our contradiction hypothesis and the quota $q-$separability of P imply that $Ch(T \backslash w_2, P') \subseteq Ch(T, P')$. Since $w_2 \notin Ch(T \backslash w_2, P')$, $Ch(T \backslash w_2, P') \subseteq$

$Ch(T,P')\backslash w_2 \subseteq T\backslash w_2$. Therefore, Remark 1 implies $Ch(T\backslash w_2,P') = Ch(T,P')\backslash w_2$, which is a contradiction because $w_1 \in Ch(T,P')\backslash w_2$ and $w_1 \notin Ch(T\backslash w_2, P')$.

CASE 2. $|Ch(T,P')| = q$. Then, by Remark 6, part (1), $Ch(T,P') = Ch(T,P) = Ch(T \cup \mu, P)$. Therefore, by hypothesis, $w_1 \in Ch(T \cup \mu, P)$. Then, by the substitutability of P, $w_1 \in Ch(w_1 \cup \mu, P)$. Hence, by property (INT) in the definition of P', $w_1 P' \emptyset$. Therefore,

$$Ch(T\backslash w_2, P')P'\emptyset. \tag{81}$$

By the substitutability of P, the fact that $Ch(T\backslash w_2, P') \cup w_1 \subseteq T$, and $w_1 \in Ch(T \cup \mu, P)$, we have

$$w_1 \in Ch(Ch(T\backslash w_2, P') \cup w_1 \cup \mu, P). \tag{82}$$

The hypothesis $w_1 \notin Ch(T\backslash w_2, P')$, $w_1 \in T\backslash w_2$, and conditions (81) and (82) contradict Claim 2.5.

(2) Assume otherwise. Then, by Claim 2.3,

$$Ch(Ch(T,P') \cup Ch(T\backslash w_2, P'), P) = Ch(Ch(T,P') \cup Ch(T\backslash w_2, P'), P').$$

Remark 1 and the equality above imply that

$$Ch(Ch(T,P') \cup Ch(T\backslash w_2, P'), P) = Ch(T,P'),$$

since $Ch(T,P') \subseteq Ch(T,P') \cup Ch(T\backslash w_2, P') \subseteq T$. But this contradicts part (1) of this Claim. ■

Proof of Lemma 2. Assume P' is not substitutable; that is, there exist $w_1, w_2 \in T$ such that $w_1 \neq w_2$, $w_1 \in Ch(T,P')$, and $w_1 \notin Ch(T\backslash w_2, P')$. By Claim 2.6, part (2),

$$Ch(Ch(T,P') \cup Ch(Ch(T\backslash w_2, P'), P) \neq Ch(Ch(T,P') \cup Ch(T\backslash w_2, P') \cup \mu, P).$$

We will distinguish between the following two cases:

CASE 1. $Ch(Ch(T,P') \cup Ch(T\backslash w_2, P') \cup \mu, P) \neq Ch(Ch(T,P') \cup \mu, P)$. Remember that, in Subsection 5.2, we are assuming $|\mu| = q$. By the quota $q-$separability of P, the cardinality of the two sets is equal to q. Since, by assumption, the two sets are not equal, there exists w such that

$$w \in Ch(Ch(T,P') \cup Ch(T\backslash w_2, P') \cup \mu, P) \tag{83}$$

and

$$w \notin Ch(Ch(T,P') \cup \mu, P). \tag{84}$$

Condition (83) implies, by the substitutability of P,

$$w \in Ch(Ch(T, P') \cup w \cup \mu, P). \tag{85}$$

But, conditions (84) and (85) imply

$$w \notin Ch(T, P') \tag{86}$$

and $w \in T$. Conditions (85) and (86) contradict Claim 2.5.

CASE 2. $Ch(Ch(T, P') \cup Ch(T \backslash w_2, P') \cup \mu, P) = Ch(Ch(T, P') \cup \mu, P)$. Again, by property (INT) in the definition of P', $Ch(T, P') = Ch(T, P') \cap Ch\,(Ch(T, P') \cup \mu, P)$. Therefore,

$$Ch(Ch(T, P') \cup \mu, P) \supseteq Ch(T, P').$$

By hypothesis,

$$w_1 \notin Ch(T \backslash w_2, P') \tag{87}$$

and $w_1 \in Ch(T, P')$. Hence, by the substitutability of P, we have

$$w_1 \in Ch(Ch(T \backslash w_2, P') \cup w_1 \cup \mu, P). \tag{88}$$

By the substitutability of P, $w_1 \in Ch\,(w_1 \cup \mu, P)$. Hence, $w_1 = w_1 \cap Ch\,(w_1 \cup \mu, P)$, implying, by property (INT) in the definition of P', that $w_1 P' \emptyset$. But this, together with $w_1 \in T \backslash w_2$, implies $Ch(T \backslash w_2, P') P' \emptyset$. Therefore, conditions (87) and (88) contradict Claim 2.5. ■

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