A Generalized Assignment Game*

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Abstract

The proposed game is a natural extension of the Shapley and Shubik Assignment Game to the case where each seller owns a set of different objects instead of only one indivisible object. We propose definitions of pairwise stability and group stability that are adapted to our framework. Existence of both pairwise and group stable outcomes is proved. We study the structure of the group stable set and we finally prove that the set of group stable payoffs forms a complete lattice with one optimal group stable payoff for each side of the market.

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1 Introduction

We study the interaction of a finite number of sellers and buyers in a market, with the characteristic that each seller owns a set of possibly different objects and each buyer wants to buy, at most, one object. The fact that agents in these markets belong, from the outset, to one of two disjoint sets, sellers and buyers, and the bilateral nature of exchange that exists, allows us to study them as "two-sided matching markets".

The worth of a potential transaction is given by a nonnegative real number associated with each possible pair of a buyer and an object. An outcome of the game specifies a matching between buyers and sellers and the price that each buyer pays to the owner of the object she is buying.[1] Therefore, we study a *many-to-one matching model with money*, in which each buyer is matched with at most one seller, but each seller can be matched with as many buyers as objects he owns, and prices are determined as part of the outcome of the game. This is a natural extension of the *one-to-one* Shapley and Shubik [9] Assignment Game, also studied in Roth and Sotomayor [8].

Two natural questions are what partnerships we can expect to observe in the market and how the agents will divide their gain. The answers to these questions involve the choice of an appropriate concept of equilibrium that in this class of games is called *stability*. There are two different ways of defining stability in a many-to-one matching model: considering deviations only of pairs of agents (*pairwise stability*) or deviations of groups of agents (*group stability*). We say that an outcome is *pairwise stable* if it is individually rational and there is no partnership and a price so that, at this price, both buyer and seller are better off under this partnership than under the previous outcome. If such a pair exists, we say that it blocks the outcome. A *group stable* outcome is one that is not blocked by any possible coalition of players. In the one-to-one Shapley and Shubik Assignment Game these two concepts coincide, since the only sensible coalition that can block a given outcome is formed by, at most, two agents. Sotomayor [11] studies a special case of our model for the case where all the objects that a seller owns are equal (in her model the set of sellers is seen as a set of firms with a number of equal vacancies and the set of buyers as workers). In her framework, the set of pairwise stable outcomes also coincides with the set of group stable outcomes. However, in our case, this equivalence does not hold, since group stability is a sufficient, but not a necessary condition for pairwise stability. This relationship has to do with the fact that the total gain that a seller and a

[1] The set of sellers in our model can also be seen as multiproduct firms or as multidivisional firms with one vacancy per division, and the set of buyers as workers, where the salaries are determined explicitly in the model.

buyer can share is not always the same, but depends on the object bought.

Shapley and Shubik [9] show that every Assignment Game (or one-to-one buyer-seller market) has at least one stable outcome, and that the set of stable outcomes is the Cartesian product of the set of stable payoffs and the set of optimal matchings. They further prove that every one-to-one buyer-seller market has a seller-optimal stable payoff and a buyer-optimal stable payoff. For the many-to-one case in which all the objects owned by a seller are equal, the existence of stable outcomes and of an optimal stable payoff for each side of the market is also proved (see Sotomayor [10]). Sotomayor [11] proves that these optimal stable payoffs are unique.

We also show existence of the group stable (and hence the pairwise stable) set in our model. We then focus the study on the analysis of the structure of the group stable set, since it is a more adequate concept of stability for our framework. We prove that any matching which is compatible with a given group stable payoff is optimal in the sense that it maximizes the gain of the whole set of players. On the other hand, the set of group stable outcomes is the Cartesian Product of the set of group stable payoffs and the set of optimal matchings. That is, for any group stable payoff and any optimal matching, there exists a vector of prices that makes the outcome group stable. Moreover, it forms a complete lattice with an optimal group stable payoff for each side of the market. We also study the set of competitive equilibria and prove that it is a proper subset of the group stable set, while these two sets coincide in the Assignment Game.[2]

The paper is organized as follows: Section 2 presents the formal model. Section 3 defines the concept of pairwise stability and group stability and establishes their relationship. Section 4 connects our model with the Assignment Game and proves existence of stable outcomes, and Section 5 studies the structure of the group stable set. Section 7 concludes.

2 The model

We consider a buyer-seller market consisting of m buyers and t sellers. Each seller owns a number of possibly different objects, and each buyer wants to buy at most one object.

[2]Pérez-Castrillo and Sotomayor [6] have proposed a simple sequential mechanism to implement the sellers' optimal payoff in the Assignment Game, using the fact that the stable set is equivalent to the set of competitive equilibria. For other mechanisms to implement stable solutions in matching markets with money see also Demange *et al.* [2], Kameche [4] and Alcalde *et al.* [1].

Formally, there are two finite disjoint sets of players, P and S, containing m and t players, respectively, and a set Q of n objects. Let $P = \{p_1, p_2, ..., p_m\}$ be the set of buyers. Generic buyers will be denoted by p_i and p_k. The payoff of buyer $p_i \in P$ will be denoted by u_i. Let $S = \{s_1, s_2, ..., s_t\}$ be the set of sellers. Generic sellers will be denoted by s_r and s_d, and the payoff of seller $s_r \in S$ by w_r. Let $Q = \{q_1, q_2, ..., q_n\}$ be the set of indivisible objects. Generic objects will be denoted by q_j and q_h, and the price of object $q_j \in Q$ by v_j.

We also define a function $f : Q \to S$ that assigns each object to the seller who owns it, i.e., $f(q_j) = s_r$ if and only if seller s_r owns object q_j. We denote by $Q_r \equiv \{q_j \in Q : f(q_j) = s_r\}$ the set of objects that seller s_r owns, and by $|Q_r|$ the quota of seller $s_{r,}$ that is, the number of objects he owns.

Associated with each possible pair $(p_i, q_j) \in P \times Q$ there is a nonnegative real number, α_{ij}, which denotes the maximum price that buyer p_i is willing to pay for object q_j, that is, her reservation value. We may interpret it as if she had in hand an offer of α_{ij} from a client who will purchase the object from the buyer at that price. For simplicity, we assume w.l.o.g. that the reservation price of seller s_r for every object $q_j \in Q_r$ is zero (that is, if the seller offers any of his objects to an outside party, he will obtain zero). Therefore, α_{ij}, denotes the potential gains from trade between the buyer p_i and the seller $f(q_j)$ if the object sold is q_j. We will denote by α the $m \times n$ matrix $(\alpha_{ij})_{i=1,...m;j=1,...n}$. We also assume that there are no monetary transfers among agents of the same side, which is a natural assumption since we are studying a buyer-seller market, so the model allows only the conventional transfer of the purchase price from the successful buyer to the seller. Thus, if buyer p_i buys the object q_j at a price v_j then the resulting payoffs are $u_i = \alpha_{ij} - v_j$ for the buyer and v_j for seller $s_r = f(q_j)$. The total payoff of seller s_r, denoted by w_r, is the sum of all the prices of the objects he sells.[3] Agents' preferences are concerned only with their monetary payoffs. That implies that for any pair of objects and a buyer, there is a pair of prices that makes the buyer indifferent between purchasing either of the objects.

Therefore, a market M is determined by (P, S, Q, f, α). We call it the Generalized Assignment Game.

[3]If we allowed to have reservation prices of sellers different from zero, say, c_j for object $q_j \in Q_r$, then the potential gains from trade would be: $\max\{0, \alpha_{ij} - c_j\}$. Like this, we are normalizing each seller's utility of keeping one of his own objects q_j to zero rather that c_j.

3 Feasibility and stability

In this section we define what is a feasible outcome of this game. Our main concern is to predict which of those outcomes are likely to occur and which are not. For that purpose, we define an appropriate concept of stability.

For technical convenience, we introduce one artificial null object, q_0, and one dummy player, seller s_0. Several buyers may buy this null object. This convention allows us to treat a buyer p_i that does not buy any object as if she bought q_0. We assume that $f(q_0) = s_0$, (and $Q_0 = \{q_0\}$), so p_i will be matched to the dummy player s_0 if she buys no object. We also assume that the value α_{i0} is zero to all buyers, and the price of the object q_0 is always zero, $v_0 = 0$. Hence, if buyer p_i buys q_0 she obtains a utility $u_i = \alpha_{i0} - v_0 = 0$.

3.1 Feasibility

An outcome of this game specifies a matching between buyers and sellers and the price that each buyer pays to the owner of the object she is buying. First, we define a feasible matching as a function that matches buyers with objects, with the possibility that some object remains unsold. We also define a correspondence associated with each feasible matching, that matches buyers and sellers with an agent from the opposite side of the market, with the characteristic that each seller can be matched with as many buyers as objects he owns.

Definition 1 *A **feasible matching** μ for a market $M \equiv (P, S, Q, f, \alpha)$ is a function from the set $P \cup Q$ into the set $P \cup Q \cup \{q_0\}$ such that:*

(i) For any $p_i \in P$, $\mu(p_i) \in Q \cup \{q_0\}$.

(ii) For any $q_j \in Q$, either $\mu(q_j) \in P$ or $\mu(q_j) = q_j$.

(iii) For any $(p_i, q_j) \in P \times Q$, $\mu(p_i) = q_j$ if and only if $\mu(q_j) = p_i$.

We say that buyer p_i is unmatched if $\mu(p_i) = q_0$. Similarly, we say that object q_j is unsold if $\mu(q_j) = q_j$.

Definition 2 *For any given feasible matching μ, we define its **associated matching** μ_s as a correspondence from the set $P \cup S$ into the set of subsets of $P \cup S \cup \{s_0\}$, such that:*

(I) $\mu_s(p_i) = f(q_j)$ if and only if $\mu(p_i) = q_j$

(II) $\mu_s(s_r) = \{p_i \in P : \mu(p_i) \in Q_r\}$.

Given a feasible matching μ, a vector $(u, w, v) \in \mathbb{R}_+^m \times \mathbb{R}_+^t \times \mathbb{R}_+^n$ of utilities for the players and prices is *compatible with* μ if:[4]

(i) $u_i = \alpha_{i\mu(p_i)} - v_{\mu(p_i)}$, for every $p_i \in P$, and

(ii) $w_r = \sum_{\substack{q_j \in Q_r \\ \mu(q_j) \in P}} v_j = \sum_{\substack{q_j \in Q_r \\ \mu(q_j) \in P}} \left(\alpha_{\mu(q_j)j} - u_{\mu(q_j)}\right)$, for every $s_r \in S$.

Definition 3 *A* ***feasible outcome****, denoted by* $(u, w, v; \mu)$ *, is a vector of utilities (or payoff vector)* $(u, w) \in \mathbb{R}_+^m \times \mathbb{R}_+^t$*, a price vector* $v \in \mathbb{R}_+^n$*, and a feasible matching* μ*, such that the vector* (u, w, v) *of utilities and prices is compatible with* μ*. If* $(u, w, v; \mu)$ *is a feasible outcome, then* (u, w) *is called a* ***feasible payoff****.*

Note that a feasible outcome is always individually rational.

Definition 4 *A feasible matching* μ *is* ***optimal*** *for a market* M *if it maximizes the gain of the whole set of players. That is, if for all feasible matchings* μ' *we have:*

$$\sum_{\substack{p_i \in P \\ q_j = \mu(p_i)}} \alpha_{ij} \geq \sum_{\substack{p_i \in P \\ q_j = \mu'(p_i)}} \alpha_{ij}.$$

3.2 Stability

A feasible outcome is pairwise stable if there is no pair that can block the outcome, that is, there is no pair of a seller and a buyer that can generate together a gain from trade that leaves both of them better off.

Definition 5 *A feasible outcome* $(u, w, v; \mu)$ *is* ***pairwise stable*** *if:*

(i) For any (p_i, q_j) *such that* $f(q_j) \neq f(\mu(p_i))$ *we have:*

$u_i + v_j \geq \alpha_{ij}$, *if* $\mu(q_j) \in P$

$u_i \geq \alpha_{ij}$, *if* $\mu(q_j) = q_j$

(ii) For any (p_i, q_j) *such that* $f(q_j) = f(\mu(p_i))$ *we have:*

$\alpha_{i\mu(p_i)} + v_j \geq \alpha_{ij}$, *if* $\mu(q_j) \in P$

$\alpha_{i\mu(p_i)} \geq \alpha_{ij}$, *if* $\mu(q_j) = q_j$

[4] We sometimes abuse notation by writing $\alpha_{i\mu(p_i)}$ instead of α_{ij}, where $q_j = \mu(p_i)$. Similarly for $\alpha_{\mu(q_j)j}$.

Condition (i) is the usual requirement for pairwise stability in every matching market. Note that in our case it is a sufficient condition for all pairs formed by a buyer and an object, but it is not necessary for those pairs where the object and the mate of the buyer belong to the same seller. For these pairs we need condition (ii). This is due to the fact that a partnership formed by a buyer and a seller can generate different gains depending on the object sold. Therefore, condition (ii) implies that each buyer is buying the object that maximizes the gain that she can share with the seller she is matched with.

For an outcome to be group stable, we do not only require the non-existence of blocking pairs, but also of blocking coalitions. We denote by T a coalition of agents, and T_s and T_p will denote the sets of $S-$ and $P-$ agents in T, respectively, (i.e., the intersection of the coalition T with S and P, respectively).

Definition 6 *A feasible outcome $(u, w, v; \mu)$ is* ***group stable*** *if it is not blocked by any coalition. That is, if there does not exist any coalition $T = T_s \cup T_p$ of agents that, by matching among themselves, according to, say, μ', and setting a price v'_j for every $q_j \in \bigcup_{s_r \in T_s} Q_r$ such that $\mu(q_j) \in T_p$, all members of T prefer this new assignment to μ.*

Note that in this definition we allow each seller in T to still sell some of his objects to buyers outside T at the same prices as before. We can include these buyers in the blocking coalition T, if the seller they are buying from transfers them money by reducing the price at ε. The only case in which we must leave one of these buyers indifferent is when the price she was paying was zero. Therefore, it is clear that the set of group stable outcomes corresponds to the core of the game.

In particular, we are interested in outcomes that are not blocked by any coalition formed by one seller and a set of buyers. We call this concept *restricted group stability.*

Definition 7 *A feasible outcome $(u, w, v; \mu)$ is* ***restricted group stable*** *if it is not blocked by any coalition formed by a single seller and a set of buyers, that is, if there does not exist any coalition $T = s_r \cup T_p$ with $s_r \in S$ and $T_p \subset P$, and any feasible matching $\widehat{\mu}$, such that*

$$\sum_{\substack{p_i \in T_p \\ \widehat{\mu}(p_i) = q_j \\ q_j \in Q_r}} \alpha_{ij} > w_r + \sum_{p_i \in T_p} u_i$$

We prove that the previous two concepts coincide.

Proposition 1 *Given a market M, the set of group stable outcomes coincides with the set of restricted group stable outcomes.*

Proof. It follows directly from Definitions 6 and 7, and taking into account that we do not allow for side payments among agents in the same side of the market. ■

Given the definitions of pairwise and group stability, it is clear that if an outcome is group stable for a given market then it is pairwise stable, that is, group stability is a sufficient condition for pairwise stability. However, both definitions do not coincide in general. The following example shows that group stability is not a necessary condition for pairwise stability.

Example 1:

Let $M \equiv (P, S, Q, f, \alpha)$, with $S = \{s_1\}$, $Q = Q_1 = \{q_1, q_2\}$, $P = \{p_1, p_2\}$, and $\alpha_{11} = \alpha_{22} = 10$, $\alpha_{12} = \alpha_{21} = 8$. Taking $(u, w, v; \mu) = ((2,2), 12, (6,6); \mu(q_1) = p_2, \mu(q_2) = p_1)$ is pairwise stable ($\alpha_{11} = \alpha_{22} = 10 < 8 + 6 = \alpha_{12} + v_1 = \alpha_{21} + v_2$, see Definition 5). But it is not group stable. Indeed the grand coalition $T = \{s_1, p_1, p_2\}$ can be matched as follows: $\mu'(q_1) = p_1, \mu'(q_2) = p_2$, and by setting, for example, $v_1' = v_2' = 7$, all agents win more than in outcome $(u, w, v; \mu)$ since the new payoffs are: $(u', w') = ((3,3), 14) > ((2,2), 12) = (u, w)$.

We have proven that, given a market M, the set of group stable outcomes can be strictly contained in the set of pairwise stable outcomes, that is, group stability is a stronger condition than pairwise stability. This result differs from the usual one found for the many-to-one models previously studied, where group stability was equivalent to pairwise stability. The difference is due to the fact that we allow the objects to be different. If all the objects a seller owns were equal, that is, $\alpha_{ij} = \alpha_{ik}$ if $f(q_j) = f(q_k)$, group stability and pairwise stability would be equivalent (see Sotomayor [11]). In that case, if an outcome $(u, w, v; \mu)$ is not group stable, it means that there exists a coalition of a seller, say s_r, and a set of buyers that blocks it, and for sure that at least one of the buyers, say p_i, is not buying from s_r under μ. But this means that $u_i + v_j < \alpha_{ij}$ for some $q_j \in Q_r$ and $\mu(q_j) \in P$, or $u_i < \alpha_{ij}$ for some $q_j \in Q_r$ unsold, because otherwise there is no additional gain that the coalition can share. But then outcome $(u, w, v; \mu)$ is not pairwise stable.

Therefore, in the Generalized Assignment Game, if an outcome $(u, w, v; \mu)$ is pairwise stable but not group stable it is because there exists a coalition, formed by a seller and the buyers buying form him under μ, that can reorganize themselves in a strictly profitable way for all of them. Then, we can state the following proposition.

Proposition 2 *Given a market M, an outcome $(u, w, v; \mu)$ is group stable if and only if the following two properties hold:*

(a) $(u, w, v; \mu)$ is pairwise stable, and

(b) the set of buyers buying from the same seller are optimally allocated, i.e., for every seller $s_r \in S$, the buyers buying from him, P_r, are such that $\sum_{\substack{p_i \in P_r \\ q_j = \mu(p_i)}} \alpha_{ij} \geq \sum_{\substack{p_i \in P_r \\ q_j = \mu'(p_i)}} \alpha_{ij}$, *for all feasible matchings μ'.*

Proof. The *only if* part trivially holds good.

For the *if* part, just note that the only cases in which a pairwise stable outcome is not group stable is when a group of buyers buying from the same seller are not optimally allocated. ■

4 Connection with the Assignment Game

Given a market $M \equiv (P, S, Q, f, \alpha)$, we can define the "one-to-one" market (an Assignment Game), $M' \equiv (P, S', Q, f', \alpha)$, as follows:

$Q = \{q_1, q_2, ..., q_n\}$, set of objects
$S' = \{s'_1, s'_2, ..., s'_n\}$, set of sellers with $f'(q_j) = s'_j$, for all $j = 1, ..., n$.
$P = \{p_1, p_2, ..., p_m\}$, set of buyers.

Given a feasible outcome $(u, w, v; \mu)$ in the Generalized Assignment Game, the *transformed* outcome in the Assignment Game is given by $(u, w', v; \mu)$ with:

$$w'_j = \begin{cases} v_j, & \text{if } \mu(q_j) \in P \\ 0, & \text{if } \mu(q_j) = q_j, \end{cases}$$

for all $j = 1, ..., n$.

We can define feasibility and stability for these *transformed* markets in the same way as before, since they are a subset of our generalized markets. Note that the pairwise stable set and the group stable set coincide in the one-to-one market, but the concept of pairwise stability used is different to that used for the many-to-one market, since now condition (i) in Definition 5 is needed for all pairs $(p_i, q_j) \in P \times Q$.

The following proposition states the relationship between group stable outcomes of a given market M and (pairwise) stable outcomes in the corresponding Assignment Game M'.

Proposition 3 *Take a market M, and its corresponding one-to-one market M'. If the outcome $(u, w', v; \mu)$ is (pairwise) stable for M', then $(u, w, v; \mu)$ is a group stable outcome for M, where $w_r = \sum_{\substack{q_j \in Q_r \\ \mu(q_j) \in P}} v_j$ for every $s_r \in S$.*

Proof. By contradiction, suppose that the outcome $(u, w, v; \mu)$ is not group stable for market M. We prove that the outcome $(u, w', v; \mu)$ is also not (pairwise) stable for M'.

Since $(u, w, v; \mu)$ is not group stable, there exists a coalition T formed by, say, seller s_r and a subset of buyers T_p and a feasible matching μ', such that, $w'_r > w_r$, i.e.,

$$\sum_{\substack{q_j \in Q_r \\ \mu'(q_j) \in P}} v'_j > \sum_{\substack{q_j \in Q_r \\ \mu(q_j) \in P}} v_j$$

and

$$\alpha_{i\mu'(p_i)} - v'_{\mu'(p_i)} > \alpha_{i\mu(p_i)} - v_{\mu(p_i)}, \text{ for every } p_i \in T_p,$$

where v' is the new vector of prices.

This means that there exists $q_j \in Q_r$ with $\mu'(q_j) \in T_p$ such that, either $v'_j > 0$ and $\mu(q_j) = q_j$, or $v'_j > v_j$, $\mu(q_j) \in P$, and $\mu'(q_j) \neq \mu(q_j)$. In both cases, we must have $u'_{\mu'(q_j)} > u_{\mu'(q_j)}$. Therefore, in M', the pair $(\mu'(q_j), q_j)$ blocks the outcome $(u, w', v; \mu)$. ■

Proposition 3 proves that (pairwise) stability in the Assignment Game is a sufficient condition for group stability. We show that, in general, it is not a necessary condition with the following example:

Example 2:

Take market $M \equiv (P, S, Q, f, \alpha)$ with $S = \{s_1\}$, $Q = Q_1 = \{q_1, q_2\}$, $P = \{p_1\}$, and $\alpha_{11} = 5, \alpha_{12} = 4$. (Note that in this case, pairwise and group stable outcomes coincide since there is only one seller and one buyer.) The outcome $(u, w, v; \mu) = (1, 4, (4, 0); \mu(p_1) = q_1)$ is pairwise stable ($\alpha_{12} = 4 < \alpha_{11} = 5$, see Definition 5) but the corresponding one-to-one outcome $(u, w', v; \mu) = (1, (4, 0), (4, 0); \mu(p_1) = q_1)$, is blocked by the pair $(p_1, q_2) = (p_1, s'_2)$ since $u_1 = 1 < \alpha_{12} = 4$, so there exist a price for q_2, for example, $v'_2 = 1$, that gives a utility of $3 > 1$ to buyer p_1 and a utility of $1 > 0$ to seller s'_2.

The importance of Proposition 3 is the following theorem that proves the existence of group stable outcomes for any given market M. Taking into account that existence in the Assignment Game is proved (see Shapley and Shubik [9]) we can say the following:

Theorem 1 *A group stable outcome exists for every given market* $M \equiv (P, S, Q, f, \alpha)$.

Note that we also have existence of pairwise stable outcomes using Proposition 3 and our previous result that the set of group stable outcomes is contained in the set of pairwise stable outcomes for any given market M. With both we have that the set of pairwise stable outcomes for a market M contains the set of stable outcomes for the transformed market M'. Hence, we have the following corollary:

Corollary 1 *A pairwise stable outcome exists for every given market* $M \equiv (P, S, Q, f, \alpha)$.

5 Structure of the group stable set

We are particularly interested in matchings that maximize the gain of the whole set of agents, that is, *optimal matchings* as defined in Definition 4. Therefore, it is clear that the optimal matchings for a given market M coincide with the optimal matchings for the *transformed* one-to-one market M'.

Next proposition is the first step in the characterization of the mathematical structure of the set of group stable payoffs. It shows that, at any group stable outcome, the associated matching is optimal. This property is also shared with the Assignment Game (see Shapley and Shubik [9]), and with the case where each seller owns a set of equal objects studied in Sotomayor [11].

Proposition 4 *Let* $(u, w, v; \mu)$ *be a group stable outcome for a given market* M. *Then,* μ *is an optimal matching.*

Proof. By contradiction, suppose μ is not optimal. Then, there exists a feasible matching μ' such that:

$$\sum_{\substack{p_i \in P \\ q_j = \mu'(p_i)}} \alpha_{ij} > \sum_{\substack{p_i \in P \\ q_j = \mu(p_i)}} \alpha_{ij} = \sum_{p_i \in P} u_i + \sum_{s_r \in S} w_r$$

Rewrite:

$$\sum_{\substack{p_i \in P \\ q_j = \mu'(p_i)}} \alpha_{ij} = \sum_{s_r \in S} \left(\sum_{\substack{p_i \in P \\ \mu'(p_i) = q_j \\ q_j \in Q_r}} \alpha_{ij} \right)$$

and

$$\sum_{p_i \in P} u_i + \sum_{s_r \in S} w_r = \sum_{s_r \in S} \left[w_r + \sum_{\substack{p_i \in P \\ \mu'(p_i)=q_j \\ q_j \in Q_r}} u_i \right] + \sum_{\substack{p_i \in P \\ \mu'(p_i)=q_0}} u_i.$$

Then,

$$\sum_{s_r \in S} \left(\sum_{\substack{p_i \in P \\ \mu'(p_i)=q_j \\ q_j \in Q_r}} \alpha_{ij} \right) > \sum_{s_r \in S} \left[w_r + \sum_{\substack{p_i \in P \\ \mu'(p_i)=q_j \\ q_j \in Q_r}} u_i \right] + \sum_{\substack{p_i \in P \\ \mu'(p_i)=q_0}} u_i \geq \sum_{s_r \in S} \left[w_r + \sum_{\substack{p_i \in P \\ \mu'(p_i)=q_j \\ q_j \in Q_r}} u_i \right]$$

and this implies that there exists a seller $s_{\widehat{r}}$ such that $\sum_{\substack{p_i \in P \\ \mu'(p_i)=q_j \\ q_j \in Q_{\widehat{r}}}} \alpha_{ij} > w_{\widehat{r}} + \sum_{\substack{p_i \in P \\ \mu'(p_i)=q_j \\ q_j \in Q_{\widehat{r}}}} u_i$. But this means that the coalition formed by $s_{\widehat{r}}$ and those buyers matched under μ' with an object in $Q_{\widehat{r}}$ can gain more by reorganizing among themselves. Therefore, $(u, w, v; \mu)$ is not group stable and we have a contradiction. ■

This is an important result for the model. Now we can concentrate on optimal matchings, since there is no vector of prices that can support any not optimal matching as a group stable outcome.

Remark 1 *Pairwise stability does not imply efficiency as group stability does. See Example 1.*

In the Assignment Game (see Shapley and Shubik [9]) the set of (pairwise) stable outcomes is the Cartesian product of the set of (pairwise) stable payoffs and the set of optimal matchings. This means that if we take a (pairwise) stable payoff, in the sense that there exist a feasible matching that makes the outcome (pairwise) stable, then the same is true for any optimal matching. In the following proposition we prove that the result for the Assignment Game is also true in the Generalized Assignment Game, and therefore the set of group stable outcomes is the Cartesian Product of the set of group stable payoffs and the set of optimal matchings, for a compatible price vector. This implies, as we will see later, that the group stable set forms a complete lattice, which is a nice structure to deal with. Our proofs differ technically from the Shapley and Shubik's [9] and Sotomayor's [11] proofs. They base their results in the Duality Theorem and in a central theorem [11, Theorem 1], respectively, that has no parallel in our model, since they depend critically on the fact that the object (s) a seller owns is (are) equal.

Definition 8 *A* ***payoff vector*** (u, w) *is* ***group stable*** *for a market* M *if there exists a vector of prices* $v \in \mathbb{R}^n$ *and a feasible matching* μ *such that* $(u, w, v; \mu)$ *is a group stable outcome.*

Proposition 5 *Take any group stable payoff* (u, w) *and any optimal matching* μ'. *Then there exists a vector of prices* v' *such that* $(u, w, v'; \mu')$ *is a group stable outcome.*

Proof. Since (u, w) is a group stable payoff, there exists a vector of prices v and a feasible matching μ such that $(u, w, v; \mu)$ is group stable. By Proposition 4 we know that this matching μ is optimal. Since both μ and μ' are optimal,

$$\sum_{\substack{p_i \in P \\ q_j = \mu'(p_i)}} \alpha_{ij} = \sum_{\substack{p_i \in P \\ q_j = \mu(p_i)}} \alpha_{ij} = \sum_{p_i \in P} u_i + \sum_{s_r \in S} w_r. \tag{1}$$

Define v' as follows:
$v'_j = \alpha_{ij} - u_i$, if $\mu'(q_j) = p_i$, and
$v'_j = 0$, if $\mu'(q_j) = q_j$.

To check whether $(u, w, v'; \mu')$ is a group stable outcome, we first check its feasibility (see Definition 3). By definition of v', $u_i + v'_j = \alpha_{ij}$, if $\mu'(p_i) = q_j$. Then, the only thing left to check is whether $w_r = \sum_{q_j \in Q_r} v'_j$, for every seller $s_r \in S$. By feasibility of $(u, w, v; \mu)$, we know that $w_r = \sum_{q_j \in Q_r} v_j$. Denote $w'_r = \sum_{q_j \in Q_r} v'_j$. We want to prove that $w_r = w'_r$, for every $s_r \in S$. We prove it by contradiction.

Eq. (1) implies that $\sum_{s_r \in S} w_r = \sum_{s_r \in S} w'_r$. Suppose that there exist seller s_r and s_d such that $w'_r < w_r$ and $w'_d > w_d$. Then seller s_d can form a coalition with the buyers buying from him under μ' (we call this group of buyers T_p), and be all strictly better off than under $(u, w, v; \mu)$:

$$w'_d = \sum_{q_j \in Q_d} v'_j = \sum_{q_j \in Q_d} \alpha_{\mu'(q_j)j} - \sum_{q_j \in Q_d} u_{\mu'(q_j)} > w_d.$$

Then,

$$\sum_{\substack{p_i \in T_p \\ \mu'(p_i) = q_j \\ q_j \in Q_d}} \alpha_{ij} = \sum_{\substack{q_j \in Q_d \\ \mu'(q_j) \in P}} \left(v'_j + u_{\mu'(q_j)}\right) = w'_d + \sum_{p_i \in T_p} u_i > w_d + \sum_{p_i \in T_p} u_i.$$

But this means that $(u, w, v; \mu)$ is not group stable so we have the contradiction.

Now that we know that $(u, w, v'; \mu')$ is feasible, we prove that there is no blocking coalition. We do it by contradiction. Suppose that there exists a coalition $T = s_{\widehat{r}} \cup T_p$ and a feasible matching $\widehat{\mu}$ such that

$$\sum_{\substack{p_i \in T_p \\ \widehat{\mu}(p_i) = q_j \\ q_j \in Q_{\widehat{r}}}} \alpha_{ij} > w_{\widehat{r}} + \sum_{p_i \in T_p} u_i.$$

But this means that $(u, w, v; \mu)$ is also blocked by coalition T, which is a contradiction. A similar argument can be used to prove that the prices v_j' we have defined are greater or equal to zero for every object. If one of these prices was negative, the seller that owns that object could form a blocking coalition by not selling that object. ■

We provide the following example to show that we can not state a similar result as in Proposition 5 if we not only specify the payoff vector but also a vector of prices.

Example 3:

Let $M \equiv (P, S, Q, f, \alpha)$ be $S = \{s_1\}$, $Q = Q_1 = \{q_1, q_2\}$, $P = \{p_1, p_2\}$, and $\alpha_{11} = \alpha_{12} = \alpha_{22} = \alpha_{21} = 5$. The outcome $(u, w, v; \mu) = ((2, 3), 5, (3, 2); \mu(p_1) = q_1, \mu(p_2) = q_2)$ is group stable. Take μ' such that $\mu'(p_1) = q_2, \mu'(p_2) = q_1$. Both μ' and μ are optimal matchings, but $(u, w, v; \mu')$ is not group stable since it is not feasible: $u_1 + v_2 = 4 < 5 = \alpha_{12}$ and $u_2 + v_1 = 6 > 5 = \alpha_{21}$. We need to set $v_1' = 2$ and $v_2' = 3$, and then $(u, w, v'; \mu')$ is group stable.

In what follows, we analyze the lattice structure of the group stable set. A lattice is a partially ordered set any two of whose elements have a (least) upper bound and a (greatest) lower bound in the set. When each of the possible subsets of the set has a (least) upper bound and a (greatest) lower bound in the set, we say that the lattice is complete.

Let us define the partial orders $\geq_P$ and $\geq_S$: for any two group stable payoffs (u, w) and (u', w'), $(u, w) \geq_P (u', w')$ if $u_i \geq u_i'$ for all p_i in P, and $(u, w) \geq_S (u', w')$ if $w_r \geq w_r'$ for all s_r in S.

From now on we concentrate on those stable outcomes where the unsold objects have zero price. We start with the following definition.

Definition 9 *Take (u, w) and (u', w') group stable payoffs. We define $\overline{u}$ and $\underline{w}(\mu)$ as follows:*

(i) for every $p_i \in P$, $\overline{u}_i = \max\{u_i, u'_i\}$.

(ii) for every $s_r \in S$, $\underline{w}_r(\mu) = \sum_{q_j \in Q_r} \left(\min\{v_j, v'_j\}\right)$, *where* v_j *and* v'_j *are, respectively, the compatible prices for* (u, w) *and* (u', w') *for the optimal matching* μ. [5]

Similarly, we define $\underline{u}$ *and* $\overline{w}$.

Proposition 6 *Let* (u, w) *and* (u', w') *be two group stable payoffs. Then, the payoffs* $(\overline{u}, \underline{w}(\mu))$ *and* $(\underline{u}, \overline{w}(\mu))$ *defined for an optimal matching* μ *are group stable.*

Proof. Take the optimal matching μ and the vector of prices $\underline{v}$ such that $\underline{v}_j = \min\{v_j, v'_j\}$, where v and v' are the compatible price vectors for (u, w) and (u', w'), respectively. We prove that $(\overline{u}, \underline{w}(\mu), \underline{v}; \mu)$ is a group stable outcome. First, in (a), we prove that it is a feasible outcome, using the fact that $(u, w, v; \mu)$ and $(u', w', v'; \mu)$ are feasible since they are group stable by Proposition 5. Then, in (b), we prove that the outcome cannot be blocked.

(a) For every (p_i, q_j) such that $\mu(p_i) = q_j$, either $\overline{u}_i = u_i$ or $\overline{u}_i = u'_i$. In the first case, by feasibility of $(u, w, v; \mu)$ and $(u', w', v'; \mu)$, $u_i + v_j = u'_i + v'_j = \alpha_{ij}$. Therefore, $u_i \geq u'_i$ implies that $v_j \leq v'_j$, and we must have $\underline{v}_j = v_j$. Hence, $\overline{u}_i + \underline{v}_j = u_i + v_j = \alpha_{ij}$. The proof for the second case is similar.

(b) For every $s_r \in S$, $\underline{w}_r(\mu) = \sum_{q_j \in Q_r} \left(\min\{v_j, v'_j\}\right) = \sum_{q_j \in Q_r} \underline{v}_j$, by definition. Now we check that there does not exist any coalition that blocks the outcome $(\overline{u}, \underline{w}(\mu), \underline{v}; \mu)$. By Proposition 2, an outcome is group stable if and only if it is pairwise stable and the clients of the same seller are optimally allocated. The second condition holds since $(u, w, v; \mu)$ and $(u', w', v'; \mu)$ are group stable by Proposition 2. Therefore, the only property left to prove is that $(\overline{u}, \underline{w}(\mu), \underline{v}; \mu)$ is pairwise stable. Following Definition 5:

(i) For every (p_i, q_j) with $f(q_j) \neq f(\mu(p_i))$, either

$$\overline{u}_i + \underline{v}_j = \overline{u}_i + v_j \geq u_i + v_j \geq \alpha_{ij},$$

where the last inequality is due to the pairwise stability of $(u, w, v; \mu)$, or

$$\overline{u}_i + \underline{v}_j = \overline{u}_i + v'_j \geq u'_i + v'_j \geq \alpha_{ij},$$

where the last inequality is due to the pairwise stability of $(u', w', v'; \mu)$.

(ii) For every (p_i, q_j) with $f(q_j) = f(\mu(p_i))$, either

$$\alpha_{i\mu(p_i)} + \underline{v}_j = \alpha_{i\mu(p_i)} + v_j \geq \alpha_{ij},$$

where the last inequality is due to the pairwise stability of $(u, w, v; \mu)$, or

$$\alpha_{i\mu(p_i)} + \underline{v}_j = \alpha_{i\mu(p_i)} + v'_j \geq \alpha_{ij},$$

[5] We will prove later (Lemma 1) that this sum of prices coincides for any optimal matching.

where the last inequality is due to the pairwise stability of $(u', w', v'; \mu)$.

Similarly, we can prove the properties for $(\underline{u}, \overline{w}(\mu), \overline{v}; \mu)$. ■

The following lemma shows that the definition of $\underline{w}_r(\mu)$ for any two group stable payoffs does not depend on the matching.

Lemma 1 *Take any two group stable payoffs, (u, w) and (u', w'), and any two optimal matchings, μ and $\widehat{\mu}$. Let v and v' be the respective compatible prices for (u, w) and (u', w') given μ, and let $\widehat{v}$ and $\widehat{v}'$ denote the same for $\widehat{\mu}$. Then, for every seller s_r in S,*

$$\sum_{q_j \in Q_r} \left(\min\left\{v_j, v'_j\right\}\right) = \sum_{q_j \in Q_r} \left(\min\left\{\widehat{v}_j, \widehat{v}'_j\right\}\right)$$

and

$$\sum_{q_j \in Q_r} \left(\max\left\{v_j, v'_j\right\}\right) = \sum_{q_j \in Q_r} \left(\max\left\{\widehat{v}_j, \widehat{v}'_j\right\}\right)$$

Proof. Denote $\underline{w}'_r = \sum_{q_j \in Q_r} \left(\min\left\{v_j, v'_j\right\}\right)$ and $\underline{w}''_r = \sum_{q_j \in Q_r} \left(\min\left\{\widehat{v}_j, \widehat{v}'_j\right\}\right)$. Following the proof of Proposition 6, $(\overline{u}, \underline{w}', v^m; \mu)$ and $(\overline{u}, \underline{w}'', \widehat{v}^m; \widehat{\mu})$ are group stable outcomes, with $v^m = \min\left\{v_j, v'_j\right\}$ and $\widehat{v}^m = \min\left\{\widehat{v}_j, \widehat{v}'_j\right\}$. Also, by Proposition 5, there exists a vector of prices v^* such that $(\overline{u}, \underline{w}', v^*; \widehat{\mu})$ is group stable. But by feasibility of $\widehat{\mu}$, it must be that $\overline{u}_i = \alpha_{i\widehat{\mu}(p_i)} - v^*_{\widehat{\mu}(p_i)}$, for every $p_i \in P$. This implies that $v^*_j = \widehat{v}^m_j$ for every $q_j \in Q$, and therefore, $\underline{w}'_r = \underline{w}''_r$ for every $s_r \in S$.

Similarly, we can prove the property for the maximum prices. ■

Now, for any two group stable payoffs (u, w) and (u', w'), we can properly denote by $\underline{w}_r$ the minimum gain that seller s_r can get at any optimal matching, and by $\overline{u}_i$ the maximum payoff for buyer p_i.

The previous properties allow us to state the main result of the paper in the following theorem.

Theorem 2 *The set of group stable payoffs forms a complete lattice under the partial orders $\geq_P$ and $\geq_S$.*

Proof. By Proposition 6, every two group stable payoffs (u, w) and (u', w'), have a supremum, denoted $(\overline{u}, \underline{w})$, and an infimum, denoted $(\underline{u}, \overline{w})$, under the partial order $\geq_P$, and $(\underline{u}, \overline{w})$ and $(\overline{u}, \underline{w})$, respectively, under the partial order $\geq_S$. This directly proves that the set of group stable payoffs is a lattice under the partial orders $\geq_P$ and $\geq_S$. To prove that it is a complete lattice, we show that this set is convex and compact. By Proposition 5, the set of group stable payoffs is the same for any optimal matching. Let μ be an optimal matching. The set of group stable payoffs is the solution of a system of linear non strict inequalities associated with μ, so it is closed and convex. That it is bounded follows from the fact that for all matched pairs (p_i, q_j) under μ, $0 \leq u_i \leq \alpha_{ij}$ and $0 \leq v_j \leq \alpha_{ij}$. Hence, the set of group stable payoffs is convex and compact, and therefore it forms a complete lattice under the partial order $\geq_P$, dual to the complete lattice with ordering $\geq_S$. ■

Definition 10 *A group stable payoff is called a* ***P-optimal group stable payoff*** *if every player in P weakly prefers it to any other group stable payoff. Similarly, we define an* ***S-optimal group stable payoff.***

Proposition 7 *There exists one and only one P-optimal group stable payoff* (u^*, w_*) *with the property that for any group stable payoff* (u, w), $u^* \geq_P u$ *and* $w_* \leq_S w$. *There exists one and only one S-optimal group stable payoff* (u_*, w^*) *with symmetrical properties.*

Proof. The proof is direct using the fact that every complete lattice has one and only one maximal element and noting the duality of both partial orders $\geq_P$ and $\geq_S$. This duality comes from the definition of $(\overline{u}, \underline{w})$ and $(\underline{u}, \overline{w})$ for any two group stable payoffs (u, w) and (u', w'). ■

Finally, we relate the optimal group stable payoffs for each side of the market in the Generalized Assignment Game with the optimal stable payoffs in the corresponding Assignment Game. At first, it is intuitive to think that the P-optimal group stable payoff for a market M will coincide with the P-optimal stable payoff in the corresponding one-to-one market M', and that this will not be the case, in general, for the S-optimal group stable payoff, since the sellers gain market power in the Generalized Assignment Game. We prove in the following remark that, there exist markets, where neither of the optimal group stable payoffs for each side of the market in the Generalized Assignment Game, coincide with the optimal stable payoff in the corresponding Assignment Game. We think that this can be due to the fact that sellers not only gain market power, but they also lose blocking power when we have markets with more than one seller.

Remark 2 *The P-optimal (S-optimal) group stable payoff for a given market* M *does not coincide, in general, with the P-optimal (S-optimal) stable payoff in the corresponding one-to-one market* M'.

To see that the P-optimal group stable payoff in the Generalized Assignment Game does not necessarily coincide with the P-optimal stable payoff in the corresponding Assignment Game, take the following example:

Example 4:

Let $M \equiv (P, S, Q, f, \alpha)$ with $S = \{s_1, s_2\}$, $Q = \{q_1, q_2, q_3\}$, $Q_1 = \{q_1, q_2\}$, $Q_2 = \{q_3\}$,
$P = \{p_1, p_2, p_3\}$, and $\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} = \begin{pmatrix} 12 & 10 & 4 \\ 10 & 13 & 4 \\ 18 & 12 & 12 \end{pmatrix}$.

In this market the only optimal matching is $\mu(p_1) = q_1$, $\mu(p_2) = q_2$ and $\mu(p_3) = q_3$. The payoff vector $(u, w) = ((6, 10, 12), (9, 0))$ is group stable (it is pairwise stable and the buyers p_1 and p_2 are optimally allocated with s_1), but it is not stable in the one-to-one case, since $u_1 + v_2 < \alpha_{12}$. We can prove that it is not possible to find a payoff vector that gives more or equal utility to all the buyers and that can be sustained as a stable outcome in the one-to-one market. Suppose we leave p_1 and p_3 with the same utility level (note that p_3 could never obtain more) and make p_2 better off. For this, it is necessary to set $v_2' = \alpha_{22} - u_2 < v_2$. But again $u_1' + v_2' = u_1 + v_2' < u_1 + v_2 < \alpha_{12}$, so it is not stable in the one-to-one market. If we try to make p_2 and (or) p_1 better off it is necessary to set $v_1' = \alpha_{11} - u_1 < v_1$, and now $u_3' + v_1' = u_3 + v_1' < u_3 + v_1 = \alpha_{31}$, so they form a blocking coalition and it is not pairwise stable.

To check that the S-optimal group stable payoff also does not coincide with the S-optimal stable payoff in the one-to-one market, see the following example:

Example 5:

Let $M \equiv (P, S, Q, f, \alpha)$ with $S = \{s_1\}$, $Q = Q_1 = \{q_1, q_2\}$, $P = \{p_1\}$, and $\alpha_{11} = 5$, $\alpha_{12} = 4$. For M the set of group stable payoffs is $G = \left\{(u_1, w_1) \in \mathbb{R}_+^2 / u_1 + w_1 = 5\right\}$, and the seller's optimal is $(0, 5)$. But $(0, 5)$ is not a stable payoff in the one-to-one market M' as defined in Section 3. In fact, the set of stable payoffs in M' are all feasible payoffs where $v_1 \in [0, 1]$, and the seller's optimal is $v_1 = 1$.

To end up this section, we comment on the relationship between the set of stable outcomes and the set of competitive equilibria. This relationship has been often studied in matching markets. In particular, for the Assignment Game, the set of (pairwise) stable allocations coincides with the set of competitive equilibria. Moreover, the two extreme allocations (the sellers' optimal stable payoff and the buyers' optimal stable payoff) correspond to the maximum and to the minimum equilibrium prices, respectively. Since in our model we allow each seller to have a set of different objects, we do not have this equivalence between the set of group stable outcomes and the set of competitive equilibria that is usual in matching markets. To study this relationship we briefly define a competitive equilibrium following the definition used for the Assignment Game.

Let $D_i(v)$ denote the *demand set of buyer p_i given a vector of prices $v \in \mathbb{R}_+^n$*, defined as the non empty set of all objects that maximize p_i's utility given v, that is,

$D_i(v) = \{q_j \in Q; \alpha_{ij} - v_j \geq \alpha_{ih} - v_h, \forall q_h \in Q\}$.

The price vector $v \in \mathbb{R}^n_+$ is *competitive* if each buyer can be matched with an object in her demand set, that is, if there exists a feasible matching μ such that $\mu(p_i) \in D_i(v)$ for all p_i in P. Such a matching μ is said to be *competitive* for the prices v.

The pair (v, μ) is a **competitive equilibrium** if v is competitive, μ is competitive for v, and $v_j = 0$ for any unsold object q_j. We call v an **equilibrium price vector**.

Following the result of the Assignment Game that the (pairwise) stable set coincides with the set of competitive equilibria, we can say the following in our Generalized Assignment Game. Given a market M and its corresponding one-to-one market M', to each competitive equilibrium (v, μ) we can associate a (pairwise) stable outcome in M'. Then, by Proposition 3, the set of competitive equilibria is contained in the set of group stable outcomes for a given market M. In particular, by Remark 2, the optimal group stable payoffs for each side of the market are not competitive in general.

6 Concluding Remarks

We have proposed a Generalized Assignment Game, that is, a many-to-one matching market with money. This market is composed by heterogeneous buyers and sellers, where each seller owns a set of *possibly different* objects and each buyer wants to buy, at most, one of the objects. This is a generalization of the case where each seller owns a single indivisible object (Shapley and Shubik [9] Assignment Game). The main difference with the many-to-one models studied in the literature is that the gain that a seller and a buyer can share is not independent on the object bought. The extension also applies to situations like a labor market with multidivisional firms with one (or more) vacancies per division, or the labor market for medical interns where each hospital has a number of different internships to offer.

We propose appropriate definitions of pairwise and group stability for these markets and prove the existence of both sets of outcomes. We concentrate on the group stable set, since it is a more adequate concept of stability for this model, and we study its structure. We prove that the set of group stable payoffs forms a complete lattice with one optimal group stable payoff for each side of the market. We observe a polarization of interests between the two sides of the matching within the set of group stable payoffs. We show that the optimal group stable payoff for each side of the market in the Generalized Assignment Game does not coincide, in general, with the respective optimal payoffs in the Assignment Game. Moreover, the set of competitive equilibria is a proper subset of the set of group stable allocations.

The extension of the model to a many-to-many market, where each buyer can buy more than one object, is a topic subject to further research. The fact that a seller and a buyer can share different gains depending on the object(s) bought remains, but the analysis of the stable set becomes more complicated and it is not a direct generalization.

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