Manipulation via Endowments in Exchange Markets with Indivisible Goods*

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Abstract

We consider exchange markets with heterogeneous indivisible goods. We are interested in exchange rules that are efficient and immune to manipulations via endowments (either with respect to hiding or destroying part of the endowment or transferring part of the endowment to another trader). We consider three manipulability axioms: hiding-proofness, destruction-proofness, and transfer-proofness. We prove that no rule satisfying efficiency and hiding-proofness (which implies individual rationality) exists. For two-agent exchange markets with separable and responsive preferences, we show that efficient, individually rational, and destruction-proof rules exist. However, for separable preferences, no rule satisfies efficiency, individual rationality, and destruction-proofness. In the case of transfer-proofness the compatibility with efficiency and individual rationality for the two-agent case extends to the unrestricted domain. For exchange markets with separable preferences and more than two agents no rule satisfies efficiency, individual rationality, and transfer-proofness.

Keywords: Hiding-proofness, destruction-proofness, transfer-proofness, exchange markets with heterogeneous indivisible objects.

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1 Introduction

We consider exchange markets with heterogeneous indivisible objects where each agent is endowed with a set of objects. As an example, one may think of markets where people trade collectibles, for instance stamps, Pockeymon cards, *etc.*. Other applications (see also Pápai, 2003) are exchanges of equipment or tasks among workers or departments of a firm or an organization. A well-known special case of our exchange model are so-called housing markets (Shapley and Scarf, 1974) where each agent is endowed with exactly one object. For housing markets, the so-called top trading rule that assigns the unique core allocation to each housing market satisfies many appealing properties. In particular, the top trading rule is *efficient* and *strategy-proof* [no agent ever benefits from misrepresenting his preferences] (Roth, 1982). Moreover, it is the only rule satisfying *efficiency*, *strategy-proofness*, and *individual rationality* [no agent is worse off after trading with other agents] (Ma, 1994). However, this compatibility result does not extend to "multiple object" exchange markets (Sönmez, 1999; Klaus and Miyagawa, 2000).[1]

We are interested in *efficient* and *individually rational* exchange rules. In addition, we do not want any traders to be able to successfully manipulate the outcome to his advantage by hiding or destroying part of his endowment or transferring part of it to another trader who experiences the transfer as endowment improving. We call an exchange rule that is immune to this type of manipulation *hiding-proof*, *destruction-proof*, and *transfer-proof*, respectively.

In the context of classical exchange economies, Postlewaite (1979) is the first to introduce and study *hiding-proofness* and *destruction-proofness*. He shows that, when preferences are continuous, strictly increasing and strictly convex, *hiding-proofness* is incompatible with *efficiency* and *individual rationality*. He also shows that *destruction-proofness* is compatible with *efficiency* and *individual rationality*.[2] For reallocation problems with single-peaked preferences, Klaus, Peters and Storcken (1997) characterize *hiding-proof* rules satisfying various fairness and/or consistency properties. In the context of two-sided matching with endowments, Sertel and Özkal-

[1]Some recent studies for exchange markets with indivisibilities and multiple assignment problems without endowments that consider *strategy-proofness* in combination with other properties are Ehlers and Klaus (2003), Klaus and Miyagawa (2001), and Pápai (2002, 2003).

[2]Thomson (1987) strengthens the former result by showing that the incompatibility persists on the restricted domain of homothetic preferences even if *hiding-proofness* is replaced by a weaker notion at which agents can consume only a positive percentage of what they hide no matter how small that percentage is.

Sanver (2002) and Fiestras-Janeiro, Klijn and Sánchez (2003) analyze the manipulability of men- (women-) optimal matching rules via endowments[3]. *Transfer-proofness* is also related to the so-called "transfer paradox" (a trader can be hurt by accepting a predonation). Leontief (1936) is the first to demonstrate that the Walrasian rule is not immune to the transfer paradox for two-agent exchange economies. For two-agent economies, any *efficient* rule is *transfer-proof* if and only if it is immune to the transfer-paradox.

We demonstrate that, similarly as in other models, *efficient* and *individually rational* rules are generally not immune to manipulations via endowments (Theorems 1, 2, and 3). However, we also identify some subclasses of exchange markets where these incompatibilities do not apply: for two-agent exchange markets with separable and responsive preferences *destruction-proofness* is compatible with *efficiency* and *individual rationality* (Proposition 1), and for two-agent exchange markets with unrestricted preferences, *transfer-proofness* is compatible with *efficiency* and *individual rationality* (Proposition 2).

2 The Model

2.1 Exchange Markets with Indivisible Objects

Let K be a *set of heterogeneous objects* containing at least two objects (we allow $|K| = \infty$). Let 2^K denote the set of all (possibly empty) subsets of K. To simplify notation, we omit the brackets when denoting subsets of K and write, for instance, xyz instead of $\{x, y, z\}$. Let $N \equiv \{1, \ldots, n\}$ be a *finite set of agents* containing at least two agents. Each agent $i \in N$ is endowed with a finite (possibly empty) set of objects $E_i \in 2^K$. No two agents own the same object(s). So, an *endowment distribution* $E \equiv (E_1, \ldots, E_n)$ is defined by **(i)** for all $i \in N$, $|E_i| < \infty$, **(ii)** $\bigcup_{i=1}^{n} E_i \in 2^K$, and[4] **(iii)** if $i, j \in N$ are such that $i \neq j$, then $E_i \cap E_j = \emptyset$. We denote the *set of all endowment distributions* by $\mathcal{E}$.

Each agent $i \in N$ has complete and transitive preferences R_i over 2^K. The associated strict preference relation is denoted by P_i. Moreover, preferences are strict, that is, for all distinct subsets $S, S' \in 2^K$, either $S \; P_i \; S'$ or $S' \; P_i \; S$. Thus, $S \; R_i \; S'$ means that either $S \; P_i \; S'$ or $S = S'$. Agent i's preferences are *separable* whenever he prefers x to nothing if and only if for any set S not containing x he prefers $S \cup x$ to S: for all $S \subseteq K$ and

[3]Their non-manipulability by predonation corresponds to our *transfer-proofness* condition.

[4]Note that $\bigcup_{i=1}^{n} E_i \subsetneq K$ is possible.

all $x \in K \setminus S$, $x \, P_i \, \emptyset \Leftrightarrow (S \cup x) \, P_i \, S$. Together with strictness and completeness of preferences, this implies that for all $S \subseteq K$ and all $x \in K \setminus S$, $\emptyset \, P_i \, x \Leftrightarrow S \, P_i \, (S \cup x)$.[5] Let $\mathcal{R}_s$ be the set of separable preference relations over 2^K. At various points, we also consider the following three domains of preferences: the (unrestricted) domain of all strict preferences $\mathcal{R}_u$; the domain of separable and responsive[6] preferences $\mathcal{R}_{sr}$; and the domain of additive[7] preferences $\mathcal{R}_a$. Clearly, $\mathcal{R}_a \subsetneq \mathcal{R}_{sr} \subsetneq \mathcal{R}_s \subsetneq \mathcal{R}_u$. Whenever we introduce notation or concepts that apply to all preference domains, we use the generic preference domain $\mathcal{R}$. We denote a typical *preference profile* by $R = (R_1, R_2, \ldots, R_n)$ and the *set of preference profiles* by $\mathcal{R}^N$.

Thus, given a preference profile $R \in \mathcal{R}^N$ and an endowment distribution $E \in \mathcal{E}$, we denote an *exchange market (with indivisible objects)* by (R, E). An *allocation* for an exchange market $(R, E) \in \mathcal{R}^N \times \mathcal{E}$ is a list $(S_1, \ldots, S_n)$ such that **(i)** each agent $i \in N$ receives some subset $S_i \subseteq \bigcup_{i=1}^n E_i$ and **(ii)** no two agents receive the same object: if $i, j \in N$ are such that $i \neq j$, then $S_i \cap S_j = \emptyset$. Note that we allow for *free disposal*, that is, $\bigcup_{i=1}^n S_i \subsetneq \bigcup_{i=1}^n E_i$ is possible. All our results remain valid without free disposal.

2.2 Exchange Rules and their Properties

An *(exchange) rule* is a function φ that associates with each exchange market $(R, E) \in \mathcal{R}^N \times \mathcal{E}$ an allocation $\varphi(R, E) = (S_i)_{i \in N}$. Given $i \in N$, we call $\varphi_i(R, E)$ the *allotment* of agent i at $\varphi(R, E)$.

We assume that a rule only chooses (Pareto) *efficient* allocations: for all $(R, E) \in \mathcal{R}^N \times \mathcal{E}$ there is no allocation $(S_i)_{i \in N}$ such that for all $i \in N$, $S_i \, R_i \, \varphi_i(R, E)$, with strict preference holding for some $j \in N$.

To express voluntary participation or *individual rationality*, we assume that agents find their allotments at least as good as their endowments: for all $(R, E) \in \mathcal{R}^N \times \mathcal{E}$ and all $i \in N$, $\varphi_i(R, E) \, R_i \, E_i$.

For all $(R, E) \in \mathcal{R}^N \times \mathcal{E}$, we denote the set of *efficient* allocations by $\mathcal{P}(R, E)$, the set of *individually rational* allocations by $\mathcal{I}(R, E)$, and the set of *efficient* and *individually rational* allocations by $\mathcal{PI}(R, E)$.

[5] For the notion of separability we use here, we refer to Barberà, Sonnenschein and Zhou (1991).

[6] Agent i's preferences are *responsive* if, for any two sets that differ only in one object, agent i prefers the set containing the more preferred object: for all $S \subseteq K$ and all $x, y \in K \setminus S$, $x \, P_i \, y \Rightarrow (S \cup x) \, P_i \, (S \cup y)$. Roth (1985) introduces this notion of responsiveness for college admission problems.

[7] Agent i's preferences are *additive* if there exists a function $u_i \colon K \to \mathbb{R}$ such that for all $S, S' \in 2^K$, $S \, R_i \, S' \Leftrightarrow \sum_{k \in S} u_i(k) \geq \sum_{k \in S'} u_i(k)$.

Given that individual endowments are private information, an agent may manipulate the outcome to his advantage by hiding, destroying, or transferring part of his endowment.

Given an endowment distribution $E \in \mathcal{E}$, an agent $i \in N$, and a subset $E_i' \subsetneq E_i$, we obtain the new endowment distribution (E_i', E_{-i}) where agent i hides part of his endowment by replacing agent i's endowment E_i with E_i'. Let $(R, E) \in \mathcal{R}^N \times \mathcal{E}$. We denote the economy that is obtained after agent i hides part of his endowment by $(R, (E_i', E_{-i}))$.

First, we consider *hiding-proofness*: if agent i hides part of his endowment E_i and pretends to only own $E_i' \subsetneq E_i$, then he finds his original allotment $\varphi_i(R, E)$ at least as good as the set of objects $\varphi_i(R, (E_i', E_{-i})) \cup (E_i \backslash E_i')$ he finally can consume.

Hiding-Proofness: For all $(R, E) \in \mathcal{R}^N \times \mathcal{E}$, all $i \in N$, and all $E_i' \subsetneq E_i$, $\varphi_i(R, E)\ R_i\ [\varphi_i(R, (E_i', E_{-i})) \cup (E_i \backslash E_i')]$.

Since an agent could hide all of his endowment ($E_i' = \emptyset$), we deduce the following:

Lemma 1. *Efficiency and hiding-proofness imply individual rationality.*

Lemma 1 applies to any preference domain, particularly to $\mathcal{R}_a$, $\mathcal{R}_{sr}$, $\mathcal{R}_s$, and $\mathcal{R}_u$. Also, Lemma 1 holds without *efficiency* (that is, *hiding-proofness* implies *individual rationality*) if each object is *desirable* for each agent, that is, for all $i \in N$, and all $x \in K$, $x\ P_i\ \emptyset$.[8]

Second, we consider *destruction-proofness*: if an agent i destroys part of his endowment E_i, thereby reducing it to $E_i' \subsetneq E_i$, then he finds his original allotment $\varphi_i(R, E)$ at least as good as his new allotment $\varphi_i(R, (E_i', E_{-i}))$.

Destruction-Proofness: For all $(R, E) \in \mathcal{R}^N \times \mathcal{E}$, all $i \in N$, and all $E_i' \subsetneq E_i$, $\varphi_i(R, E)\ R_i\ \varphi_i(R, (E_i', E_{-i}))$.

Given an endowment distribution $E \in \mathcal{E}$, agents $i, j \in N$, and a subset $E_i' \subsetneq E_i$, we obtain the new endowment distribution (E_i', E_j', E_{-ij}) where agent i transfers part of his endowment, namely $E_i \backslash E_i'$, to agent j by replacing agent i's endowment E_i with E_i' and agent j's endowment E_j with $E_j' \equiv E_j \cup E_i \backslash E_i'$. Let $(R, E) \in \mathcal{R}^N \times \mathcal{E}$. We denote the economy that is obtained after agent i transfers $E_i \backslash E_i'$ to agent j by $(R, (E_i', E_j', E_{-ij}))$.

[8] If each object is desirable to each agent, separability is equivalent to monotonicity, that is, for all $i \in N$, and all $S, S' \in 2^K$, if $S \supsetneq S'$, then $S\ P_i\ S'$. In fact, "*hiding-proofness* implies *individual rationality*" is a model-free observation if preferences are monotonic.

Third, we consider *transfer-proofness*: if agent i transfers part of his endowment E_i to another agent, say agent j, who experiences the transfer as endowment improving, thereby reducing his endowment to $E'_i \subsetneq E_i$, and expanding agent j's endowment to $E'_j \supsetneq E_j$ such that $E'_j \; R_j \; E_j$, then agent i finds his original allotment $\varphi_i(R, E)$ at least as good as his new allotment $\varphi_i(R, (E'_i, E'_j, E_{-ij}))$.

Transfer-Proofness: For all $(R, E) \in \mathcal{R}^N \times \mathcal{E}$, all $i, j \in N$, all $E'_i \subsetneq E_i$, and $E'_j \equiv E_j \cup E_i \backslash E'_i$, if $E'_j \; R_j \; E_j$, then $\varphi_i(R, E) \; R_i \; \varphi_i(R, (E'_i, E'_j, E_{-ij}))$.

As the following examples demonstrate, no direct relationship exists between *hiding-proofness*, *destruction-proofness*, and *transfer-proofness*.

Example 1. ***No-Trade Rule***
On $\mathcal{R}_a$, $\mathcal{R}_{sr}$, $\mathcal{R}_s$, and $\mathcal{R}_u$, the no-trade rule that assigns to each agent his endowment is *hiding-proof* and *individually rational*, but neither *destruction-proof*, nor *transfer-proof*, nor *efficient*.[9] $\diamond$

Example 2. ***Serial Dictatorship Rule***
On $\mathcal{R}_a$, $\mathcal{R}_{sr}$, and $\mathcal{R}_s$, any serial dictatorship rule that assigns to each agent in a serial way his most preferred set of objects (among the remaining objects) is *destruction-proof*, *transfer-proof*, and *efficient*, but neither *hiding-proof*, nor *individually rational*.[10] $\diamond$

Example 3. ***Conditional Serial Dictatorship Rule*** $\varphi^{csd(x,E)}$
A conditional serial dictatorship rule $\varphi^{csd(x,E)}$ is defined as follows: Let $x \in K$ and φ^d, $\varphi^{d'}$ be serial dictatorship rules such that for φ^d lower-indexed agents come first and for $\varphi^{d'}$ higher-indexed agents come first. For all E such that $x \in \bigcup_{i \in N} E_i$, let $\varphi^{csd(x,E)}(R, E) \equiv \varphi^d(R, E)$. For all E such that $x \notin \bigcup_{i \in N} E_i$, let $\varphi^{csd(x,E)}(R, E) \equiv \varphi^{d'}(R, E)$. On $\mathcal{R}_a$, $\mathcal{R}_{sr}$, and $\mathcal{R}_s$, $\varphi^{csd(x,E)}$ is *efficient* and *transfer-proof*, but neither *hiding-proof* (*individually rational*), nor *destruction-proof*. $\diamond$

[9]Since later we show that no rule satisfying *efficiency* and *hiding-proofness* exists, it is not possible to find an example of independence satisfying *efficiency*, *hiding-proofness*, but not *destruction-proofness* or *transfer-proofness*.

[10]We refer to Klaus and Miyagawa (2001) for a precise definition of serial dictatorship rules. On preference domain $\mathcal{R}_u$ a serial dictatorship may not satisfy *destruction-proofness* (*e.g.*, destroying an object may cause a predecessor to abstain from consuming other objects that he considers complementary to the destroyed one).

Example 4. ***Conditional Serial Dictatorship Rule*** $\varphi^{csd(x,E_1)}$
A conditional serial dictatorship rule $\varphi^{csd(x,E_1)}$ is defined as follows: Let $x \in K$ and φ^d, $\varphi^{d'}$ be serial dictatorship rules such that for φ^d lower-indexed agents come first and for $\varphi^{d'}$ higher-indexed agents come first. For all E such that $x \in E_1$, let $\varphi^{csd(x,E_1)}(R,E) \equiv \varphi^d(R,E)$. For all E such that $x \notin E_1$, let $\varphi^{csd(x,E_1)}(R,E) \equiv \varphi^{d'}(R,E)$. On $\mathcal{R}_a$, $\mathcal{R}_{sr}$, and $\mathcal{R}_s$, if $n \geq 3$, $\varphi^{csd(x,E_1)}$ is *efficient* and *destruction-proof*, but neither *hiding-proof* (*individually rational*), nor *transfer-proof*.[11] ◇

3 Results

Throughout this section, whenever R is fixed, we simply denote an exchange market by its endowment distribution E.

3.1 Hiding-Proofness

Theorem 1. *For exchange markets with additive preferences, no rule is efficient and hiding-proof.*

Theorem 1 holds on any domain that includes the domain of additive preferences $\mathcal{R}_a$. In particular, Theorem 1 applies to $\mathcal{R}_a$, $\mathcal{R}_{sr}$, $\mathcal{R}_s$, and $\mathcal{R}_u$.

Proof: Suppose that φ is *efficient* and *hiding-proof*. Hence, by Lemma 1, φ is *individually rational*. Let $N = \{1,2\}$, $E = (E_1, E_2)$ such that $E_1 = ab$, $E_2 = cd$, and $(R_1, R_2) \in \mathcal{R}_a^N$ with utility representation

$u_1(a) = 5,$	$u_2(a) = 6,$
$u_1(b) = 2.1,$	$u_2(b) = 3,$
$u_1(c) = 3,$	$u_2(c) = 1.1,$
$u_1(d) = 4,$	$u_2(d) = 4.$

The only *efficient* and *individually rational* allocations are $A = (ac, bd)$ and $B = (bcd, a)$. Hence, $\varphi(E) \in \{A, B\}$.

Case 1: $\varphi(E) = A$. If agent 1 hides object b, the endowment distribution becomes $E^1 = (a, cd)$ and the only *efficient* and *individually rational* allocation for the resulting exchange market is $A^1 = (cd, a)$. So, $\varphi(E^1) = A^1$. Hence, agent 1 consumes bcd, which he prefers to ac, his allotment at A, in violation of *hiding-proofness*. Thus, $\varphi(E) \neq A$.

[11] If $n = 2$, $\varphi^{csd(x,E_1)}$ is *transfer-proof* as well.

Case 2: $\varphi(E) = B$. If agent 2 hides object d, the endowment distribution becomes $E^2 = (ac, c)$ and the only *efficient* and *individually rational* allocation for the resulting exchange market is $B^1 = (ad, b)$. So, $\varphi(E^2) = B^1$. Hence, agent 2 consumes bd, which he prefers to bc, his allotment at B, in violation of *hiding-proofness*. Thus, $\varphi(E) \neq B$.

Cases 1 and 2 together show that for $n = 2$, *efficiency* and *hiding-proofness* are incompatible. For $n > 2$, we simply add agents who prefer their endowments to any possible trade. Since only agents 1 and 2 trade with each other as specified above, the incompatibility of *efficiency* and *hiding-proofness* persists for $n > 2$. □

3.2 Destruction-Proofness

If we replace *hiding-proofness* by *destruction-proofness*, compatibility with *efficiency* and *individual rationality* is possible for two-agent exchange markets with separable and responsive preferences.

Let $N = \{1, 2\}$ and $(R, E) \in \mathcal{R}_{sr}^N \times \mathcal{E}$. In order to present a rule satisfying the properties listed above, we introduce some notation. First, for $i \in N$ we obtain $\bar{E}_i$ by discarding all *undesirable objects* x, that is, objects $x \in E_i$ such that $\emptyset \, P_i \, x$. Second, in order to preserve *efficiency*, we define the set $\tilde{E}_i$ by adding to $\bar{E}_i$ all objects that agent $j \neq i$ discarded, and that agent i likes, that is, $\tilde{E}_i = \bar{E}_i \cup \{x \in E_j \backslash \bar{E}_j : x \; P_i \; \emptyset\}$. Note that $\mathcal{PI}(R, \tilde{E}) \subseteq \mathcal{PI}(R, E)$.

Example 5. ***Restricted Dictatorship Rule***[12] $\varphi^{rd(i)}$
Let $N = \{1, 2\}$ and $i \in N$. For all $(R, E) \in \mathcal{R}_{sr}^N \times \mathcal{E}$, $\varphi^{rd(i)}$ picks the unique best allocation for agent i in $\mathcal{PI}(R, \tilde{E})$. We call agent i the *dictator*. By construction, $\varphi^{rd(i)}$ is *efficient* and *individually rational*. ◇

Next, we show that when preferences are separable and responsive, $\varphi^{rd(i)}$ is *destruction-proof*.[13]

[12] For $n > 2$ we can define *restricted serial dictatorship rules* $\tilde{\varphi}^{rd(\pi)}$, where π denotes the ordering of "dictators." Similarly as before, we can derive an economy $(R, \tilde{E})$ by first letting all agents discard of undesirable objects and then distributing them among the agents who would like to consume them (this distribution can, for instance, be done sequentially using π). Then, for all $(R, E) \in \mathcal{R}_{sr}^N \times \mathcal{E}$, the first dictator restricts the set $\mathcal{PI}(R, \tilde{E})$ to all allocations where he receives his best allotment. Next, if several allocations are left over, the second dictator restricts the remaining set to all allocations where he receives his best allotment, *etc.*. In order to adjust restricted serial dictatorship rules if free disposal is not allowed, we simply assume that an agent has to keep any object that is undesirable for all agents.

[13] One can easily show that $\varphi^{rd(i)}$ does not satisfy *hiding-proofness*.

Proposition 1. *For two-agent exchange markets with separable and responsive preferences, restricted dictatorship rules are destruction-proof.*

Proposition 1 only remains valid on $\mathcal{R}_a$ and $\mathcal{R}_{sr}$, but not on $\mathcal{R}_s$ and $\mathcal{R}_u$ (see Theorem 2).[14]

Proof: Let $N = \{1, 2\}$, $\varphi^r = \varphi^{rd(1)}$, and $(R, E) \in \mathcal{R}_{sr}^N \times \mathcal{E}$. Note that by definition, no agent i can benefit by destroying an undesirable object $x \in E_i$. Hence, it is without loss of generality to assume that $(R, E) = (R, \tilde{E})$. We prove that neither agent can benefit from destroying one of his objects. The proof that neither agent can benefit from destroying several objects follows by applying the "one-object-argument" for each object and invoking transitivity of preferences.

Case 1: Agent 1 destroys $x \in E_1$. Let $A = \varphi^r(E)$ and $B = \varphi^r(E_1 \backslash x, E_2)$. Suppose $B_1 \; P_1 \; A_1$. Note that by separability, $(B_1 \cup x) \; P_1 \; B_1$ and $(B_1 \cup x, B_2) \in \mathcal{I}(E)$. Hence, there exists $C \in \mathcal{PI}(E)$ such that $C_1 \; R_1 \; (B_1 \cup x)$. Thus, $C_1 \; P_1 \; A_1$, which contradicts the assumption that A is the best allocation for agent 1 in $\mathcal{PI}(E)$.

Case 2: Agent 2 destroys $x \in E_2$. Let $A = \varphi^r(E)$ and $B = \varphi^r(E_1, E_2 \backslash x)$. Suppose $B_2 \; P_2 \; A_2$. If $x \in A_2$, then $B = (A_1, A_2 \backslash x)$ and $A_2 \; P_2 \; B_2$; a contradiction. Thus, $x \in A_1$. Since $A \in \mathcal{P}(E)$, $A_1 \; P_1 \; (B_1 \cup x)$. By responsiveness, $A_1 \backslash x \; P_1 \; B_1$. Note that $(A_1 \backslash x, A_2) \in \mathcal{I}(E_1, E_2 \backslash x)$. Hence, there exists $C \in \mathcal{PI}(E_1, E_2 \backslash x)$ such that $C_1 \; P_1 \; B_1$, which contradicts the assumption that B is the best allocation for agent 1 in $\mathcal{PI}(E_1, E_2 \backslash x)$. $\square$

The class of rules that are *efficient*, *individually rational*, and *destruction-proof* for two-agent exchange markets with separable and responsive preferences is very large. The following example serves to illustrate the largeness of this class of rules.

Example 6. Let $N = \{1, 2\}$, $E = (E_1, E_2)$ such that $E_1 = abc$, $E_2 = d$, and $(R_1, R_2) \in \mathcal{R}_a^N$ with utility representation

$u_1(a) = 1,$	$u_2(a) = 1,$
$u_1(b) = 3,$	$u_2(b) = 3,$
$u_1(c) = 5,$	$u_2(c) = 5,$
$u_1(d) = 10,$	$u_2(d) = 0.1.$

[14] For $\mathcal{R}_u$, it is easy to see that destroying an object which is considered complementary by a previous dictator, may induce this dictator to choose for a trade that is more advantageously for the agent who destroyed the object.

The only *efficient* and *individually rational* allocations are $A = (bcd, a)$, $B = (acd, b)$, $C = (cd, ab)$, $D = (abd, c)$, $F = (bd, ac)$, $G = (ad, bc)$, and $H = (d, abc)$. If agent 1 is the dictator, then the restricted dictatorship rule picks allocation A. If agent 2 is the dictator, then the restricted dictatorship rule picks allocation H. Moreover, allocations B and C cannot be manipulated by destruction. Hence, a *destruction-proof* rule can pick any of the four allocations A, B, C, and H. Therefore, many *destruction-proof* rule can be easily constructed. $\diamond$

The next example demonstrates that for exchange markets with more than two agents, a restricted (serial) dictatorship rule may be manipulable by destruction. This manipulability result holds for any subdomain of $\mathcal{R}_s$ that includes the domain of additive preferences $\mathcal{R}_a$, in particular $\mathcal{R}_a$, $\mathcal{R}_{sr}$, and $\mathcal{R}_s$ (recall that our definition of a restricted (serial) dictatorship rules only applies to separable preferences so that we cannot make any statements on $\mathcal{R}_u$).

Example 7. Let $N = \{1, 2, 3\}$, $E = (E_1, E_2, E_3)$ such that $E_1 = a$, $E_2 = bc$, $E_3 = de$, and $(R_1, R_2, R_3) \in \mathcal{R}_a^N$ with utility representation

$u_1(a) = 1,$	$u_2(a) = 5,$	$u_3(a) = 7,$
$u_1(b) = 8,$	$u_2(b) = 4,$	$u_3(b) = 6,$
$u_1(c) = 5,$	$u_2(c) = 2,$	$u_3(c) = 1.1,$
$u_1(d) = 10.5,$	$u_2(d) = 8,$	$u_3(d) = 3,$
$u_1(e) = 0.1,$	$u_2(e) = 1.5,$	$u_3(e) = 2.3.$

If agent 1 is the dictator, then the restricted (serial) dictatorship rule picks (cd, ae, b). However, if agent 3 destroys object e, in the resulting economy the restricted (serial) dictatorship rule picks (bc, d, a). Hence, agent 3 consumes a, which he strictly prefers to b, in violation of *destruction-proofness*. $\diamond$

The previous example demonstrates that for exchange markets with more than two agents, restricted (serial) dictatorship rules may not be *destruction-proof*. At this moment, it is an open question whether for more than two agents with either additive, or separable and responsive preferences, a rule satisfying *efficiency*, *individual rationality*, and *destruction-proofness* exists. If preferences are "only" separable, then we can establish the incompatibility of *efficiency*, *individual rationality*, and *destruction-proofness* for exchange markets with any number of agents.

Theorem 2. *For exchange markets with separable preferences, no rule satisfies efficiency, individual rationality, and destruction-proofness.*

Theorem 2 holds on any domain that includes the domain of separable preferences $\mathcal{R}_s$. In particular, Theorem 2 applies to $\mathcal{R}_s$ and $\mathcal{R}_u$.

Proof: Suppose that φ is *efficient*, *individually rational*, and *destruction-proof*. Let $N = \{1, 2\}$, $E = (E_1, E_2)$ such that $E_1 = ab$, $E_2 = cde$, and $(R_1, R_2) \in \mathcal{R}_{sr}^N$ be such that

R_1	R_2
$bcde$	bde
cde	bd
ace	ab
ac	a
ab	cde

Note that R_1, R_2 can be completed in a separable way. The only *efficient* and *individually rational* allocations are $A = (bcde, a)$, $B = (cde, ab)$, $C = (ac, bde)$, and $D = (ace, bd)$. Hence, $\varphi(E) \in \{A, B, C, D\}$.

Case 1: $\varphi(E) \in \{A, B\}$. If agent 2 destroys object e, the endowment distribution becomes $E^1 = (ab, cd)$ and the only *efficient* and *individually rational* allocation for the resulting exchange market is $A^1 = (ac, bd)$. So, $\varphi(E^1) = A^1$. Hence, agent 2 consumes bd, which he prefers to a, his allotment at A and ab, his allotment at B, in violation of *destruction-proofness*. Thus, $\varphi(E) \notin \{A, B\}$.

Case 2: $\varphi(E) \in \{C, D\}$. If agent 1 destroys object b, the endowment distribution becomes $E^2 = (a, cde)$ and the only *efficient* and *individually rational* allocation for the resulting exchange market is $C^1 = (cde, a)$. So, $\varphi(E^2) = C^1$. Hence, agent 1 consumes cde, which he prefers to ac, his allotment at C and ace, his allotment at D, in violation of *destruction-proofness*. Thus, $\varphi(E) \notin \{C, D\}$.

Cases 1 and 2 together show that for $n = 2$, *efficiency*, *individual rationality*, and *destruction-proofness* are incompatible. For $n > 2$, we simply add agents that prefer their endowments to any possible trade. Since only agents 1 and 2 trade with each other as specified above, the incompatibility of *efficiency*, *individual rationality*, and *destruction-proofness* persists for $n > 2$. $\square$

3.3 Transfer-Proofness

For two-agent exchange markets *transfer-proofness* is compatible with *efficiency* and *individual rationality*. In fact, restricted serial dictatorship rules (defined in the previous section on the domain of separable and responsive preferences) satisfy *transfer-proofness*. We first extend the definition of restricted serial dictatorship rules to the domain of unrestricted preferences $\mathcal{R}_u$.

Let $N = \{1,2\}$ and $(R,E) \in \mathcal{R}_u^N \times \mathcal{E}$. In order to adjust restricted serial dictatorship rules to the domain $\mathcal{R}_u$, for $j \in N$, $\bar{E}_j$ is the most preferred subset of E_j for agent j, that is, for all $S \subseteq E_j$, $\bar{E}_j \; R_j \; S$.

Example 8. *Restricted Dictatorship Rule* $\varphi^{rd(i)}$
Let $N = \{1,2\}$ and $i \in N$. For all $(R,E) \in \mathcal{R}_u^N \times \mathcal{E}$, $\varphi^{rd(i)}$ picks the unique best allocation for agent i in $\mathcal{PI}(R,E)$ that is *individually rational* for agent $j \neq i$ with respect to $\bar{E}_j$, that is $\varphi_j^{rd(i)}(R,E) \; R_j \; \bar{E}_j$. By construction, $\varphi^{rd(i)}$ is *efficient* and *individually rational*. $\diamond$

Next, we show that $\varphi^{rd(i)}$ is *transfer-proof*.

Proposition 2. *For two-agent exchange markets with unrestricted preferences, restricted dictatorship rules are transfer-proof.*

Proposition 2 remains valid on $\mathcal{R}_a$, $\mathcal{R}_{sr}$, $\mathcal{R}_s$, and $\mathcal{R}_u$.

Proof: Let $N = \{1,2\}$, $\varphi^r = \varphi^{rd(1)}$, and $(R,E) \in \mathcal{R}_u^N \times \mathcal{E}$. We prove that neither agent can benefit from transferring one of his objects to the other agent. The proof that neither agent can benefit from transferring several objects follows by applying the "one-object-argument" for each object and invoking transitivity of preferences.

Case 1: Agent 1 transfers $x \in E_1$ to agent 2 such that $E_2' \equiv (E_2 \cup x) \, R_2 \, E_2$. Let $A = \varphi^r(E)$ and $B = \varphi^r(E_1 \backslash x, E_2')$. Suppose $B_1 \; P_1 \; A_1$. Note that by *individual rationality*, $B_2 \; R_2 \; \bar{E}_2' \; R_2 \; E_2'$. Note that $\bar{E}_2' \; R_2 \; \bar{E}_2$. Then, by transitivity, $B_1 \; P_1 \; \bar{E}_1$ and $B_2 \; R_2 \; \bar{E}_2$. Hence, by *individual rationality*, there exists $C \in \mathcal{PI}(E)$ such that $C_1 \; R_1 \; B_1$ and $C_2 \; R_2 \; B_2$. Thus, $C_1 \; P_1 \; A_1$ and $C_2 \; R_2 \; \bar{E}_2$, which contradicts the assumption that A is the best available allocation for agent 1 at (R,E).

Case 2: Agent 2 transfers $x \in E_2$ such that $(E_1 \cup x) \, R_1 \, E_1$ and $E_2' \equiv E_2 \backslash x$. Let $A = \varphi^r(E)$ and $B = \varphi^r(E_1 \cup x, E_2')$. Suppose $B_2 \; P_2 \; A_2$. Then, by *efficiency*, $A_1 \; P_1 \; B_1$. Note that by *individual rationality*, $B_1 \; R_1 \; (E_1 \cup x)$. By the definition of φ^r, $A_2 \; R_2 \; \bar{E}_2 \; R_2 \; E_2'$. Note that $\bar{E}_2 \; R_2 \; \bar{E}_2'$. Then, by transitivity, $A_1 \; P_1 \; (E_1 \cup x)$ and $A_2 \; R_2 \; \bar{E}_2'$. Hence, by *individual rationality*, there exists $C \in \mathcal{PI}(E_1 \cup x, E_2')$ such that $C_1 \; P_1 \; B_1$ and $C_2 \; R_2 \; \bar{E}_2'$, which contradicts the assumption that B is the best available allocation for agent 1 at $(E_1 \cup x, E_2')$. $\square$

The following is an example of a rule that is *efficient*, *individually rational*, and *transfer-proof*, but not *destruction-proof*.

Example 9. ***Restricted Conditional Dictatorship Rule*** $\varphi^{rcd(x,\tilde{E})}$
Let $N = \{1,2\}$ and $x \in K$. For all $(R,E) \in \mathcal{R}_{sr}^N \times \mathcal{E}$ such that $x \in \bigcup_{i \in N} \tilde{E}_i$, $\varphi^{rcd(x,\tilde{E})}(R,E) = \varphi^{rd(1)}(R,E)$. For all $(R,E) \in \mathcal{R}_{sr}^N \times \mathcal{E}$ such that $x \notin \bigcup_{i \in N} \tilde{E}_i$, $\varphi^{rcd(x,\tilde{E})}(R,E) = \varphi^{rd(2)}(R,E)$. Then, $\varphi^{rcd(x,\tilde{E})}$ is *efficient*, *individually rational*, and *transfer-proof*, but not *destruction-proof*. $\diamond$

The next example demonstrates that for exchange markets with more than two agents, a restricted (serial) dictatorship rule may be manipulable by transfers. This manipulability result holds for any subdomain of $\mathcal{R}_u$ that includes the domain of additive preferences $\mathcal{R}_a$, in particular $\mathcal{R}_a$, $\mathcal{R}_{sr}$, $\mathcal{R}_s$, and $\mathcal{R}_u$.

Example 10. Let $N = \{1,2,3\}$, $E = (E_1, E_2, E_3)$ such that $E_1 = a$, $E_2 = bc$, $E_3 = de$, and $(R_1, R_2, R_3) \in \mathcal{R}_a^N$ be the same as in Example 7.

If agent 1 is the dictator, then the restricted (serial) dictatorship rule picks (cd, ae, b). However, if agent 3 transfers object e to agent 2, in the resulting economy the restricted (serial) dictatorship rule picks (bce, d, a). Hence, agent 3 consumes a, which he prefers to b, in violation of *transfer-proofness*. $\diamond$

We next prove that for three or more agents with separable preferences *efficiency*, *individual rationality*, and *transfer-proofness* are not compatible. At this moment, it is an open question whether for more than two agents with either additive, or separable and responsive preferences, a rule satisfying *efficiency*, *individual rationality*, and *transfer-proofness* exists.

Theorem 3. *For exchange markets with separable preferences and at least three agents, no rule is efficient, individual rational, and transfer-proof.*

Theorem 3 holds on any domain that includes the domain of separable preferences $\mathcal{R}_s$. In particular, Theorem 3 applies to $\mathcal{R}_s$ and $\mathcal{R}_u$.

Proof: Suppose that φ is *efficient*, *individually rational*, and *transfer-proof*. Let $N = \{1,2,3\}$, $E = (E_1, E_2, E_3)$ such that $E_1 = ab$, $E_2 = cd$, $E_3 = ef$, and $(R_1, R_2, R_3) \in \mathcal{R}_s^N$ be such that

R_1	R_2	R_3
df	bf	cd
abe	acd	cef
de	ab	bc
ef	ae	ac
ab	cd	ef
b	d	f

Note that R_1, R_2, R_3 can be completed in a separable way. The only *efficient* and *individually rational* allocations are $A = (df, ae, bc)$, $B = (de, bf, ac)$, and $C = (ef, ab, cd)$. Hence, $\varphi(E) \in \{A, B, C\}$.

Case 1: $\varphi(E) = A$. If agent 2 transfers object c to agent 3, the endowment distribution becomes $E^1 = (ab, d, cef)$ and the only *efficient* and *individually rational* allocation for the resulting exchange market is C. So, $\varphi(E^1) = C$. Hence, agent 2 consumes ab, which he prefers to ae, his allotment at A, in violation of *transfer-proofness*. Thus, $\varphi(E) \neq A$.

Case 2: $\varphi(E) = B$. If agent 3 transfers object e to agent 1, the endowment distribution becomes $E^2 = (abe, cd, f)$ and the only *efficient* and *individually rational* allocation for the resulting exchange market is A. So, $\varphi(E^2) = A$. Hence, agent 3 consumes bc, which he prefers to ac, his allotment at B, in violation of *transfer-proofness*. Thus, $\varphi(E) \neq B$.

Case 3: $\varphi(E) = C$. If agent 1 transfers object a to agent 3, the endowment distribution becomes $E^3 = (b, acd, ef)$ and the only *efficient* and *individually rational* allocation for the resulting exchange market is B. So, $\varphi(E^3) = B$. Hence, agent 1 consumes de, which he prefers to ef, his allotment at C, in violation of *transfer-proofness*. Thus, $\varphi(E) \neq C$.

Cases 1, 2, and 3 together show that *efficiency*, *individual rationality*, and *transfer-proofness* are incompatible for exchange economies with three agents. For $n > 3$, we simply add agents that prefer their endowments to any possible trade. Since only agents 1, 2, and 3 trade with each other as specified above, the incompatibility of *efficiency*, *individual rationality*, and *transfer-proofness* persists for $n > 3$. □

As in Example 9, one can condition any rules that are *transfer-proof* on the set of objects collectively owned by the agents. Any such rule is *transfer-proof*. Hence, the class of rules that are *efficient*, *individually rational*, and *transfer-proof* for two-agent exchange markets with separable and responsive preferences is very large.

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