Income Taxation with Habit Formation and Consumption Externalities*

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July 2001

Abstract

We analyze the dynamic behavior and the welfare properties of the equilibrium path of a growth model where both habits and consumption externalities affect the utility of consumers. We discuss the effects of flat rate income taxes and characterize the optimal income taxation policy. We show that, when consumption externalities and habit adjusted consumption are not perfect substitutes, a counter-cyclical income tax rate allows the competitive equilibrium to replicate the efficient path. Our analysis highlights the crucial role played by complementarities between externalities and habits in order to generate an inefficient dynamic equilibrium.

JEL classification codes: D91, E62, O40.

Keywords: Habit Formation, Consumption Externalities, Optimal Taxation.

* Financial support from the Spanish Ministry of Science and Technology through grants SEC99-1094 and SEC2000-0684, and from the Generalitat of Catalonia through grant SGR2000-00054 is gratefully acknowledged.

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1. Introduction

In this paper we analyze the dynamic behavior and the welfare properties of a deterministic endogenous growth model where individual preferences are subjected to a process of habit formation and the average level of consumption of the economy affects individuals' felicity. These two departures from standard specifications of preferences have been introduced in several models in order to account for some empirical phenomena that cannot be explained under more traditional forms of the utility function.

On the one hand, our consumers will form habits so that they will not derive utility from the absolute level of their consumption but from the comparison of the level of current consumption with that in the previous period. The presence of this process of habit formation has qualitative consequences for the dynamic optimization problem faced by consumers since, when they choose their current consumption, they are also selecting a standard of living that will be compared with the level of future consumption. A quite obvious implication brought about by the presence of habits is that individuals will dislike more to experience changes along the consumption path. This is so because, as habits decrease the utility derived from a given amount of current consumption, consumers are less willing to substitute consumption across periods. In other words, the intertemporal elasticity of substitution turns out to be lower in equilibrium. Moreover, since past consumption becomes now a state variable, the dynamic behavior of the economy will be also qualitatively affected by the introduction of habits.

Some of the implications of habit formation for the process of capital accumulation have been discussed in the seminal contribution of Ryder and Heal (1973) and in the more recent papers of Carroll et al. (1997, 2000). Concerning asset pricing models, the decrease in the intertemporal elasticity of substitution due to habit formation is in fact equivalent to an increase in the equilibrium value of the index of relative risk aversion. According to Mehra and Prescott (1985), the empirical difference between average returns to stocks and average returns to Treasury bills would mean that investors are implausibly averse to risk. The higher risk aversion implied by the presence of habits has been used by several authors to generate higher risk premia in equilibrium and, hence, to serve as a potential resolution of that equity premium puzzle.[1]

On the other hand, the consumers' utility will depend on the average level of consumption in the economy. These spillovers from the others' consumption may either increase or decrease the marginal utility of own (habit adjusted) consumption. In the first case, preferences display the typical "keeping up with the Joneses" feature

[1]See, for instance, Abel (1990) and Boldrin et al. (2001). Other asset pricing models, like those of Constantinides (1990), Abel (1999), and Campbell and Cochrane (1999) consider preferences displaying "external" habits, that is, individuals use as a standard of living the past levels of the average consumption instead of the own past consumption. External habits are mathematically easier to treat since individuals disregard the effect of current consumption on their future standard of living. However, these models of external habits also exhibit high risk premia in equilibrium.

since consumption of other individuals makes more valuable a marginal increase of own consumption. Galí (1994) has shown in a stochastic context that, when average consumption exhibits negative externalities (i.e., it decreases the marginal utility of own consumption) the equilibrium risk premium increases. Therefore, negative consumption externalities also provide a potential resolution of the aforementioned equity premium puzzle. This effect on the attitude of individuals towards risk has also implications for the equilibrium value of the intertemporal elasticity of substitution and, thus, on the dynamic behavior of deterministic models of capital accumulation.

The growth model we will use in this paper is a very stylized one. Growth of income per capita will arise from an Ak-type production function as in Rebelo (1991). Under standard preferences, the growth rate of this model displays no transition. This is so because the interest rate is constant and, thus, the rate of consumption growth immediately jumps to its stationary value. However, when habit formation is present, the stock of past consumption at a given period is fixed and, thus, the process of capital accumulation leads to a non-instantaneous adjustment of such a consumption reference. Therefore, in our model transitional dynamics will be exclusively driven by preferences.

As can be easily deduced from our previous discussion, changes in the parameters measuring the strength both of habits and of consumption spillovers will affect the equilibrium value of the intertemporal elasticity of substitution and, thus, the speed at which the economy converges towards its stationary equilibrium. In particular, we will show that both stronger habits and stronger spillovers will reduce the rate of convergence. Several growth models found in the literature, which assume standard inter-temporally independent preferences, exhibit an abnormally high speed of convergence. For instance, Ortigueira and Santos (1997) report that reasonable calibrations of several standard growth models display rates of convergence that lie around 7%. Therefore, the introduction of either habits or consumption spillovers makes the theoretical value of the convergence rate closer to that supported by empirical evidence, which is around 2%.

Consumption externalities constitute an obvious potential source of inefficiency since individuals do not take them into account when they choose their individual consumption paths. In a centralized economy a social planner internalizes those consumption spillovers and the resulting consumption path could not coincide with the competitive one. However, if both the competitive economy and the socially planned economy have balanced growth paths, then the competitive and the socially planned paths of consumption coincide. Thus, consumption externalities turn out to be irrelevant in terms of the welfare properties of the competitive equilibrium. The reason for this irrelevance is that, if there exist competitive and efficient paths for which consumption is growing at a constant rate, then the functional form of the marginal rate of substitution between consumption at different dates of an individual behaving competitively must be the identical to that of the social planner. However, we will see that when we add a process of habit formation to individual preferences, the competitive equilibrium might fail to be efficient. In fact, even if we preserve the existence of competitive and efficient balance growth paths, inefficiencies arise whenever habit adjusted consumption and average consumption enter as not perfect substitutes in the utility function of individuals (like, for instance, in the multiplicative specification of Carroll et al., 1997). In this context we can characterize the income tax rate that allows to implement the socially planned solution. This optimal tax rate turns out to be counter-cyclical since the efficient path exhibits a rate of convergence to the stationary equilibrium that is higher than the competitive one. Moreover, this tax rate tends to zero in the long run since no inefficiencies appear along a balanced growth path.

The plan of the paper is the following. Section 2 presents the endogenous growth model with only consumption spillovers. It is shown that no sub-optimality arises when both the competitive economy and the socially planned economy have a balanced growth path. Section 3 adds to the previous model a simple process of habit formation in consumption and derives the difference equations defining the dynamic competitive equilibrium. In section 4 we analyze the dynamics of the competitive equilibrium around its steady state, while in Section 5 we characterize the short run and the long run effects of changes in the income tax rate and in the preference parameters measuring the importance of habits and spillovers. Sections 6 and 7 replicate the analysis of Sections 3 and 4 for the socially planned economy. Section 8 characterizes the optimal income taxation policy and discusses the role played by the different assumptions of the model in order to obtain such a characterization. Section 9 concludes the paper. Some lengthy proofs are contained in the appendix.

2. Consumption Externalities and Balanced Growth

Let us consider an infinite horizon economy in discrete time. The economy is populated by a continuum of identical dynasties facing also an infinite horizon. All the members of a dynasty are identical. Each dynasty maximizes the discounted sum of instantaneous utilities of one of its representative members. The rate of time discount is $\frac{1-\beta}{\beta}$ with $\beta \in (0,1)$ and the net rate of population growth is $n > -1$. Individual preferences exhibit consumption externalities so that the average consumption in the economy affects the utility of agents as in Galí (1994), Harbaugh (1996), Abel (1999) and Ljungqvist and Uhlig (2000), among many others. Therefore, each dynasty chooses the sequence of per capita consumption $\{c_t\}_{t=0}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t u\left(c_t, \overline{c}_t\right), \tag{2.1}$$

where $\overline{c}_t$ is the consumption per capita of the economy at period t. The utility function u is twice continuously differentiable and satisfies $u_c\left(c, \overline{c}\right) > 0$, $u_{cc}\left(c, \overline{c}\right) < 0$, where the subindexes denote the variable with respect to which the partial derivative is taken. Moreover, the following Inada conditions hold: $\lim_{c \to 0} u_c\left(c, \overline{c}\right) = \infty$ and $\lim_{c \to \infty} u_c\left(c, \overline{c}\right) = 0$, for all $\overline{c} > 0$.

In this economy, there is a firm per dynasty. Following Rebelo (1991), we will assume that the net production function per capita is

$$y_t = Ak_t \quad \text{with } A > 0,$$

where k_t is the capital per capita and y_t is the corresponding net output. This production function can be justified by assuming, for instance, that output can be saved in the form of either physical capital k_p or human capital k_h. Aggregate capital is $k = k_p + k_h$. Both kinds of capital enter in the net production function $F(k_p, k_h)$, which is linearly homogenous, strictly increasing, strictly concave, and satisfies the standard Inada conditions both at the origin and at infinity. In this case, the arbitrage conditions between the two inputs implies that their marginal productivities are equal, $\frac{\partial F(k_p, k_h)}{\partial k_p} = \frac{\partial F(k_p, k_h)}{\partial k_h}$. Since these partial derivatives are homogeneous of degree zero, the previous arbitrage condition can be written as

$$\frac{\partial F\left(1, \frac{k_h}{k_p}\right)}{\partial k_p} = \frac{\partial F\left(1, \frac{k_h}{k_p}\right)}{\partial k_h}. \tag{2.2}$$

Therefore, $F(k_p, k_h) = \hat{A}k_p$ holds in equilibrium, where $\hat{A} = F(1, q)$ and q is the value of the ratio $\frac{k_h}{k_p}$ that solves equation (2.2). Taking into account that $k = (1 + q)k_p$, the production function in equilibrium becomes $F(k_p, k_h) = Ak$ with $A = \frac{\hat{A}}{1+q}$.

The government sets a flat rate tax on net output (or net income). The proceeds from this proportional tax are remitted in a lump-sum fashion to consumers. Hence, the budget constraint of the government is

$$\tau_t A k_t = S_t, \tag{2.3}$$

where τ_t is the income tax rate at time t and S_t is the corresponding lump-sum transfer per capita. The budget constraint of a dynasty is thus

$$c_t = (1 - \tau_t) A k_t + S_t - (1 + n) k_{t+1} + k_t. \tag{2.4}$$

Taking as given the initial capital per capita k_0 and the sequence $\bar{c} = \{\bar{c}_t\}_{t=0}^{\infty}$ of average consumption, each dynasty maximizes (2.1) subject to the budget constraint (2.4). The Lagrangian associated with the problem of the dynasty is the following:

$$L(c, k, \mu) = \sum_{t=0}^{\infty} \left\{\beta^t u(c_t, \bar{c}_t) + \mu_t \left[(1 + (1 - \tau_t) A) k_t - c_t + S_t - (1 + n) k_{t+1}\right]\right\},$$

where $c = \{c_t\}_{t=0}^{\infty}$, $k = \{k_t\}_{t=0}^{\infty}$, and $\mu = \{\mu_t\}_{t=0}^{\infty}$ is the infinite sequence of positive Lagrange multipliers associated with the budget constraint of each period. The first order conditions (or Euler equations) of the dynamic problem are

$$\frac{\partial L}{\partial c_t} = \beta^t u_c(c_t, \bar{c}_t) - \mu_t = 0, \tag{2.5}$$

$$\frac{\partial L}{\partial k_{t+1}} = \mu_{t+1}(1 + (1 - \tau_{t+1}) A) - (1 + n)\mu_t = 0, \tag{2.6}$$

for $t = 0, 1, ...$ The corresponding transversality condition is

$$\lim_{t \to \infty} \mu_t k_{t+1} = 0. \tag{2.7}$$

Combining equations (2.5) and (2.6), and using the fact that in equilibrium $c_t = \bar{c}_t$, we get

$$\frac{u_c(c_{t+1}, c_{t+1})}{u_c(c_t, c_t)} = \frac{1+n}{\beta[1+(1-\tau_{t+1})A]}. \tag{2.8}$$

We can also combine the consumers budget constraint (2.4) and the government budget constraint (2.3) to obtain

$$\frac{k_{t+1}}{k_t} = \left(\frac{1+A}{1+n}\right) - \frac{c_t}{k_t}\left(\frac{1}{1+n}\right). \tag{2.9}$$

Using (2.5), the transversality condition (2.7) can be written as

$$\lim_{t\to\infty} \beta^t u_c(c_t, c_t) k_{t+1} = 0. \tag{2.10}$$

The competitive equilibrium is thus given by the sequence $\{c_t, k_t\}_{t=0}^{\infty}$ satisfying (2.8), (2.9) and the transversality condition (2.10) with the initial capital per capita k_0 exogenously given.

Let us characterize now the solution that a benevolent social planer would implement in this economy. This social planer internalizes the spillovers from average consumption so that he is facing the instantaneous utility function $\hat{u}(c) \equiv u(c,c)$. We will assume that $\hat{u}'(c) > 0$, $\hat{u}''(c) < 0$, and the Inada conditions $\lim_{c\to 0} \hat{u}'(c) = \infty$ and $\lim_{c\to\infty} \hat{u}'(c) = 0$. The resource constraint of the planer's problem is

$$c_t = Ak_t - (1+n)k_{t+1} + k_t, \tag{2.11}$$

which is equivalent to (2.9). Following the same steps as before, it is straightforward to see that the optimality conditions are given by

$$\frac{\hat{u}'(c_{t+1})}{\hat{u}'(c_t)} = \frac{1+n}{\beta(1+A)}, \tag{2.12}$$

the resource constraint (2.9), and the transversality condition

$$\lim_{t\to\infty} \beta^t \hat{u}'(c_t) k_{t+1} = 0. \tag{2.13}$$

The social planner solution is given by the sequence $\{c_t, k_t\}_{t=0}^{\infty}$ satisfying (2.12), (2.9) and the transversality condition (2.13) with k_0 exogenously given. The path chosen by the social planner is also called the efficient path.

At a balanced growth path (BGP) the output per capita grows at a constant rate, which implies that the gross rate of growth of capital $\frac{k_{t+1}}{k_t}$ is constant. Hence, we see from (2.9) that the ratio $\frac{c_t}{k_t}$ is also constant and that both consumption and capital grow at the same rate along a BGP. As is customary in the economic growth literature, we will assume that both the competitive economy and the socially planned economy have a BGP. Regarding the competitive economy, this assumption means that there exists a sequence $\{c_t, k_t\}_{t=0}^{\infty}$ satisfying (2.8), (2.9) and (2.10) along which the variables c_t and k_t grow at constant rates. The existence of a BGP for the socially planned economy means that there exists a sequence $\{c_t, k_t\}_{t=0}^{\infty}$ satisfying

(2.12) , (2.9) and (2.13) along which the variables c_t and k_t grow also at constant rates. Obviously, the BGP's of these two economies are not necessarily equal.

On the one hand, the assumption of existence of a BGP for the competitive economy stems from the fact that, if the tax rate τ_t were time invariant, then the equilibrium path should be consistent with Kaldor's stylized facts. In particular, if the government sets a constant tax rate, the economy should exhibit a constant rate of growth in the long run. On the other hand, the requirement of existence of a BGP for the socially planned economy is usually justified by an argument running in the opposite direction, namely, that tax rates aimed to implement the efficient path should become stationary in the long run.

Note that, if the competitive economy has a BGP, we must impose that $v_1(c) \equiv u_c(c,c)$ be an homogeneous function in order to satisfy the Euler equation (2.8) with τ_t constant when c_t is growing at a constant rate. Similarly, the existence of a BGP for the socially planned economy implies that the function $\hat{u}'(c)$ must be also homogeneous so as to allow the Euler equation (2.12) to hold when c_t is growing at a constant rate. Let us first establish the following technical lemma relating the functions v_1 and $\hat{u}'$ with the function $v_2(c) \equiv u_{\bar{c}}(c,c)$.

Lemma 2.1. *If the functions v_1 and $\hat{u}'$ are both homogeneous and $v_2(c) \neq 0$ for all c, then v_1, v_2 and $\hat{u}'$ are all homogeneous of the same degree.*

Proof. Note that

$$\hat{u}'(c) = u_c(c,c) + u_{\bar{c}}(c,c) = v_1(c) + v_2(c), \tag{2.14}$$

so that, if the functions v_1 and $\hat{u}'$ are homogeneous of degree κ_1 and κ_2, respectively, we have that

$$x^{\kappa_2}\hat{u}'(c) = \hat{u}'(xc) = v_1(xc) + v_2(xc) = \\ x^{\kappa_1}v_1(c) + v_2(xc), \quad \text{for all } x \in \mathbb{R}_{++} \text{ and } c \in \mathbb{R}_{++}$$

Let us proceed by contradiction and assume that $\kappa_1 \neq \kappa_2$. Then, after dividing by x^{κ_2} and rearranging, the previous expression becomes

$$\hat{u}'(c) - x^{\kappa_1-\kappa_2}v_1(c) = \frac{1}{x^{\kappa_2}}v_2(xc), \qquad \text{for all } x \in \mathbb{R}_{++} \text{ and } c \in \mathbb{R}_{++} \tag{2.15}$$

Hence, for any arbitrarily given value $c \in \mathbb{R}_{++}$, there exists a value $x^* \in \mathbb{R}_{++}$ such that $\hat{u}'(c) - (x^*)^{\kappa_1-\kappa_2}v_1(c) = 0$, which in turn implies that $\frac{1}{x^{\kappa_2}}v_2(x^*c) = 0$, and this is impossible by assumption. Thus, $\kappa_1 = \kappa_2$, so that (2.15) becomes

$$\hat{u}'(c) - v_1(c) = \frac{1}{x^{\kappa_1}}v_2(xc), \qquad \text{for all } x \in \mathbb{R}_{++} \text{ and } c \in \mathbb{R}_{++}$$

which combined with (2.14) implies that

$$v_2(xc) = x^{\kappa_1}v_2(c),$$

and this is the desired conclusion. ■

The fact that both v_1 and v_2 are homogenous of the same degree has some surprising implications for the welfare properties of the competitive solution under zero taxes, as the following proposition shows:

Proposition 2.2. *Assume that both the competitive economy and the socially planned economy have a BGP and that the initial capital k_0 is the same for both economies . Then, the paths of consumption and capital $\{c_t, k_t\}_{t=0}^{\infty}$ for the socially planned economy and for the competitive economy with $\tau_t = 0$ coincide.*

Proof. As follows from our previous discussion, the assumption of existence of a BGP for both economies implies that the functions v_1 and $\hat{u}'$ must be homogeneous of the same degree κ in order to satisfy the Euler equations (2.8) and (2.12) along a BGP. Moreover, Lemma 2.1 implies that the function v_2 is also homogeneous of degree κ. Therefore, the ratio $\frac{v_2(c)}{v_1(c)}$ is constant for all c, since for all pairs $(c, c') \in \mathbb{R}^2_{++}$ we have that

$$\frac{v_2(c')}{v_1(c')} = \frac{\left(\frac{c'}{c}\right)^{\kappa} v_2(c)}{\left(\frac{c'}{c}\right)^{\kappa} v_1(c)} = \frac{v_2(c)}{v_1(c)}. \tag{2.16}$$

Let us define the constant $\varsigma = \dfrac{v_2(c)}{v_1(c)}$. Note that $\varsigma > -1$ since $\hat{u}'(c) > 0$, $v_1(c) > 0$ and

$$\hat{u}'(c) = v_1(c) + v_2(c) = (1 + \varsigma) v_1(c). \tag{2.17}$$

By imposing $\tau_t = 0$, we see that the right hand sides of the Euler equations (2.8) and (2.12) are identical. Moreover, their left hand sides have also the same functional form since

$$\frac{\hat{u}'(c_{t+1})}{\hat{u}'(c_t)} = \frac{v_1(c_{t+1}) + v_2(c_{t+1})}{v_1(c_t) + v_2(c_t)} = \frac{(1+\varsigma) v_1(c_{t+1})}{(1+\varsigma) v_1(c_t)} =$$

$$\frac{v_1(c_{t+1})}{v_1(c_t)} = \frac{u_c(c_{t+1}, c_{t+1})}{u_c(c_t, c_t)}.$$

Furthermore, the transversality conditions (2.10) and (2.13) are also equivalent as can be seen from (2.17). Therefore, given the same initial condition on k_0, the path $\{c_t, k_t\}_{t=0}^{\infty}$ that solves the social planner's problem constitutes a competitive equilibrium with no taxes. ■

It should be pointed out that our previous proposition also holds if we had assumed a standard neoclassical net production function per capita $f(k)$ instead of one of the Ak-type. To see this, we just have to observe that the constant A appearing in the right hand sides of (2.12) and (2.8) with $\tau_t = 0$ should be replaced by $f'(k_{t+1})$. Therefore, the only difference between the two Euler equations will be in their left hand sides. Since along a BGP the marginal productivity of capital is constant, the existence of a BGP for both the competitive and the socially planned solution requires again the homogeneity of both v_1 and $\hat{u}'$. Moreover, from Proposition 2.2, this homogeneity condition implies that the left hand sides of equation (2.12) and of equation (2.8) with $\tau_t = 0$ have both the same functional form. Finally, the transversality conditions (2.10) and (2.13) are also equivalent in this case.

We are thus arrived at a paradoxical result according to which the existence of BGP's for the competitive economy and for the socially planned one leads to the efficiency of the competitive accumulation path even if consumption externalities are present. This means that public intervention is not needed in order to implement an efficient path. Note that the assumed homogeneity of both v_1 and v_2 implies that the function $u(c, \bar{c})$ is homothetic with respect to its two arguments along the 45°-degree line, i.e., when $c = \bar{c}$ (see (2.16)). This kind of "restricted homotheticity" constitutes in fact the necessary and sufficient condition discussed in Fisher and Hof (2000) for having a competitive solution identical to its socially planned counterpart when consumption spillovers affect the utility of individuals.

We will modify in the next section our setup by assuming that private consumption is subjected to a process of habit formation. With this modification the existence of BGP's is not longer incompatible with inefficiencies in the capital accumulation process when consumption externalities are present.

3. The Model with Externalities and Habit Formation.

We will introduce in our model the assumption that individuals will not derive utility from their absolute level of consumption at a given period but from the change of consumption with respect to their past experience. Therefore, individuals care about the lagged values of their own consumption, as in the seminal paper of Ryder and Heal (1973) and the models with rational addiction of Becker (1992), Becker and Murphy (1988) and Orphanides and Zervos (1995). In particular, we will assume that the instantaneous utility function of individuals is $u(h_t, \overline{c}_t)$, where $h_t = c_t - \gamma c_{t-1}$ with $\gamma \in (0, 1)$. This means that consumption in the previous period becomes a standard of living that is used to evaluate the utility accruing from current consumption. The parameter γ measures thus how important is the reference set by past consumption. As follows from our discussion in the previous section, we will assume that the partial derivatives of u with respect to its two arguments are homogeneous in order to guarantee the existence of BGP's for the competitive economy and for the socially planned one.

We will use a specification of preferences that involves complementarities between the two arguments of the utility function so that the marginal rate of substitution between average consumption $\overline{c}_t$ and the habit adjusted private consumption h_t will not be constant. We generalize thus the parametrization in Galí (1994), who only considered externalities in consumption, by positing the instantaneous utility function

$$u(h_t, \bar{c}_t) = \frac{(h_t)^{1-\sigma} (\overline{c}_t)^{\theta\sigma}}{1-\sigma}, \qquad \sigma > 0. \tag{3.1}$$

This formulation implies the following properties:

$$u_h(h, \bar{c}) > 0,$$

$$u_{hh}(h, \bar{c}) < 0,$$

$$-\frac{u_{hh}(h, \bar{c}) h}{u_h(h, \bar{c})} = \sigma, \tag{3.2}$$

and

$$\frac{u_{h\bar{c}}(h, \bar{c}) h}{u_h(h, \bar{c})} = \theta\sigma. \tag{3.3}$$

The first three properties are standard. In particular, condition (3.2) would allow the existence of a balanced growth path if the spillover effects of consumption were absent ($\theta = 0$). Condition (3.3) implies that the marginal utility of habit adjusted consumption increases (decreases) with average consumption whenever $\theta > 0$ ($\theta < 0$). Thus, in the case $\theta > 0$ average consumption displays positive externalities and corresponds to the typical "keeping up with the Joneses" formulation since the consumption of other households makes more valuable an additional unit of own (habit adjusted) consumption. In the case $\theta < 0$ average consumption displays negative externalities since the others' consumption lowers the marginal utility of own consumption. We see thus that the consumption externality introduces a scale factor to the marginal utility derived from present consumption (once it has been adjusted by the corresponding past reference).[2] Note also that the concavity of u and the linearity of h_t imply the joint concavity with respect to c_t and c_{t-1} of the function $u\left(c_t - \gamma c_{t-1}, \cdot\right)$, which is the relevant concavity needed to solve the consumer's problem in a competitive economy.

Since we will also analyze the social planner solution, we impose the conditions $\theta < 1$ and $\dfrac{\theta}{1-\sigma} \geq 0$ which guarantee that the utility function perceived by the social planner,

$$\hat{u}\left(c_t, c_{t-1}\right) \equiv u(c_t - \gamma c_{t-1}, c_t) = \frac{\left(c_t - \gamma c_{t-1}\right)^{1-\sigma} \left(c_t\right)^{\theta\sigma}}{1-\sigma}, \tag{3.4}$$

is increasing in c_t and jointly concave with respect to c_t and c_{t-1} (see the appendix).

Note that we use a subtractive form for modelling habit formation instead of the multiplicative form suggested by recent authors like Abel (1990, 1999), Carroll et al. (1997, 2000) and Carroll (2000). Under multiplicative habits the functional form of habit adjusted consumption would be the following:

$$h_t = \frac{c_t}{\left(c_{t-1}\right)^{\gamma}}, \qquad \text{with } \gamma \in (0,1). \tag{3.5}$$

In order to ensure the concavity of the function (3.1) from the social planner's viewpoint when habits are multiplicative, we should restrict the spillover parameter θ to be positive so that negative externalities cannot be examined in this context.[3] Moreover, under the preferences represented by the utility function (3.1) with $\theta > 0$ and habit adjusted consumption satisfying (3.5), only the case $\sigma \geq 1$ yields an interior solution for the competitive consumption path. Obviously, when $\sigma < 1$ solutions involving zero consumption in some (but not all) periods give rise to an

[2] The functional form of u given in (3.1) could be written as

$$u(h_t, \bar{c}_t) = \frac{\left(h_t\right)^{\sigma_1} \left(\bar{c}_t\right)^{\sigma_2}}{\sigma_1}, \quad \sigma_1 < 1.$$

As in Galí (1994), we make $\sigma_1 = 1 - \sigma$ and $\sigma_2 = \theta\sigma$ so that σ can be interpreted as the intertemporal elasticity of substitution of consumption if both habits and consumption spillovers were absent, and θ is the ratio of the elasticities of marginal utility of habit adjusted consumption with respect to average consumption and with respect to habit adjusted consumption (see (3.2) and (3.3)).

[3] The necessary conditions for the joint concavity with respect to c_t and c_{t-1} of the function (3.1) when (3.5) holds are $\theta \leq 1,\ 1 + \gamma(1-\sigma) \leq 0$ and $\gamma + \sigma(1-\gamma) \leq \theta\sigma$. Clearly, the last inequality implies that $\theta > 0$.

infinite value of the discounted sum of utilities from the private viewpoint, since in this case $h_t = \infty$ in some periods and the instantaneous utility function is bounded below. Such a kind of non-interior solution is no longer optimal when $\sigma \geq 1$ since then a period of zero consumption yields an instantaneous utility equal to minus infinity (see Alonso-Carrera et al., 2001). Our formulation avoids this problem at the cost of having a different one, namely, that the term $c_t - \gamma c_{t-1}$ appearing as an argument of u could be negative and, hence, the utility function would not be well defined. This problem is easily solved in our deterministic model by asking for conditions that make the economy exhibit a positive rate of consumption growth.[4] Nevertheless, all the results of our paper hold under multiplicative habits when the corresponding utility function of the social planner is concave and both the competitive and the efficient paths are interior.

Taking as given k_0, c_{-1}, and the sequence $\bar{c} = \{\bar{c}_t\}_{t=0}^{\infty}$ of average consumption, each dynasty chooses the sequence of per capita consumption $\{c_t\}_{t=0}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t u\left(c_t - \gamma c_{t-1}, \overline{c}_t\right),$$

subject to the budget constraint (2.4). The Lagrangian associated with this problem is

$$L\left(c,k,\mu\right) = \sum_{t=0}^{\infty} \left\{\beta^t u\left(h_t, \overline{c}_t\right) + \mu_t \left[\left(1 + \left(1 - \tau_t\right) A\right) k_t - c_t + S_t - \left(1+n\right) k_{t+1}\right]\right\},$$

where again $c = \{c_t\}_{t=-1}^{\infty}$, $k = \{k_t\}_{t=0}^{\infty}$, and $\mu = \{\mu_t\}_{t=0}^{\infty}$ is the infinite sequence of positive Lagrange multipliers. To ease the notation we define $u\left(t\right) = u\left(h_t, \overline{c}_t\right)$ and $u_h\left(t\right) = u_h\left(h_t, \overline{c}_t\right)$. The first order conditions of that problem are thus

$$\frac{\partial L}{\partial c_t} = \beta^t u_h\left(t\right) - \beta^{t+1}\gamma u_h\left(t+1\right) - \mu_t = 0, \tag{3.6}$$

$$\frac{\partial L}{\partial k_{t+1}} = \mu_{t+1}\left(1 + \left(1 - \tau_{t+1}\right) A\right) - \left(1+n\right)\mu_t = 0, \tag{3.7}$$

for $t = 0, 1, ...$ Under the assumptions we have spelled out before, the previous first order conditions (or Euler equations) turn out to be sufficient for characterizing the paths of c_t, k_t, and μ_t when they are combined with the initial conditions on k_0 and c_{-1}, the budget constraint (2.4), and the following transversality conditions:

$$\lim_{t\to\infty} \mu_t k_{t+1} = 0, \tag{3.8}$$

$$\lim_{t\to\infty} \beta^t u_h(t) c_t = 0. \tag{3.9}$$

[4]In stochastic models either of real business cycle or of asset pricing, like those of Constantinides (1990), Allessie and Lusardi (1997), Campbell and Cochrane (1999) and Ljungqvist and Uhlig (2000), standard modelizations of the stochastic processes of either technological shocks or dividends could yield nonpositive equilibrium realizations of the term $c_t - \gamma c_{t-1}$. In this case, the process of habit formation should be specified in such a way that habits be decreasing in consumption when the habit adjusted consumption gets too close to zero.

Combining equations (3.6) and (3.7), we get

$$\frac{\beta^{t+1}u_h(t+1)-\beta^{t+2}\gamma u_h(t+2)}{\beta^t u_h(t)-\beta^{t+1}\gamma u_h(t+1)}=\frac{1+n}{1+(1-\tau_{t+1})A}. \tag{3.10}$$

Note that the previous equation differs from the Euler equation appearing in standard models of capital accumulation in the fact that individuals take into account the effect that present consumption has in setting the reference for next period consumption. Equation (3.10) can be rewritten as

$$\left(\frac{1-\gamma\beta\left(\frac{u_h(t+2)}{u_h(t+1)}\right)}{1-\gamma\beta\left(\frac{u_h(t+1)}{u_h(t)}\right)}\right)\left(\frac{u_h(t+1)}{u_h(t)}\right)=\frac{1}{\beta\varphi_{t+1}}, \tag{3.11}$$

where

$$\varphi_{t+1}=\frac{1+(1-\tau_{t+1})A}{1+n}. \tag{3.12}$$

Since in a symmetric equilibrium $c_t=\bar{c}_t$ for all t, the marginal utilities appearing in the previous expression become in equilibrium

$$u_h(t)=(c_t-\gamma c_{t-1})^{-\sigma}c_t^{\theta\sigma}. \tag{3.13}$$

Let us define the gross rate of growth of the marginal utility of habit adjusted consumption,

$$f_t=\frac{u_h(t+1)}{u_h(t)}, \tag{3.14}$$

so that (3.11) simplifies to

$$\left(\frac{1-\gamma\beta f_{t+1}}{1-\gamma\beta f_t}\right)f_t=\frac{1}{\beta\varphi_{t+1}}.$$

Solving for f_{t+1}, we get

$$f_{t+1}=\frac{1}{\beta\varphi_{t+1}}\left(1-\frac{1}{\beta\gamma f_t}\right)+\frac{1}{\beta\gamma}. \tag{3.15}$$

Using the equilibrium value of the marginal utility in (3.13), we have that

$$f_t=\frac{u_h(t+1)}{u_h(t)}=\frac{(h_{t+1})^{-\sigma}(c_{t+1})^{\theta\sigma}}{(h_t)^{-\sigma}(c_t)^{\theta\sigma}}=$$

$$\frac{\left(\frac{u(t+1)}{h_{t+1}}\right)}{\left(\frac{u(t)}{h_t}\right)}=\left(\frac{u(t+1)}{u(t)}\right)\left(\frac{c_t-\gamma c_{t-1}}{c_{t+1}-\gamma c_t}\right). \tag{3.16}$$

Therefore, the variable f_t can also be interpreted as the gross rate of growth of the average utility of habit adjusted consumption. Equation (3.16) can be rewritten as

$$f_t=\left(\frac{c_t}{c_{t-1}}\right)^{-\sigma}\left(\frac{\left(\frac{c_{t+1}}{c_t}\right)-\gamma}{\left(\frac{c_t}{c_{t-1}}\right)-\gamma}\right)^{-\sigma}\left(\frac{c_{t+1}}{c_t}\right)^{\theta\sigma}. \tag{3.17}$$

Let us define now the gross rate of growth of consumption $x_{t+1} = \frac{c_{t+1}}{c_t}$. Then, (3.17) becomes

$$f_t = (x_t)^{-\sigma} \left(\frac{x_{t+1} - \gamma}{x_t - \gamma} \right)^{-\sigma} (x_{t+1})^{\theta\sigma},$$

which can also be written as

$$g(x_{t+1}, x_t, f_t) \equiv \left(\frac{x_t - \gamma}{x_{t+1} - \gamma} \right) (x_{t+1})^{\theta} - x_t (f_t)^{\frac{1}{\sigma}} = 0. \tag{3.18}$$

Recall that, combining the government and the consumer budget constraints (2.3) and (2.4), we had already obtained the resource constraint (2.9). Defining $z_{t+1} = \frac{k_{t+1}}{c_t}$, equation (2.9) becomes

$$z_{t+1} = \left(\frac{z_t}{x_t} \right) \left(\frac{1 + A}{1 + n} \right) - \left(\frac{1}{1 + n} \right). \tag{3.19}$$

The system of first order difference equations (3.15), (3.18) and (3.19), together with the initial condition $z_0 = \frac{k_0}{c_{-1}}$ and the transversality conditions (3.8) and (3.9), fully describes the equilibrium path of the variables f_t, x_t, and z_t. The system has two control variables, f_t and x_t, and one state variable, z_t.

4. The Dynamics around the Stationary Competitive Equilibrium

Let us assume now that the government follows a stationary tax policy, that is, $\tau_t = \tau$ for all t. Therefore,

$$\varphi_t = \frac{1 + (1 - \tau)A}{1 + n} \equiv \varphi. \tag{4.1}$$

Recall that along a BGP consumption and capital grow at constant rates and, thus, it follows from (2.9) that the ratio $\frac{c_t}{k_t}$ should be constant. Hence, capital, consumption and income per capita must all grow at the same rate along a BGP. Let x be this common stationary rate of growth. From the definition of z_t, if follows that z_t is constant along a BGP. Finally, it is also clear from (3.18) that f_t is also constant along a BGP. Let f and z be the steady state values of f_t and z_t. Making $x_t = x$, $f_t = f$, and $z_t = z$ for all t in the system of equations (3.15), (3.18) and (3.19), and solving for f, x and z, we get the following steady state values of the new variables of the model:[5]

$$f = \frac{1}{\beta\varphi}, \tag{4.2}$$

$$x = f^{\frac{-1}{\sigma(1-\theta)}}, \tag{4.3}$$

and

$$z = \frac{x}{(1 + A) - (1 + n)\, x}. \tag{4.4}$$

[5] The autonomous difference equation (3.15) has in fact two steady states: the one given by (4.2) and another equal to $\frac{1}{\beta\gamma}$. However, the latter steady state violates the positiveness condition on Lagrange multipliers along a BGP (see (4.10)).

It is important to remark that the combinations of the parameter values σ and θ yielding the same value for $\sigma(1-\theta)$ give rise to the same stationary rate of growth (see (4.3)). However, we will see that, even if $\sigma(1-\theta)$ remained constant, the transitional dynamics of the economy would be affected by the particular combination of values taken by the parameters σ and θ. Note that, if we define the stationary intertemporal elasticity of substitution as the elasticity of the stationary rate of growth with respect to the return to capital net of taxes, such an elasticity is given by $\frac{\partial \ln x}{\partial \ln(1+(1-\tau)A)} = \frac{1}{\sigma(1-\theta)}$. As usual, an increase in the parameter σ yields a lower value of the stationary intertemporal elasticity of substitution, while an increase in the parameter θ yields a higher stationary intertemporal elasticity of substitution.[6]

A well defined BGP displaying positive growth requires a series of additional conditions on the parameters of the model, as the following lemma shows:

Lemma 4.1. *If a competitive BGP with $x > 1$ exists, then the following three inequalities must hold:*

$$\beta\varphi > 1, \tag{4.5}$$

$$\beta\varphi^{1-\sigma(1-\theta)} < 1, \tag{4.6}$$

and

$$\frac{1+A}{1+n} > (\beta\varphi)^{\frac{1}{\sigma(1-\theta)}}. \tag{4.7}$$

Proof. Since $\sigma(1-\theta) > 0$, we see from (4.2) and (4.3) that $x > 1$ if and only if $f < 1$, which is in turn equivalent to inequality (4.5).

The transversality condition (3.9) at a BGP requires that $\beta f x < 1$, as dictated by the definitions of f and x. This inequality becomes (4.6), as implied by (4.2) and (4.3).

Finally, as $z > 0$ by definition, we need that

$$\frac{1+A}{1+n} > x, \tag{4.8}$$

as can be seen from (4.4). Using (4.2) and (4.3), inequality (4.8) becomes (4.7). ■

Note then that (4.5), (4.6) and (4.7) are necessary and sufficient conditions for $x > 1$, for the transversality condition (3.9) at a BGP and for $z > 0$, respectively.

Inequalities (4.6) and (4.7) are closely related. On the one hand, it is straightforward to check that inequality (4.7) is obtained whenever both (4.6) and $\tau > 0$ hold. To see this, note that (4.6) is equivalent to $(\beta\varphi)^{\frac{1}{\sigma(1-\theta)}} < \varphi$, and $\varphi \leq \frac{1+A}{1+n}$ when $\tau > 0$. On the other hand, (4.7) implies (4.6) when the tax rate τ takes a value lying in a sufficiently small neighborhood around zero.

We will see next that the conditions established in Lemma 4.1 have a series of implications that confirm that the BGP is well defined. Note first that under

[6] In stochastic models the intertemporal elasticity of substitution is the inverse of the index of relative risk aversion. Galí (1994) shows that negative externalities raise the equilibrium value of the index of relative risk aversion and this leads to a higher equity premium.

(4.5), the term $c_t - \gamma c_{t-1}$ appearing as an argument in the instantaneous utility u is always strictly positive in the long run since $x = \frac{c_t}{c_{t-1}} > 1$ and $\gamma < 1$. Therefore, the objective function of the consumer is well defined at (and around) a competitive BGP.

The first order condition (3.6) along a BGP becomes

$$\mu_t = \beta^t u_h(t) - \beta^{t+1}\gamma u_h(t+1) = \beta^t u_h(t)(1 - \beta\gamma f), \tag{4.9}$$

where the second equality comes from the fact that f is the gross rate of growth of $u_h(t)$ at a BGP. Note that $\mu_t > 0$ if and only if

$$\beta\gamma f < 1. \tag{4.10}$$

Therefore, since f satisfies (4.2), the previous inequality is equivalent to

$$\varphi > \gamma, \tag{4.11}$$

which always holds as inequalities (4.5) and $\beta < 1$ imply that $\varphi > 1$, while $\gamma < 1$ holds by assumption. Note also that since $\mu_t > 0$, the discounted sum of utilities is increasing in the amount of current consumption c_t (see (4.9)). Finally, plugging (4.9) in the transversality condition (3.8) and using the fact that condition (4.6) is equivalent to $\beta f x < 1$, we immediately conclude that the transversality condition (3.8) is also satisfied at a BGP.

We will assume from now on that (4.5), (4.6) and (4.7) hold in order to allow for a well defined BGP displaying sustained growth of income per capita in the long run.

Let us linearize around its steady state the system formed by the difference equations (3.15) with $\varphi_t = \varphi$, (3.18) and (3.19) . From inspection of these equations, this linearized system will be of the following form:

$$\begin{bmatrix} f_{t+1} - f \\ x_{t+1} - x \\ z_{t+1} - z \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ \lambda_{21} & \lambda_2 & 0 \\ 0 & \lambda_{32} & \lambda_3 \end{bmatrix} \begin{bmatrix} f_t - f \\ x_t - x \\ z_t - x \end{bmatrix}. \tag{4.12}$$

The block recursiveness of the 3×3 matrix of partial derivatives appearing in (4.12) implies that the elements along its diagonal coincide with its eigenvalues. Differentiating (3.15), (3.18) and (3.19) we get

$$\lambda_1 = \frac{\partial f_{t+1}}{\partial f_t} = \frac{1}{\beta^2\gamma f^2\varphi} = \frac{\varphi}{\gamma},$$

$$\lambda_2 = \frac{\partial x_{t+1}}{\partial x_t} = -\frac{\frac{\partial g}{\partial x_t}}{\frac{\partial g}{\partial x_{t+1}}} = \frac{\gamma}{x - \theta(x - \gamma)}, \tag{4.13}$$

$$\lambda_3 = \frac{\partial z_{t+1}}{\partial z_t} = \left(\frac{1}{x}\right)\left(\frac{1 + A}{1 + n}\right). \tag{4.14}$$

It follows from (4.11) that $\lambda_1 > 1$. Moreover, since $\theta < 1$, $\gamma \in (0,1)$ and $x > 1$, we have that $\lambda_2 \in (0,1)$. Finally, (4.8) implies that $\lambda_3 > 1$. Therefore, we can immediately conclude that the steady state of the previous system of difference equations is locally saddle path stable.

The difference equation (3.15) governs autonomously the dynamic behavior of the control variable f_t. Hence, because of the instability of its steady state, the control variable f_t immediately jumps to its stationary value and displays no transition. Therefore, we can substitute f_t by f in equation (3.18) and then we can analyze the dynamic behavior of the variables x_t and z_t by just looking at the sub-system formed by equations (3.18) and (3.19) with $f_t = f$. We can then characterize precisely the shape of the local saddle path (or stable manifold) of the previous sub-system in the plane (x_t, z_t). The saddle path of the linearized sub-system will be

$$x_t = \upsilon\,(z_t - z) + x, \tag{4.15}$$

where the scalar υ is such that the vector $(\upsilon, 1)$ is an eigenvector associated with the eigenvalue λ_2 of the sub-matrix

$$\begin{bmatrix} \lambda_2 & 0 \\ \lambda_{32} & \lambda_3 \end{bmatrix}.$$

Therefore, $(\upsilon, 1)$ must be orthogonal to the vector $(\lambda_{32}, \lambda_3 - \lambda_2)$, which means that $\upsilon = \dfrac{\lambda_2 - \lambda_3}{\lambda_{32}}$. Differentiating the right hand side of (3.19) with respect to x_t at a steady state and using (4.14), we get that $\lambda_{32} = \dfrac{\partial z_{t+1}}{\partial x_t} = -\lambda_3\left(\dfrac{z}{x}\right)$. Therefore,

$$\upsilon = \left(1 - \frac{\lambda_2}{\lambda_3}\right)\left(\frac{x}{z}\right) > 0\,, \tag{4.16}$$

where the inequality follows since $\lambda_2 \in (0,1)$ and $\lambda_3 > 1$. This means that the saddle path in the plane (x_t, z_t) is increasing around the steady state. Figure 1 displays the corresponding phase diagram in which the growth rate x_t is a control variable, whereas z_t is a state variable. This saddle path is in fact the policy function that assigns to each value of the state variable z_t the optimal value of the control variable x_t. Such a policy function is thus monotonically increasing.

(Insert Figure 1)

5. Dynamic Effects of Changes in Income Taxation and in Preferences.

The long run effects of changes in the tax rate τ follow immediately from just looking at the expressions for f, x, and z in (4.2), (4.3) and (4.4), and from the fact that φ is strictly decreasing in the tax rate (see (4.1)).

Proposition 5.1. *The stationary rate of growth x is decreasing in the tax rate τ on income. Moreover, f is increasing, while z is decreasing in τ.*

Regarding the short run effects of tax changes on the control variables f_t and x_t, we should first note that the short run effects on f_t coincide with those in the long run since this variable displays no transition. For the marginal effects on the rate of growth x_t, we should look at the locally stable manifold given in (4.15). The following proposition characterizes the short run dynamics:

Proposition 5.2. *Assume that the economy is initially at its steady state and that the tax rate τ suffers an unexpected and permanent marginal increase. Then the growth rate x_t increases in the short run and it converges monotonically towards its new steady state value, which is lower than the initial one. The variable f_t jumps instantaneously up to its new steady state value, whereas the variable z_t moves continuously (without jumps) and monotonically towards its lower steady state value.*

Proof. The effects on f_t and z_t are straightforward. To evaluate the instantaneous effect on the control variable x_t when the economy is initially at its steady state, we just need to perform the differentiation of (4.15) at $t = 0$ when $z_0 = z$ so as to get the following derivative:

$$\frac{\partial x_0}{\partial \tau} = -v\left(\frac{\partial z}{\partial \tau}\right) + \left(\frac{\partial x}{\partial \tau}\right). \tag{5.1}$$

It can be checked from (4.4) and (4.14) that $\frac{\partial z}{\partial \tau} = (1 + A)\left(\frac{z}{x}\right)^2\left(\frac{\partial x}{\partial \tau}\right)$ and $\lambda_3 = 1 + \frac{1}{(1+n)z}$. Hence, after using (4.16), (5.1) becomes

$$\frac{\partial x_0}{\partial \tau} = (1+n)\, z\, (\lambda_2 - 1)\left(\frac{\partial x}{\partial \tau}\right). \tag{5.2}$$

Therefore, $\frac{\partial x_0}{\partial \tau} > 0$ since $\frac{\partial x}{\partial \tau} < 0$ and $\lambda_2 \in (0, 1)$. ■

The previous proposition tells us that the short run and the long run effects on the rate of growth of a marginal change in the tax rate are opposite. The phase diagram in Figure 2 shows both effects for a marginal increase in the tax rate τ. The original steady state was at point A, whereas the steady state associated with the higher tax rate is at point C. The transition involves an instantaneous jump to point B.

(Insert Figure 2)

It is worth to emphasize that a model without habit formation, i.e., with $\gamma = 0$, displays no transition after a change in the tax rate. This can be seen by noticing that the expression (4.13) for the eigenvalue λ_2 becomes equal to zero in this case. In fact, equation (3.18) becomes simply $x_{t+1} = (f_t)^{\frac{-1}{\sigma(1-\theta)}}$ so that the lack of transition of f_t is immediately inherited by the rate of growth x_t.

We can also ask ourselves how the speed of convergence to the steady state is affected by changes in the structural parameters of the model. Throughout our analysis of the dynamic behavior of the economy, we will keep constant the stationary value $\sigma(1-\theta)$ corresponding to the inverse of the intertemporal elasticity of substitution in a long run equilibrium. Therefore, the changes in the spillover parameter θ will be accompanied with changes in σ that will leave unchanged the long run values of the variables x_t, f_t and z_t. The evolution of the state variable z_t around the steady state can be approximated by the solution to the linearization of the system (3.15), (3.18) and (3.19). Therefore, the local behavior of the state variable z_t around z is given by:

$$z_t = (z_0 - z)(\lambda_2)^t + z. \tag{5.3}$$

The speed of convergence is inversely related to the eigenvalue λ_2 of the matrix associated with the linearized dynamic system. Since (5.3) implies that

$$1 - \lambda_2 = \frac{z_{t+1} - z_t}{z - z_t},$$

we see that $1-\lambda_2$ measures the fraction of the gap between the current value of the state variable and its stationary value which is closed in one period.

Proposition 5.3. *Assume that the stationary growth rate x is fixed. The speed of convergence is decreasing both in the parameter γ measuring the intensity of habit formation and in the parameter θ measuring the intensity of consumption externalities.*

Proof. From (4.13) we can compute the following two partial derivatives:

$$\frac{\partial(\ln\lambda_2)}{\partial\gamma} = \frac{(1-\theta)\lambda_2 x}{\gamma^2} > 0,$$

$$\left.\frac{\partial(\ln\lambda_2)}{\partial\theta}\right|_{\sigma(1-\theta)=\text{ constant}} = \frac{\lambda_2(x-\gamma)}{\gamma} > 0. \tag{5.4}$$

Hence, the speed of convergence decreases with both γ and θ for a given stationary growth rate. ■

The fact that the speed of convergence decreases as the value of the parameter γ increases is a quite intuitive result since, as past consumption becomes more important, individuals face in the short run a utility function that is more concave. To see this, note that we can view the habit stock at a given period as fixed, so that the standard measure of concavity of u with respect to present consumption is given by the index of relative risk aversion,

$$-\frac{c_t u_{c_t c_t}(c_t - \gamma c_{t-1}, \bar{c}_t)}{u_{c_t}(c_t - \gamma c_{t-1}, \bar{c}_t)} = -\frac{c_t u_{h_t h_t}(h_t, \bar{c}_t)}{u_{h_t}(h_t, \bar{c}_t)} = \frac{\sigma c_t}{c_t - \gamma c_{t-1}}, \tag{5.5}$$

which is increasing in γ. Therefore, individuals dislike more to experience changes in consumption along the equilibrium path, and this results in a lower speed of adjustment. It also should be pointed out in this respect that v is clearly decreasing in γ (see (4.16)), which means that the policy function (4.15) becomes "flatter" as habits become more important. Since consumption growth is less sensitive to changes in the state variable, a flatter policy function directly results in a lower speed of convergence.

The speed of convergence is also decreasing in θ. Clearly, for a constant rate of long run growth, as the value of the parameter θ increases, the value of the parameter σ must also increase so as to keep $\sigma(1-\theta)$ constant. Therefore, the concavity of u increases with σ (see (5.5)), which implies in turn that the adjustment of the economy takes place at a lower speed since, again, consumers are less willing to substitute consumption across periods.

As a by-product of Proposition 5.3, we can characterize how the short run response of the economy to changes in the tax rate is affected by the preference parameters. From (5.2) , we can compute

$$\frac{\partial x_0}{\partial \tau \partial \gamma} = (1+n)\, z \left(\frac{\partial x}{\partial \tau} \right) \left(\frac{\partial \lambda_2}{\partial \gamma} \right) < 0,$$

and

$$\left. \frac{\partial x_0}{\partial \tau \partial \theta} \right|_{\sigma(1-\theta)= \text{ constant}} = (1+n)\, z \left(\frac{\partial x}{\partial \tau} \right) \left(\frac{\partial \lambda_2}{\partial \theta} \right) < 0.$$

Therefore, the growth rate of consumption is less sensible in the short run to unanticipated permanent changes in the tax rate τ when the parameters γ and θ exhibit higher values. This is again a direct consequence of the increasing sluggishness of the consumption policy triggered by either stronger habits or stronger spillovers.

6. The Socially Planned Solution

In this section we are going to characterize the solution that a time-consistent social planner would implement. This planner maximizes the same objective function as the individuals but he internalizes the spillovers from average consumption. Moreover, the planner is just facing the aggregate resource constraint per capita (2.9) .

Therefore, we can write the following Lagrangian for the social planner's problem:

$$\hat{L}(c,k,\hat{\mu}) = \sum_{t=0}^{\infty} \beta^t u\left(c_t - \gamma c_{t-1}, c_t\right) + \hat{\mu}_t \left((1+A)\, k_t - c_t - (1+n)\, k_{t+1}\right),$$

where $\hat{\mu} = \{\hat{\mu}_t\}_{t=0}^{\infty}$ is the infinite sequence of positive Lagrange multipliers. According to (3.4), let us define

$$\hat{u}\,(t) \equiv \hat{u}\left(c_t, c_{t-1}\right) = u\left(c_t - \gamma c_{t-1}, c_t\right),$$

$\hat{u}_1\,(t) = \dfrac{\partial \hat{u}\left(c_t, c_{t-1}\right)}{\partial c_t}$ and $\hat{u}_2\,(t) = \dfrac{\partial \hat{u}\left(c_t, c_{t-1}\right)}{\partial c_{t-1}}$. The first order conditions for the social planner problem are thus

$$\frac{\partial \hat{L}}{\partial c_t} = \beta^t \hat{u}_1\,(t) + \beta^{t+1} \hat{u}_2\,(t+1) - \hat{\mu}_t = 0, \tag{6.1}$$

$$\frac{\partial \hat{L}}{\partial k_{t+1}} = (1+A)\, \hat{\mu}_{t+1} - (1+n)\, \hat{\mu}_t = 0, \tag{6.2}$$

for $t=0,1,...$ The previous Euler equations combined with the resource constraint (2.9), the transversality conditions

$$\lim_{t\to\infty}\hat{\mu}_t k_{t+1}=0, \tag{6.3}$$

$$\lim_{t\to\infty}\beta^t\hat{u}_1(t)c_t=0, \tag{6.4}$$

and the initial conditions on k_0 and c_{-1} fully characterize the paths of c_t, k_t, and $\hat{\mu}_t$ that solve the planner's problem.

Combining equations (6.1) and (6.2), we obtain

$$\frac{\beta^{t+1}\hat{u}_1(t+1)+\beta^{t+2}\hat{u}_2(t+2)}{\beta^t\hat{u}_1(t)+\beta^{t+1}\hat{u}_2(t+1)}=\frac{1}{\hat{\varphi}}, \tag{6.5}$$

where $\hat{\varphi}=\dfrac{1+A}{1+n}$. Equation (6.5) simplifies to

$$\frac{\hat{u}_1(t+1)+\beta\hat{u}_2(t+2)}{\hat{u}_1(t)+\beta\hat{u}_2(t+1)}=\frac{1}{\beta\hat{\varphi}}. \tag{6.6}$$

It can be shown that

$$\hat{u}_1(t)=\hat{u}(t)\left(\frac{(1-\sigma+\theta\sigma)c_t-\gamma\theta\sigma c_{t-1}}{c_t(c_t-\gamma c_{t-1})}\right), \tag{6.7}$$

and

$$\hat{u}_2(t)=-\hat{u}(t)\left(\frac{(1-\sigma)\gamma}{c_t-\gamma c_{t-1}}\right). \tag{6.8}$$

Plugging (6.7) and (6.8) in (6.6), we obtain

$$\frac{\hat{u}(t+1)\left[\dfrac{(1-\sigma+\theta\sigma)c_{t+1}-\gamma\theta\sigma c_t}{c_{t+1}(c_{t+1}-\gamma c_t)}\right]-\beta\hat{u}(t+2)\left[\dfrac{(1-\sigma)\gamma}{c_{t+2}-\gamma c_{t+1}}\right]}{\hat{u}(t)\left[\dfrac{(1-\sigma+\theta\sigma)c_t-\gamma\theta\sigma c_{t-1}}{c_t(c_t-\gamma c_{t-1})}\right]-\beta\hat{u}(t+1)\left[\dfrac{(1-\sigma)\gamma}{c_{t+1}-\gamma c_t}\right]}=\frac{1}{\beta\hat{\varphi}},$$

which can be rewritten as

$$\left(\frac{\dfrac{1-\sigma+\theta\sigma}{\beta(1-\sigma)\gamma}-\dfrac{\theta\sigma c_t}{\beta(1-\sigma)c_{t+1}}-\dfrac{\hat{u}(t+2)}{\hat{u}(t+1)}\left(\dfrac{c_{t+1}-\gamma c_t}{c_{t+2}-\gamma c_{t+1}}\right)}{\dfrac{1-\sigma+\theta\sigma}{\beta(1-\sigma)\gamma}-\dfrac{\theta\sigma c_{t-1}}{\beta(1-\sigma)c_t}-\dfrac{\hat{u}(t+1)}{\hat{u}(t)}\left(\dfrac{c_t-\gamma c_{t-1}}{c_{t+1}-\gamma c_t}\right)}\right)\times$$

$$\left(\frac{\hat{u}(t+1)}{\hat{u}(t)}\right)\left(\frac{c_t-\gamma c_{t-1}}{c_{t+1}-\gamma c_t}\right)=\frac{1}{\beta\hat{\varphi}}. \tag{6.9}$$

Let us now define the gross rate of growth of the average utility of habit adjusted consumption,

$$\hat{f}_t=\frac{\left(\frac{\hat{u}(t+1)}{h_{t+1}}\right)}{\left(\frac{\hat{u}(t)}{h_t}\right)}=\left(\frac{\hat{u}(t+1)}{\hat{u}(t)}\right)\left(\frac{c_t-\gamma c_{t-1}}{c_{t+1}-\gamma c_t}\right). \tag{6.10}$$

Note that the definition of $\hat{f}_t$ coincides with that of f_t given in (3.16) for the competitive economy. However, while f_t was also the gross rate of growth of the marginal utility of consumption in the competitive economy (see (3.14)), $\hat{f}_t$ is not necessarily equal to the gross rate of growth of the marginal utility of consumption in the socially planned economy, as can be seen after computing $\frac{\hat{u}_1(t+1)}{\hat{u}_1(t)}$ from (6.7).[7]

Let $\hat{x}_t = \frac{c_t}{c_{t-1}}$ be the gross rate of consumption growth of the social planner's solution. Therefore, (6.9) becomes

$$\left(\frac{\varepsilon - \dfrac{\eta}{\hat{x}_{t+1}} - \hat{f}_{t+1}}{\varepsilon - \dfrac{\eta}{\hat{x}_t} - \hat{f}_t} \right) \hat{f}_t = \frac{1}{\beta \hat{\varphi}}, \tag{6.11}$$

where

$$\varepsilon = \frac{1 - \sigma + \theta\sigma}{\beta\,(1-\sigma)\,\gamma} \quad \text{and} \quad \eta = \frac{\theta\sigma}{\beta\,(1-\sigma)} \tag{6.12}$$

Note that ε and η are both non-negative since the joint concavity of $\hat{u}\,(c_t, c_{t-1})$ with respect to its two arguments requires that $\dfrac{\theta}{1-\sigma} \geq 0$ (see the appendix). We can rewrite (6.11) as

$$\hat{f}_{t+1} = \varepsilon - \frac{\eta}{\hat{x}_{t+1}} + \frac{1}{\beta\varphi} + \left(\frac{1}{\hat{f}_t \beta \hat{\varphi}} \right) \left(\frac{\eta}{\hat{x}_t} - \varepsilon \right) \equiv M\left(\hat{x}_{t+1}, \hat{x}_t, \hat{f}_t \right). \tag{6.13}$$

Next, combining the definition of $\hat{f}_t$ in (6.10) with that of $\hat{x}_t$, we immediately get

$$\hat{f}_t = (\hat{x}_t)^{-\sigma} \left(\frac{\hat{x}_t - \gamma}{\hat{x}_{t+1} - \gamma} \right)^{\sigma} (\hat{x}_{t+1})^{\theta\sigma}.$$

We can also write the previous equation as

$$g\left(\hat{x}_{t+1}, \hat{x}_t, \hat{f}_t \right) \equiv \left(\frac{\hat{x}_t - \gamma}{\hat{x}_{t+1} - \gamma} \right) (\hat{x}_{t+1})^{\theta} - \hat{x}_t \left(\hat{f}_t \right)^{\frac{1}{\sigma}} = 0, \tag{6.14}$$

which is equivalent to the difference equation (3.18) characterizing the competitive equilibrium.

Finally, from the resource constraint (2.9), we obtain,

$$\hat{z}_{t+1} = \left(\frac{\hat{z}_t}{\hat{x}_t} \right) \left(\frac{1+A}{1+n} \right) - \left(\frac{1}{1+n} \right), \tag{6.15}$$

where $\hat{z}_t = \frac{k_t}{c_{t-1}}$.

The difference equations (6.13), (6.14), and (6.15), together with the initial condition $z_0 = \frac{k_0}{c_{-1}}$ and the transversality conditions (6.3) and (6.4) fully characterize the dynamics of the variables $\hat{f}_t$, $\hat{x}_t$, and $\hat{z}_t$.

[7] We will see later on that f_t and $\hat{f}_t$ only coincide at a BGP.

7. The Dynamics around the Stationary Socially Planned Path.

In order to find the steady state of the dynamic system formed by the difference equations we have just described at the end of the previous section, we evaluate these equations at $\hat{x}_t = \hat{x}$, $\hat{f}_t = \hat{f}$, and $\hat{z}_t = \hat{z}$ for all t, and solve the corresponding system of equations to obtain

$$\hat{f} = \frac{1}{\beta \hat{\varphi}}, \tag{7.1}$$

$$\hat{x} = \hat{f}^{\frac{-1}{\sigma(1-\theta)}}, \tag{7.2}$$

and

$$\hat{z} = \frac{\hat{x}}{(1+A) - (1+n)\,\hat{x}}. \tag{7.3}$$

We see from looking at (4.2), (4.3) and (4.4) , that $\hat{f} = f$, $\hat{x} = x$ and $\hat{z} = z$ whenever $\hat{\varphi} = \varphi$, that is, when $\tau_t = 0$. Therefore, when the tax rate on income is zero, no differences arise at the steady state between the competitive and the efficient solution.

As we did for the competitive solution, we are going to introduce some additional assumptions that are necessary for the existence of a socially planned BGP displaying positive growth.

Lemma 7.1. *If a socially planned BGP with $\hat{x} > 1$ exists, then the following two inequalities must hold:*

$$\beta \hat{\varphi} > 1, \tag{7.4}$$

$$\beta \hat{\varphi}^{1-\sigma(1-\theta)} < 1. \tag{7.5}$$

Proof. As follows from (7.1) and (7.2), the stationary solution to the planner's problem exhibits positive sustained growth, $\hat{x} > 1$, if and only if (7.4) holds.

In order to check whether the transversality condition (6.4) holds, we should observe first that equation (6.7) evaluated at a BGP becomes

$$\hat{u}_1(t) = \frac{\hat{u}(t)}{c_t} \left(\frac{(1-\sigma+\theta\sigma)\,\hat{x} - \gamma\theta\sigma}{\hat{x} - \gamma} \right).$$

Therefore,

$$\frac{\hat{u}_1(t+1)}{\hat{u}_1(t)} = \left(\frac{\hat{u}(t+1)}{\hat{u}(t)} \right) \left(\frac{1}{\hat{x}} \right) = \hat{f},$$

where the last equality comes immediately from (6.10). Hence, we can then conclude that the transversality condition (6.4) at a BGP requires that $\beta \hat{f} \hat{x} < 1$. As follows from (7.1) and (7.2), this inequality is equivalent to (7.5). ■

Let us point out that the two conditions (7.4) and (7.5) needed to obtain a socially planned solution displaying positive growth in the long run are the exact counterparts of conditions (4.5) and (4.6), which were imposed for the competitive economy. Moreover, (7.4) and (7.5) constitute necessary and sufficient conditions for $\hat{x} > 1$ and for the transversality condition (6.4) at a BGP, respectively. We will next show some implications of these two conditions, which confirm that the efficient BGP is well defined.

Note first that, since (7.4) implies that $\hat{x} > 1$, the term $c_t - \gamma c_{t-1}$ is strictly positive so that the instantaneous utility function is well defined at (and around) an efficient BGP. Observe also that (7.3) requires that

$$\frac{1+A}{1+n} > \hat{x}, \tag{7.6}$$

as $\hat{z} > 0$. From (7.1) and (7.2), inequality (7.6) becomes

$$\frac{1+A}{1+n} > (\beta\hat{\varphi})^{\frac{1}{\sigma(1-\theta)}}, \tag{7.7}$$

which is equivalent to the condition (4.7) imposed for the competitive economy. It is straightforward to check that inequality (7.7) holds whenever (7.5) is assumed.

Note also that $\hat{u}_1(t) > 0$ (the instantaneous utility function faced by the social planner is increasing in current consumption) since (6.7) is positive if and only if

$$\left(1 + \frac{\theta\sigma}{1-\sigma}\right) c_t > \frac{\gamma\theta\sigma}{1-\sigma} c_{t-1}. \tag{7.8}$$

Since $c_t - \gamma c_{t-1} > 0$ and the concavity of $\hat{u}$ implies that $\frac{\theta}{1-\sigma} \geq 0$, inequality (7.8) automatically holds. It is also obvious that, under the same conditions, (6.8) implies that $\hat{u}_2(t) < 0$.

Moreover, using (6.7) and (6.8) we obtain

$$\frac{\hat{u}_2(t+1)}{\hat{u}_1(t)} = -\hat{f}_t\left(\frac{(1-\sigma)\gamma c_t}{(1-\sigma+\theta\sigma)c_t - \gamma\theta\sigma c_{t-1}}\right),$$

which at a BGP becomes

$$\frac{\hat{u}_2(t+1)}{\hat{u}_1(t)} = -\frac{\hat{f}}{\beta}\left(\frac{\hat{x}}{\varepsilon\hat{x} - \eta}\right).$$

Hence, the first order condition (6.1) becomes at a BGP

$$\hat{\mu}_t = \beta^t \hat{u}_1(t) + \beta^{t+1}\hat{u}_2(t+1) = \beta^t \hat{u}_1(t)\left[1 - \hat{f}\left(\frac{\hat{x}}{\varepsilon\hat{x} - \eta}\right)\right]. \tag{7.9}$$

We see that $\hat{\mu}_t > 0$ if and only if

$$\hat{f}\left(\frac{\hat{x}}{\varepsilon\hat{x} - \eta}\right) < 1. \tag{7.10}$$

From (7.1) and the definitions of $\hat{\varphi}$, ε, and η, inequality (7.10) becomes

$$\frac{\hat{x}\gamma}{\hat{x} + \left[\frac{\theta\sigma(\hat{x}-\gamma)}{1-\sigma}\right]} < \varphi, \tag{7.11}$$

and, since $\hat{x}-\gamma>0$ and $\frac{\theta}{1-\sigma}\geq 0$, we can conclude that the denominator of the left hand side of inequality (7.11) is positive. Therefore, we can rewrite (7.11) as

$$\hat{x}\left(\hat{\varphi}-\gamma\right)+\frac{\varphi\theta\left(\hat{x}-\gamma\right)\sigma}{\left(1-\sigma\right)}>0. \tag{7.12}$$

The previous inequality always holds under the set of assumption we are making. To see this, note that $\hat{x}>1$ is equivalent to $\beta\hat{\varphi}>1$, which in turn implies that $\hat{\varphi}>1$ and, hence, $\hat{\varphi}-\gamma>0$ and $\hat{x}-\gamma>0$. Moreover, because of the concavity of $\hat{u}$, we have $\frac{\theta}{1-\sigma}\geq 0$. The last three inequalities readily imply that (7.12) holds. Therefore, since $\hat{\mu}_t>0$, the discounted sum of utilities faced by the social planner is increasing in the amount of current consumption c_t (see (7.9)).

Finally, plugging inequality (7.9) in the transversality condition (6.3) and using the fact that (7.5) is equivalent to $\beta\hat{f}\hat{x}<1$, we can immediately conclude that the transversality condition (6.3) is satisfied at a BGP.

We will assume from now on that conditions (7.4) and (7.5) hold in order to allow for a well defined efficient BGP displaying a positive rate of growth.

In order to study the local dynamics of the optimal path selected by the planner, we should linearize the system of difference equations (6.13), (6.14) and (6.15) around its steady state. Such a linearized dynamic system has the form

$$\begin{bmatrix}\hat{f}_{t+1}-\hat{f}\\ \hat{x}_{t+1}-\hat{x}\\ \hat{z}_{t+1}-\hat{z}\end{bmatrix}=\begin{bmatrix}\hat{\lambda}_{11} & \hat{\lambda}_{12} & 0\\ \hat{\lambda}_{21} & \hat{\lambda}_{22} & 0\\ 0 & \hat{\lambda}_{32} & \hat{\lambda}_{33}\end{bmatrix}\begin{bmatrix}\hat{f}_{t}-\hat{f}\\ \hat{x}_{t}-\hat{x}\\ \hat{z}_{t}-\hat{z}\end{bmatrix}. \tag{7.13}$$

Lemma 7.2. *The stationary equilibrium of the dynamic system (7.13) is saddle path stable.*

Proof. See the appendix. ■

As shown in the proof of this lemma, the dynamic system (7.13) has three eigenvalues satisfying $\hat{\lambda}_1>1$, $\hat{\lambda}_2\in(0,1)$ and $\hat{\lambda}_3>1$. Thus, we can write the stable solution path of the linearized system as

$$\hat{f}_t=A_1\hat{\lambda}_2^t+\hat{f},$$

$$\hat{x}_t=A_2\hat{\lambda}_2^t+\hat{x}$$

$$\hat{z}_t=A_3\hat{\lambda}_2^t+\hat{z}. \tag{7.14}$$

with $A_3=(\hat{z}_0-\hat{z})$. Hence,

$$\hat{f}_t=k_1\left(\hat{z}_0-\hat{z}\right)\hat{\lambda}_2^t+\hat{f}, \tag{7.15}$$

$$\hat{x}_t=k_2\left(\hat{z}_0-\hat{z}\right)\hat{\lambda}_2^t+\hat{x}, \tag{7.16}$$

where the vector $(k_1, k_2, 1)$ is the eigenvector associated with the eigenvalue $\hat{\lambda}_2$ of the matrix of partial derivatives appearing in (7.13) . Therefore,

$$k_2 = \frac{\hat{\lambda}_2 - \hat{\lambda}_{33}}{\hat{\lambda}_{32}} > 0, \tag{7.17}$$

where the inequality arises since $\hat{\lambda}_{32} < 0$ and $\hat{\lambda}_{33} > 1$ (see (A.9) and (A.11) in the appendix).

We can make $k_1 = k_2 \hat{v}$, where $(\hat{v}, 1)$ is an eigenvector associated with the eigenvalue $\hat{\lambda}_2$ of the sub-system formed by the linearization of (6.14) and (6.15). Therefore,

$$\hat{v} = \frac{\hat{\lambda}_2 - \hat{\lambda}_{22}}{\hat{\lambda}_{21}} = \frac{\hat{\lambda}_2 - \left[\frac{\gamma}{\hat{x} - \theta(\hat{x} - \gamma)}\right]}{\hat{\lambda}_{21}}. \tag{7.18}$$

In order to find the sign of $\hat{v}$ we should observe that the denominator of the previous expression is negative (see (A.12) in the appendix). Concerning the sign of the numerator, we just have to compute

$$\left(\hat{\lambda}_1 - \frac{\gamma}{\hat{x} - \theta(\hat{x} - \gamma)}\right)\left(\hat{\lambda}_2 - \frac{\gamma}{\hat{x} - \theta(\hat{x} - \gamma)}\right) =$$

$$\hat{\lambda}_1 \hat{\lambda}_2 + \left(\frac{\gamma}{\hat{x} - \theta(\hat{x} - \gamma)}\right)\left(\frac{\gamma}{\hat{x} - \theta(\hat{x} - \gamma)} - \left(\hat{\lambda}_1 + \hat{\lambda}_2\right)\right) =$$

$$-\left(\frac{\varphi}{\hat{x}}\right)\left(\frac{\theta(1-\theta)(\hat{x} - \gamma)^2}{(1-\sigma)(\hat{x} - \theta(\hat{x} - \gamma))^2}\right) < 0,$$

where the second equality comes from some simplifications after using the expressions for $\hat{\lambda}_1 \hat{\lambda}_2$ and $\hat{\lambda}_1 + \hat{\lambda}_2$, which can be found in equations (A.13) and (A.14) of the appendix. The final inequality follows from the concavity condition $\frac{\theta}{1-\sigma} \geq 0$ and the inequalities $\hat{x} - \gamma > 0$ and $\theta < 1$. Thus, the eigenvalues $\hat{\lambda}_1$ and $\hat{\lambda}_2$ satisfy

$$\hat{\lambda}_1 > \frac{\gamma}{\hat{x} - \theta(\hat{x} - \gamma)} > \hat{\lambda}_2, \tag{7.19}$$

and this means that $\hat{v} > 0$ and, hence, that $k_1 = k_2 \hat{v} > 0$ (see (7.17) and (7.18)). Therefore, there is a positive relationship between the state variable z_t and the control variables f_t and x_t along the saddle path around the stationary solution to the planner's problem. In particular, we obtain from (7.14) and (7.16) the following linearized policy function for the growth rate of consumption:

$$\hat{x}_t = k_2 (\hat{z}_t - \hat{z}) + \hat{x}, \tag{7.20}$$

with $k_2 > 0$ (see (7.17)). Note also that the variable f_t did not display transition in the competitive equilibrium, whereas it does in the planner's solution as can be seen from (7.15).

In the next proposition we compare the rate of convergence of the competitive economy with that of the socially planned economy:

Proposition 7.3. *The rate of convergence of the competitive economy with no taxes is lower than that of the socially planned economy.*

Proof. Recall from (4.13) that $\lambda_2 = \dfrac{\gamma}{\hat{x} - \theta(\hat{x} - \gamma)}$, since $\hat{x} = x$ when $\tau = 0$. Therefore, since $\lambda_2 > \hat{\lambda}_2$ (see (7.19)), the competitive economy converges towards its steady state at a slower speed than its socially planned counterpart. ■

The intuition behind Proposition 7.3 lies in the fact that, if the growth rate is initially above its steady state value, then the marginal rate of substitution (MRS) in the socially planned solution is smaller than the MRS faced by the consumers along the competitive solution. Because of the saddle path stability of the steady states of both economies, the previous fact can be reformulated by saying that, if the growth rate is decreasing in time then the MRS of the social planner is smaller than the MRS associated with the competitive equilibrium. A smaller value of the MRS means that the corresponding decision makers are less willing to substitute present consumption by future consumption, and this in turn implies that they dislike more to follow consumption paths exhibiting high rates of growth. Obviously, if the initial rate of growth is higher than the stationary one and the MRS of the social planner is lower than that of the consumers in the competitive economy, the speed of convergence in the socially planned economy will be higher than in the competitive economy, since the planner is more willing to reduce immediately the rate of consumption growth. The argument is symmetric for the case where the initial growth rate is below its steady state.

To see that a decreasing path of growth rates is associated with a socially planned MRS that is smaller than the competitive MRS, we assume that

$$x_{t+2} < x_{t+1}, \tag{7.21}$$

and check whether the following inequality holds:

$$\frac{u_h(t+1) - \beta\gamma u_h(t+2)}{u_h(t) - \beta\gamma u_h(t+1)} > \frac{u_h(t+1) + u_{\overline{c}}(t+1) - \beta\gamma u_h(t+2)}{u_h(t) + u_{\overline{c}}(t) - \beta\gamma u_h(t+1)}. \tag{7.22}$$

From the Euler equations (3.10) and (6.6), and since $\hat{u}_1(t) = u_h(t) + u_{\overline{c}}(t)$ and $\hat{u}_2(t+1) = -\gamma u_h(t+1)$, we see that the left-hand side of the previous inequality is the MRS faced by the consumers in the competitive economy, whereas the right-hand side is the MRS of the social planner. Rearranging terms in (7.22), we obtain

$$\frac{u_h(t+1) - \beta\gamma u_h(t+2)}{u_h(t) - \beta\gamma u_h(t+1)} > \frac{u_{\overline{c}}(t+1)}{u_{\overline{c}}(t)}. \tag{7.23}$$

All the partial derivatives appearing in the previous expression are homogenous of degree $\sigma(\theta - 1)$ and, hence, (7.23) becomes

$$(x_{t+1})^{\sigma(\theta-1)} \left(\frac{1 - (x_{t+2})^{\sigma(\theta-1)}\beta\gamma}{1 - (x_{t+1})^{\sigma(\theta-1)}\beta\gamma} \right) > (x_{t+1})^{\sigma(\theta-1)},$$

which simplifies to

$$\left(\frac{x_{t+2}}{x_{t+1}} \right)^{\sigma(\theta-1)} < 1.$$

The previous inequality follows from (7.21) since $\sigma(\theta-1)<0$.

We will see in the next section that, if the two arguments of the function u are perfect substitutes, then the efficient path coincides with the competitive one as the competitive and socially planned MRS's are always identical in this case.

8. Optimal Income Taxation

The design of a tax rate on income geared towards implementing the efficient solution turns out to be quite simple in our context. First of all, observe that, given an initial value z_0 of the state variable z_t, we just have to select a sequence of tax rates on income $\{\tau_t\}_{t=1}^{\infty}$ such that the path of the control variable f_t in the competitive equilibrium coincides with that of the variable $\hat{f}_t$ in the socially planned solution, for all $t\geq 0$. This is so because the functional form of the equation relating f_t and x_t in the competitive economy coincides with that of the equation relating $\hat{f}_t$ and $\hat{x}_t$ in the socially planned economy (see (3.18) and (6.14)). Moreover, the same identity holds between the equation relating z_t and x_t and the equation relating $\hat{z}_t$ and $\hat{x}_t$ (see (3.19) and (6.15)). The competitive path of the variable f_t is given by the solution to the difference equation (3.15). Using (3.12), the solution to this equation is

$$f_t=\frac{1+n}{\left(1+\left(1-\tau_{t+1}\right)A\right)\beta}, \tag{8.1}$$

as follows from the fact that the difference equation (3.15) is autonomous and, thus, does not depend on the state variable z_t, and the stationary value of the control variable f_t is unstable. The socially planned path of $\hat{f}_t$ is obtained from solving the system of difference equations (6.13), (6.14), and (6.15) for a given initial value of z_0. Then, from plugging the efficient sequence $\left\{\hat{f}_t\right\}_{t=0}^{\infty}$ in (8.1) and solving for τ_{t+1}, we obtain the optimal sequence of tax rates on income,

$$\tau_{t+1}=\frac{\beta(1+A)\hat{f}_t-(1+n)}{\beta A\hat{f}_t}. \tag{8.2}$$

In order to characterize this optimal fiscal policy around the steady state, we see from (7.14) and (7.15) that the socially planned path of $\hat{f}_t$ around its steady state satisfies

$$\hat{f}_t=k_1\left(\hat{z}_t-\hat{z}\right)+\hat{f}, \tag{8.3}$$

where $\hat{f}$ is the stationary value both of the variable f_t corresponding to the competitive path with zero taxes and of the variable $\hat{f}_t$ corresponding to the efficient path. Similarly, $\hat{z}$ is the stationary value of both z_t and $\hat{z}_t$. Plugging (8.3) in (8.2), we get that the optimal tax rate τ_{t+1} around the steady state satisfies

$$\tau_{t+1}=\frac{\beta(1+A)\left[\left(\hat{z}_t-\hat{z}\right)k_1+\hat{f}\right]-(1+n)}{\beta A\hat{f}_t}=\frac{\beta(1+A)\left(\hat{z}_t-\hat{z}\right)k_1}{\beta A\hat{f}_t}, \tag{8.4}$$

where the second equality comes from the fact that

$$\hat{f}=\frac{1}{\beta\hat{\varphi}}=\frac{1+n}{\beta(1+A)}.$$

Clearly, the denominator of (8.4) is positive by definition. Therefore, since $k_1 > 0$, the sign of the tax rate τ_{t+1} around the steady state coincides with the sign of $\hat{z}_t - \hat{z}$, where $\hat{z}_t$ is sufficiently close to the stationary value $\hat{z}$. Therefore, we can state the following proposition describing the relation between the optimal tax and the current value of the state variable z_t:

Proposition 8.1. *Assume that $\theta \neq 0$. The optimal tax rate on income around a steady state satisfies*

$$\tau_{t+1} > 0 \qquad \text{if} \qquad z_t > z,$$

$$\tau_{t+1} < 0 \qquad \text{if} \qquad z_t < z,$$

and

$$\lim_{t \to \infty} \tau_t = 0.$$

Proof. Obvious from (8.4) , since $\hat{z} = z$, while we can make $\hat{z}_t = z_t$ as both variables are predetermined at period t. ■

The policy prescription arising from the previous proposition has a counter-cyclical flavor since, when the variable z_t lies below its stationary value, the rate of growth x_t is also below its stationary value (see (4.15)). In this case, since the competitive economy converges towards its steady state at a lower speed than the socially planned economy, an optimal policy should consist on accelerating the rate of convergence. For economies that are displaying rates of growth below the stationary one, this optimal policy involves to accelerate the rate of capital accumulation and this is achieved by a proportional subsidy on capital income, $\tau_{t+1} < 0$. Obviously, for economies that are growing faster than in their BGP, the speed of convergence is accelerated by a positive tax rate aimed to disincentive capital accumulation.

The previous taxation policy also resembles in some respect the optimal policy discussed by Chamley (1986) in a quite different context. According to Chamley, optimal taxation on the income generated by accumulable inputs should tend to zero in the long run, which agrees with the last part of Proposition 8.1.

In order to obtain the local characterization of the sequence of optimal tax rates given in Proposition 8.1, we have made use of two crucial assumptions: 1) the instantaneous utility function u has partial derivatives that are homogeneous of the same degree, and 2) the two arguments of u are not perfect substitutes. To highlight the role played by these two assumptions we can state the following proposition:

Proposition 8.2. *Assume that the instantaneous utility function $u(h, \bar{c})$ has partial derivatives that are homogeneous of the same degree and that $\dfrac{1+A}{1+n} > 1$. Then,*

a) The optimal tax rate on income is equal to zero along the BGP.

b) The optimal tax rate on income is equal to zero off the BGP whenever the two arguments of $u(h, \bar{c})$ are perfect substitutes.

Proof. See the appendix. ■

This proposition tells us that, under the standard homogeneity condition aimed to allow the existence of BGP's, inefficiencies vanish as the competitive path with zero taxes approaches its BGP. Moreover, the proposition also shows that some kind of complementarity between habit adjusted consumption and consumption externalities is necessary to generate inefficiency during the transition towards the steady state. It is shown in the proof of the proposition (see (A.20)) that the necessary and sufficient condition for the efficiency of the competitive equilibrium is that

$$\frac{u_{\bar{c}}(h_t, c_t)}{u_h(h_t, c_t)} = \vartheta, \tag{8.5}$$

for some constant ϑ, along the competitive equilibrium path. In fact, condition (8.5) extends the "restricted homotheticity" condition (2.16) to a situation where habits are present.

Consider now an instantaneous utility function $u(h_t, \bar{c}_t)$ in which its two arguments are perfect substitutes. This is formalized without loss of generality by the following parametrization of the utility function:

$$u(h_t, \bar{c}_t) = \frac{(h_t - \theta \bar{c}_t)^{1-\sigma}}{1-\sigma}, \qquad \sigma > 0. \tag{8.6}$$

According to the functional form (8.6), the utility of an individual remains unchanged when an increase in c units of habit adjusted private consumption is accompanied by an increase of c/θ units of average present consumption. Note that no restriction is imposed on the sign of the parameter θ so that if $\theta > 0$ ($\theta < 0$) average consumption increases (decreases) the marginal utility of an additional unit of an individual's habit adjusted consumption. It should also be pointed out that the functional form (8.6) collapses in a single function both the additive specification of consumption externalities found in Ljungqvist and Uhlig (2000) and the traditional specification of additive habit formation. Finally, note that the utility function (8.6) satisfies

$$\frac{u_{\bar{c}}(h_t, c_t)}{u_h(h_t, c_t)} = -\theta,$$

so that condition (8.5) always holds and, thus, the competitive equilibrium is efficient.

The functional form (3.1) of the instantaneous utility that we have considered throughout the present paper does exhibit complementarities between its two arguments and, thus, condition (8.5) does not longer hold. Clearly, in this case,

$$\frac{u_{\bar{c}}(h_t, c_t)}{u_h(h_t, c_t)} = \left(\frac{\theta\sigma}{1-\sigma}\right)\left(\frac{h_t}{c_t}\right) = \frac{\theta\sigma}{1-\sigma}\left(1 - \frac{\gamma}{x_t}\right).$$

Therefore, since the gross rate of growth x_t is not constant off the BGP (see the analysis in Section 4), we can conclude that the competitive path without taxes is not efficient during the transition. However such an inefficiency vanishes in the long run as x_t approaches its stationary value x.

9. Conclusion

In this paper we have presented an exhaustive characterization of the dynamic competitive equilibrium of an economy displaying endogenous growth. In this economy habits and contemporaneous consumption spillovers appear in the consumers' utility function. The technological side is modelled in an extremely simple way in order to allow for a transitional dynamics entirely driven by the specification of individual preferences. Besides the consequences for the dynamics of the consumption and capital paths brought about by those two departures from traditional formulations of preferences, our analysis shows that consumption externalities are not necessarily a source of inefficiency. In particular, when habits are not present and both the competitive and the socially planned economy exhibit a BGP, consumption spillovers do not generate any kind of sub-optimality. This is so because the existence of a BGP's makes the functional form of the competitive marginal rate of substitution of consumption between two periods identical to the efficient marginal rate of substitution. When habits are introduced in the individuals' utility function in such a way that habit adjusted consumption is a perfect substitute for the average consumption in the economy, the previous identity between the two marginal rates of substitution is preserved and, again, no public intervention is needed to restore efficiency. However, such an identity between marginal rates of substitution is not longer obtained when habit adjusted consumption and average consumption are not perfect substitutes. In this context we have shown that, even if the competitive and the efficient path share the same stationary equilibrium, the latter converges faster towards its steady state than the former. Therefore, this discrepancy in the speed of convergence calls for some public intervention aimed to raise the competitive rate of convergence. Clearly, a counter-cyclical income taxation policy serves this purpose since this policy accelerates the rate of capital accumulation for economies growing slowly, while disincentives capital accumulation for fast growing economies.

Our model has just focused on the interaction between consumption externalities and habits. We have assumed throughout the paper that habits are additive so that the argument appearing in the utility function is the difference between present and past consumption. As we have pointed out, all our results could be extended to the case with multiplicative habits where the ratio between present and past consumption is the relevant argument of the utility function (as in Carroll et al., 1997).[8] In particular, if consumption spillovers are not perfect substitutes for habit adjusted consumption, the social planner will also be able to affect the importance of the habit stock for current consumption. Therefore, the socially planned economy will exhibit a higher speed of convergence than the competitive economy.

Another possible extension of our analysis will be the introduction of "external habits". Under this kind of habits the average past consumption of the economy becomes the relevant standard of living that is used to evaluate the utility accruing from present consumption.[9] Preferences subjected to a process of external habit formation are also said to display a "catching up with the Joneses" feature. However,

[8]When habits have the multiplicative form given in (3.5), the stationary rate of growth is not longer independent of the parameter γ measuring the importance of the previous standard of living. In this case, to analyze the effects of changes in γ on the rate of convergence, we should modify also the value of the parameter σ in order to keep invariant the steady state.

[9]External habits are used in the stochastic models of Constantinides (1990), Abel (1999), Campbell and Cochrane (1999), and Ljungkvist and Uhlig (2000). Moreover, the social norms appearing in the capital accumlation model of de la Coix (1998) play also the role of external habits.

it can be shown that, if we had replaced our additive "internal" habits by external ones, all the results of our analysis would still remain valid.

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A. Appendix

Conditions for the monotonicity and concavity of the function û.

We will next derive the conditions under which the utility function $\hat{u}(c_t, c_{t-1})$ faced by the social planner is increasing in present consumption c_t and jointly concave with respect to c_t and c_{t-1}. These conditions will guarantee that the Euler equations we derive for the socially planned economy are in fact characterizing a maximum. First rewrite

$$\hat{u}(c_t, c_{t-1}) = \frac{(c_t - \gamma c_{t-1})^{1-\sigma} (c_t)^{\theta\sigma}}{1-\sigma} = \frac{\left((c_t)^{1+j} - \gamma c_{t-1} (c_t)^j\right)^{1-\sigma}}{1-\sigma} = \frac{\psi^{1-\sigma}}{1-\sigma},$$

with $\sigma > 0$ and $\gamma \in (0,1)$, where

$$j = \frac{\theta\sigma}{1-\sigma}, \qquad \text{and} \qquad \psi = (c_t)^{1+j} - \gamma c_{t-1} (c_t)^j .$$

Hence, the function $\hat{u}$ is well defined whenever

$$c_t - \gamma c_{t-1} > 0, \tag{A.1}$$

which holds when consumption grows at a positive rate. Moreover, ψ is strictly positive whenever (A.1) holds.

Let us now compute the following partial derivatives

$$\frac{\partial \hat{u}}{\partial c_t} = \psi^{-\sigma}\psi_1, \tag{A.2}$$

$$\frac{\partial^2 \hat{u}}{\partial (c_t)^2} = -\sigma\psi^{-(\sigma+1)} (\psi_1)^2 + \psi^{-\sigma}\psi_{11},$$

$$\frac{\partial \hat{u}}{\partial c_{t-1}} = \psi^{-\sigma}\psi_2,$$

$$\frac{\partial^2 \hat{u}}{\partial (c_{t-1})^2} = -\sigma\psi^{-(\sigma+1)} (\psi_2)^2 + \psi^{-\sigma}\psi_{22}, \tag{A.3}$$

$$\frac{\partial^2 \hat{u}}{\partial c_t c_{t-1}} = -\sigma\psi^{-(\sigma+1)}\psi_1\psi_2 + \psi^{-\sigma}\psi_{12},$$

where the subindex 1 denotes the partial derivative with respect to c_t, while the subindex 2 denotes the partial derivative with respect to c_{t-1}. Hence,

$$\psi_1 = (1+j)(c_t)^j - j\gamma c_{t-1} (c_t)^{j-1}, \tag{A.4}$$

$$\psi_{11} = (1+j)\, j\, (c_t)^{j-1} + (1-j)\, j\gamma c_{t-1} (c_t)^{j-2},$$

$$\psi_2 = -\gamma c_{t-1} (c_t)^j < 0,$$

$$\psi_{22} = 0, \tag{A.5}$$

$$\psi_{12} = -\gamma j c_{t-1} (c_t)^{j-1} = \frac{j\psi_2}{c_t}. \tag{A.6}$$

It is immediate to see from (A.2) and (A.4) that the function $\hat{u}(c_t, c_{t-1})$ is increasing with respect to c_t whenever both (A.1) and $j \geq 0$ hold. Note that $j \geq 0$ is equivalent to

$$\frac{\theta}{1-\sigma} \geq 0. \tag{A.7}$$

Furthermore, from (A.3) and (A.5) it is obvious that

$$\frac{\partial^2 \hat{u}}{\partial (c_{t-1})^2} < 0.$$

Moreover, concavity of $\hat{u}$ requires that

$$\left(\frac{\partial \hat{u}}{\partial c_t c_t}\right)\left(\frac{\partial \hat{u}}{\partial c_{t-1} c_{t-1}}\right) \geq \left(\frac{\partial \hat{u}}{\partial c_t c_{t-1}}\right)^2,$$

which becomes

$$\left[-\sigma\psi^{-(\sigma+1)}(\psi_1)^2 + \psi^{-\sigma}\psi_{11}\right]\left[-\sigma\psi^{-(\sigma+1)}(\psi_2)^2 + \psi^{-\sigma}\psi_{22}\right] \geq \left(-\sigma\psi^{-(\sigma+1)}\psi_1\psi_2 + \psi^{-\sigma}\psi_{12}\right)^2.$$

Assume that (A.1) holds and divide the previous inequality by $\psi^{-(\sigma+1)}$ to get

$$\left[-\sigma(\psi_1)^2 + \psi\psi_{11}\right]\left[-\sigma(\psi_2)^2 + \psi\psi_{22}\right] \geq (-\sigma\psi_1\psi_2 + \psi\psi_{12})^2,$$

which, using (A.5) and (A.6), becomes

$$\left[-\sigma(\psi_1)^2 + \psi\psi_{11}\right]\left[-\sigma(\psi_2)^2\right] \geq \psi_2^2\left(-\sigma\psi_1 + \frac{\psi j}{c_t}\right)^2,$$

which simplifies to

$$-\sigma\psi_{11} \geq \psi\left(\frac{j}{c_t}\right)^2 - \frac{2\sigma\psi_1 j}{c_t},$$

and which in turn becomes

$$-\sigma\left[(1+j)\,j\,(c_t)^{j-1} + (1-j)\,j\gamma c_{t-1}(c_t)^{j-2}\right] \geq$$

$$j^2\left[(c_t)^{j-1} - \gamma c_{t-1}(c_t)^{j-2}\right] - 2\sigma j\left[(1+j)(c_t)^{j-1} - j\gamma c_{t-1}(c_t)^{j-2}\right].$$

Rearranging and simplifying the previous inequality, we get

$$j\,(c_t)^{j-1}\left[\sigma(1+j) - j\right] \geq j\gamma c_{t-1}(c_t)^{j-2}\left[-j + \sigma(1+j)\right],$$

which can also be written as

$$j\left[(c_t)^{j-1} - \gamma c_{t-1}(c_t)^{j-2}\right]\left[\sigma(1+j) - j\right] \geq 0,$$

or

$$\frac{j\psi}{(c_t)^2}\left[\sigma(1+j) - j\right] \geq 0.$$

Dividing by $\dfrac{\psi}{(c_t)^2}$ and using the definition of j, the previous weak inequality becomes

$$\frac{\theta\sigma^2(1-\theta)}{1-\sigma} \geq 0. \tag{A.8}$$

Note that condition (A.8) is equivalent to $\theta < 1$ since $\dfrac{\theta}{1-\sigma} \geq 0$ is required for $\hat{u}(c_t, c_{t-1})$ to be increasing in c_t (see (A.7)).

Summing up, conditions (A.1), (A.7) and $\theta < 1$ guarantee that the function $\hat{u}(c_t, c_{t-1})$ is well defined, strictly increasing in c_t, and jointly concave with respect to c_t and c_{t-1}.

Proof of Lemma 7.2.

The elements of the 3×3 matrix of partial derivatives appearing in the system (7.13) are

$$\hat{\lambda}_{11} = \frac{\partial \hat{f}_{t+1}}{\partial \hat{f}_t} = \frac{\partial M}{\partial \hat{f}_t} + \frac{\partial M}{\partial \hat{x}_{t+1}} \frac{\partial \hat{x}_{t+1}}{\partial \hat{f}_t},$$

$$\hat{\lambda}_{12} = \frac{\partial \hat{f}_{t+1}}{\partial \hat{x}_t} = \frac{\partial M}{\partial \hat{x}_t} + \frac{\partial M}{\partial \hat{x}_{t+1}} \frac{\partial \hat{x}_{t+1}}{\partial \hat{x}_t},$$

$$\hat{\lambda}_{21} = \frac{\partial \hat{x}_{t+1}}{\partial \hat{f}_t},$$

$$\hat{\lambda}_{22} = \frac{\partial \hat{x}_{t+1}}{\partial \hat{x}_t},$$

$$\hat{\lambda}_{32} = \frac{\partial \hat{z}_{t+1}}{\partial \hat{x}_t} = -\left(\frac{\hat{z}}{\hat{x}^2}\right)\left(\frac{1+A}{1+n}\right) < 0, \tag{A.9}$$

$$\hat{\lambda}_{33} = \frac{\partial \hat{z}_{t+1}}{\partial \hat{z}_t}. \tag{A.10}$$

Let $\hat{\lambda}_1, \hat{\lambda}_2,$ and $\hat{\lambda}_3$ be the eigenvalues of that matrix. Given the triangular nature of the 2×2 matrix of the linearized sub-system composed just of equations (6.14), and (6.15), we see from (A.10) that one of the eigenvalues is

$$\hat{\lambda}_3 = \hat{\lambda}_{33} = \frac{\partial \hat{z}_{t+1}}{\partial \hat{z}_t} = \left(\frac{1}{\hat{x}}\right)\left(\frac{1+A}{1+n}\right) > 1, \tag{A.11}$$

as follows from differentiating (6.15) and from condition (7.6). Next, looking just at the sub-system composed of the linearization of equations (6.13) and (6.14), we obtain that the other two eigenvalues of the original 3×3 matrix satisfy

$$\hat{\lambda}_1 + \hat{\lambda}_2 = \hat{\lambda}_{11} + \hat{\lambda}_{22} = \frac{\partial M}{\partial \hat{f}_t} + \frac{\partial M}{\partial \hat{x}_{t+1}} \frac{\partial \hat{x}_{t+1}}{\partial \hat{f}_t} + \frac{\partial \hat{x}_{t+1}}{\partial \hat{x}_t},$$

and

$$\hat{\lambda}_1 \hat{\lambda}_2 = \hat{\lambda}_{11}\hat{\lambda}_{22} - \hat{\lambda}_{12}\hat{\lambda}_{21} =$$

$$\left(\frac{\partial M}{\partial \hat{f}_t}+\frac{\partial M}{\partial \hat{x}_{t+1}}\frac{\partial \hat{x}_{t+1}}{\partial \hat{f}_t}\right)\frac{\partial \hat{x}_{t+1}}{\partial \hat{x}_t}-\left(\frac{\partial M}{\partial \hat{x}_t}+\frac{\partial M}{\partial \hat{x}_{t+1}}\frac{\partial \hat{x}_{t+1}}{\partial \hat{x}_t}\right)\frac{\partial \hat{x}_{t+1}}{\partial \hat{f}_t}=$$
$$\frac{\partial \hat{x}_{t+1}}{\partial \hat{x}_t}\frac{\partial M}{\partial \hat{f}_t}-\frac{\partial \hat{x}_{t+1}}{\partial \hat{f}_t}\frac{\partial M}{\partial \hat{x}_t}.$$

We can compute now the following partial derivatives appearing in the previous expressions,

$$\frac{\partial M}{\partial \hat{f}_t}=\left(\varepsilon-\frac{\eta}{x}\right)\left(\frac{1}{\varphi\beta\hat{f}^2}\right)=\left(\frac{1}{\beta\hat{f}\,(1-\sigma)}\right)\left(\frac{1+\sigma\,(\theta-1)}{\gamma}-\frac{\theta\sigma}{\hat{x}}\right),$$

$$\frac{\partial M}{\partial \hat{x}_t}=-\left(\frac{\eta}{\hat{x}^2}\right)=-\frac{\theta\sigma}{\beta\,(1-\sigma)\,\hat{x}^2},$$

$$\frac{\partial M}{\partial \hat{x}_{t+1}}=\left(\frac{\eta}{\hat{x}^2}\right)=\frac{\theta\sigma}{\beta\,(1-\sigma)\,\hat{x}^2},$$

$$\frac{\partial \hat{x}_{t+1}}{\partial \hat{x}_t}=-\frac{\dfrac{\partial g}{\partial \hat{x}_t}}{\dfrac{\partial g}{\partial \hat{x}_{t+1}}}=\frac{\gamma}{\hat{x}-\theta\,(\hat{x}-\gamma)},$$

$$\hat{\lambda}_{21}=\frac{\partial \hat{x}_{t+1}}{\partial \hat{f}_t}=-\frac{\dfrac{\partial g}{\partial \hat{f}_t}}{\dfrac{\partial g}{\partial \hat{x}_{t+1}}}=-\left(\frac{\hat{x}}{\sigma\hat{f}}\right)\left(\frac{\hat{x}-\gamma}{\hat{x}-\theta\,(\hat{x}-\gamma)}\right)<0. \qquad \text{(A.12)}$$

Therefore,

$$\hat{\lambda}_1\hat{\lambda}_2=\frac{\partial \hat{x}_{t+1}}{\partial \hat{x}_t}\frac{\partial M}{\partial \hat{f}_t}-\frac{\partial \hat{x}_{t+1}}{\partial \hat{f}_t}\frac{\partial M}{\partial \hat{x}_t}=$$
$$\left(\frac{\gamma}{\hat{x}-\theta\,(\hat{x}-\gamma)}\right)\left(\frac{1}{\beta\hat{f}\,(1-\sigma)}\right)\left(\frac{1+\sigma\,(\theta-1)}{\gamma}-\frac{\theta\sigma}{\hat{x}}\right)-$$
$$\left(\frac{\hat{x}}{\sigma\hat{f}}\right)\left(\frac{\hat{x}-\gamma}{\hat{x}-\theta\,(\hat{x}-\gamma)}\right)\frac{\theta\sigma}{\beta\,(1-\sigma)\,\hat{x}^2}.$$

Using the equilibrium values of $\hat{x}$ and $\hat{f}$, the previous messy expression simplifies dramatically and becomes simply

$$\hat{\lambda}_1\hat{\lambda}_2=\left(\frac{\varphi}{\hat{x}}\right)>0. \qquad \text{(A.13)}$$

Concerning the sum of the eigenvalues $\hat{\lambda}_1$ and $\hat{\lambda}_2$, we get after some tedious computations,

$$\hat{\lambda}_1+\hat{\lambda}_2=\frac{\partial M}{\partial \hat{f}_t}+\frac{\partial M}{\partial \hat{x}_{t+1}}\frac{\partial \hat{x}_{t+1}}{\partial \hat{f}_t}+\frac{\partial \hat{x}_{t+1}}{\partial \hat{x}_t}=$$
$$\frac{\gamma}{\hat{x}-\theta\,(\hat{x}-\gamma)}+\left(\frac{\varphi}{\gamma}\right)\left[\frac{\hat{x}-\theta\,(\hat{x}-\gamma)}{\hat{x}}+\frac{\theta\,(1-\theta)\,(\hat{x}-\gamma)^2}{(1-\sigma)\,\hat{x}\,(\hat{x}-\theta\,(\hat{x}-\gamma))}\right]>0. \qquad \text{(A.14)}$$

The inequality follows since $\hat{x} - \gamma > 0$ and $\theta < 1$. Therefore, the first term of the left hand side of the previous inequality and the first term inside the square brackets are both positive. Moreover, the concavity of $\hat{u}$ requires that $\frac{\theta}{(1-\sigma)} \geq 0$, which with the previous parametric conditions allows us to immediately conclude that the second term inside the square brackets is also positive.

In order to characterize the local dynamic behavior of the system around its steady-state, we should also compute

$$\left(1-\hat{\lambda}_1\right)\left(1-\hat{\lambda}_2\right) = 1 + \hat{\lambda}_1\hat{\lambda}_2 - \left(\hat{\lambda}_1 + \hat{\lambda}_2\right) =$$

$$-\underbrace{\left[\frac{(\hat{x}-\gamma)(1-\theta)}{\gamma\hat{x}(\hat{x}-\theta(\hat{x}-\gamma))}\right]}_{p} \times \underbrace{\left[\hat{x}(\varphi-\gamma) + \frac{\varphi\theta(\hat{x}-\gamma)\sigma}{(1-\sigma)}\right]}_{q} < 0,$$

where the second equality comes, after some simplification, from (A.13) and (A.14). The negative sign of $\left(1-\hat{\lambda}_1\right)\left(1-\hat{\lambda}_2\right)$ is a consequence of the fact that the terms p and q are both positive. Clearly, p is positive since $\theta < 1$ and we have imposed that $\hat{x} - \gamma > 0$. Concerning the term q, we just have to observe that the inequality $q > 0$ is equivalent to inequality (7.12), which always holds under our parametric conditions.

Since $\hat{\lambda}_1\hat{\lambda}_2 > 0$, $\hat{\lambda}_1 + \hat{\lambda}_2 > 0$ and $\left(1-\hat{\lambda}_1\right)\left(1-\hat{\lambda}_2\right) < 0$, we can conclude that $\hat{\lambda}_1 > 1$ and $\hat{\lambda}_2 \in (0,1)$. Therefore, recalling that $\hat{\lambda}_3 > 1$, the linearized dynamic system displays saddle path stability. ■

Proof of Proposition 8.2.

a) On the one hand, the Euler equation (3.10) for the competitive economy without taxes becomes

$$\frac{u_h(t+1) - \beta\gamma u_h(t+2)}{u_h(t) - \beta\gamma u_h(t+1)} = \frac{1+n}{\beta(1+A)}. \tag{A.15}$$

On the other hand, the Euler equation (6.6) for the socially planned economy becomes

$$\frac{u_h(t+1) + u_{\bar{c}}(t+1) - \beta\gamma u_h(t+2)}{u_h(t) + u_{\bar{c}}(t) - \beta\gamma u_h(t+1)} = \frac{1+n}{\beta(1+A)}, \tag{A.16}$$

since $\hat{u}_1(t) = u_h(t) + u_{\bar{c}}(t)$, $\hat{u}_2(t+1) = -\gamma u_h(t+1)$ and $\hat{\varphi} = \frac{1+A}{1+n}$. As the right hand sides of the two previous Euler equations are identical, the competitive allocation will coincide with the one selected by the social planner if and only if the left hand sides of (A.15) and (A.16) have the same functional form along the competitive consumption path. Therefore, taking into account that in equilibrium $\bar{c}_t = c_t$, the competitive path of consumption $\{c_t\}_{t=0}^{\infty}$ is efficient if and only if

$$\frac{u_h(h_{t+1}, c_{t+1}) - \beta\gamma u_h(h_{t+2}, c_{t+2})}{u_h(h_t, c_t) - \beta\gamma u_h(h_{t+1}, c_{t+1})} = \frac{u_h(h_{t+1}, c_{t+1}) + u_{\bar{c}}(h_{t+1}, c_{t+1}) - \beta\gamma u_h(h_{t+2}, c_{t+2})}{u_h(h_t, c_t) + u_{\bar{c}}(h_t, c_t) - \beta\gamma u_h(h_{t+1}, c_{t+1})}.$$

for all t. The previous expression simplifies to

$$\frac{u_{\bar{c}}(h_{t+1}, c_{t+1})}{u_{\bar{c}}(h_t, c_t)} = \frac{u_h(h_{t+1}, c_{t+1}) - \beta\gamma u_h(h_{t+2}, c_{t+2})}{u_h(h_t, c_t) - \beta\gamma u_h(h_{t+1}, c_{t+1})}.$$

That is, the competitive solution will be efficient if and only if

$$u_{\bar{c}}(h_t, c_t) = \varsigma \left[u_h(h_t, c_t) - \gamma\beta u_h(h_{t+1}, c_{t+1})\right], \tag{A.17}$$

for all t and for some constant ς along the competitive equilibrium path of consumption. Recalling that $f_t = \frac{u_h(h_{t+1}, c_{t+1})}{u_h(h_t, c_t)}$ and dividing by $u_h(h_t, c_t)$, condition (A.17) becomes

$$\frac{u_{\bar{c}}(h_t, c_t)}{u_h(h_t, c_t)} = \varsigma \left[1 - \gamma\beta f_t\right]. \tag{A.18}$$

Moreover, we can rewrite the Euler equation (A.15) as

$$f_{t+1} = \frac{1+n}{\beta(1+A)}\left(1 - \frac{1}{\beta\gamma f_t}\right) + \frac{1}{\beta\gamma} \equiv \Psi(f_t). \tag{A.19}$$

This difference equation has two stationary equilibria: $f = \frac{1+n}{\beta(1+A)}$ and $\breve{f} = \frac{1}{\beta\gamma}$, with $\breve{f} > f$ since $\dfrac{1+A}{1+n} > 1 > \gamma$. From Figure 3, that displays the mapping $f_{t+1} = \Psi(f_t)$, we can infer the global dynamics of the paths solving the difference equation (A.19). On the one hand, the stationary equilibrium $\breve{f}$ is locally stable but violates the positiveness condition on the Lagrange multipliers (see condition (4.10) , which also holds for a general utility function). On the other hand, the stationary equilibrium f is unstable and, thus, the equilibrium path of the variable f_t exhibits no transition.

(Insert Figure 3)

Since $f_t = f$ for all t, condition (A.18) becomes

$$\frac{u_{\bar{c}}(h_t, c_t)}{u_h(h_t, c_t)} = \varsigma \left[1 - \beta\gamma f\right] \equiv \vartheta, \tag{A.20}$$

for some constant ϑ. If all the partial derivatives of u are homogeneous of degree κ then, along a BGP with a gross rate of growth x (and, thus, with $f = x^\kappa$), it holds that

$$\frac{u_{\bar{c}}(h_t, c_t)}{u_h(h_t, c_t)} = \frac{(c_t)^{-\kappa} u_{\bar{c}}\left(\left(\frac{h_t}{c_t}\right), 1\right)}{(c_t)^{-\kappa} u_h\left(\left(\frac{h_t}{c_t}\right), 1\right)} = \frac{u_{\bar{c}}\left(\left(\frac{h_t}{c_t}\right), 1\right)}{u_h\left(\left(\frac{h_t}{c_t}\right), 1\right)} = \frac{u_{\bar{c}}\left(1 - \frac{\gamma}{x}, 1\right)}{u_h\left(1 - \frac{\gamma}{x}, 1\right)} = \vartheta,$$

for all t and for some constant ϑ. Therefore, condition (A.18) holds at a BGP. Note that the homogeneity of the partial derivatives of u implies that the stationary competitive equilibrium is efficient and, thus, that optimal tax rates along a BGP should be set equal to zero.

b) Consider now the case where the two arguments of the function u are perfect substitutes. Under this assumption the marginal rate of substitution $\dfrac{u_{\bar{c}}(h_t, c_t)}{u_h(h_t, c_t)}$ is constant and, thus, condition (A.20) always holds. Therefore, when consumption externalities interact additively with the habit adjusted consumption, the competitive equilibrium with zero income tax rates is efficient. ■

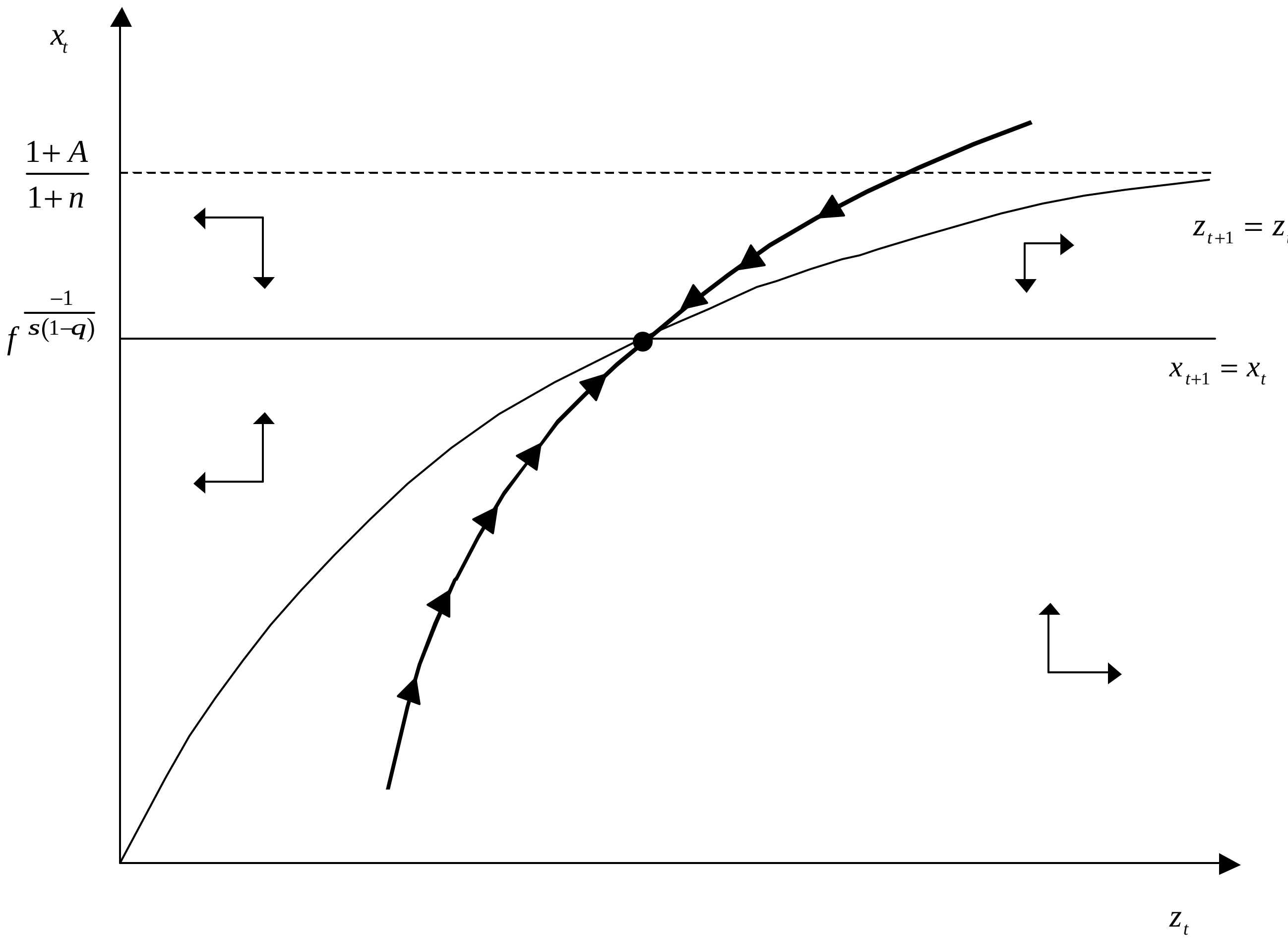

Figure 1. The Dynamics of the Competitive Equilibrium

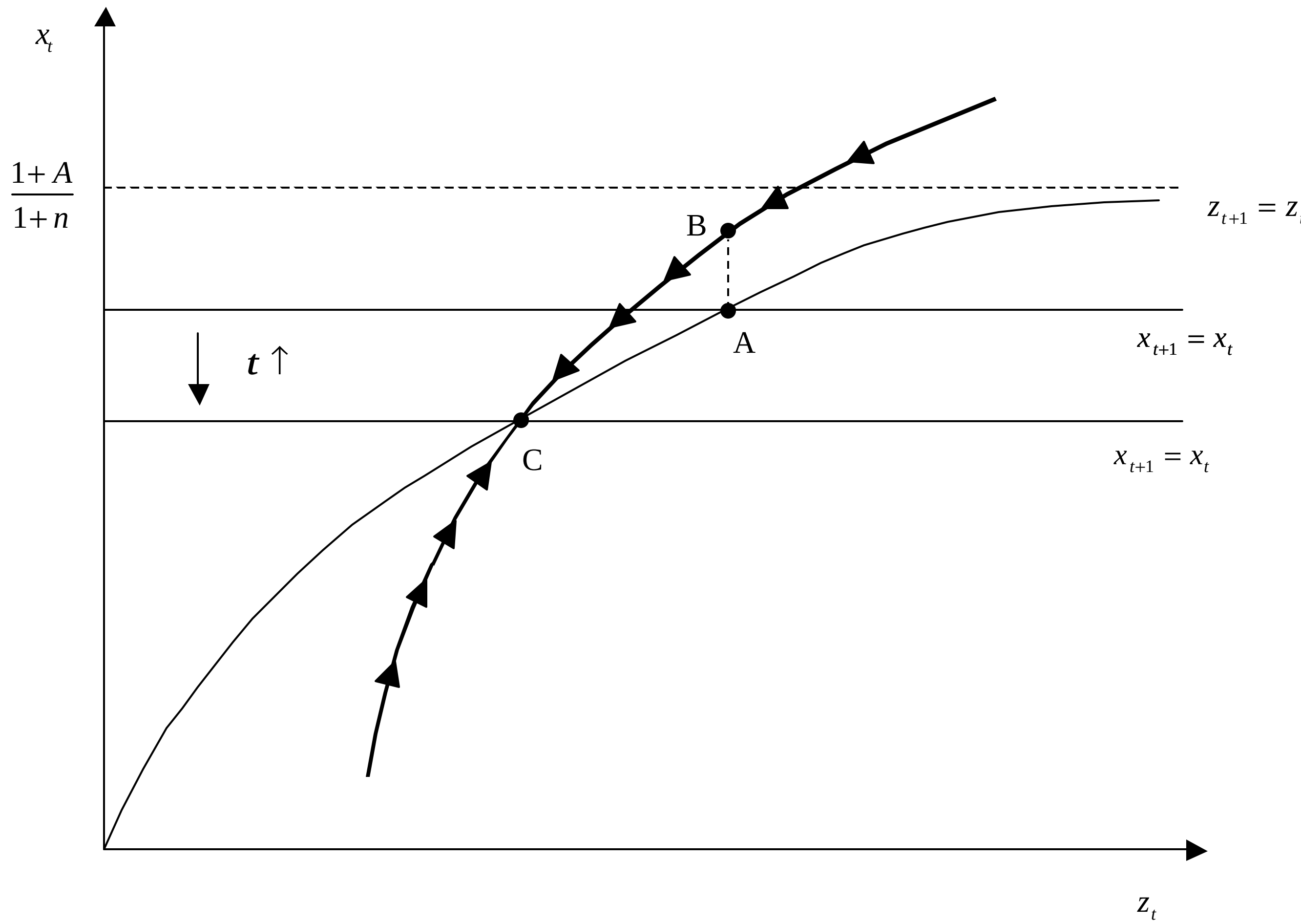

Figure 2. Dynamic Effects of an Increase of the Income Tax Rate

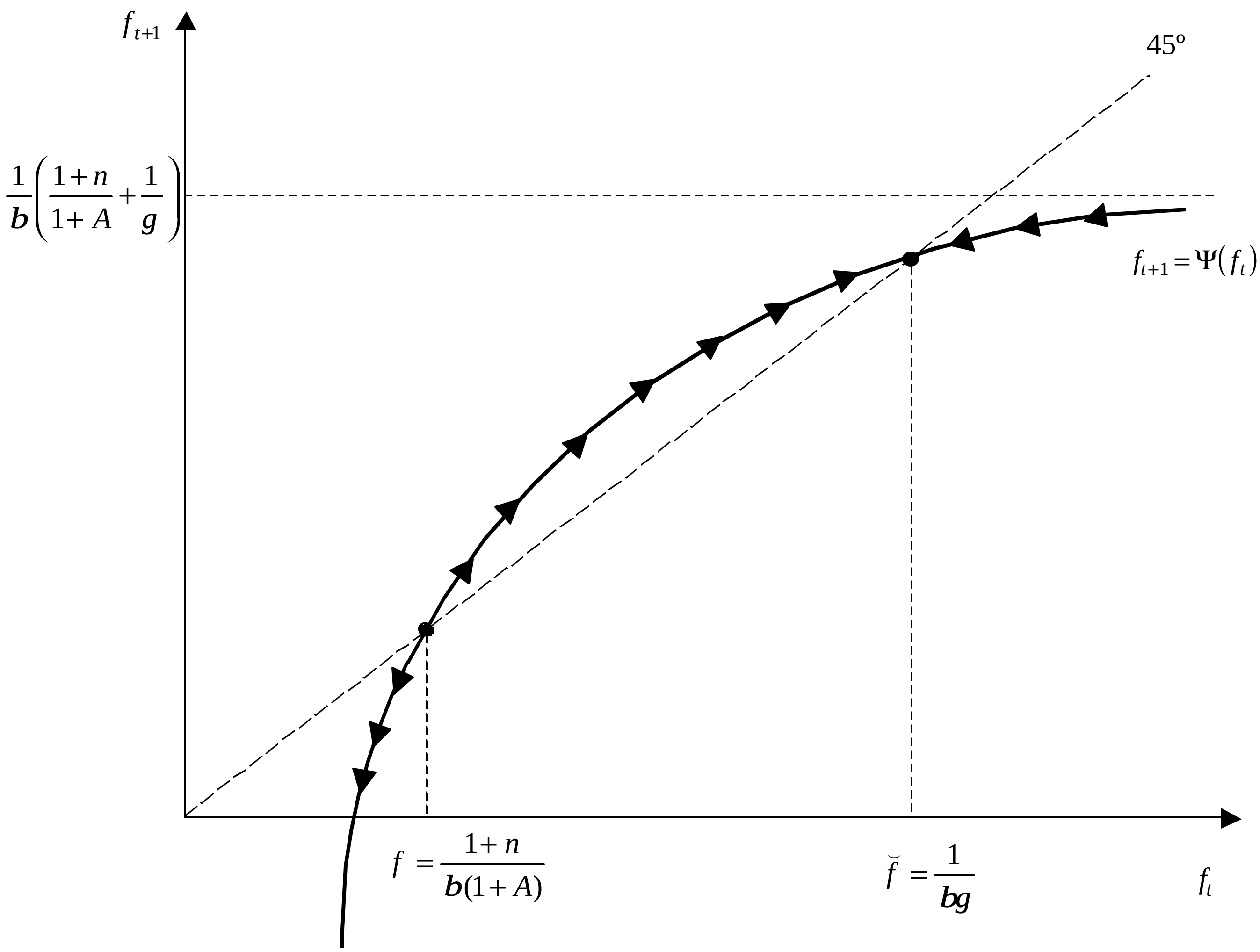

Figure 3. The mapping $f_{t+1} = \Psi(f_t)$