A Foundation Model for Marxian Breakdown Theories Based on a New Falling Rate of Profit Mechanism (Long Version)

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July 2002

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Abstract: The paper presents a foundation model for Marxian theories of the breakdown of capitalism based on a new falling rate of profit mechanism. All of these theories are based on one or more of "the historical tendencies": a rising capital-wage bill ratio, a rising capitalist share and a falling rate of profit. The model is a foundation in the sense that it generates these tendencies in the context of a model with a constant subsistence wage. The newly discovered generating mechanism is based on neo-classical reasoning for a model with land. It is non-Ricardian in that land augmenting technical progress can be unboundedly rapid. Finally, since the model has no steady state, it is necessary to use a new technique, Chaplygin's method, to prove the result.

JEL Classification: B24,E11,O41

I. Introduction.

1. The Point of the Paper.

The point of the paper is to supply a foundation model for Marxian theories of the breakdown of capitalism. All of these breakdown theories are based on one or more of what Dumenil and Levy (2000) have dubbed the historical tendencies, that is a rising ratio of capital to the wage bill, a rising share of capital[1] and a falling rate of profit. The model is a foundation in the sense that it generates each of these tendencies. Of course the model would have to be added to in different ways in order to supply the analytic counterpart of each of the theories.

2. The Model.

* This paper first appeared as a working paper of the UAB in 1992 and has accumulated numerous debts in its development. Thanks go to Hamid Azari, Jordi Brandts, Roberto Burguet, Ramon Caminal, Gerard Duménil, Simon Emsley, Ducan Foley, Alan Freeman, John Hamilton, and Carmen Matutes. Thanks for their comments go also to the participants of the following meetings where it was presented: The Bellaterra Seminar, the Macro Workshop at the UAB, The Atlantic International Economic Conference, El Simposio de Análisis Económico, the IWGVT Conference, Nuevas Direcciones en el Pensamiento Económico and the conference on New and Old Growth Theories. The paper has also greatly benefited from numerous anonymous criticisms for which the author is grateful. Finally thanks go to Deirdre Herrick for her considerable help. Financial support from Dirección General de Investigación project # SEC2000-0684 is acknowledged

[1] The Marxian terms are the composition of capital and the rate of exploitation respectively.

The model has one sector with a CES production function in land and a capital-labour aggregate, each of which experiences factor augmenting technical progress at a different rate. The labour supply is infinitely elastic at a subsistence wage. Capitalists own the land as well as capital which accumulates because of capitalist savings. The result is that, if the elasticity of substitution between land and the capital-labour aggregate is less than one, then the model generates the historical tendencies.

The model has no steady state and the result refers to the general characteristics of non-steady state behaviour. Because of the absence of a steady state it is necessary to use a new technique, Chaplygin's method, in the proof.

3. The Mechanism.

The basic contribution of the paper is the discovery of a new mechanism which generates a falling rate of profit. There are two mechanisms which have been said to cause a falling rate of profit in models of this type: The Marxian explanation which involves a growing capital-wage bill ratio in the context of a model with no land[2] and a Ricardian explanation which is based on growing land scarcity and the existence of a class of landlords.[3]

The mechanism set out here is different from both of these. It is different from the Marxian one since it involves land in a crucial way. It is also different from the Ricardian one since it does not depend on technical progress in land being slower than that in the capital-labour aggregate, nor on the existence of a landlord class which soaks up profits via rents.

Roughly the mechanism works as follows. Suppose, for example, that the rate of technical progress in land is greater than that in the capital-labour aggregate. If only technical progress were taken into account, in efficiency units, land would grow faster than capital. But the more rapid growth of land causes income, savings and thus capital to grow more rapidly. The sum of these two sources, technical progress and saving, is sufficient, under the stated condition, to ensure that capital grows faster than land. This in turn causes the rate of profit to fall.

4. Breakdown Theories.[4]

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[2] This explanation is now generally acknowledged to be incorrect. See Howard and King (1992 chap. 7) for a description of the voluminos literature. Marx also propounded a secondary Ricardian-like explanation. See Petith (2001b).

[3] Ricardo (1963, Chaps II and VI). Blaug (1988, Chap. 4) gives a summary.

[4] The information in this section is taken from Part III of Petith (2001a) which describes the historical development of these theories. Part III in turn is taken from Clarke (1994) for Marx, and Howard and King (1989,1992) for Marxist writers.

The theories can be divided into those that have capitalism ending as a result of an evermore violent series of business cycles (called crises) and those that have it ending because of trends. In the business cycle group the first theory argues that the rise in the capital-wage bill ratio will cause a continual shift of demand from consumption to investment goods, that supply will not adequately react and that the resulting over supply of consumption goods will lead to increasingly violent business cycles. An additional part of this theory is that the falling rate of profit will cause increasingly risky investment ventures to be chosen which will add to the amplitude of the cycles. The second theory works through the rising capital share. Since capitalists are thought to invest rather than consume this continually increases the proportion of aggregate demand that is devoted to investment. Since investment demand is more volatile, this increases the instability of the economy and leads to more violent fluctuations.

In the trend group the first is that capitalists will put pressure on workers as they try to avoid the fall of their rate of profit and that this will increase social conflict. The second is that the rise in the capital-wage bill ratio will somehow make large holdings of capital more efficient and thus centralise ownership. This, combined with the rising share of capital, will cause an increasingly unequal distribution of income with ever fewer increasingly wealthy capitalists on the one hand and growing mass of impoverished workers on the other. The third is that the composition of capital will rise in a way to ensure that there is a sufficiently large number of unemployed workers, called the reserve army, to keep the wages from rising about subsistence level. The last is that the rate of profit will fall to such an extent that capitalism itself will not be viable.

These descriptions show that all of the breakdown theories depend on one or more of the historical tendencies.

5. The Model as a Foundation.

Since the model is meant to serve as a foundation, it is important that the model does not contradict the general idea of each of the theories and that it allows the important aspects of the theories to be set out analytically. In this, as will be seen, the model is only partially successful.

First, consider the subsistence wage assumption. Marx was ambivalent about whether the real wage would rise or not, and sections of his writings can be cited to support either position. A number of writers have reproduced some or all of the historical tendencies in models with rising wages.[5] But when one is constructing a background for breakdown theories, I think that a subsistence wage is a far better assumption for two reasons: First, the basic Marxian notion that capitalism will breakdown because of class antagonism is much less convincing in the context of rising wages; and second, three out of four of the tendency theories actually postulate impoverishment of the working class.

Second, consider the assumption that capitalists own the land. Marx's writing contain large sections which describe the actions of landlords and there can be no doubt that he viewed the economy as divided into three classes as did Ricardo.[6] But since none of the breakdown theories involves landlords, the assumption seems harmless.

Third, consider the theories one by one. Clearly the model is unsuitable for the third trend group theory: One can not have a reserve army when the infinitely elastic supply of labour implies there will be no unemployment. A different approach is needed. The two business cycle theories need a two sector model, but it would seem that the falling rate of profit mechanism would work in this case so that only a modification would be needed. Finally the remaining three trend theories would seem to fit into the unmodified model.

Thus, while it is not perfect, the model seems well suited to its task.

6. The Importance of the Model.

At first glance, it would seem idle to construct breakdown models when capitalism is in obvious good health, but a deeper look shows that this is not the case. First, the models would be directly relevant for third world capitalist countries where the instability and conflictive class relations that these theories describe reflect the actual conditions. Second, the models could serve as an analytic framework for the descriptions of actual revolution s have followed from Skocpol's seminal work (1979).[7] And finally, with respect to current first world capitalism, models of contingent breakdown could be constructed[8] which would explore the fragility of its current success.[9]

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5 See Laibman (1977, 1997), Duménil and Lévy (1993,1995,2000), Skott (1992), Michl (1994, 1999) and Skillman (1997).

6 See Marx (1954 Parts V and VI).

7 Skocpol developed an analytic methodology and used it to analyse the French, Russian and Chinese revolutions. This line has been developed further by Popkin (1988) for Vietnam, by Wickham-Crowley (1992) for Latin America and Renner (1997) for southern Mexico and Zaire among others.

8 Petith (2000) and Foley (2000) are examples: In the first case, a fall in the rate of technical progress and in the second, a failure of account for environmental externalities switch the paths of the models from a

7. The Relation to the Literature.

The model of this paper together with those of Petith (2000) and Foley (2000) are the only ones to have generated the historical tendencies in the context of a non-rising wage. The 1970's and early 1980's saw the construction of a number of formal Marxian models, but the absence of land and the infinitely elastic supply of labour robbed them of any dynamics.[10] In the 1990's a number of Marxist writers developed models of the falling rate of profit but always in the context of a rising wage. [11] Petith (2000) has a constant capital share and a profit rate that falls only on the approach to steady state. Foley (2000) has a simulation model which appears to exhibit the historical tendencies together with a falling wage, but this aspect is not emphasised. Thus the present model appears to be somewhat of a break-though.

The rest of the paper is organised as follows: Section II sets out the model and a statement of the result, Section III gives a heuristic explanation of the mechanism and Section IV provides the proof.

II. The Model and the Result.

The specific production function that produces output *Y* is CES in a Cobb-Douglas capital/labour aggregate and land, where *K, L,* and *M* are capital, labour and land.

(1) $$Y=[\alpha(K^{\beta}L^{1-\beta}e^{\gamma t})^{-\rho}+(1-\alpha)(Me^{\delta t})^{-\rho}]^{-1/\rho}$$

The aggregate experiences factor augmenting technical progress at rate $\gamma > 0$ while land experiences it at rate $\delta \geq 0$. The elasticity of substitution between the aggregate and land is $\sigma = \dfrac{1}{1+\rho}$, $-1 \leq \rho \leq \infty$, where α, $\beta \varepsilon [0, 1]$ are parameters and t is time. The constant real wage *w* and the rental on capital are equal to their respective marginal products,

(2) $$w = \partial Y/\partial L$$

(3) $$r = \partial Y/\partial K.$$

The rate of return on investment in land, which is its marginal product plus the capital gain $\dot{P}$ divided by the price, is equal to the return on capital,[12]

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neo-classical to a Marxian mode. Galor and Moave (2000) are a reverse example: They have capitalist self interested provision of education saving capitalism from the Marxist scenario.

[9] Another important aspect of the model which falls outside the thrust of the paper is that it provides an explanation for an important empirical regularity. The full result states that the rate of profit will rise or fall as the elasticity of substitution is greater or less that one. All models that I know of have the rate of profit converging to a steady state value. But the empirical reality is that, at least for the United States, the rate of profit has experienced large long term fluctuations. See Dumenil, Glick and Rangel (1987).

[10] See Morishima (1973), Steadman (1977), Roemer (1981) and Marglin(1984).

[11] See footnote 5 for references.

[12] $\dot{x}$ is the derivative of x with respect to time.

(4) $$r = \left(\frac{\partial Y}{\partial M} + \dot{P}\right) / P,$$

where P is the price of land in terms of the good. Capitalists are assumed to own the land as well as capital. Their rate of profit R is defined as

(5) $$R = \frac{Y + \dot{P}M - wL}{K + PM} \quad .$$

It is easily shown that $r = R$ and from this point on r will be called the rate of profit. Savings are provided only by capitalists who save all their income[13]. Their savings are equal to the accumulation of wealth,

(6) $$\dot{P}M + \dot{K} = Y + \dot{P}M - wL \, .$$

The assumption that all profits are saved removes $\dot{P}M$ from the accumulation equation and immensely simplifies the model. Finally the definitions of the capital-wage bill ratio k and a measure of the capitalist share (or rate of exploitation) e are given by[14]

$$k = \frac{K}{wL}, \quad e = \frac{Y}{wL} - 1.$$

This concludes the presentation of the model.

The model yields a single differential equation in K in the following manner. (2) determines L as a function of K and t, $L(K,t)$. Thus output also depends on these two variables, $Y(K,L,t) = \underline{Y}(K,L(K,t),t)$. Substituting these into (6) gives the non-autonomous differential equation

(7) $$\dot{K} = f(K,t) \qquad K(t_0) = K_0 > 0$$

where K_0, the initial capital stock, is assumed to be positive. The initial-value problem (7) has a solution $K(t)$. Taking account of the dependence of L on K and t, this solution implies the time paths of the key variables, $k(t)$, $r(t)$ and $e(t)$ as well as the extensive ones Y, K and L.

The characteristics of these time paths are given by the following theorem.

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[13] This assumption is discussed in the following section.
[14] e without an exponent is the measure of the capitalist share. e with an exponent is the ordinary mathematical symbol.

<u>Theorem</u> : *For the model of equations (1)-(6), there exists a t*, such that, for t > t* :*[15]

a) If $\rho > 0$ *(*$\sigma < 1$*) then* $k \to$ *and* $\dot{k} > 0$, $r \to 0$ *and* $\dot{r} < 0$ *and* $e \to$ *and* $\dot{e} > 0$.

b) If $\rho = 0$ *, (*$\sigma = 1$*) then* $k \to cst > 0$, $r \to cst > 0$ *and* $e = cst > 0$.

c) If $\rho < 0$ *(*$\sigma > 1$*) then* $k \to 0$ *and* $\dot{k} < 0$, $r \to$ *and* $\dot{r} > 0$ *and* $e \to cst > 0$ *and* $\dot{e} < 0$.

In addition, for factors and output

a') If $\rho > 0$, $\hat{Y}, \hat{K} \to \delta$; $\hat{L} \to \delta^*$, *where* $\delta^* < \delta$ *and* δ^* *can be positive or negative.*

b') If $\rho = 0$, $\hat{Y}, \hat{K}, \hat{L} \to \frac{\alpha}{1-\alpha}\gamma + \delta$.

c') If $\rho < 0$, *the rate of growth of* $\ln Y$, $\ln K$, *and* $\ln L \to \frac{\gamma}{\beta}$.

III. A Heuristic Description of the Falling Rate Mechanism.

This section provides a heuristic description of the falling rate of profit mechanism. It also describes the movements of the variables generally and closes with a few comments on the assumptions.

1. The Description.

The model has no steady state, but it does approach a quasi steady state[16]. The description concerns this approach.

The first step is to understand that the capital-labour aggregate grows faster than land, basically to ensure the constancy of the marginal product of labour. One can see this as follows: Suppose the aggregate grew at the same rate as land, $\hat{A} = \delta$.

a) Then $\hat{Y} = \delta$ and this implies that $\hat{K} = \delta$ asymptotically because of the constant savings ratio.

b) Then $\hat{L} = \delta - \frac{\gamma}{1-\beta}$, that is labour must grow more slowly to compensate for the technical progress.

c) Thus *K/L* grows at $\frac{\gamma}{1-\beta}$.

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[15] $\hat{x}$ is the percentage change in *x* with respect to time, and $\to$ indicates the limit of a variable as t approaches infinity.

[16] The model has no steady state in the sense that the intensive variables of interest like *k* do not approach finite positive limits. But it has a quasi steady state in the sense that they do approach limits. One can find similar quasi steady states in, for example, Stiglitz (1974).

d) Thus the marginal product of labour, which is proportional to $\left(\frac{K}{L}\right)^{\beta} e^{\gamma t}$, grows at $\frac{\beta}{1-\beta}\gamma + \gamma = \frac{\gamma}{1-\beta}$ > *0*, which is impossible.

e) Thus labour must grow faster than the expression given in b) and the aggregate grows faster than land, $\hat{A} > \delta$

Note that the rate of profit only depends on *K/L* . This because the ratio of the marginal products depends only on *K/L* and the wage is fixed. Thus a rise in *K/L* implies that the rate of profit falls.

Consider what happens when $\sigma<1$. In this case the slowest growing factor dominates so that $\hat{Y} = \hat{K} = \delta$. Suppose that *K/L* is constant, then $\hat{A} = \delta + \gamma$. Then the ratio of the aggregate to land rises at γ. Because $\sigma<1$, the marginal product of labour falls at more that γ but only rises at γ. Thus the marginal product of labour falls which is impossible. Labour must grow more slowly, *K/L* rises and the rate of profit falls. To complete this case note that the capitalist share approaches 1 because the share of land approaches 1 and the capitalists own it. Finally note that this is an exogenous growth model in the sense that output grows at δ, a rate determined by the exogenous technical progress.

Now consider what happens when $\sigma>1$. In this case the fastest growing factor dominates and the aggregate essentially detaches itself from land:

$$w = \left(\frac{K}{L}\right)^{\beta} (1-\beta) e^{\gamma t}.$$

K/L has to fall to keep the wage constant, the rate of profit rises and the share of capital falls to that given by the Cobb-Douglas. Finally, because of the infinitely elastic labour supply, production is linear in capital alone so that the model is, asymptotically, one of endogenous growth.[17] However, because there is also exogenous technical progress, it is the rate of growth of output that grows at a constant rate.

2. Comments.

The basic characteristic of the mechanism is the rapid growth of the aggregate. This, in turn, depends on the assumptions of the infinitely elastic supply of labour and the

[17] For example, if all capital gains were saved but only a proportion of profits, then the rate of growth would depend on that proportion.

constant saving proportion.[18] It would be good to know if the result would survive the weakening of these assumptions.

With respect to the second, a natural weakening would be the introduction of utility maximisation. If an overlapping generations framework was adopted, it seems likely that the result could be preserved. On the other hand, if an infinite horizon framework was used,[19] the outcome would be more problematic. It is unlikely that a general result would come out of a more complicated model so simulation would have to be resorted to[20]. In this case the present paper can be thought of as the first prong of a two pronged attack: a general result for a simple model plus the investigation of more general models in terms of specific examples.

IV. The Proof (Referee A).[21]

1. The Differential Equation.

It seems impossible to write the differential equation (7) without side conditions. But this can be done for the variable x.

(8) $$x = K^{\beta} L^{1-\beta} e^{(\gamma-\delta)t}.$$

The differential equation is then

(9) $$\dot{x} = e^{\phi t} F(x) + G(x) \quad \text{with initial condition} \quad x(t_0) = x_0 > 0$$

where $F(x) = \dfrac{a\mu x\left(x^{\rho} + \beta c_2\right)}{\left(c_2 + x^{\rho}\right)^{\mu/\rho}\left(c_2 + \mu x^{\rho}\right)}$, $G(x) = (\phi - \delta)\dfrac{x\left(c_2 + x^{\rho}\right)}{c_2 + \mu x^{\rho}}$, $\mu = \dfrac{1 + \rho - \beta\rho}{\beta}$

and a, c_1 and c_2 are positive constants. The variables of interest depend on x as follows:

(10) $$Y = e^{\delta t} f(x)$$

(11) $$L = \frac{1-\beta}{w} e^{\delta t} x f'(x)$$

[18] As noted in the previous footnote, the savings proportion out of profits need not be 1 so that capitalists may eat. It is the constancy that is important.

[19] It might seem that the perfect foresight that is usually assumed is a bad assumption for a model of the end of capitalism. But if this impedes the functioning of the mechanism, then capitalism will not end and the assumption will be justified.

[20] Foley (2000) is an example of this approach.

[21] This proof, somewhat modified, is the one worked out by referee A of JEBO for the paper Petith (2001a). The referee's exact proof is set out in that paper. The proofs are different here because, at the time, no one realised that the theorem was true without a restriction on the rate of technical progress in land. I saw this while studying the referee's proof.

(12) $$K = \left(\frac{w}{1-\beta}\right)^{\frac{1-\beta}{\beta}} e^{(\delta-\phi)t} \frac{x}{\left(f'(x)\right)^{(1-\beta)/\beta}}$$

(13) $$r = \beta\left(\frac{1-\beta}{w}\right)^{(1-\beta)/\beta} e^{\phi t}\left(f'(x)\right)^{1/\beta}$$

(14) $$\mathrm{k} = \frac{\beta}{1-\beta}\frac{1}{r}$$

(15) $$e = \frac{1}{1-\beta}\frac{c_2 + x^{\rho}}{c_2} - 1$$

where $f(x) = \dfrac{c_1 x}{\left(c_2 + x^{\rho}\right)^{1/\rho}}$ and $\phi = \dfrac{\gamma}{\beta}$.[22] Once the asymptotic behaviour of x is found, the theorem follows from equations (10)-(15).

First, Lemma 1 gives some general characteristics of the solution to (9). Then the proof of the theorem is given only for the case where $\rho>1$ and $\varphi-\delta<0$. (The proofs for the other cases are given in Petith (2002).) This is the most important case since it shows that the historical tendencies emerge in spite of rapid technical progress in agriculture. It is also the more difficult case.

2. Lemma 1.

When $\phi-\delta<0$ the characteristics and even the existence of a solution to (9) are problematic. Let $x(t)$ be this solution. In this case it may be that $\dot{x}(t)<0$ so that $x(t)<0$ is a possibility. This, in turn, would lead to three difficulties: First, $x(t)<0$ has no economic meaning. To see the remaining difficulties which are technical, let $\rho=a/b$, where a and b are real numbers. If a is odd and b is even then $x(t)^{\rho}$ is imaginary and $F(x)$ is complex. If both a and b are odd then $x^{r}<0$, $F(x) \rightarrow -\infty$ as $x \rightarrow -c_2^{\frac{1}{\rho}}$ and the existence of a solution on $[t_0, \infty)$ is doubtful. Lemma 1 shows that $x(t)>0$, that $x(t)$ is defined on $[t_0, \infty)$ and, in addition, demonstrates a few other useful characteristics of the solution.

It uses the following results from Grimshaw (1990).

<u>Lemma</u> (page 8). If $g(x,t)$ is a continuous function of x and t in the product domain for which $x \varepsilon D$ and $t \varepsilon I$ and the partial derivative $\partial g/\partial x$ exists and is bounded for all x in the convex domain D and all t ε I, then g satisfies a Lipschitz condition with the Lipshitz constant

[22] These calculations are set out in Appendix I.

$$L = \max_{D,I}\left|\frac{\partial g}{\partial x}\right|.$$

Here and below *I* is an open interval and *D* is an open connected set.

Theorem 1.4 (Picard iteration). Let *g(x,t)* be continuous for

$$|t - t_0| \le \alpha,\ |x - x_0| \le \beta$$

and satisfy a Lipschitz condition with constant *L* in this region. Let $|g(x,t)| \le M$ there and let $\delta = \min[\alpha, \beta/M]$. Then the initial value problem

$$\dot{x} = g(x,t),\ x(t_0) = x_0$$

has a unique solution for $|t - t_0| \le \delta$.

Theorem 1.7. Let *g(x,t)* be continuous for *x* in the domain *D* and *t* in the open interval *I*, and satisfy a Lipschitz condition there. Then, for any point x_o in *D* and t_0 in *I*, the initial value problem

$$\dot{x} = g(x,t),\ x(t_0) = x_0$$

has a unique solution *x(t)* which is defined for $t_0 \le t < T$ ($T \le \infty$) and is such that, if $T<\infty$, then either $|x(t)| \to \infty$ as $t \to T$ or *(x(t),t)* approaches the boundry of the product domain *(D,I)* as $t \to T$.

These results are now used to prove Lemma 1.

Lemma 1. The solution to (9) exists and is unique on $[t_0, \infty)$. Let *x(t)* be the solution, then $x(t)>0,\ t \ge t_0$ and *x(t)* is unbounded above.

Proof: Modify (9) in two steps. The first step is to write (9) as

(9') $$\dot{x} = e^{\phi t}F(|x|) + G(|x|) \qquad x(t_0) = x_0.$$

Note that this is different from (9), but only when $x<0$. The second step is to change the variable to $y = xe^{-\phi t}$. Then we have

(9'') $$\dot{y} = g(y,t) \qquad y(t_0) = y_0$$

where $g(y,t) = F\left(\left|ye^{\phi t}\right|\right) + e^{-\phi t}G\left(\left|ye^{\phi t}\right|\right) - \phi\left|ye^{-\phi t}\right|$ and $y_0 = x_0 e^{-\phi t}$. *g(y, t)* is continuous on $(-\infty, \infty)x(0,\infty)$ and $\partial g/\partial y$ exists and is bounded on this domain. It follows from the lemma (page 8) that *g(y, t)* satisfies a Lipshitz condition on this domain. Thus it follows from Theorem 1.7 that (9'') has a unique solution *y(t)* for $t_0 \le t < T$.

Suppose that $y(t) = 0$ for some values of *t*. Then, since *y(t)* is continuous and $y(t_0)>0$, there exists a $\bar{t}$ such that $y(\bar{t})=0$ and $y(t)>0$ for $t_0 \le t < \bar{t}$. Note that *y(t)* is also a solution to the initial value problem

(16) $$\dot{y} = g(y,t),\ y(\bar{t}) = 0.$$

Now choose $\underline{\alpha}$ and $\underline{\beta}$ so that the conditions $|t-\bar{t}|\le\underline{\alpha}$, $|y-0|\le\underline{\beta}$ imply that *(y, t)εDxI*. Then the conditions of theorem 1.4 are satisfied on this region and *y(t)*, $|t-\bar{t}|\le\underline{\delta}$ is the unique solution to (16). But by observation, $\bar{y}(t), \bar{y}(\bar{t})=0$, $|t-\bar{t}|\le\underline{\delta}$ is also a solution to (16). Since $\bar{y}(t)\neq y(t)$, this contradicts the uniqueness. Thus $y(t)=0$ for no value of *t* and by continuity $y(t)>0$ for $t_0\le t<T$.

It follows that (9') has a unique solution $x(t)=y(t)e^{\phi t}$ defined on $t_0\le t<T$ and that $x(t)>0$. Since (9') is identical to (9) for $x>0$, it follows that (9) has the same solution as (9') on the same domain of t.

Next to show that *x(t)* is defined on $[t_0, \infty)$, write *g(.)* as

$$g(x,t)=\frac{a}{x^{\mu}}xe^{\phi t}\tilde{F}(x)+\frac{\phi-\delta}{\mu}x\tilde{G}(x) \tag{17}$$

where $\tilde{F}(x)=\dfrac{\mu x^{\mu}\left(x^{\rho}+\beta c_2\right)}{\left(c_2+x^{\rho}\right)^{\frac{\mu}{\rho}}\left(c_2+\mu x^{\rho}\right)}$ and $\tilde{G}(x)=\dfrac{\mu\left(c_2+x^{\rho}\right)}{c_2+\mu x^{\rho}}$. Note that

$$\lim_{x\to\infty}\tilde{F}(x)=\lim_{x\to\infty}\tilde{G}(x)=1. \tag{18}$$

Suppose that $T<\infty$, then by theorem 1.7, $x(t)\to\infty$ as $t\to T$, since it is impossible that *(x, t)* approaches the boundary of the product domain. Choose $\tilde{t}$ such that $x(t)>1$ for $t\ge\tilde{t}$. Then one can find *A* such that

$$A>\frac{a}{x^{\mu}}e^{\phi t}\tilde{F}(x)+\frac{\phi-\delta}{\mu}\tilde{G}(x) \qquad t\le T$$

so that $\dot{x}=g(x,t)<Ax$. Integrating gives $\ln x(T)<A\left(T-\tilde{t}\right)+\ln x\left(\tilde{t}\right)$ which contradicts the approach of x to infinity. Thus $x(t)$ is defined on $[t_0, \infty)$.

Finally suppose that *x(t)* is bounded above, $x(t)<x^*$. First suppose that $\phi-\delta<0$. Clearly there is a $\bar{G}$ such that $\bar{G}>\tilde{G}(x(t), t)$ for all *t*. Furthermore *x* does not approach 0 as $t\to\infty$ since, if it did, $\lim_{t\to\infty}\dot{x}\le 0$; but in this case, i.e. when $x\to 0$, $\lim_{t\to\infty}g\left(x(t),t\right)>0$. Thus, since *x(t)* is continuous and $x(t)>0$, it is possible to find $\underline{x}>0$ such that $x(t)>\underline{x}$, $t_0\le t<\infty$. Thus one can find $\underline{F}<\tilde{F}(x(t), t)$, $t_o\le t<\infty$. The existence of $\underline{F}$, $\bar{G}$ and x^* allows one to write

$$g\left(x(t),t\right)>\left[\frac{a}{\left(x^*\right)^{\mu}}e^{\phi t}\underline{F}+\frac{\phi-\delta}{\mu}\bar{G}\right]x(t).$$

Let t^* such that the expression in the square brackets for this value of *t, A,* is positive. Thus $g(x(t),t)>Ax(t)$, $t\ge t^*$. Integration from $\bar{t}$ to t^* gives $\ln x(\bar{t})>\ln x(t^*)+A(\bar{t}-t^*)$, $x(\bar{t})\to\infty$ as $\bar{t}\to\infty$ which contradicts the assumption that *x(t)* was bounded above.

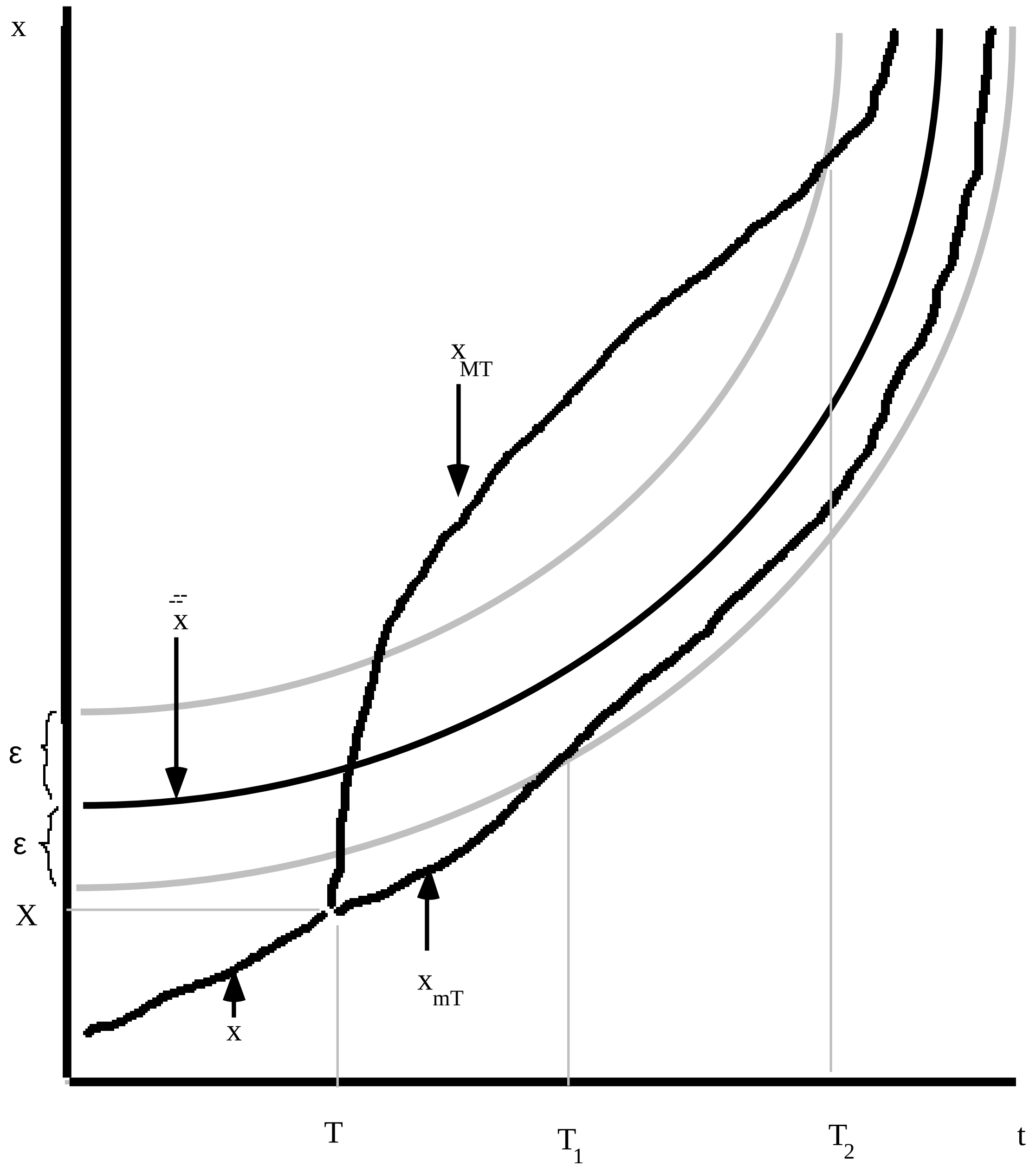

Figure 1. The illustration of lemmas 2 and 3.

Second let $\phi-\delta>0$. Then a similar argument can be constructed by finding $\underline{G}$ such that $G(x(t), t)>\underline{G}$ for $t^*\leq t<\infty$.

3. The Proofs of the Theorem for the Three Cases.

a. $\rho>0$ **and** $\phi-\delta<0$**.**

The structure of the argument can be read from figure 1. Lemma 2 uses Chaplygin's theorem to establish that x eventually lies between two bounding functions, x_{MT} and x_{mT}.

Lemma 3 then shows that the bounding functions eventually enter the ε tube surrounding a known path $\bar{\bar{x}}$. Thus x is asymptotically equivalent to $\bar{\bar{x}}$.

<u>Theorem</u> (Chaplygin)[23] For an equation of the form $\dot{x}=g(x,t)$, $x(T)=X$, if the differential inequalities

$$\dot{x}_{mT}(t)-g(x_{mT}(t),t)<0$$

$$\dot{x}_{MT}(t)-g(x_{MT}(t),t)>0$$

hold with $t>T$ and $x_{mt}(T)=x_{MT}(T)=X$, then

$$x_{mT}(t)<x(t)<x_{MT}(t)$$

holds for all $t>T$.

A few preliminaries are necessary. It is clear that

(19) $$\tilde{G}(x)/\tilde{F}(x)>1,$$

(20) there exists $\bar{x}$ such that $\tilde{F}'(x)<0$, $\tilde{G}'(x)<0$, $(\tilde{G}(x)/\tilde{F}(x))'<0$ for $x>\bar{x}$.

Next consider the equation that is the limit of (9) as $x\to\infty$. Using (17) and (18), this is

(21) $$\dot{x}=\frac{a}{x^{\mu-1}}e^{\phi t}+\frac{\phi-\delta}{\mu}x,\quad x(T)=X.$$

This has the solution[24]

(22) $$\bar{x}(t;a,\delta,T,X)=\left\{\frac{a\mu}{\delta}e^{\phi t}\left[1-e^{\delta t}C(a,\delta,X,T)\right]\right\}^{\frac{1}{\mu}}$$

where $C(.)=e^{\delta T}\left(1-\frac{\delta X^{\mu}}{a\mu e^{\phi T}}\right)$.

(21) also has an asymptotic solution:

(23) $$\bar{\bar{x}}(t)=\left(\frac{a\mu}{\delta}e^{\phi t}\right)^{\frac{1}{\mu}}.$$

Differentiating (22) gives

(24) $$\dot{\bar{x}}(t)=\frac{1}{\mu}\bar{x}^{\frac{1}{\mu}-1}\frac{a\mu}{\delta}e^{\phi t}\left[\phi+(\delta-\phi)C(.)e^{-\delta t}\right].$$

It is possible to modify equation (21) in two ways:

(25) $$\begin{aligned}\dot{x}&=\frac{a}{x^{\mu-1}}e^{\phi t}+m\frac{\phi-\delta}{\mu}x,\qquad x(t)=X\\&=\frac{a}{x^{\mu-1}}e^{\phi t}+\frac{\phi-\delta_m}{\mu}x\end{aligned}$$

[23] See Mikhlin and Smolitskiy (1967) pp.9-12 or Zwillinger (1989) pp. 388-91.
[24] Set out in Appendix II.

where $\delta_m=\delta m+(1-m)\phi$; and

(26)
$$\dot{x} = M\left[\frac{a}{x^{\mu-1}}e^{\phi t} + \frac{\phi-\delta}{\mu}x\right], \qquad x(T) = X$$
$$= \frac{a_M}{x^{\mu-1}}e^{\phi t} + \frac{\phi-\delta_M}{\mu}x$$

where $a_M=Ma$ and $\delta_M=\delta M+(1-M)\phi$. Equations (25) and (26) have the solution given by (22) with a and δ modified appropriately.

Lemma 2. Let $x(t)$ be the solution to (9) and let $\rho>0$ and $\phi-\delta<0$. For any $\tilde{x}$ there exists a T such that $x(T)>\tilde{x}$ and

$$\bar{x}_{mT}(t)<x(t)<\bar{x}_{MT}(t), \qquad t>T$$

where
$$\bar{x}_{mT}(t)=\bar{x}(t;a,\delta_m,x(T),T), \quad m=\tilde{G}(x(T))/\tilde{F}(x(T))$$
$$\bar{x}_{MT}(t)=\bar{x}(t;a_M,\delta_M,x(T),T), \quad M=\tilde{G}(T).$$

Proof: For the given $\tilde{x}$ choose T so that $x(T)>\tilde{x}$, so that $x(T)>\bar{x}$ of (20), and so that $\dot{x}(T)>0$. This is possible since, by lemma 1, $x(t)$ is unbounded above. Since $\dot{x}(T)>0$, looking at (17),

(27)
$$\frac{a}{x^{\mu-1}}e^{\phi t}\tilde{F}(x) + \frac{\phi-\delta}{\mu}x\tilde{G}(x) > 0, \quad x = x(T),$$

by (20)

$$\dot{\bar{x}}_{mT}(T) = \frac{a}{x^{\mu-1}}e^{\phi T} + \frac{\phi-\delta}{\mu}x\frac{\tilde{G}(x)}{\tilde{F}(x)} > 0, \quad x = x_{mT}(T)$$

and from (24)

(28)
$$\dot{\bar{x}}_{mT}(t) > 0, \quad t \geq T.$$

Also from (27)

$$\dot{\bar{x}}_{MT}(T) = \tilde{G}(x)\left[\frac{a}{x^{\mu-1}}e^{\phi t} + \frac{\phi-\delta}{\mu}x\right] > \tilde{G}(x)\left[\frac{\tilde{F}(x)}{\tilde{G}(x)}\frac{a}{x^{\mu-1}}e^{\phi t} + \frac{\phi-\delta}{\mu}x\right] > 0 \quad \text{for } x = x_{MT}(T)$$

by (19), and from (24)

(29)
$$\dot{\bar{x}}_{MT}(t) > 0, \quad t \geq T.$$

Now it is shown that the conditions of Chaplygin's theorem are satisfied:

$$\bar{x}_{mT}(T)=x(T)=\bar{x}_{MT}(T)$$

by construction.

(30)
$$\bar{x}_{mT}(t)>\bar{x}_{mT}(T), \quad \bar{x}_{MT}(t)>\bar{x}_{MT}(T) \qquad \text{for } t>T$$

by (28) and (29).

$$\dot{x}_{mT}(t) = \frac{a}{x^{\mu-1}} e^{\phi t} + m\frac{\phi-\delta}{\mu} x < \left[\frac{a}{x^{\mu-1}} e^{\phi t} + \frac{\tilde{G}(x)}{\tilde{F}(x)}\frac{\phi-\delta}{\mu} x\right]\tilde{F}(x)$$

$$= \frac{a}{x^{\mu-1}} e^{\phi t}\tilde{F}(x) + \frac{\phi-\delta}{\mu} x\tilde{G}(x) \quad \text{for } x = x_{mT}(t),\ t > T$$

by (18), (20), (28) and (30).

$$\dot{x}_{MT}(t) = M\left[\frac{a}{x^{\mu-1}} e^{\phi t} + \frac{\phi-\delta}{\mu} x\right] > \left[\frac{\tilde{F}(x)}{\tilde{G}(x)}\frac{a}{x^{\mu-1}} e^{\phi t} + \frac{\phi-\delta}{\mu} x\right]\tilde{G}(x)$$

$$= \frac{a}{x^{\mu-1}} e^{\phi t}\tilde{F}(x) + \frac{\phi-\delta}{\mu} x\tilde{G}(x), \quad \text{for } x = x_{MT}(t),\ t > T$$

by (19), (20), (29) and (30). This completes the proof.

<u>Lemma</u> 3. Let $x(t)$ be the solution to (9) and let $\rho>0$ and $\phi-\delta<0$. Then

$$\lim_{t\to\infty}\frac{x(t)}{\bar{\bar{x}}(t)} = 1.$$

<u>Proof:</u> Choose $\varepsilon>0$ arbitrary but with $\varepsilon<1$, $\varepsilon < 4/\left(\frac{\delta}{\phi}-1\right)$. It must be shown that there is a T_ε such that

$$(1-\varepsilon)\bar{\bar{x}}(t) < x(t) < (1+\varepsilon)\bar{\bar{x}}(t), \quad t > T_\varepsilon.$$

Choose $\tilde{x}$ large enough so that for the T given by lemma 2

(31) $$\frac{\tilde{G}(x)}{\tilde{F}(x)} < 1 + \frac{\varepsilon/2}{1-\phi/\delta},\ \tilde{G}(x) < 1 + \frac{1}{\frac{\phi}{\delta}\frac{1+\varepsilon/4}{\varepsilon/4}-1} \quad \text{for } x > x(T).$$

Now apply lemma 2. The proof is completed by showing that there exists a T_ε such that

$$(1-\varepsilon)\bar{\bar{x}}(t) < x_{mT}(t),\ x_{MT}(t) < (1+\varepsilon)\bar{\bar{x}}(t),\ t > T_\varepsilon.$$

Choose $T_1 > T$ so that

$$(1-\varepsilon/2) < 1 - e^{\delta_m t}C(a,\delta_m,x(T),T),\ t > T_1.$$

(31) implies

$$\frac{1}{1+\varepsilon/2} < \frac{1}{m+(1-m)\phi/\delta}.$$

$$(1-\varepsilon)^{\mu} < (1-\varepsilon) < \frac{1}{1+\varepsilon/2}(1-\varepsilon/2) < \frac{1}{\delta_m/\delta}(1-\varepsilon/2)$$

$$(1-\varepsilon)^{\mu}\frac{a\mu}{\delta}e^{\phi t} < \frac{a\mu}{\delta_m}e^{\phi t}\left[1-e^{\delta_m t}C(a,\delta_m,x(T),T)\right]$$

$$(1-\varepsilon)\bar{\bar{x}}(t) < x_{mT}(t) \quad \text{for } t > T_1.$$

Choose $T_2 > T$ such that

$$1 - e^{\delta_M t}C(a_M,\delta_M,x(T),T) < 1 + \varepsilon/4.$$

(31) implies

$$\frac{M}{M+(1-M)\phi/\delta} < 1+\varepsilon/4\,.$$

$$\frac{a_M}{\delta_M} < \frac{a}{\delta}(1+\varepsilon/4)$$

$$\frac{a_M\mu}{\delta_M}e^{\phi t}\left[1-e^{\delta_M t}C\left(a_M,\delta_M,x(T),T\right)\right] < \frac{a\mu}{\delta}e^{\phi t}(1+\varepsilon/4)^2$$

$$< \frac{a\mu}{\delta}e^{\phi t}(1+\varepsilon) < \frac{a\mu}{\delta}e^{\phi t}(1+\varepsilon)^{\mu}, \quad t>T_2\,.$$

$$x_{MT}(t) < (1+\varepsilon)\bar{\bar{x}}(t) \quad t>T_2\,.$$

The proof is completed by taking $T_\varepsilon = max(T_1, T_2)$.

<u>Proof of the theorem for the case $\rho>0$, ϕ-$\delta<0$:</u> From lemma 3

$$x(t) = \left(\frac{a\mu}{\delta}\right)^{\frac{1}{\mu}} e^{\frac{\phi}{\mu}t}$$

asymptotically. The proof is completed by substituting this into equations (10)-(15).

b. $\rho>0$ **and** ϕ-$\delta>0$.

The proof follows the lines of the previous case but is more simple. Again it is possible to modify equation (21) as

(32) $$\dot{x} = \lambda\left(\frac{ae^{\phi t}}{x^{\mu-1}} + \frac{\phi-\delta}{\mu}x\right) \text{ with the initial condition } x(T)=X.$$

$$= \frac{a_\lambda e^{\phi t}}{x^{\mu-1}} + \frac{\phi-\delta_\lambda}{\mu}x$$

with $a_\lambda = \lambda a$ and $\delta_\lambda = \lambda\delta + (1-\lambda)\phi$ which has the solution given by (22) with a and δ modified appropriately.

Next one has the equivalent of Lemma 2.

Lemma 2'. Let $\rho>0$ and $\phi-\delta>0$. If $x(t)$ is the solution of equation (9), if $T > t_0$, and if m and M satisfy:

(33) $$m < \tilde{F}(x) < M \text{ and } m < \tilde{G}(x) < M \text{ if } x > x(T)$$

then

(34) $$\bar{x}_{m,T}(t) < x(t) < \bar{x}_{M,T}(t) \text{ if } t>T$$

with $\bar{x}_{m,T}(t) = \bar{x}\left(t;a_m,\delta_m,T,x(T)\right)$ and $\bar{x}_{M,T}(t) = \bar{x}\left(t;a_M,\delta_M,T,x(T)\right)$.

Proof:

Since $x(t)$ satisfies equation (9) with $x(t_0)=x_0 > 0$ and $\phi\text{-}\delta>0$, $\dot{x}(t)$ is always positive and $x(t)$ is increasing and positive. The same is true of $\bar{x}_{m,T}$ and $\bar{x}_{M,T}$ which satisfy equation (32) with the initial condition $\bar{x}_{m,T}(T) = \bar{x}_{M,T}(T) = x(T) > 0$, which are also increasing functions. It follows that $\bar{x}_{m,T}(t)$ and $\bar{x}_{M,T}(t)$ are strictly larger than $x(T)$ for $t > T$. Consequently, using equations (32), (33) and (17), one has

$$\dot{\bar{x}}_{m,T} = m\left(\frac{ae^{\phi t}}{\left(\bar{x}_{m,T}\right)^{\mu-1}} + \frac{\phi-\delta}{\mu}\bar{x}_{m,T}\right) < \frac{ae^{\phi t}}{\left(\bar{x}_{m,T}\right)^{\mu-1}}\tilde{F}\left(\bar{x}_{m,T}\right) + \frac{\phi-\delta}{\mu}\bar{x}_{m,T}\tilde{G}\left(\bar{x}_{m,T}\right) = e^{\phi t}F\left(\bar{x}_{m,T}\right) + G\left(\bar{x}_{m,T}\right)$$

as well as:

$$\dot{\bar{x}}_{M,T} > e^{\phi t}F\left(\bar{x}_{M,T}\right) + G\left(\bar{x}_{M,T}\right).$$

Chaplygins's conditions are satisfied, and this brings the proof of equation (34) to completion.

Finally one has the equivalent of Lemma 3.

Lemma 3'. Let $\rho>0$ and $\phi\text{-}\delta>0$, if $x(t)$ is the solution of (16), then:

$$\lim_{x\to\infty}\frac{x(t)}{\bar{\bar{x}}(t)} = 1.$$

Proof:

We want to prove that, for any ε (which can be assumed to be smaller than 1), one can find T_ε such that

$$(1-\varepsilon)\bar{\bar{x}}(t) < x(t) < (1+\varepsilon)\bar{\bar{x}}(t) \qquad t > T_\varepsilon .$$

We define:

(35) $$\nu = 3 + 4\frac{\phi-\delta}{\delta}.$$

Since $\tilde{F}(x)$ and $\tilde{G}(x)$ tend to 1 when x tends to infinity, X_ε exists such that:

$$1-\frac{\varepsilon}{\nu} < \tilde{F}(x) < 1+\frac{\varepsilon}{\nu} \quad \text{and} \quad 1-\frac{\varepsilon}{\nu} < \tilde{G}(x) < 1+\frac{\varepsilon}{\nu} \qquad x > X_\varepsilon .$$

We define T_ε^1 by $x\left(T_\varepsilon^1\right) = X_\varepsilon$, and apply the lemma for $T = T_\varepsilon^1$ and for :

(36) $$m = 1-\frac{\varepsilon}{\nu} \quad \text{and} \quad M = 1+\frac{\varepsilon}{\nu}.$$

We must now prove that a value of T_ε can be determined such that:

(37) $$(1-\varepsilon)\bar{\bar{x}}(t) < \bar{x}_{m,T}(t) \quad \text{and} \quad \bar{x}_{M,T}(t) < (1+\varepsilon)\bar{\bar{x}}(t) \qquad t > T_\varepsilon .$$

1. With ν defined by equation (35), $\phi-\delta$ larger than 0, m and M defined by equation (36) and ε smaller than 1, one can prove that:

$$\left(1-\frac{\varepsilon}{3}\right)\frac{a}{\delta}<\frac{a_m}{\delta_m} \quad \text{and} \quad \frac{a_M}{\delta_M}<\left(1+\frac{\varepsilon}{3}\right)\frac{a}{\delta}.$$

2. If $C_{m,T}$ and $C_{M,T}$ are the two constants in the expressions of $\bar{x}_{m,T}(t)$ and $\bar{x}_{M,T}(t)$:

$$C_{m,T} = C(a_m, \delta_m\, T, x(T)) \text{ and } C_{M,T} = C(a_M, \delta_M\, T, x(T))$$

and defining T_ε^2 and T_ε^3 by:

$$\left|C_{m,T_\varepsilon^1}\right| e^{-\delta_m T_\varepsilon^2} = \frac{\varepsilon}{3} \quad \text{and} \quad \left|C_{M,T_\varepsilon^1}\right| e^{-\delta_M T_\varepsilon^3},$$

$$\left|C_{m,T_\varepsilon^1}\right| e^{-\delta_m T_\varepsilon^2} = \frac{\varepsilon}{3} \quad \text{and} \quad \left|C_{M,T_\varepsilon^1}\right| e^{-\delta_M T_\varepsilon^3} = \frac{\varepsilon}{3}$$

then one has:

$$1-\frac{\varepsilon}{3}<1-C_{m,T_\varepsilon^1}e^{-\delta_m t} \quad \text{and} \quad 1-C_{M,T_\varepsilon^1}e^{-\delta_M t}<1+\frac{\varepsilon}{3} \qquad t>max\left(T_\varepsilon^2, T_\varepsilon^3\right)$$

.

3. Since μ is larger than 1, or $1/\mu$ smaller than 1, and still assuming that ε is smaller than 1, one has:

$$1-\varepsilon<(1-\varepsilon)^{1/\mu}<\left(\left(1-\frac{\varepsilon}{3}\right)^2\right)^{1/\mu} \quad \text{and} \quad \left(\left(1+\frac{\varepsilon}{3}\right)^2\right)^{1/\mu}<(1+\varepsilon)^{1/\mu}<1+\varepsilon.$$

Choosing $T_\varepsilon = max\left(T_\varepsilon^1, T_\varepsilon^2, T_\varepsilon^3\right)$, the inequalities (37) are satisfied, and this brings the proof of Lemma 3' to completion.

Proof of the theorem for the case $\rho>0$, $\varphi-\delta>0$: From lemma 3'

$$x(t)=\left(\frac{a\mu}{\delta}\right)^{\frac{1}{\mu}} e^{\frac{\varphi}{\mu}t}$$

asymptotically. The proof is completed by substituting this into equations (10)-(15).

c. $\rho<$***0*****.**

This case differs from the preceding two in that it is the rate of growth of the logarithm of *x* rather than of *x* itself that approaches a constant. Thus the proof is in terms of *y=lnx.* Other than this the structure of the proof is similar.

Equation (9) can be written as

(38) $$\dot{y}=e^{\varphi t}\frac{a\mu\beta}{c_2^{\mu/\rho}}\tilde{F}(y)+(\varphi-\delta)\tilde{G}(y) \qquad y(t_0)=y_0$$

where $\tilde{F}(y)=\frac{1}{\beta}c_2^{\mu/\rho}\frac{e^{\rho y}+\beta c_2}{\left(c_2+e^{\rho y}\right)\left(c_2+\mu e^{\rho y}\right)}$, $\tilde{G}(y)=\frac{c_2+e^{\rho y}}{c_2+\mu e^{\rho y}}$ and $y_0=\ln x_0$. As before

(39) $$\lim_{y\to\infty}\tilde{F}(y)=\lim_{y\to\infty}\tilde{G}(y)=1.$$

Note that the limit equation

(40) $$\dot{y} = e^{\phi t}\frac{a\mu\beta}{c_2^{\mu/\rho}} + (\phi - \delta) \qquad y(T) = Y$$

has the solution

(41) $$\bar{y}(t; a, \delta, Y, T) = \frac{a\mu\beta}{\phi c_2^{\mu/\rho}} e^{\phi t} S(\delta, t)$$

where $S(\delta, t) \equiv 1 - e^{-\phi t} t(\phi - \delta) - e^{-\phi t}\left[e^{\phi T} - (\phi - \delta)T - Y\right]$.

It also has the asymptotic solution

(42) $$\bar{\bar{y}}(t) = \frac{a\mu\beta}{\phi c_2^{\mu/\rho}} e^{\phi t}.$$

The limit equation can be modified as follows:

(43) $$\dot{y} = \lambda e^{\phi t}\frac{a\mu\beta}{c_2^{\mu/\rho}} + \lambda'(\phi - \delta) \qquad y(T) = Y$$

$$= e^{\phi t}\frac{a_\lambda \mu\beta}{c_2^{\mu/\rho}} + (\phi - \delta_{\lambda'})$$

where $a_\lambda = \lambda a$ and $\delta_{\lambda'} = \lambda'\delta + (1 - \lambda')\phi$. This has the solution

(44) $$\bar{y}(t; a_\lambda, \delta_{\lambda'}, Y, T)$$

Note that $\dot{\bar{y}}$ may be negative if $\delta > \phi$. But from (40)

(45) $$\dot{\bar{y}}(t) > 0 \text{ if } \phi - \delta > 0 \text{ or if } \phi - \delta < 0 \text{ and } t > \frac{1}{\phi}\ln\frac{\delta - \phi}{a\mu\beta c_2^{\mu/\rho}} \equiv T(\delta, a).$$

The equivalent of Lemma 2 holds in this case as well.

<u>Lemma 2".</u> Let $y(t)$ be the solution to (38). Choose m and M so that

(46) $$m < \tilde{F}(y) < M, \quad \text{and } m < \tilde{G}(y) < M$$

for $y>y(T)$, $T>T^*$, where

(47) $$T^* = max[t_0, T(a_m, \delta_M), T(a_M, \delta_m)].$$

Then for $T>T^*$

$$\bar{y}_{mMT}(t) < y(t) < \bar{y}_{MmT}(t) \text{ if } t > T.$$

Where, if $\delta > \phi$, $\bar{y}_{mMT}(t) = \bar{y}(t; a_m, \delta_M, T, y(T))$, $\bar{y}_{MmT}(t) = \bar{y}(t; a_M, \delta_m, T, y(T))$ and if $\delta < \phi$, $\bar{y}_{mMT}(t) = \bar{y}(t; a_m, \delta_m, T, y(T))$, $\bar{y}_{MmT}(t) = \bar{y}(t; a_M, \delta_M, T, y(T))$.

<u>Proof:</u> The idea is to show that $\bar{y}_{mMT}(t)$ and $\bar{y}_{MmT}(t)$ satisfy the conditions of Chaplygin's theorem. Since $\bar{y}_{mMT}(T) = \bar{y}_{MmT}(T) = y(T)$; and $\dot{\bar{y}}_{mMT}(t) > 0$ and $\dot{\bar{y}}_{MmT}(t) > 0$ for $t>T$ by (45) and (47), both of these functions are greater than $y(T)$ for $t>T$ so that the inequalities (46) are satisfied for these functions. If $\delta > \phi$,

$$\dot{\bar{y}}_{mMT}(t) = e^{\phi t}\frac{a\mu\beta}{c_2^{\mu/\rho}}\, m + (\phi - \delta)M < e^{\phi t}\frac{a\mu\beta}{c_2^{\mu/\rho}}\tilde{F}\big(y_{mMT}(t)\big) + (\phi - \delta)\tilde{G}\big(y_{mMT}(t)\big)$$

for $t>T$, $T>T^*$. Similarly

$$\dot{\bar{y}}_{MmT}(t) > e^{\phi t}\frac{a\mu\beta}{c_2^{\mu/\rho}}\tilde{F}\big(y_{MmT}(t)\big) + (\phi - \delta)\tilde{G}\big(y_{MmT}(t)\big)$$

for $t>T$, $T>T^*$. In the same way these inequalities hold if $\phi \geq \delta$. Thus the conditions of Chaplygin's theorem are satisfied and the proof is complete.

Finally the equivalent of Lemma 3 holds.

<u>Lemma 3".</u> If $y(t)$ is the solution to (38), then

$$\lim_{y \to \infty}\frac{y(t)}{\bar{\bar{y}}(t)} = 1.$$

<u>Proof:</u> We want to show that there is a T_ε such that

(48) $$(1-\varepsilon)\bar{\bar{y}}(t) < y(t) < (1+\varepsilon)\bar{\bar{y}}(t), \quad t > T_\varepsilon$$

for ε arbitrary but $\varepsilon<1$. First choose m and M so that

$$\left(1-\frac{\varepsilon}{2}\right) < m < 1 \text{ and } 1 < M < \left(1+\frac{\varepsilon}{2}\right).$$

From Lemma 1 and equations (39) and (45) it is possible to choose T_ε^1 so that inequalities (46) are satisfied for $t> T_\varepsilon^1$, Next choose $T_\varepsilon^2 = max(T_\varepsilon^1, T^*)$. Then from Lemma 2"

(49) $$\bar{y}_{mMT}(t) < y(t) < \bar{y}_{MmT}(t), \qquad t > T,\ T > T_\varepsilon^2.$$

The final step is to show that there exists a T_ε^3 such that

(50) $$(1-\varepsilon)\bar{\bar{y}}(t) < \bar{y}_{mMT}(t),\ \ \bar{y}_{MmT}(t) < (1+\varepsilon)\bar{\bar{y}}(t) \quad \text{for } t > T_\varepsilon^3.$$

Consider the first inequality. Since $S(\delta, t) \to 1$ as $t \to \infty$, choose $T_\varepsilon^{3'}$ such that $S(\delta_m, t) > (1-\varepsilon/2)$ for $t> T_\varepsilon^{3'}$. Remember that $a_m = am$. From (41) and (42) we must have that

$$(1-\varepsilon) < mS(\delta_m, t) \quad \text{for } t > T_e^{3'}.$$

But $(1-\varepsilon) < \left(1-\frac{\varepsilon}{2}\right)\left(1-\frac{\varepsilon}{2}\right) < mS(\delta_m, t)$ for $t> T_\varepsilon^{3'}$ so that the first inequality is demonstrated. The second follows in a similar fashion with a $T_\varepsilon^{3''}$ in the role of $T_\varepsilon^{3'}$. Finally let $T_\varepsilon^3 = max(T_\varepsilon^{3'}, T_\varepsilon^{3''})$.

Now let $T_\varepsilon = max(T_\varepsilon^2, T_\varepsilon^3)$. Then (48) follows from (49) and (50) and the proof of Lemma 3" is complete.

Proof of the theorem for the case $\rho<0$. From lemma 3”

$$y(t)=\frac{a\mu\beta}{\phi c_2^{\mu/\rho}}e^{\phi t}$$

asymptotically. From the definition of *y*, $x(t)=e^{y(t)}$. The proof is completed by substituting the expression for *x(t)* implied by these two equations into equations (10)-(15).

Appendix I.

The definitions of *x* and *f(x)* allow equation (1) to be written as equation (10) where

$$c_1=\frac{M}{(1-\alpha)^{1/\rho}},\quad \text{and}\quad c_2=\alpha\left(c_1\right)^{\rho}.$$

A few properties of *f(x)* will be useful:

$$f'(x)=\frac{c_1 x}{\left(c_2+x^{\rho}\right)^{1+1/\rho}},\quad f''(x)=-\frac{c_1c_2(1+\rho)x^{\rho-1}}{\left(c_2+x^{\rho}\right)^{2+1/\rho}}$$

$$f(x)-(1-\beta)xf'(x)=\frac{c_1x\left(\beta c_2+x^{\rho}\right)}{\left(c_2+x^{\rho}\right)^{1+1/\rho}}$$

$$g(x)=\frac{f'(x)-\dfrac{1-\beta}{\beta}xf''(x)}{\left(f(x)\right)^{1/\beta}}=\frac{1}{\left(c_1c_2\right)^{(1-\beta)/\beta}}\frac{c_2+\mu x^{\rho}}{\left(c_2+x^{\rho}\right)^{1+(1-\mu)/\rho}}\quad.$$

Equations (2) and (3) can be written as

(A1) $\qquad w=e^{\delta t}f'(x)(1-\beta)\dfrac{x}{L}$

(A2) $\qquad r=e^{\delta t}f'(x)\beta\dfrac{x}{K}.$

One can now express the main variables as functions of *x*: (11) follows from (A1); (12) from the definition of *x* and (11); (A1) and (A2) imply (14); (11), (12) and (14) imply (13); and (10) and (11) imply (15),

The differential equation in terms of *x* is got as follows: Profits Π are

$$\Pi=Y-wL=e^{\delta t}\left(f(x)-(1-\beta)xf'(x)\right)=c_1e^{\delta t}\frac{x\left(\beta c_2+x^{\rho}\right)}{\left(c_2+x^{\rho}\right)^{1+1/\rho}}\quad.$$

Differentiating equation (12)

$$\dot{K}=\left(\frac{w}{1-\beta}\right)^{\frac{1-\beta}{\beta}}e^{(\delta-\phi)t}\left((\delta-\phi)\frac{x}{\left(f'(x)\right)^{(1-\beta)/\beta}}+\dot{x}g(x)\right).$$

Then from $\dot{K} = \Pi$ one gets equation (9) where $a = \frac{\left(c_1 c_2^{1-\beta}\right)}{\mu}\left(\frac{1-\beta}{w}\right)^{(1-\beta)/\beta}$.

Appendix II.

Let $y(t) = a\mu \frac{e^{\phi t}}{\left(x(t)\right)^{\mu}}$. Then, upon substitution, equation (21) becomes

$$\dot{y} = y(\delta - y), \qquad y(T)=Y.$$

This can be integrated to give

$$\bar{y}(t) = \frac{\delta}{1 - \tilde{C}e^{-\delta t}}$$

where $\tilde{C} = e^{\delta T}\left(\frac{\delta}{y(T)} - 1\right)$. Writing this in terms of x give the solution (22).

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