Manipulation of Optimal Matchings
via Predonation of Endowment*

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Abstract: In this paper we answer a question posed by Sertel and Özkal-Sanver (2002) on the manipulability of optimal matching rules in matching problems with endowments. We characterize the classes of consumption rules under which optimal matching rules can be manipulated via predonation of endowment.

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1   Introduction

In a recent paper, Sertel and Özkal-Sanver (2002), hereafter S & Ö-S, explained that in two-sided matching models "... the consumption possibilities of an agent may depend on the respective endowments of the pair in which the agent ends up under a matching" (S &

Ö-S, p. 66). They endowed the classical marriage matching model (cf. Gale and Shapley (1962)) with two additional features: endowments of some resource and consumption rules which determine how endowments are consumed jointly, i.e., within a matched pair.

In their study of this extended model, S & Ö-S focused on the manipulability of the man optimal matching rule φ_M (for the woman optimal matching rule one simply changes the roles of men and women).[4] More specifically, they considered four types of manipulation: destruction, hiding, perfect hiding, and predonation of endowments. Mostly under the assumption that there are at least three agents on each side of the market, they characterized the class of consumption rules under which φ_M is vulnerable to manipulation of any of the first three types. Regarding manipulation by predonation of endowment, they only provided examples of matching markets to show that under several 'natural' consumption rules φ_M is prone to manipulation. Subsequently, S & Ö-S (p. 80) posed a research question: Which are the maximal classes of consumption rules under which φ_M can be manipulated via predonation of endowment?

In this paper we answer this question. We first characterize for any of the four cases of predonation (i.e., man to woman, man to man, woman to man, and woman to woman) the class of consumption rules under which φ_M is manipulable. We use conditions that are very similar to the ones used by S & Ö-S, which make a clear comparison with their characterizations possible. Finally, we derive from our characterization results that φ_M is only non-manipulable under the trivial consumption rule, where each agent consumes exactly his own endowment. (S & Ö-S already noted that at least under this consumption rule, φ_M is non-manipulable.) The proofs of our characterizations are considerably different from the proofs in S & Ö-S regarding destruction or (perfect) hiding of endowment. This is a consequence of the fact that predonation of endowment typically affects the preferences of many more agents than destruction or (perfect) hiding of endowment.

Predonation of endowment can be related to the concept of bribing on which only very recently some work has been done. Schummer (2000) studied bribe-proof rules in a very broad class of economies. He showed that if the domain of preferences is sufficiently rich then any bribe-proof rule is a constant function. Eső and Schummer (2003) studied the equilibria of a game in the context of a two-bidder, second-price auction where one bidder may bribe the other to commit to stay away from the auction. Massó and Neme (2003) characterized bribe-proof rules in the context of a division problem with one divisible good and single-peaked preferences. The two main differences between these papers and ours are the following. In the first place, a bribe is not shared with other agents, while in our model a predonation is typically also allocated to a third agent according to a consumption rule. In the second place, a bribe only leads to a change of the type of the receiving agent, whereas in our model a predonation may affect other agents' preferences as well.

The paper is organized as follows. In Section 2, we recall the model and notation. Moreover, we recall two results on the manipulability of optimal matching rules in the marriage model, which facilitate the exposition of our proofs. In Section 3, we provide

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[4]For a comprehensive account on manipulation in the marriage model we refer to Roth and Sotomayor (1990), and for a short review on manipulation of endowments in other contexts we refer to S & Ö-S.

our characterization results. Finally, in Section 4, we summarize our results and conclude with some examples and remarks.

2   The model

The model and most of the notation presented next are due to S & Ö-S. We consider two finite and disjoint sets $M = \{m_1, \ldots, m_r\}$ of men and $W = \{w_1, \ldots, w_s\}$ of women where possibly $r \neq s$. Since S & Ö-S need the assumption $r, s \geq 3$ for most of their results we make this assumption from the start. The set $A = M \cup W$ is called a society. For each agent $i \in A$, we define the potential set of mates as

$$A(i) := \begin{cases} W \cup \{i\} & \text{if } i \in M; \\ M \cup \{i\} & \text{if } i \in W. \end{cases}$$

Let $\mathcal{R}_+ = \{x \in \mathcal{R} : x \geq 0\}$. For each $i \in A$, let $v_i : A(i) \to \mathcal{R}_+$ be a map that totally orders $A(i)$ meaning that for each $i \in A$, $v_i(j) = v_i(k)$ if and only if $j = k \in A(i)$. A valuation profile is a set of maps $v = \{v_i\}_{i \in A}$.

   We will assume that each agent $i \in A$ has an initial endowment $e_i \in \mathcal{R}_+$. A vector $e = (e_i)_{i \in A} \in E := \mathcal{R}_+^A$ is called an endowment profile. Endowments are allocated between matched agents according to some exogenous consumption rule, which for agent $i \in A$ is a function $c_i : A(i) \times E \to \mathcal{R}_+$. For any $j \in A(i)$ and any $e \in E$, $c_i(j, e)$ denotes the consumption of i when matched to j. We require that the functions $\{c_i\}_{i \in A}$ satisfy the following three conditions. First, we require that no matter who i's mate is, he/she does not consume more than their total endowment, i.e.,

C_1: $c_i(j, e) \leq e_i + e_j$ for all $j \in A(i) \setminus \{i\}$ and $e \in E$.

Second, every self-matched agent consumes his own endowment, i.e.,

C_2: $c_i(i, e) = e_i$ for all $e \in E$.

The third condition is a combined monotonicity and independency property:

C_3: $c_i(j, e) \leq c_i(j, e')$ for all $j \in A(i)$ and $e, e' \in E$ with $e_i \leq e'_i$ and $e_j \leq e'_j$.

Note that for $i \in A$, $j \in A(i) \setminus \{i\}$ we do not demand feasibility in the sense that $c_i(j, e) + c_j(i, e) \leq e_i + e_j$.

Lemma 1 *For all $i \in A$ and $j \in A(i) \setminus \{i\}$ there exists a non-decreasing function $\gamma^{ij} : \mathcal{R}_+ \times \mathcal{R}_+ \to \mathcal{R}_+$ such that $\gamma^{ij}(0,0) = 0$ and $\gamma^{ij}(e_i, e_j) = c_i(j, e)$ for any $e \in E$.*

Proof. It is straightforward from C_1 and C_3. ∎

In view of Lemma 1 and condition C_2 we conveniently define for any $i \in A$ a function $\gamma^{ii} : \mathcal{R}_+ \times \mathcal{R}_+ \to \mathcal{R}_+$ as $\gamma^{ii}(e_i, e_i) := c_i(i, e) = e_i$ for any $e \in E$. For each agent $i \in A$, we identify the consumption rule c_i with the associated set of functions $\gamma = \{\gamma^{ij}\}_{j \in A(i)}$. We denote by Γ the collection of these associated set of functions γ.

S & Ö-S provided the following examples of consumption rules, which will be discussed later,

1. Trivial: $\gamma_t^{ij}(e_i, e_j) = e_i$ for all $e \in E$, $i \in A$, and $j \in A(i)$.

2. Reciprocal: $\gamma_r^{ij}(e_i, e_j) = e_j$ for all $e \in E$, $i \in A$, and $j \in A(i)$.

3. Equal share: $\gamma_e^{ij}(e_i, e_j) = \frac{1}{2}(e_i + e_j)$ for all $e \in E$, $i \in A$, and $j \in A(i)$.

4. Maximal: $\gamma_{\max}^{ij}(e_i, e_j) = \max\{e_i, e_j\}$ for all $e \in E$, $i \in A$, and $j \in A(i)$.

5. Minimal: $\gamma_{\min}^{ij}(e_i, e_j) = \min\{e_i, e_j\}$ for all $e \in E$, $i \in A$, and $j \in A(i)$.

6. Public: $\gamma_p^{ii}(e_i, e_i) = e_i$, $\gamma_p^{ij}(e_i, e_j) = e_i + e_j$ for all $e \in E$, $i \in A$, and $j \in A(i) \setminus \{i\}$.

A matching problem is given by a quadruple $\alpha = (A, e, v, \gamma)$ where A is a society, e is an endowment profile, v is a valuation profile, and γ a consumption rule. Note that by taking $e_i = 0$ for every $i \in A$ we obtain a classical marriage problem (cf. Gale and Shapley (1962)).

Given a matching problem $\alpha = (A, e, v, \gamma)$, each agent $i \in A$ consumes a pair $(j, \gamma^{ij}(e_i, e_j))$ which consists of a mate $j \in A(i)$ and some amount $\gamma^{ij}(e_i, e_j) \in \mathcal{R}_+$. The sum $v_i(j) + \gamma^{ij}(e_i, e_j)$ represents i's quasi-linear utility if he were matched to agent $j \in A(i)$. For each agent $i \in A$, we define a strict rank order, R_i, on the set $A(i) \times \mathcal{R}_+$ as follows. Let $j, k \in A(i)$, then $(j, \gamma^{ij}(e_i, e_j)) R_i (k, \gamma^{ik}(e_i, e_k))$ if

$$v_i(j) + \gamma^{ij}(e_i, e_j) > v_i(k) + \gamma^{ik}(e_i, e_k), \quad \text{or}$$
$$v_i(j) + \gamma^{ij}(e_i, e_j) = v_i(k) + \gamma^{ik}(e_i, e_k) \quad \text{and} \quad v_i(j) > v_i(k).$$

With a slight abuse of notation we sometimes write $jR_i k$. Furthermore, for all $i \in A$, P_i denotes the strict relation of R_i (we need the notation R_i because we will be considering situations where $jR_i k$ and $j = k$).

It is clear that every matching problem $\alpha = (A, e, v, \gamma)$ induces a matching market (M, W, P) (cf. Gale and Shapley (1962)) where P is the profile of strict rank orders as derived above. Let $\alpha = (A, e, v, \gamma)$ be a matching problem. An outcome or matching for α is a bijection $\mu : A \to A$ such that for all $i \in A$, $\mu(i) \in A(i)$ and for all $i, j \in A$, $\mu(i) = j$ implies that $\mu(j) = i$. Given $i \in A$ and a matching μ, $\mu(i)$ is called the mate of agent i under matching μ. The set of all matchings for A is denoted by $\mathcal{M}^A$. We say that a matching $\mu \in \mathcal{M}^A$ is individually rational for α if and only if for all $i \in A$ we have that $\mu(i) R_i i$. A pair of agents (i, j) blocks a matching $\mu \in \mathcal{M}^A$ under α if $jP_i\mu(i)$ and

$iP_j\mu(j)$. A matching $\mu \in \mathcal{M}^A$ is stable for α if and only if it is individually rational for α and there is no blocking pair (i, j) for μ under α.

Gale and Shapley (1962) showed the existence of a stable matching for every matching market. Moreover, they proved that there is a stable matching, μ_M, in which every man (woman) gets his best (her worst) mate under stable matchings, and there is a, generally different, stable matching, μ_W, in which every man (woman) gets his worst (her best) optimal outcome under stable matchings. Both can be obtained by the so-called deferred acceptance algorithm (cf. Gale and Shapley (1962)).

A matching rule φ is a map that associates with each matching problem $\alpha = (A, e, v, \gamma)$ a matching $\varphi[\alpha] = \mu \in \mathcal{M}^A$. We define the man optimal matching rule φ_M as the matching rule that associates with each matching problem $\alpha = (A, e, v, \gamma)$ the man optimal matching μ_M of the corresponding market, i.e., $\varphi_M[\alpha] = \mu_M$. With the obvious change of roles we define the woman optimal matching rule φ_W.

We use the following two results to establish our characterizations. Theorem 2 tells us that the man optimal matching mechanism in a matching market makes it a (weakly) dominant strategy for each man to state his true preferences. Theorem 3 points out what happens to the optimal matchings in a matching market if some men extend their list of acceptable women. (Clearly, a similar result to Theorem 3 can be obtained by switching the roles of men and women.)

Theorem 2 *(Dubins and Freedman (1981); Roth (1982); cf. Theorem 4.7 in Roth and Sotomayor (1990)) Let (M, W, P) be a matching market. Let P' be a profile of rank orders*[5] *such that for some man $m^* \in M$ it holds that $P'_i = P_i$, $i \neq m^*$. Let μ_M and μ'_M be the man optimal matchings with respect to P and P', respectively. Then, $\mu_M(m^*)R_{m^*}\mu'_M(m^*)$.*

Theorem 3 *(Gale and Sotomayor (1985); cf. Theorem 2.24 in Roth and Sotomayor (1990)) Let (M, W, P) be a matching market with strict rank orders P. Let $M' \subseteq M$. Let P' be a profile of rank orders with $P'_i = P_i$ for all $i \in W \cup (M \backslash M')$ and where for any $m' \in M'$, $P'_{m'}$ is a strict rank order obtained by adding women to the end of the list of acceptable women in P_m. Let μ_M and μ'_M (μ_W and μ'_W) be the man (woman) optimal matchings with respect to P and P', respectively. Then,*

$$\mu_M(m)R_m\mu'_M(m) \text{ and } \mu_W(m)R_m\mu'_W(m) \text{ for all } m \in M, \text{ and}$$
$$\mu'_M(w)R_w\mu_M(w) \text{ and } \mu'_W(w)R_w\mu_W(w) \text{ for all } w \in W.$$

3 Manipulation via predonation

Throughout this section we consider a fixed set of agents A. Moreover, we assume that any consumption rule satisfies conditions C_1, C_2, and C_3. Finally, we focus on the man optimal matching rule φ_M, as symmetric results for the woman optimal matching rule φ_W can easily be obtained by changing the roles of men and women.

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[5]Rank orders need not be strict for this result.

Definition 4 *(Sertel and Özkal-Sanver (2002)) A matching rule φ is manipulable via predonation by some agent i to some other agent $j \in A\backslash\{i\}$ if and only if there exist two problems $\alpha = (A, e, v, \gamma)$ and $\alpha' = (A, e', v, \gamma)$ where $e'_i < e_i$, $e'_j = e_j + e_i - e'_i$ and $e'_r = e_r$ for all $r \in A\backslash\{i, j\}$ such that*

$$(i_{\alpha'}, \gamma^{ii_{\alpha'}}(e'_i, e'_{i_{\alpha'}}))P_i\left(i_\alpha, \gamma^{ii_\alpha}(e_i, e_{i_\alpha})\right)$$

and

$$(j_{\alpha'}, \gamma^{jj_{\alpha'}}(e'_j, e'_{j_{\alpha'}}))R_j\left(j_\alpha, \gamma^{jj_\alpha}(e_j, e_{j_\alpha})\right)$$

where $i_\alpha = \varphi\left[\alpha\right](i)$, $j_\alpha = \varphi\left[\alpha\right](j)$, $i_{\alpha'} = \varphi\left[\alpha'\right](i)$, and $j_{\alpha'} = \varphi\left[\alpha'\right](j)$.

In order to characterize the class of consumption rules under which the man optimal matching rule can be manipulated via predonation of endowment we recall[6] from S & Ö-S the following two slightly technical, but very weak conditions on consumption rules. For an interpretation of the conditions consider an agent i. For expositional convenience we assume that $i = m \in M$. The first condition, Relevancy of Endowment (RE(m) for short), says that m's endowment matters to some woman, while keeping her endowment constant. The second condition, Reflexive Relevancy of Endowment (RRE(m) for short), says that m's endowment matters to himself, (a) when comparing between the women, or (b) when comparing between the women on the one side and m being single on the other side.

Formally, a consumption rule γ satisfies for some $i \in A$ condition

RE(i) if there exist $j \in A(i)\backslash\{i\}$ and $x, y, z \in \mathcal{R}_+$ such that

$$\gamma^{ji}(x, y) \neq \gamma^{ji}(x, z).$$

RRE($i, \sim$) if (a) there exist $j, j' \in A(i)\backslash\{i\}$, $j \neq j'$, and $x, y, z, s \in \mathcal{R}_+$ such that[7]

$$\gamma^{ij}(z, x) - \gamma^{ij}(y, x) \neq \gamma^{ij'}(z, s) - \gamma^{ij'}(y, s)$$

or (b) there exist $j \in A(i)\backslash\{i\}$ and $x, y, z \in \mathcal{R}_+$ with $y < z$ such that

$$\gamma^{ij}(z, x) - \gamma^{ij}(y, x) \sim z - y.$$

The symbol $\sim$ in the last condition will stand for $>$, $<$, or $\neq$, depending on the type of predonation, i.e., man to woman, man to man, etc. This turns out not only to be a concise and convenient way to adapt conditions that S & Ö-S used in their characterization results, but it also facilitates a comparison of our results with theirs. For instance, they showed that the man optimal matching rule φ_M is prone to manipulability by a woman

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[6]More precisely, the only condition from S & Ö-S that we generalize is RRE($i, >$)(*b*) by replacing $>$ with $\sim$ which will stand for $>$, $<$, or $\neq$. In S & Ö-S conditions RE and RRE were introduced without further interpretation and called $\overline{C_3}$ and C_3^*, respectively.

[7]For convenience, we will assume $\gamma^{ij}(z, x) - \gamma^{ij}(y, x) > \gamma^{ij'}(z, s) - \gamma^{ij'}(y, s)$ and $y < z$.

hiding (or destroying) her endowment if and only if γ satisfies $\text{RE}(w)$ or $\text{RRE}(w, >)$ for some $w \in W$.

S & Ö-S showed that under consumption rules γ_r, γ_e, $\gamma_{\max}$, $\gamma_{\min}$, and γ_p the man optimal matching rule φ_M is manipulable by predonation of endowment from a man to a woman, a man to another man, a woman to a man, and a woman to another woman. As we will see, this is a consequence of the fact that each of these consumption rules satisfies $\text{RE}(i)$ for each $i \in A$ (as can be verified easily).

Next we will characterize for any of the four cases of predonation the class of consumption rules under which φ_M is manipulable.

3.1 Predonation from man to woman

Let Γ_{MW} be the class of consumption rules $\gamma \in \Gamma$ for which at least one of the conditions $\text{RE}(w)$ or $\text{RRE}(w, >)$ for some $w \in W$ holds.

Proposition 5 φ_M *is manipulable by some man via predonation to some woman if and only if $\gamma \in \Gamma_{MW}$.*

Proof. We will first prove the 'if'-part by providing examples that show that if one of the conditions is satisfied then φ_M is manipulable via predonation of endowment by a man to a woman. All our examples can simply be extended to situations with more men and women by choosing preferences appropriately. We elaborate only the first example. Similar procedures for the other examples are left to the reader.

Example $\text{RE}(w)$, i.e., $\gamma \in \Gamma_{MW}$ because there is some $w \in W$ for which $\text{RE}(w)$ holds. Without loss of generality we assume $w = w_1$. Then, there is a man, $m_2 \in M$, say, and $x, y, z \in \mathcal{R}_+$ with $z > y$, such that

$$\gamma^{m_2 w_1}(x, z) > \gamma^{m_2 w_1}(x, y). \tag{1}$$

We now construct two matching problems $\alpha = (A, e, v, \gamma)$ and $\alpha' = (A, e', v, \gamma)$ where α' is obtained by means of a predonation from m_1 to w_1. Let the endowments e and e' be defined by:

- $e_{m_2} = x$, $e_{w_1} = y$, $e_{m_1} = z - y$, and $e_i \in \mathcal{R}_+$ for all $i \in A \backslash \{m_1, m_2, w_1\}$,

- $e'_{m_1} = 0$, $e'_{w_1} = z$, and $e'_j = e_j$ for all $j \in A \backslash \{m_1, w_1\}$.

Choose the valuation profile v such that it satisfies the following conditions:[8]

$$v_{m_1}(w_1) > \max_{j \in A(m_1) \backslash \{w_1, w_2\}} \left\{ v_{m_1}(j) + \gamma^{m_1 j}(e_{m_1}, e_j) \right\} - \gamma^{m_1 w_1}(e_{m_1}, e_{w_1}), \tag{2}$$

$$v_{m_1}(w_2) > v_{m_1}(w_1) + \max \left\{ \gamma^{m_1 w_1}(e_{m_1}, e_{w_1}), \gamma^{m_1 w_1}(e'_{m_1}, e'_{w_1}) \right\} - \gamma^{m_1 w_2}(e'_{m_1}, e'_{w_2}), \tag{3}$$

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[8]To see that such a choice of v_{m_1} is possible, note that we can first choose any value for $v_{m_1}(m_1)$ and $v_{m_1}(w_r)$ ($r \neq 1, 2$), then $v_{m_1}(w_1)$ sufficiently large, and finally $v_{m_1}(w_2)$, also sufficiently large. The feasibility of the choice of the other valuations can be seen in a similar way. Note also that we can find $v_{m_2}(w_2)$ satisfying (4) and (5) since (1) holds.

$$v_{m_2}(w_1) > \max_{j \in A(m_2) \setminus \{w_1, w_2\}} \left\{ v_{m_2}(j) + \gamma^{m_2 j}(e_{m_2}, e_j) \right\} - \gamma^{m_2 w_1}(e_{m_2}, e_{w_1}),$$

$$\gamma^{m_2 w_1}(e_{m_2}, e_{w_1}) - \gamma^{m_2 w_2}(e_{m_2}, e_{w_2}) < v_{m_2}(w_2) - v_{m_2}(w_1), \tag{4}$$

$$v_{m_2}(w_2) - v_{m_2}(w_1) < \gamma^{m_2 w_1}(e'_{m_2}, e'_{w_1}) - \gamma^{m_2 w_2}(e_{m_2}, e_{w_2}), \tag{5}$$

$$v_{m_k}(m_k) > \max_{j \in A(m_k) \setminus \{m_k\}} \left\{ v_{m_k}(j) + \gamma^{m_k j}(e'_{m_k}, e'_j) \right\} - e_{m_k}, \ m_k \in M \setminus \{m_1, m_2\},$$

$$v_{w_1}(m_1) > \max_{j \in A(w_1) \setminus \{m_1, m_2\}} \left\{ v_{w_1}(j) + \gamma^{w_1 j}(e'_{w_1}, e'_j) \right\} - \gamma^{w_1 m_1}(e_{w_1}, e_{m_1}),$$

$$v_{w_1}(m_2) > v_{w_1}(m_1) + \max \left\{ \gamma^{w_1 m_1}(e_{w_1}, e_{m_1}), \gamma^{w_1 m_1}(e'_{w_1}, e'_{m_1}) \right\} - \gamma^{w_1 m_2}(e_{w_1}, e_{m_2}), \tag{6}$$

$$v_{w_2}(m_1) > \max_{j \in A(w_2) \setminus \{m_1, m_2\}} \left\{ v_{w_2}(j) + \gamma^{w_2 j}(e_{w_2}, e_j) \right\} - \gamma^{w_2 m_1}(e_{w_2}, e'_{m_1}),$$

$$v_{w_2}(m_2) > v_{w_2}(m_1) + \gamma^{w_2 m_1}(e_{w_2}, e_{m_1}) - \gamma^{w_2 m_2}(e_{w_2}, e_{m_2}),$$

$$v_{w_k}(w_k) > \max_{j \in A(w_k) \setminus \{w_k\}} \left\{ v_{w_k}(j) + \gamma^{w_k j}(e_{w_k}, e_j) \right\} - e_{w_k}, \ w_k \in W \setminus \{w_1, w_2\}.$$

Then the first positions in the initial preference system can be concisely depicted as follows:

m_1	m_2	w_1	w_2
w_2	$\mathbf{w_2}$	m_2	m_2
$\updownarrow$	$\mathbf{w_1}$	$\updownarrow$	m_1
w_1	m_2	m_1	$\updownarrow$
m_1		w_1	w_2

$$(\mathrm{RE}(w_1))$$

Here and henceforth we indicate in boldface which positions definitely switch going from α to α' due to the choice of e and v. The arrows indicate 'sufficient distance' in the valuations so that after a predonation 1) no agent from the upper (lower) part has moved to the lower (upper) part, and 2) the utility levels of the upper part are still higher than the pre-predonation utility levels of the lower part. For instance, the arrow in the first column depicts conditions (2) and (3), which not only guarantee that w_2 remains agent m_1's favorite mate, but also make sure that the utility from w_2 after predonation is still higher than the utility from any other agent before the predonation. We have left out any agent not in $\{m_1, m_2, w_1, w_2\}$ because their positions do not influence in the changes of the man optimal matching.

In matching problem α, the man optimal stable matching μ_M^α is given by

$$\begin{aligned}
\mu_M^\alpha(m_1) &= w_1, \ \mu_M^\alpha(m_2) = w_2, \\
\mu_M^\alpha(m_k) &= m_k, \ \text{for all } m_k \in M \setminus \{m_1, m_2\}, \ \text{and} \\
\mu_M^\alpha(w_k) &= w_k, \ \text{for all } w_k \in W \setminus \{w_1, w_2\},
\end{aligned}$$

whereas in α', the man optimal stable matching $\mu_M^{\alpha'}$ is given by

$$\begin{aligned}
\mu_M^{\alpha'}(m_1) &= w_2, \ \mu_M^{\alpha'}(m_2) = w_1, \\
\mu_M^{\alpha'}(m_k) &= m_k, \ \text{for all } m_k \in M \setminus \{m_1, m_2\}, \ \text{and} \\
\mu_M^{\alpha'}(w_k) &= w_k, \ \text{for all } w_k \in W \setminus \{w_1, w_2\}.
\end{aligned}$$

From (3), (6), and C_3 it follows that,

$$v_{m_1}(w_2) + \gamma^{m_1 w_2}\left(e'_{m_1}, e'_{w_2}\right) > v_{m_1}(w_1) + \gamma^{m_1 w_1}\left(e_{m_1}, e_{w_1}\right) \text{ and}$$
$$v_{w_1}(m_2) + \gamma^{w_1 m_2}\left(e'_{w_1}, e'_{m_2}\right) > v_{w_1}(m_1) + \gamma^{w_1 m_1}\left(e_{w_1}, e_{m_1}\right).$$

So the predonation from m_1 to w_1 is indeed profitable:

$$\left(w_2, \gamma^{m_1 w_2}\left(e'_{m_1}, e'_{w_2}\right)\right) P_{m_1}\left((w_1, \gamma^{m_1 w_1}\left(e_{m_1}, e_{w_1}\right)\right) \text{ and}$$
$$\left(m_2, \gamma^{w_1 m_2}\left(e'_{w_1}, e'_{m_2}\right)\right) R_{w_1}\left((m_1, \gamma^{w_1 m_1}\left(e_{w_1}, e_{m_1}\right)\right).$$

The remaining cases $\text{RRE}(w, >)(a)$ and $\text{RRE}(w, >)(b)$ can be analyzed extending the examples below in the same way as we previously have done for $\text{RE}(w)$.

Examples $\text{RRE}(w, >)(a)$ and $\text{RRE}(w, >)(b)$. Without loss of generality suppose that $w = w_1$. Let a man, m_1, say, predonate to w_1. Then the following tables show that this is profitable for both m_1 and w_1 if we maintain appropriate distances:

m_1	m_2	m_3	w_1	w_2
w_2	w_1	w_1	$\mathbf{m_3}$	m_2
↕	w_2	m_3	$\mathbf{m_2}$	m_1
m_1	m_2	w_2	↕	↕
w_1			w_1	w_2
			m_1	m_3

$$\text{RRE}(w_1, >)(a)$$
$$j = m_2,\ j' = m_3$$

m_1	m_2	w_1	w_2
w_1	w_1	$\mathbf{w_1}$	m_2
w_2	w_2	$\mathbf{m_2}$	m_1
↕	m_2	↕	↕
m_1		m_1	w_2

$$\text{RRE}(w_1, >)(b)$$
$$j = m_2$$

Now we will prove the 'only if'-part. Let $\gamma \notin \Gamma_{MW}$. Then for any $w \in W$ none of the conditions $\text{RE}(w)$ and $\text{RRE}(w, >)$ holds.

Consider a problem $\alpha = (A, e, v, \gamma)$. Suppose some man predonates some amount of his endowment to some woman. Without loss of generality we assume that m_1 predonates to w_1. Let $\alpha' = (A, e', v, \gamma)$ be the resulting problem. We are done if we prove that m_1 is not strictly better off at α'.

The predonation causes the following changes in the preferences. It follows from [not $\text{RE}(w_1)$] that $\gamma^{m_1 w_1}(y, x) \leq \gamma^{m_1 w_1}(z, t)$ for all $x, y, z, t \in \mathcal{R}_+$ with $y < z$ and $y + x = z + t$. Hence,

$$v_{m_1}(w_1) + \gamma^{m_1 w_1}(e'_{m_1}, e'_{w_1}) \leq v_{m_1}(w_1) + \gamma^{m_1 w_1}(e_{m_1}, e_{w_1}). \tag{7}$$

From C_2 and C_3 it follows immediately that

$$v_{m_1}(i) + \gamma^{m_1 i}(e'_{m_1}, e'_i) \leq v_{m_1}(i) + \gamma^{m_1 i}(e_{m_1}, e_i) \text{ for all } i \in A(m_1) \setminus \{w_1\}. \tag{8}$$

It also follows from [not $\text{RE}(w)$] that

$$v_m(i) + \gamma^{mi}(e'_m, e'_i) = v_m(i) + \gamma^{mi}(e_m, e_i) \text{ for all } m \in M \setminus \{m_1\} \text{ and all } i \in A(m). \tag{9}$$

From [not RRE($w_1, >$)] it follows that $z - y \geq \gamma^{w_1 m}(z, s) - \gamma^{w_1 m}(y, s) \geq \gamma^{w_1 m_1}(z, t) - \gamma^{w_1 m_1}(y, x)$ for all $m \in M \backslash \{m_1\}$ and $x, y, z, t, s \in \mathcal{R}_+$ with $y < z$ and $y + x = z + t$. Using these inequalities and again [not RRE($w_1, >$)] we obtain

$$
\begin{aligned}
(v_{w_1}(w_1) + e'_{w_1}) \quad &- \quad (v_{w_1}(w_1) + e_{w_1}) \geq && (10) \\
(v_{w_1}(\bar{m}) + \gamma^{w_1 \bar{m}}(e'_{w_1}, e'_{\bar{m}})) \quad &- \quad (v_{w_1}(\bar{m}) + \gamma^{w_1 \bar{m}}(e_{w_1}, e_{\bar{m}})) = && \\
(v_{w_1}(\tilde{m}) + \gamma^{w_1 \tilde{m}}(e'_{w_1}, e'_{\tilde{m}})) \quad &- \quad (v_{w_1}(\tilde{m}) + \gamma^{w_1 \tilde{m}}(e_{w_1}, e_{\tilde{m}})) \geq && \\
(v_{w_1}(m_1) + \gamma^{w_1 m_1}(e'_{w_1}, e'_{m_1})) \quad &- \quad (v_{w_1}(m_1) + \gamma^{w_1 m_1}(e_{w_1}, e_{m_1})) \text{ for all } \bar{m}, \tilde{m} \in A(w_1) \backslash \{w_1, m_1\}. &&
\end{aligned}
$$

Condition C_3 yields

$$
v_w(m_1) + \gamma^{w m_1}(e'_w, e'_{m_1}) \leq v_w(m_1) + \gamma^{w m_1}(e_w, e_{m_1}) \text{ for all } w \in W \backslash \{w_1\}. \tag{11}
$$

Note also that

$$
v_w(i) + \gamma^{wi}(e'_w, e'_i) = v_w(i) + \gamma^{wi}(e_w, e_i) \text{ for all } w \in W \backslash \{w_1\} \text{ and all } i \in A(w) \backslash \{m_1\}. \tag{12}
$$

Next let us turn to an interpretation of inequalities (7)-(12). Inequalities (7) and (8) show us that the utility levels of m_1 all may drop, but it does not provide any information on whether the positions of mates interchange. Equality (9) shows that the rank order of any other man does not change. From inequality (10) the only possible changes in w_1's rank order are a descent of m_1 and an ascent of w_1. Inequalities (11) and (12) show that the only possible change in a woman's $w \neq w_1$ rank order is a descent of m_1.

Let P_i^0 and P_i^3 denote agent i's rank order in problems α and α', respectively. Note that $P_{w_1}^3$ can be obtained from $P_{w_1}^0$ in two steps: first we put m_1 in a (weakly) lower position, and subsequently we put w_1 in a (weakly) higher position. Let Q be the rank order of agent w_1 after the first step.

We consider two auxiliary intermediate profiles of rank orders P^1 and P^2 for all agents. Define $P_{m_1}^1 := P_{m_1}^0$ and $P_i^1 := P_i^3$ for $i \in A \backslash \{m_1, w_1\}$, and $P_{w_1}^1 := Q$. Define $P_{m_1}^2 := P_{m_1}^0$ and $P_i^2 := P_i^3$ for $i \in A \backslash \{m_1\}$. Note that all rank orders are strict; there are no ties. Let μ^0, μ^1, μ^2, and μ^3 denote the man optimal matchings for rank order profiles P^0, P^1, P^2, and P^3, respectively.

Let $m^* := m_1$ and denote $w^* := \mu^0(m^*) \in A(m^*)$. In view of inequality (8) we are done if we prove that $w^* R_{m^*}^0 \mu^3(m^*)$. This will be done in three steps. First we show $w^* R_{m^*}^0 \mu^1(m^*)$, then $\mu^1(m^*) R_{m^*}^0 \mu^2(m^*)$, and finally $\mu^2(m^*) R_{m^*}^0 \mu^3(m^*)$. The result then follows from the transitivity of $R_{m^*}^0$.

STEP 1: Suppose $\mu^1(m^*) P_{m^*}^0 w^*$. Then by individual rationality of μ^0 at P^0, $\tilde{w} := \mu^1(m^*) \in W$ So, $\tilde{w} P_{m^*}^0 w^* R_{m^*}^0 m^*$ and $\tilde{w} \neq w^*$.

Define a new profile of strict rank orders $\bar{P}^0$ as follows. Define $\bar{P}_i^0 := P_i^0$ for $i \neq m^*$, and let $\bar{P}_{m^*}^0$ be such that $\tilde{w} \bar{P}_{m^*}^0 w^* \bar{R}_{m^*}^0 m^*$, $[w^* P_{m^*}^0 m^*$ implies $w^* \bar{P}_{m^*}^0 m^*]$, and $m^* \bar{P}_{m^*}^0 w$ for all $w \in A(m^*) \backslash \{m^*, \tilde{w}, w^*\}$. Similarly, define a profile of rank orders $\bar{P}^1$ by $\bar{P}_i^1 := P_i^1$ for $i \neq m^*$ and $\bar{P}_{m^*}^1 := \bar{P}_{m^*}^0$.

Let $\bar{\mu}^0$ and $\bar{\mu}^1$ denote the man optimal matchings for rank order profiles $\bar{P}^0$ and $\bar{P}^1$, respectively. Recall that for the man optimal matching mechanism it is always a weakly

dominant strategy for m^* to state his true rank order (Theorem 2). Assume $\bar{\mu}^0(m^*) \neq w^*$. If $\bar{\mu}^0(m^*)\bar{P}^0_{m^*}w^*$ then $\bar{\mu}^0(m^*) = \tilde{w}$. Hence, at P^0 man m^* can profit by reporting $\bar{P}^0_{m^*}$ instead of $P^0_{m^*}$, a contradiction to truth-telling being weakly dominant. If $w^*\bar{P}^0_{m^*}\bar{\mu}^0(m^*)$ then $\bar{\mu}^0(m^*) = m^*$ by individual rationality of $\bar{\mu}^0$ at $\bar{P}^0$. Hence, at $\bar{P}^0$ man m^* can profit by reporting $P^0_{m^*}$ instead of $\bar{P}^0_{m^*}$, again a contradiction to truth-telling being weakly dominant. Hence, $\bar{\mu}^0(m^*) = w^*$. Analogously it follows that $\bar{\mu}^1(m^*) = \tilde{w}$.

Since $\tilde{w}\bar{P}^0_{m^*}w^*$ and $\bar{\mu}^0$ is the man optimal matching at $\bar{P}^0$ it follows that $\bar{\mu}^1$ is not stable at $\bar{P}^0$.

Suppose $\bar{\mu}^1$ is not individual rational at $\bar{P}^0$. Then there is an agent $i \neq m^*$ such that $i\bar{P}^0_i\bar{\mu}^1(i)$. If $i \in M$ then matching $\bar{\mu}^1$ is also not individual rational at $\bar{P}^1$ since $\bar{P}^0_i = \bar{P}^1_i$, a contradiction. If $i \in W$, then $i\bar{P}^0_i\bar{\mu}^1(i)$ implies $i\bar{P}^1_i\bar{\mu}^1(i)$, and again we find that $\bar{\mu}^1$ is also not individual rational at $\bar{P}^1$, a contradiction.

Hence, $\bar{\mu}^1$ is not stable at $\bar{P}^0$ because there exists a blocking pair $(\bar{m}, \bar{w}) \in M \times W$, say. So,

$$\bar{m}\bar{P}^0_{\bar{w}}\bar{\mu}^1(\bar{w}) \qquad \text{and} \tag{13}$$
$$\bar{w}\bar{P}^0_{\bar{m}}\bar{\mu}^1(\bar{m}). \tag{14}$$

On the other hand, $\bar{\mu}^1$ is stable at $\bar{P}^1$. Hence,

$$\bar{\mu}^1(\bar{w})\bar{P}^1_{\bar{w}}\bar{m} \qquad \text{or} \tag{15}$$
$$\bar{\mu}^1(\bar{m})\bar{P}^0_{\bar{m}}\bar{w}. \tag{16}$$

(Notice that $\bar{P}^0_{\bar{m}} = \bar{P}^1_{\bar{m}}$ by construction.) Since (16) contradicts (14) it follows that (15) holds.

If both $\bar{m}, \bar{\mu}^1(\bar{w}) \neq m^*$, then (15) implies $\bar{\mu}^1(\bar{w})\bar{P}^0_{\bar{w}}\bar{m}$, contradicting (13). If $\bar{\mu}^1(\bar{w}) = m^*$ and $\bar{m} \neq m^*$, then (13) implies $\bar{m}\bar{P}^1_{\bar{w}}m^*$, contradicting (15).

Hence, $\bar{m} = m^*$. From (14) it follows that $\bar{w}\bar{P}^0_{m^*}\bar{\mu}^1(m^*)$, contradicting that $\bar{\mu}^1(m^*) = \tilde{w}$ is m^*'s best choice. This completes the first step.

STEP 2: In view of the only difference between the rank orders P^1 and P^2 (less men acceptable for w_1), it follows from Theorem 3 (switching the roles of men and women) that $\mu^1(m^*)R^1_{m^*}\mu^2(m^*)$. Since $R^1_{m^*} = R^0_{m^*}$ we conclude $\mu^1(m^*)R^0_{m^*}\mu^2(m^*)$.

STEP 3: It follows from Theorem 2 that $\mu^2(m^*)R^2_{m^*}\mu^3(m^*)$. Since $R^2_{m^*} = R^0_{m^*}$ it follows that $\mu^2(m^*)R^0_{m^*}\mu^3(m^*)$, completing the proof. ∎

3.2 Predonation from man to man

Let Γ_{MM} be the class of consumption rules $\gamma \in \Gamma$ for which at least one of the conditions $\text{RE}(m)$ or $\text{RRE}(m, <)$ for some $m \in M$ holds.

Proposition 6 φ_M *is manipulable by some man via predonation to some other man if and only if $\gamma \in \Gamma_{MM}$.*

Proof.   We will first prove the 'if'-part by providing examples that show that if one of the conditions is satisfied then φ_M is manipulable via predonation of endowment by a man to some other man. All our examples can simply be extended to situations with more men and women by choosing preferences appropriately.

Examples $\mathrm{RE}(m)$, $\mathrm{RRE}(m, <)(a)$, and $\mathrm{RRE}(m, <)(b)$. Suppose without loss of generality $m = m_2$. Let m_1 predonate some endowment to m_2. Then the following tables show that this is profitable for both m_1 and m_2 if we maintain appropriate distances:

m_1	m_2	w_1	w_2
w_1	w_1	$\mathbf{w_1}$	m_2
w_2	$\updownarrow$	$\mathbf{m_2}$	m_1
$\updownarrow$	w_2	m_1	$\updownarrow$
m_1	m_2		w_2

$\mathrm{RE}(m_2)$

$j = w_1$

m_1	m_2	w_1	w_2
w_1	$\mathbf{w_1}$	m_2	m_2
$\updownarrow$	$\mathbf{w_2}$	m_1	m_1
w_2	$\updownarrow$	$\updownarrow$	w_2
m_1	m_2	w_1	

$\mathrm{RRE}(m_2, <)(a)$

$j = w_2, j' = w_1$

m_1	m_2	w_1
w_1	$\mathbf{w_1}$	m_2
$\updownarrow$	$\mathbf{m_2}$	m_1
m_1		$\updownarrow$
		w_1

$\mathrm{RRE}(m_2, <)(b)$

$j = w_1$

Now we will prove the 'only if'-part. Let $\gamma \notin \Gamma_{MM}$. Then for any man $m \in M$ none of the conditions $\mathrm{RE}(m)$ and $\mathrm{RRE}(m, <)$ holds.

Consider a problem $\alpha = (A, e, v, \gamma)$. Suppose some man predonates some amount of his endowment to some other man. Without loss of generality we assume that m_1 predonates to m_2. Let $\alpha' = (A, e', v, \gamma)$ be the resulting problem. We are done if we prove that m_1 is not strictly better off at α'.

The predonation causes the following changes in the preferences. It follows from C_2 and C_3 that

$$v_{m_1}(i) + \gamma^{m_1 i}(e'_{m_1}, e'_i) \leq v_{m_1}(i) + \gamma^{m_1 i}(e_{m_1}, e_i) \text{ for all } i \in A(m_1). \tag{17}$$

It follows from [not $\mathrm{RE}(m)$] that

$$v_w(i) + \gamma^{wi}(e'_w, e'_i) = v_w(i) + \gamma^{wi}(e_w, e_i) \text{ for all } w \in W \text{ and all } i \in A(w). \tag{18}$$

Note that

$$v_m(i) + \gamma^{mi}(e'_m, e'_i) = v_m(i) + \gamma^{mi}(e_m, e_i) \text{ for all } m \in M \backslash \{m_1, m_2\} \text{ and all } i \in A(m). \tag{19}$$

It follows from [not $\mathrm{RRE}(m_2, <)$] that

$$
\begin{aligned}
\left(v_{m_2}(\bar{w}) + \gamma^{m_2 \bar{w}}(e'_{m_2}, e'_{\bar{w}})\right) &- \left(v_{m_2}(\bar{w}) + \gamma^{m_2 \bar{w}}(e_{m_2}, e_{\bar{w}})\right) = \\
\left(v_{m_2}(\tilde{w}) + \gamma^{m_2 \tilde{w}}(e'_{m_2}, e'_{\tilde{w}})\right) &- \left(v_{m_2}(\tilde{w}) + \gamma^{m_2 \tilde{w}}(e_{m_2}, e_{\tilde{w}})\right) \geq \\
\left(v_{m_2}(m_2) + e'_{m_2}\right) &- \left(v_{m_2}(m_2) + e_{m_2}\right) \text{ for all } \bar{w}, \tilde{w} \in A(m_2) \backslash \{m_2\}.
\end{aligned}
\tag{20}
$$

Inequality (17) shows that the utility levels of m_1 all may drop, but it does not provide any information on whether the positions of mates interchange. Equalities (18) and (19)

say that the rank order of any agent $i \in A\backslash\{m_1, m_2\}$ does not change. Inequality (20) establishes that the only possible change in m_2's rank order is a descent of m_2.

Let P_i^0 and P_i^2 denote agent i's rank order in problems α and α', respectively.

We consider an auxiliary intermediate profile of rank orders P^1 for all agents. Define $P_{m_1}^1 := P_{m_1}^0$ and $P_i^1 := P_i^2$ for $i \in A\backslash\{m_1\}$. Note that all rank orders are strict; there are no ties. Let μ^0, μ^1, and μ^2 denote the man optimal matchings for rank order profiles P^0, P^1, and P^2, respectively.

In view of inequality (17) we are done if we prove that $\mu^0(m_1)R_{m_1}^0\mu^2(m_1)$. This will be done in two steps. First we show $\mu^0(m_1)R_{m_1}^0\mu^1(m_1)$ and subsequently $\mu^1(m_1)R_{m_1}^0\mu^2(m_1)$. The result then follows from the transitivity of $R_{m_1}^0$.

STEP 1: In view of the only difference between the rank orders P^1 and P^0 (more women acceptable in m_2's rank order), it follows from Theorem 3 that $\mu^0(m_1)R_{m_1}^0\mu^1(m_1)$.

STEP 2: It follows from Theorem 2 that $\mu^1(m_1)R_{m_1}^1\mu^2(m_1)$. Since $R_{m_1}^1 = R_{m_1}^0$ it holds that $\mu^1(m_1)R_{m_1}^0\mu^2(m_1)$. This completes the proof. ∎

3.3 Predonation from woman to man

Let Γ_{WM} be the class of consumption rules $\gamma \in \Gamma$ for which at least one of the conditions $\mathrm{RE}(i)$ and $\mathrm{RRE}(i,>)$ for some $i \in A$ holds.

Proposition 7 *φ_M is manipulable by some woman via predonation to some man if and only if $\gamma \in \Gamma_{WM}$.*

Proof. We will first prove the 'if'-part by providing examples that show that if one of the conditions is satisfied then φ_M is manipulable via predonation of endowment by a woman to a man. All our examples can simply be extended to situations with more men and women by choosing preferences appropriately.

Examples $\mathrm{RE}(m)$, $\mathrm{RE}(w)$, $\mathrm{RRE}(m,>)(a)$, $\mathrm{RRE}(m,>)(b)$, $\mathrm{RRE}(w,>)(a)$, and $\mathrm{RRE}(w,>)$ (b). Suppose without loss of generality that $m = m_1$ and $w = w_1$. Let w_1 predonate to m_1. Then the following tables show that this is profitable for both w_1 and m_1 if we maintain appropriate distances:

m_1	m_2	w_1	w_2
w_2	w_2	m_2	**m_2**
$\updownarrow$	w_1	$\updownarrow$	**m_1**
w_1	$\updownarrow$	m_1	w_2
m_1	m_2	w_1	

$$\mathrm{RE}(m_1)$$
$$j = w_2$$

m_1	m_2	m_3	w_1	w_2
m_1	**w_1**	w_2	m_1	m_1
$\updownarrow$	**w_2**	w_1	m_3	m_2
w_1	$\updownarrow$	$\updownarrow$	$\updownarrow$	m_3
w_2	m_2	m_3	m_2	w_2
			w_1	

$$\mathrm{RE}(w_1)$$
$$j = m_2$$

m_1	m_2	w_1	w_2	w_3
$\mathbf{w_3}$	w_2	m_2	m_1	m_1
$\mathbf{w_2}$	w_1	$\updownarrow$	m_2	w_3
$\updownarrow$	$\updownarrow$	w_1	w_2	m_2
m_1	m_2	m_1		
w_1	w_3			

$$\text{RRE}(m_1,>)(a)$$
$$j = w_2, j' = w_3$$

m_1	m_2	w_1	w_2
$\mathbf{m_1}$	w_2	m_1	m_1
$\mathbf{w_2}$	w_1	m_2	m_2
$\updownarrow$	$\updownarrow$	$\updownarrow$	w_2
w_1	m_2	w_1	

$$\text{RRE}(m_1,>)(b)$$
$$j = w_2$$

m_1	m_2	m_3	w_1	w_2	w_3
w_1	w_1	w_2	m_3	m_2	m_1
$\updownarrow$	$\updownarrow$	w_1	$\updownarrow$	m_3	w_3
w_3	w_2	$\updownarrow$	$\mathbf{m_2}$	m_1	m_2
$\updownarrow$	w_3	w_3	$\mathbf{m_1}$	w_2	m_3
w_2	m_2	m_3	$\updownarrow$		
m_1			w_1		

$$\text{RRE}(w_1,>)(a)$$
$$j = m_2, j' = m_1$$

m_1	m_2	m_3	w_1	w_2
m_1	w_1	w_2	m_3	m_2
$\updownarrow$	w_2	w_1	$\updownarrow$	m_3
w_1	m_2	$\updownarrow$	$\mathbf{m_2}$	w_2
w_2		m_3	$\mathbf{w_1}$	m_1
			m_1	

$$\text{RRE}(w_1,>)(b)$$
$$j = m_2$$

Now we will prove the 'only if'-part. Let $\gamma \notin \Gamma_{WM}$. Then for any $i \in A$ none of the conditions $\text{RE}(i)$ and $\text{RRE}(i,>)$ holds. From [not $\text{RE}(m)$] it follows that $\gamma^{wm}(x,y) = \gamma^{wm}(x,z)$ for all $w \in W$, $m \in M$, and $x,y,z \in \mathcal{R}_+$. Hence, for each $w \in W$ and each $m \in M$ there exists a function $f^{wm} : \mathcal{R}_+ \to \mathcal{R}_+$ such that $f^{wm}(x) = \gamma^{wm}(x,z)$ for all $x,z \in \mathcal{R}_+$. From [not $\text{RRE}(w,>)(a)$] it follows that

$$f^{wm}(x) - f^{wm}(y) = f^{wm'}(x) - f^{wm'}(y) \text{ for all } w \in W, m, m' \in M, \text{ and } x,y \in \mathcal{R}_+. \quad (21)$$

Since $f^{wm}(0) = \gamma^{wm}(0,0) = 0$ for all $w \in W$ and $m \in M$, (21) yields

$$f^{wm}(x) = f^{wm'}(x) \text{ for all } w \in W, m, m' \in M, \text{ and } x \in \mathcal{R}_+.$$

So for each $w \in W$ there exists a function $g^w : \mathcal{R}_+ \to \mathcal{R}_+$ such that

$$g^w(x) = f^{wm}(x) = \gamma^{wm}(x,y) = \gamma^{wm}(x,0) \le x \text{ for all } m \in M \text{ and } x,y \in \mathcal{R}_+. \quad (22)$$

Similarly, [not $\text{RE}(w)$] and [not $\text{RRE}(m,>)(a)$] imply that for each $m \in M$ there exists a function $g^m : \mathcal{R}_+ \to \mathcal{R}_+$ such that

$$g^m(x) = f^{mw}(x) = \gamma^{mw}(x,y) = \gamma^{mw}(x,0) \le x \text{ for all } w \in W \text{ and } x,y \in \mathcal{R}_+. \quad (23)$$

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Consider a problem $\alpha = (A, e, v, \gamma)$. Suppose some woman predonates some amount of her endowment to some man. Without loss of generality we assume that w_1 predonates to m_1. Let $\alpha' = (A, e', v, \gamma)$ be the resulting problem. We are done if we prove that w_1 is not strictly better off at α'.

Clearly, (22) and (23) imply that the rank order of any agent $i \in A \backslash \{w_1, m_1\}$ does not change. From (22) and [not $\mathrm{RRE}(w, >)(b)$] it follows that the only possible change in w_1's rank order is a descent of herself (i.e., more men are acceptable now). From (23) and [not $\mathrm{RRE}(m, >)(b)$] it follows that the only possible change in m_1's rank order is an ascent of himself (i.e., less women are acceptable now).

Let P_i^0 and P_i^2 denote agent i's rank order in the initial and new problem, respectively.

We consider an auxiliary intermediate profile of rank orders P^1 for all agents. Define $P_{w_1}^1 := P_{w_1}^0$ and $P_i^1 := P_i^2$ for $i \in A \backslash \{w_1\}$. Note that all rank orders are strict; there are no ties. Let μ^0, μ^1, and μ^2 denote the man optimal matchings for rank order profiles P^0, P^1, and P^2, respectively.

From (22) it follows that w_1's utility levels all may drop. Hence, we are done if we prove that $\mu^0(w_1) R_{w_1}^0 \mu^2(w_1)$. This will be done in two steps. First we show $\mu^0(w_1) R_{w_1}^0 \mu^1(w_1)$ and subsequently $\mu^1(w_1) R_{w_1}^0 \mu^2(w_1)$. The result then follows from the transitivity of $R_{w_1}^0$.
STEP 1: In view of the only change in the rank orders P^0 and P^1 (less women acceptable for m_1), it follows from Theorem 3 that $\mu^0(w_1) R_{w_1}^0 \mu^1(w_1)$.
STEP 2: The only difference between the rank orders P^1 and P^2 is that more men are acceptable for w_1. From Theorem 3 (switching the roles of men and women) it follows that $\mu^1(w_1) R_{w_1}^1 \mu^2(w_1)$. Since $R_{w_1}^1 = R_{w_1}^0$ we have $\mu^1(w_1) R_{w_1}^0 \mu^2(w_1)$, completing the proof. ∎

3.4 Predonation from woman to woman

Let Γ_{WW} be the class of consumption rules $\gamma \in \Gamma$ for which at least one of the conditions $\mathrm{RE}(w)$ or $\mathrm{RRE}(w, \neq)$ for some $w \in W$ holds.

Proposition 8 φ_M *is manipulable by some woman via predonation to some other woman if and only if $\gamma \in \Gamma_{WW}$.*

Proof. We will first prove the 'if'-part by providing examples that show that if one of the conditions is satisfied then φ_M is manipulable via predonation of endowment by a woman to some other woman. All our examples can simply be extended to situations with more men and women by choosing preferences appropriately.

Examples $\mathrm{RE}(w)$, $\mathrm{RRE}(w, \neq)(a)$, $\mathrm{RRE}(w, <)(b)$, and $\mathrm{RRE}(w, >)(b)$. Let w_1 predonate to w_2. Then the following tables show that this is profitable for both w_1 and w_2 if we

maintain appropriate distances:

m_1	m_2	w_1	w_2
$\mathbf{m_1}$	w_2	m_1	m_1
$\mathbf{w_2}$	w_1	m_2	$\updownarrow$
w_1	$\updownarrow$	$\updownarrow$	m_2
	m_2	w_1	w_2

$$\mathrm{RE}(w_2)$$
$$j = m_1$$

m_1	m_2	w_1	w_2
w_2	w_2	m_1	$\mathbf{m_1}$
w_1	w_1	$\updownarrow$	$\mathbf{m_2}$
$\updownarrow$	m_2	m_2	$\updownarrow$
m_1		w_1	w_2

$$\mathrm{RRE}(w_2, \neq)(a)$$
$$j = m_2, \; j' = m_1$$

m_1	w_1	w_2
w_2	m_1	$\mathbf{m_1}$
w_1	$\updownarrow$	$\mathbf{w_2}$
$\updownarrow$	w_1	
m_1		

$$\mathrm{RRE}(w_2, <)(b)$$
$$j = m_1$$

m_1	m_2	m_3	w_1	w_2	w_3
w_2	w_2	w_2	m_1	m_1	m_1
w_1	w_1	w_3	m_3	$\updownarrow$	m_2
w_3	w_3	w_1	$\updownarrow$	m_2	m_3
m_1	m_2	$\updownarrow$	$\mathbf{m_2}$	m_3	w_3
		m_3	$\mathbf{w_1}$	w_2	

$$\mathrm{RRE}(w_1, >)(b)$$
$$j = m_2$$

Now we will prove the 'only if'-part. Let $\gamma \notin \Gamma_{WW}$. Then for any $w \in W$ none of the conditions $\mathrm{RE}(w)$ or $\mathrm{RRE}(w, \neq)$ holds.

Consider a problem $\alpha = (A, e, v, \gamma)$. Suppose some woman predonates some amount of her endowment to some other woman. Without loss of generality we assume that w_1 predonates to w_2. Let $\alpha' = (A, e', v, \gamma)$ be the resulting problem. We are done if we prove that w_1 is not strictly better off at α'.

The predonation causes the following changes in the preferences. It follows from [not $\mathrm{RE}(w)$] that

$$v_m(i) + \gamma^{mi}(e'_m, e'_i) = v_m(i) + \gamma^{mi}(e_m, e_i) \text{ for all } m \in M \text{ and all } i \in A(m). \tag{24}$$

From [not $\mathrm{RRE}(w, \neq)$] it follows that

$$
\begin{aligned}
\left(v_w(\bar{m}) + \gamma^{w\bar{m}}(e'_w, e'_{\bar{m}})\right) &- \left(v_w(\bar{m}) + \gamma^{w\bar{m}}(e_w, e_{\bar{m}})\right) = \\
\left(v_w(\tilde{m}) + \gamma^{w\tilde{m}}(e'_w, e'_{\tilde{m}})\right) &- \left(v_w(\tilde{m}) + \gamma^{w\tilde{m}}(e_w, e_{\tilde{m}})\right) = \\
\left(v_w(w) + e'_w\right) &- \left(v_w(w) + e_w\right) \text{ for all } w \in \{w_1, w_2\} \text{ and all } \bar{m}, \tilde{m} \in M.
\end{aligned}
\tag{25}
$$

Finally note that

$$v_w(i) + \gamma^{wi}(e'_w, e'_i) = v_w(i) + \gamma^{wi}(e_w, e_i) \text{ for all } w \in W \backslash \{w_1, w_2\} \text{ and all } i \in A(w). \tag{26}$$

Equality (24) shows that the utility levels (in particular the rank orders) of the men do not change. Equality (25) says that although all utility levels of agent w_2 (agent w_1)

increase (decrease), her rank order does not change. Finally, equality (26) says that the utility levels (in particular the rank order) of any woman $w \in W \setminus \{w_1, w_2\}$ do not change.

Note that the rank orders in the initial and new problem are the same. Hence, the man optimal matching at α and α' is also the same. Since all of w_1's utility levels (weakly) drop it follows immediately that w_1 is not strictly better off at α'. ∎

4 Concluding remarks

We have summarized our main results in Table 1. The result in S & Ö-S that each of the consumption rules γ_r, γ_e, $\gamma_{\max}$, $\gamma_{\min}$, and γ_p is manipulable in any of the four cases follows easily from the observation that these consumption rules satisfy RE(i) for each $i \in A$.

Notice that if a consumption rule allows manipulation by a man via predonation to a woman then it also allows manipulation by a woman via predonation to a man and by a woman via predonation to a woman, and vice versa, i.e., $\Gamma_{MW} = \Gamma_{WM} \cap \Gamma_{WW}$. Switching the roles of men and women in Table 1, we obtain a characterization of the consumption rules that allow for manipulation of the woman optimal matching rule φ_W.

Type of predonation	Domain of consumption rules under which φ_M is manipulable
man to woman	$\Gamma_{MW} = \{\gamma \in \Gamma : \gamma \text{ satisfies RE}(w) \text{ or RRE}(w, >) \text{ for some } w \in W\}$
man to man	$\Gamma_{MM} = \{\gamma \in \Gamma : \gamma \text{ satisfies RE}(m) \text{ or RRE}(m, <) \text{ for some } m \in M\}$
woman to man	$\Gamma_{WM} = \{\gamma \in \Gamma : \gamma \text{ satisfies RE}(i) \text{ or RRE}(i, >) \text{ for some } i \in A\}$
woman to woman	$\Gamma_{WW} = \{\gamma \in \Gamma : \gamma \text{ satisfies RE}(w) \text{ or RRE}(w, \neq) \text{ for some } w \in W\}$

Table 1: Domains of manipulating φ_M via predonation

In Table 2 we present for each particular type of predonation an example of a consumption rule under which the man optimal matching rule cannot be manipulated.

Type of predonation	φ_M not manipulable under consumption rule
man to woman	$\gamma^{wm} = \gamma_p^{wm}$, $\gamma^{mw} = \gamma_t^{mw}$, $w \in W, m \in M$
man to man	$\gamma^{wm} = \gamma_t^{wm}$, $\gamma^{mw} = \gamma_p^{mw}$, $w \in W, m \in M$
woman to man	$\gamma^{ij}(e_i, e_j) = e_i/2$ for all $i \in A, j \in A(i) \setminus \{i\}$
woman to woman	$\gamma^{wm} = \gamma_p^{wm}$, $\gamma^{mw} = \gamma_t^{mw}$, $w \in W, m \in M$

Table 2: Consumption rules under which φ_M is not manipulable

S & Ö-S pointed out that under the trivial consumption rule γ_t the man (or woman) optimal matching rule is non-manipulable by predonation of any agent (man or woman) to any other agent (man or woman). One may wonder whether it is the unique consumption rule with this property. The next corollary answers this question in the affirmative.

Corollary 9 *The only consumption rule under which φ_M cannot be manipulated by any agent via predonation to any other agent is the trivial consumption rule, i.e., $\gamma_t^{ij}(x, y) = x$ for all $i \in A$, $j \in A(i)$, and $x, y \in \mathcal{R}_+$.*

Proof. It remains to prove that there is no other consumption rule with this property. Let γ be a consumption rule under which φ_M is non-manipulable. Then $\gamma \notin \Gamma_{WM} \cup \Gamma_{WW} \cup \Gamma_{MM}$. Since $\gamma \notin \Gamma_{WM}$ it follows from the proof of Proposition 7 that for each $i \in A$ there exists a function $g^i : \mathcal{R}_+ \to \mathcal{R}_+$ with

$$g^i(0) = 0 \text{ and } g^i(x) = \gamma^{ij}(x, y) \leq x \text{ for all } j \in A(i) \backslash \{i\} \text{ and } x, y \in \mathcal{R}_+.$$

Moreover, as $\gamma \notin \Gamma_{WW}$, [not RRE($w, \neq$)] implies that $g^w(x) = x$ for all $w \in W$ and $x \in \mathcal{R}_+$. Similarly, as $\gamma \notin \Gamma_{MM}$, [not RRE($m, <$)] implies that $g^m(x) = x$ for all $m \in M$ and $x \in \mathcal{R}_+$. Hence, $\gamma = \gamma_t$. ∎

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