# A Dual Characterization of Incentive Efficiency*

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**Abstract**

We show that incentive efficient allocations in economies with adverse selection and moral hazard can be determined as optimal solutions to a linear programming problem and we use duality theory to obtain a complete characterization of the optima. Our dual analysis identifies welfare effects associated with the incentives of the agents to truthfully reveal their private information. Because these welfare effects may generate non-convexities, incentive efficient allocations may involve randomization. Other properties of incentive efficient allocations are also derived.

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# 1 Introduction

It is well known that informational asymmetries generate adverse selection and moral hazard problems. To take these problems into account, in addition to the standard resource constraints, an allocation must satisfy *incentive compatibility* constraints: agents must be given incentives to truthfully reveal their private characteristics and actions. Despite great progress in understanding these environments, the characterization of economies under asymmetric information remains problematic. For example, there is no clear consensus about *the* theory of competitive markets with asymmetric information. Adapting the well-established techniques of linear programming and duality theory to characterize efficient allocations under asymmetric information, we study the role of incentive compatibility in determining allocations.

The point of departure of this work is the classic contribution of Prescott and Townsend [23]. Those authors investigate the extent to which standard methods for the analysis of efficient allocations and their decentralization as competitive equilibria can be applied to environments with asymmetric information. Using a framework which embeds a large class of economies, and showing that lotteries play a role in the analysis, they demonstrate that competitive equilibria exist and are incentive efficient when trading takes place before the asymmetry of information is realized (e.g. moral hazard). However, their approach does not extend to economies where trading takes place after the asymmetric information is realized (e.g. adverse selection). Our main contribution to the insights of Prescott and Townsend is to exploit the linear structure that arises when lotteries are allowed for, and which, as Myerson [21] shows, is inherent to environments where incentive constraints are relevant. Framing the analysis in terms of linear programming allows us to obtain a complete characterization of the set of incentive efficient allocations. Also, duality emerges as a powerful tool to study these environments. Our findings point to the presence of external effects associated with the incentives of agents to reveal their private characteristics as the source of the problems encountered by Prescott and Townsend in decentralizing efficient allocations with adverse selection. Our results are reminiscent of Myerson's [22] linear programming characterization of efficient mechanisms in cooperative games with incomplete information and, in particular, of the concept of virtual utilities.

We introduce the linear programming methodology using two simple economies. The first is an adverse selection insurance economy similar to the one used by Rothschild and Stiglitz [25] and Wilson [28]. The second is a moral hazard variation of the first. In both cases, we show that incentive efficient allocations (i.e. allocations which are Pareto optimal in the set of resource feasible and incentive compatible allocations) can be determined as solutions to a linear programming problem. Then we use the primal problem, its dual, and their corresponding complementary slackness conditions to obtain a precise and simple characterization of these allocations.

The adverse selection model is a standard insurance economy with two types of agents, high-and-low risk, and two possible idiosyncratic endowment states. Following Prescott and Townsend [23], we define allocations in the space of lotteries over bundles of contingent commodities. A lottery is just a random insurance plan. Insurance claims are assumed perfectly verifiable and fully enforceable.[1] Agents have von Neumann-Morgenstern preferences. Therefore, their objective function as well as their incentive constraints are linear in the lotteries. Incentive efficient allocations can then be determined as optimal solutions to a linear programming problem; more precisely, a Linear Semi-Infinite Programming problem. We derive the "dual problem" and use the complementary slackness theorem to obtain a complete characterization of the set of incentive efficient allocations. We also show that there is no loss of generality in restricting attention to lotteries with finite support. Then we derive properties of incentive efficient allocations as well as conditions under which lotteries can be dispensed with.

The main economic insights of the model arise from the use of duality theory. In the "dual problem" we identify the welfare effects arising from constraints on the allocation. Apart from the standard welfare effects (i.e., utilities and economic costs) we find others associated with the incentives of the agents to reveal their types. Intuitively, a given allocation may be relatively costly because it gives a greater incentive to one type of agent to misrepresent their type. For instance, all actuarially fair insurance plans for the low-risk agents generate identical economic costs. However, those plans that are more attractive to the high-risk agents imply higher total welfare costs. The reason is that under such plans it becomes more costly to prevent the high-risk agents from lying about their type (i.e., more resources are needed to induce truthful revelation). That is, an external cost arises as a result of the effect of the assignment to the low-risk agents on the high-risk agents.

Incentive efficient allocations must internalize the welfare effects of incentives. Our analysis shows that these welfare effects may generate non-convexities. Hence, some of the incentive efficient allocations may be random. The source of these non-convexities lies in differences in preferences for risk across types. For instance, randomization is beneficial when low-risk agents are risk neutral and high-risk agents are risk averse. In this case, any fair insurance plan is equally good for a low-risk agent and equally

[1]Bisin and Gottardi [5] and Bisin and Guaitoli [7] depart from this "exclusive" benchmark and study competitive economies with non-verifiable trades. Dubey, Geanakoplos and Shubik [11] study environments where asymmetric information arises from the possibility of default.

costly in terms of resources. However, a random insurance plan generates lower external costs because it involves higher risk and is thus less attractive for a high-risk agent. In general, whenever the incentives of the high-risk agents are at issue, and as long as these agents are sufficiently more risk averse than low-risk agents, assigning a lottery to the latter will reduce the external cost of the assignment and may allow a welfare improvement. By contrast, if low-risk agents are at least as risk averse as high-risk agents then the optimal insurance plan of the low-risk agents is deterministic.

The analysis of the moral hazard economy is very similar. In this economy, there is a continuum of ex ante identical agents and two possible idiosyncratic endowment states. Each agent can exert either high or low effort at a direct utility cost. Higher effort reduces the probability of ending up in the poor state. The main difference compared with the adverse selection model is that allocations may now involve two kinds of randomization: not only a random insurance plan, but also a random effort level. In this economy, a given insurance plan may be relatively costly because it gives a greater incentive to the agents to deviate from an optimal high effort level. If risk aversion decreases fast enough with effort, the welfare cost of incentives may be non-convex and optimal insurance plans may be random. Effort may also be random. When their effort is high, agents have higher expected wealth, but this comes at a direct utility cost. In some instances, agents may be willing to give up some consumption to reduce their effort. The tradeoff between consumption and effort is resolved by allowing the agent to provide low effort with some positive probability at the cost of reducing his expected consumption. We find that if the agents' expected wealth is large enough, or if the cost of effort increases fast enough with consumption, incentive efficient allocations involve random effort.

*Related Literature.* A recent paper by Bisin and Gottardi [6] analyzes an adverse selection economy as one with consumption externalities . Because consumption plans must satisfy incentive constraints, the consumption plan of one agent type affects the set of admissible plans of the other type. These authors construct an appropriately enlarged market structure which allows then to decentralize incentive efficient allocations. Greenwald and Stiglitz [13] and Arnott, Greenwald and Stiglitz [2] also emphasize the importance of external effects in economies with asymmetric information, but there, the external effects are modeled as exogenous. We show that the external effects arise endogenously once incentive constraints are explicitly considered.

The idea that lotteries help separate types on the basis of their attitude towards risk is discussed by Prescott and Townsend [23, 24] and further investigated by Cole [10] and Arnott and Stiglitz [3]. Using duality theory, we can bring to light the welfare analysis underlying this discussion and establish a formal link between the separating role of lotteries and the presence of non-convexities arising from the welfare effects of incentives. In recent work, Kehoe, Levine and Prescott [18] study an exchange economy where agents trade after they learn their type. They show that if the agents' preferences display decreasing absolute risk aversion then lotteries are suboptimal. We obtain analogous results, also based on absolute risk aversion, for the case of adverse selection and moral hazard.

Bennardo and Chiappori [4] study a moral hazard economy characterized by the presence of both idiosyncratic and aggregate uncertainty where consumption and leisure are complementary goods. They show that if the cost of effort increases fast enough with consumption, incentive efficient allocations involve a random

effort level. We obtain similar results for the case of purely idiosyncratic uncertainty and show that optimal effort is random also if the aggregate endowment is sufficiently large.

Our approach is inspired by the work of Makowski and Ostroy [20]. These authors use a linear programming model to study large economies with full information. Gretsky, Ostroy and Zame (1999) present a similar linear programming treatment of large assignment economies. This work is a first step in trying to introduce incentive constraints in these models.

The structure of the paper is as follows. In section 2, we present the adverse selection model. We set up the linear programming problem and its dual. Then we use the complementary slackness theorem to characterize incentive efficient allocations and study their properties. Section 3 presents a similar analysis for the case of moral hazard. The proofs are deferred to the Appendix which involves an application of Linear Semi-Infinite Programming.

# 2 Adverse Selection

## 2.1 The Economy

Consider an exchange economy with a single consumption good and a continuum of agents of two types $i = L, H$. The fraction of agents of type $i$ is denoted by $\xi_i$.

The agents in the economy are subject to idiosyncratic endowment shocks. Specifically, each agent can be in one of two states $s = 1, 2$. At each state, the agent is endowed with a different amount $\omega_s$ of the good, where $0 < \omega_1 < \omega_2$. The probability that state 1 (the low endowment state) is realized is higher for an agent of type $H$ ("high risk") than for an agent of type $L$ ("low risk"). These probabilities will be denoted by $\theta_H$ and $\theta_L$, respectively, so that $0 < \theta_L < \theta_H < 1$. Agents of type $i$ have von Neumann-Morgenstern preferences over contingent co nsumption plans as defined by the Bernoulli utility function $U_i : \mathbb{R}_+ \to \mathbb{R}$, where $U_i$ is twice continuously differentiable, strictly increasing, and strictly concave with $\lim_{c\to 0} U_i'(c) = \infty$ and $\lim_{c\to\infty} U_i'(c) = 0$.[2]

Idiosyncratic shocks are independent across agents, rendering no uncertainty at the aggregate level.[3] Ex post, the fraction of type-$i$ agents with a low endowment is $\theta_i$, and the average endowment of the type-$i$ group is $\bar{\omega}_i = \theta_i\omega_1 + (1 - \theta_i)\omega_2$. The ex-post aggregate endowment is given by $\bar{\omega} = \xi_L\bar{\omega}_L + (1 - \xi_L)\bar{\omega}_H$.

Agents choose their contingent consumption plans before the realization of the individual shock. The structure of individual uncertainty is common knowledge and the realization of the endowment shocks is observable. State-contingent net trades are perfectly verifiable and fully enforceable ex post. However, an individual agent's type is known only to herself.

## 2.2 Allocations

In this section, we define the space of allocations and describe allocations which are physically feasible and incentive compatible. Then we define incentive efficient allocations.

Let $Z$ denote the net trade set of an agent; that is, the set of all pairs $z = (z_1, z_2) \in \mathbb{R}^2$ such that $z_s \geq -\omega_s$ for $s = 1, 2$. For any $z \in Z$, the expected net trade of an agent of type $i$ is given by

$$r_i(z_1, z_2) = \theta_i z_1 + (1 - \theta_i)z_2,$$

and her expected utility is defined as

$$EU_i(z_1, z_2) = \theta_i U_i(\omega_1 + z_1) + (1 - \theta_i)U_i(\omega_2 + z_2).$$

An allocation in this economy is a random net trade assignment for each type. That is, before the realization of individual uncertainty, each agent receives a lottery

[2] Our model is slightly more general than the Rothschild-Stiglitz [25] economy, where state utilities are type-invariant. As we will see, differences in tastes across types may have important consequences for the nature of the incentive efficient allocations. All results can be extended to state-dependent utilities and to any finite number of idiosyncratic states.

[3] The measurability problems associated with the aggregation of a continuum of independent and identically distributed random variables are well known; see Judge [17]. A way around this problem, as shown by Sun [27], is to consider a process of individual uncertainty which is measurable with respect to hyperfinite Loeb product spaces. For such processes almost sure pairwise independence guarantees the validity of the law of large numbers (see Sun [27, Theorem 3.10, p. 436]). For alternative approaches to this problem see Al-Najjar [1] and Hammond and Lisboa [15].

depending on her type. The realization of this lottery determines a net trade, and thus, a contingent consumption plan for the agent. Formally, an allocation is a pair of probability measures on $Z$. Denote the space of Borel measures on $Z$ which have compact support by $M_c(Z)$. The space of allocations is the set of pairs $(x_L, x_H) \in M_c(Z) \times M_c(Z)$ such that

$$\int_Z dx_i = 1, \quad x_i \geq 0, \quad i = L, H.^4 \tag{2.1}$$

Here, $x_i$ is a probability measure describing the lottery assigned to each agent of type $i$. For any Borel set $B \subset Z$, $x_i(B)$ is the probability that the agent is assigned a net trade $z \in B$. Under this formulation, deterministic assignments are given by degenerate measures. The law of large numbers implies that $x_i$ is also the distribution of net trades of type-$i$ agents once the outcomes of all individual lotteries are realized (e.g. $x_i(B)$ is the fraction of agents of type $i$ assigned to a net trade $z \in B$).

An allocation is *feasible* if the aggregate net trade is non-positive. The average net trade of the agents of type $i$ is given by $\langle r_i, x_i\rangle = \int_Z r_i dx_i$. Hence, the aggregate resource constraint is

$$\xi_L\langle r_L, x_L\rangle + (1-\xi_L)\langle r_H, x_H\rangle \;\; \leq \;\; 0. \tag{2.2}$$

Since types are private information, an agent of type $i$ may claim to be any of the two types $j = 1, 2$, and receive expected utility $\langle EU_i, x_j\rangle = \int_Z EU_i dx_j$. An allocation is *incentive compatible* if it is not in the interest of agents to misrepresent their type:

$$\langle EU_i, x_i\rangle \;\; \geq \;\; \langle EU_i, x_j\rangle, \quad j \neq i, \quad i = L, H. \tag{2.3}$$

An allocation is *incentive efficient* if it is feasible, incentive compatible, and there exists no other feasible and incentive compatible allocation that is weakly preferred by both types and strictly preferred by at least one type.

## 2.3 The Primal and Dual Problems

In this section, we show that every incentive efficient allocation is an optimal solution to a linear programming problem.

The problem of the planner is to find an allocation so as to maximize a weighted average of the utilities of the two types subject to the feasibility and the incentive constraints. Note that utilities are linear in the lotteries. Constraints (2.1)-(2.3) are also linear. In order to extend the inner product notation $\langle \cdot\,, \cdot\rangle$ to the adding-up constraint (2.1), we define $\mathcal{I} : Z \to \{0, 1\}$ to be the characteristic function on $Z$ and write $\langle \mathcal{I}, x_i\rangle = \int_Z dx_i$ for $i = L, H$. For given positive welfare weights $(\gamma_L, \gamma_H)$, with $\gamma_H = 1 - \gamma_L$, the problem of the planner is to find an allocation $(x_L, x_H) \in M_c(Z) \times M_c(Z)$ to solve

$$(D) \qquad \sup \quad \gamma_L \langle EU_L, x_L\rangle + (1-\gamma_L)\langle EU_H, x_H\rangle$$

s.t.

$$\langle \mathcal{I}, x_L\rangle = 1, \tag{2.4}$$

$$\langle \mathcal{I}, x_H\rangle = 1, \tag{2.5}$$

$$-\langle EU_L, x_L\rangle + \langle EU_L, x_H\rangle \leq 0, \tag{2.6}$$

$$\langle EU_H, x_L\rangle - \langle EU_H, x_H\rangle \leq 0, \tag{2.7}$$

$$\xi_L\langle r_L, x_L\rangle + (1-\xi_L)\langle r_H, x_H\rangle \leq 0, \tag{2.8}$$

$$x_L, x_H \geq 0. \tag{2.9}$$

Problem $(D)$ is a linear programming problem. Standard results in linear programming theory show that problem $(D)$ is *dual* to another linear programming problem, known as the *primal* problem or problem $(P)$. Whereas problem $(D)$ is a maximization problem with an infinite number of variables and a finite number of constraints, problem $(P)$ is a minimization problem with a finite number of variables and an infinite number of constraints. In optimization theory, these kind of problems are known as Linear Semi-Infinite Programming (LSIP) problems.[5] As we shall see, the primal and dual problems are related because the primal variables are also the shadow prices of the dual constraints, and vice versa.

Problem $(P)$, which is derived in detail in Appendix A, consists of finding a quintuple $(\alpha_L, \alpha_H, \beta_L, \beta_H, q) \in \mathbb{R}^5$ to solve

$$(P) \qquad \inf \quad \alpha_L + \alpha_H$$

s.t.

$$\alpha_L \geq \gamma_L EU_L(z) + \beta_L EU_L(z) - \beta_H EU_H(z) - q\xi_L r_L(z) \quad \forall z \in Z, \tag{2.10}$$

$$\alpha_H \geq (1-\gamma_L) EU_H(z) - \beta_L EU_L(z) + \beta_H EU_H(z) - q(1-\xi_L) r_H(z) \quad \forall z \in Z, \tag{2.11}$$

$$\beta_L, \beta_H, q \geq 0, \tag{2.12}$$

[5]An LSIP problem is an optimization problem with linear objective and linear constraints in which either number of variables or the number of constraints is finite. For an excellent survey on LSIP theory, see Goberna and López [12].

where $(\alpha_L, \alpha_H)$ are the shadow prices of the adding-up constraints (2.4) and (2.5), $(\beta_L, \beta_H)$ are the shadow prices of the incentive constraints (2.6) and (**??**), and $q$ is the shadow price of the resource constraint (2.8).

Denote the optimal values for problems $(P)$ and $(D)$ by $\nu(P)$ and $\nu(D)$, respectively. It is easy to see that both problems are consistent (i.e. their feasible sets are not empty) as well as bounded (i.e. $\nu(P)$ and $\nu(D)$ are finite).[6] However, unlike an ordinary linear program, a bounded LSIP problem need not have optimal solutions. Moreover, the primal and dual problems need not have the same optimal value, as a "positive duality gap" may occur: $\nu(P) - \nu(D) > 0$. The next two theorems show that the problems in this paper are well-behaved.

**Theorem 2.1** $\nu(D) = \nu(P)$.

**Theorem 2.2** *Problems $(P)$ and $(D)$ have optimal solutions.*

Hence, the maximum in problem $(D)$ and the minimum in problem $(P)$ are well-defined and they are equal. A nice property of the dual problem is that the space of variables can be restricted without loss of generality to *measures with finite support*. Proposition 2.1 below establishes the formal result. Let $M_F$ denote the set of finitely supported measures on $Z$. Consider the restricted dual problem, $(D_F)$, where allocations are defined in $M_F \times M_F$. Denote its optimal value by $\nu(D_F)$.

**Proposition 2.1** *Problem $(D_F)$ has optimal solutions. Further, $\nu(D_F) = \nu(D)$.*

Results of this kind are common to many LSIP programs (Goberna and López [12]).

## 2.4 Full Information

To gain some insight into the linear programming framework, we first consider the case of full information. This case provides a benchmark for the rest of the analysis.

When agent types are public information, a simpler pair of LSIP problems obtains. Theorems 2.2 and 2.1 and Proposition 2.1 extend to these problems. The dual problem $(D_0)$ is obtained by eliminating the incentive constraints in $(D)$. Every first best

[6]The allocation under autarky (where both types have zero net trade) is feasible and incentive compatible, so problem $(D)$ is consistent. In problem $(P)$, let $\beta_L = \beta_H = 0$ and $q = q^0 > 0$. Since $EU_i$ is strictly concave, the right-hand side of both (2.10) and (2.11) is bounded on $Z$. Fixing $\alpha_L = \alpha_L^0$ and $\alpha_H = \alpha_H^0$ sufficiently large ensures that (2.10) and (2.11) hold. By the weak duality theorem (see Krabs [19, Theorem I.3.1]), since problems $(P)$ and $(D)$ are consistent, they are also bounded: $\gamma_L EU_L(0) + (1-\gamma_L) EU_H(0) \le \nu(D) \le \nu(P) \le \alpha_L^0 + \alpha_H^0$.

allocation is an optimal solution to problem $(D_0)$ for some weight $\gamma_L \in (0,1)$. The primal $(P_0)$ is obtained by eliminating the shadow prices of the incentive constraints, $\beta_L$ and $\beta_H$, and all the associated terms in $(P)$. Thus, the objective function in problem $(P_0)$ is the same as in $(P)$, and the constraint systems are given by

$$\alpha_i \;\geq\; v_i(z_i;q) \quad \forall z_i \in Z \quad i = L, H; \tag{2.13}$$

where

$$v_i(z_i;q) = \gamma_i EU_i(z_i) - q\xi_i r_i(z_i). \tag{2.14}$$

Consider the terms in the function $v_i(z_i;q)$. The first term, $\gamma_i EU_i(z_i)$, is the contribution to welfare when agents of type $i$ have net trade $z_i$. The second term, $q\xi_i r_i(z_i)$, is the value of the aggregate net trade of these agents when the shadow price of the good is $q$. Thus, $v_i(z_i;q)$ represents the *net contribution to social welfare* when agents of type $i$ have net trade $z_i$ and the shadow price of resources is $q$.

According to (2.13), a feasible value of $\alpha_i$ is an upper bound of $v_i(\cdot\,;q)$ for given $q$. The *maximal* net contribution of type-$i$ agents at price $q$ is defined as

$$v_i^*(q) \equiv \sup_{z_i \in Z} v_i(z_i;q) = \sup_{z_i \in Z}\{\gamma_i EU_i(z_i) - q\xi_i r_i(z_i)\}, \tag{2.15}$$

so the primal systems (2.13) can be put in the form

$$\alpha_i \geq v_i^*(q), \quad i = L, H. \tag{2.16}$$

Because the objective of problem $(P_0)$ is to minimize the sum of the $\alpha_i$'s, the two constraints in (2.16) bind at an optimum. Thus, the optimal shadow price $q^*$ of resources minimizes the sum of the types' maximal net contributions:[7]

$$q^* = \arg\left\{ \min_{q \geq 0} \{v_L^*(q) + v_H^*(q)\} \right\},$$

while the optimal value of $\alpha_i$ is the maximal net contribution of type $i$ at price $q^*$: $\alpha_i^* = v_i(q^*)$.

*First best allocations.* The complementary slackness theorem (see, for instance, Krabs [19, Theorem I.3.3]) allows us to characterize optimal solutions for problems

[7] This full information economy is an example of the general problem studied by Makowski and Ostroy [20]. In particular, $\alpha_i^*(q)$ is the conjugate or indirect utility, redefined in its expected value form for economies with uncertainty. These authors have shown how the fact that the constraints of the primal program (the "pricing problem" in their terminology) can be incorporated into the objective function is characteristic of the LP version of General Equilibrium.

($P_0$) and ($D_0$). According to the theorem, feasible solutions $(\alpha_L, \alpha_H, q)$ and $(x_L, x_H)$ for problems ($P_0$) and ($D_0$), respectively, are optimal if and only if they satisfy the complementary slackness conditions:

$$q\big(\xi_L\langle r_L, x_L\rangle + (1-\xi_L)\langle r_H, x_H\rangle\big) = 0, \tag{2.17}$$

$$\alpha_i = v_i^*(q) = v_i(z_i; q) \quad \text{if} \;\; x_i(z_i) > 0,\; i = L, H. \tag{2.18}$$

Condition (2.17) states that the optimal shadow price $q^*$ is a complementary multiplier for the resource constraint (2.8). Since the monotonicity of preferences implies that $q^*$ is positive, (2.17) implies that the aggregate net trade is zero. Condition (2.18) states that the optimal assignments, $x_L^*$ and $x_H^*$, are complementary multiplier vectors for the respective constraint systems in (2.13). This implies that

$x_i^*$ *puts weight only on net trades* $z_i$ *that maximize the net contribution to social welfare of type* $i$ *at price* $q^*$. H owever, $v_i(\cdot\,; q^*)$ is a strictly concave function, and has at most one maximum. Thus, $x_i^*$ is a degenerate measure; that is, randomization is never optimal. Further, it is easily verified that both types are fully insured as their optimal consumption is independent of the realization of the idiosyncratic shock. In summary, conditions (2.17)-(2.18) yield standard efficiency results for convex economies with full information and no aggregate uncertainty: all agents are fully insured and the aggregate consumption equals the aggregate endowment. We now use a similar characterization to derive the more subtle properties of the optima when agent types are private information.

## 2.5 Incentive Efficiency

When types are private information, allocations must provide incentives for the agents to reveal their type. When $\gamma_L$ is very large, first best allocations assign higher consumption to type $L$ than to type $H$, so agents of type $H$ are inclined to lie. Similarly, when $\gamma_L$ is very low, first best allocations give higher consumption to type $H$, which induces the agents of type $L$ to lie. It is easily verified that, for some intermediate weight, it is optimal that both types consume the ex-post average endowment $\bar{\omega}$ with certainty. This weight is given by

$$\bar{\gamma}_L = \Big(1 + \frac{(1-\xi_L)U_L'(\bar{\omega})}{\xi_L U_H'(\bar{\omega})}\Big)^{-1},$$

and corresponds to the only first best allocation that is also incentive compatible. The next proposition describes a partitioning of the set of incentive effici ent allocations into three regions according to which incentive compatibility constraint binds.

**Proposition 2.2** *The set of incentive efficient allocations has three regions:*

*(i) When $\gamma_L = \bar{\gamma}_L$, the incentive efficient allocation assigns each type $\bar{\omega}$ units of consumption in every state, and the two incentive constraints trivially bind. Thus, $\beta_L = \beta_H = 0$.*

*(ii) When $\bar{\gamma}_L < \gamma_L < 1$, incentive efficient allocations assign higher expected consumption to type L than to type H and, only the incentive constraint of type H binds. In this case, $\beta_L = 0$ and $\beta_H > 0$ .*

*(iii) When $0 < \gamma_L < \bar{\gamma}_L$, incentive efficient allocations assign higher expected consumption to type H than to type L and, only the incentive constraint of type L binds. In this case, $\beta_L > 0$ and $\beta_H = 0$.*

Cases (*ii*) and (*iii*) in Proposition 2.2 are essentially symmetric and can be studied separately.[8]

### 2.5.1 The incentives of type-$H$ agents

In this section, we characterize incentive efficient allocations in which the incentive constraint of type $H$ binds. Throughout we let $\gamma_L \in (\bar{\gamma}_L, 1)$ and assume, without loss of generality, $\beta_L = 0$.

The first constraint system in problem $(P)$ is given by

$$\alpha_L \quad \geq \quad v_L(z_L; \beta_H, q) \quad \forall z_L \in Z, \tag{2.19}$$

where the function

$$v_L(z_L; \beta_H, q) = \gamma_L EU_L(z_L) - q\xi_L r_L(z_L) - \beta_H EU_H(z_L) \tag{2.20}$$

represents the *net contribution to social welfare of type-L agents* when types are private information. The net contribution of type $L$ is adjusted with respect to its full information version and, unlike the latter, depends on the shadow price of the incentive constraint of type $H$. Specifically, a new term arises which is not present under full information: $-\beta_H EU_H(z_L)$. This term reflects an *external cost* that arises as a result of the effect of the assignments to type $L$ on type-$H$ agents. That is, the better the assignment of type $L$ in the eyes of type-$H$ agents, the more costly it is to prevent the latter

[8] Prescott and Townsend [23, Section 3] characterize the set of incentive efficient allocations when utilities are identical across types using the first-order conditions of the planner problem. In this case, $\bar{\gamma}_L = \xi_L$.

from lying. The total cost of assignments to type $L$ is given by the sum of the resource cost and the external cost: $q\xi_L r_L(z_L) + \beta_H EU_H(z_L)$. When $\beta_H$ is positive, the external cost is positive and, for a given $q$, the total shadow cost is higher than under full information. Note that, since $EU_H(\cdot)$ is strictly concave, the total cost is *not* a convex function of $z_L$.

The second constraint system is given by

$$\alpha_H \geq v_H(z_H; \beta_H, q) \quad \forall z_H \in Z, \tag{2.21}$$

where the function

$$v_H(z_H; \beta_H, q) = (1-\gamma_L)EU_H(z_H) - q(1-\xi_L)r_H(z_H) + \beta_H EU_H(z_H) \tag{2.22}$$

represents the *net contribution to social welfare of type-H agents* at prices $\beta_H$ and $q$. The third term in the function reflects a *benefit* of assignments to type $H$ on the incentives of these agents. Clearly, the higher the utility that type-$H$ agents derive from their own assignment, the more incentives they have to report the truth. The shadow cost of assignments to type $H$ is given by the resource cost net of the benefit on incentives: $q(1-\xi_L)r_H(z_H) - \beta_H EU_H(z_H)$. When $\beta_H$ is positive, the benefit on incentives is positive and, for a given $q$, the total cost is lower than under full information. In this case, however, the cost is a convex function of $z_H$.

According to (2.19) and (2.21), a feasible value of $\alpha_i$ is an upper bound of $v_i(\cdot\,; \beta_H, q)$ for given $\beta_H$ and $q$. The *maximal* net social contribution of type-$i$ agents at prices $\beta_H$ and $q$ is defined as

$$v_i^*(\beta_H, q) \equiv \sup_{z_i \in Z} v_i(z_i; \beta_H, q). \tag{2.23}$$

Constraints (2.19)-(2.21) can be expressed as

$$\alpha_i \geq v_i^*(\beta_H, q), \quad i = L, H. \tag{2.24}$$

Thus, the optimal prices $\beta_H^*$ and $q^*$ minimize the sum of the types' maximal net contributions:

$$(q^*, \beta_H^*) \in \arg\left\{ \min_{\beta_H, q \geq 0} v_L^*(\beta_H, q) + v_H^*(\beta_H, q)\} \right\},$$

while the optimal net contributions are given by $\alpha_i^* = v_i^*(\beta_H^*, q^*)$ for $i = L, H$.

The principal result in this section is the characterization of incentive efficient allocations. According to the complementary slackness theorem, feasible solutions

$(\alpha_L, \alpha_H, \beta_H, q)$ and $(x_L, x_H)$ for problems $(P)$ and $(D)$ respectively, are optimal if and only if:

$$\beta_H(\langle EU_H, x_H\rangle - \langle EU_H, x_L\rangle) = 0, \tag{2.25}$$

$$q(\xi_L\langle r_L, x_L\rangle + (1-\xi_L)\langle r_H, x_H\rangle) = 0, \tag{2.26}$$

$$\alpha_i = v_i^*(\beta_H, q) = v_i(z_i; \beta_H, q) \quad \text{if} \;\; x_i(z_i) > 0,\; i = L, H. \tag{2.27}$$

By Proposition 2.2 we know that $\beta_H^* > 0$, and it can be verified that $q^* > 0$.[9] Thus, incentive efficient allocations satisfy the following three properties. First, an agent of type $H$ is indifferent between her assignment and that of type $L$. Second, the aggregate net trade is zero. Third, the lottery $x_i^*$ assigned to type $i$ puts weight only on net trades that maximize the net contribution of type $i$ at prices $q^*$ and $\beta_H^*$. The third property leads to the following result which stems directly from the strict concavity of $v_H(\cdot\,; \beta_H^*, q^*)$ and first-order conditions.

**Proposition 2.3** *$x_H^*$ is degenerate and provides full insurance.*

Proposition 2.3 states that agents of type $H$ should be fully insured when their average consumption is lower than $\bar{\omega}$. Full insurance increases both the utility of type $H$ for a given resource cost and her incentives to report the truth (decreasing the total cost of the assignment). Since the

incentive constraint of type $L$ is not binding, type $L$ wants to tell the truth. On the other hand, more insurance to type $L$, while increasing her own utility, may raise the incentives of type $H$ to lie. For a given resource cost, an increase in insurance to type $L$ raises the total cost. As a result, the optimal assignment to type $L$ is distorted from full insurance. Since the total cost is not convex on $z_L$, optimal assignments may even be random. If utilities are type-invariant (as in Rothschild and Stiglitz [25] and Wi lson [28]), however, optimal assignments to type $L$ are deterministic.

**Proposition 2.4** *If utilities are type-invariant, then $x_L^*$ is degenerate and assigns lower consumption in state* 1 *than in state* 2*.*

Given the consumption level of type $H$, the planner chooses $x_L$ so as to increase the utility of type $L$ as much as possible without inducing type $H$ to lie. For a given resource cost, the planner may increase the net social contribution of type $L$ (i.e., widen the gap between the utility of type $L$ and the external cost of the assignment)

[9] $q^*$ cannot be zero. If $U_H$ is unbounded then $v_H(\cdot\,;\beta^*, 0)$ is unbounded on $Z$, which contradicts (2.27). If $U_H$ is bounded, then $v_H(\cdot\,;\beta^*, 0)$ does not have a maximum (recall that $\lim_{c\to\infty} U_i(c) = 0$), a contradiction since $(D)$ is solvable and the support of $x_H^*$ is non-empty.

by exploiting differences in the preferences of the two types. Because type $L$ is more likely to be in state 2 than type $H$, assignments which give higher consumption in state 2 relative to state 1 are relatively more attractive for type $L$ than for type $H$. This explains why partial insurance to type $L$ is incentive efficient. Lottery assignments, in turn, exploit differences in *preferences for risk*. When utilities are identical across types, there are no such differences and lotteries do not help enhance efficiency. There are economies, however, where lotteries play a useful role. Consider the extreme case where agents of type $L$ are risk neutral and agents of type $H$ are risk averse. One can then easily devise a random allocation which is incentive compatible and first best efficient. First, agents announcing type $H$ are assigned their first best deterministic consumption level. Agents announcing type $L$, on the other hand, receive a non-degenerate lottery. Whereas the implied expected consumption (and, hence, the utility) of type $L$ is also the first best one, the risk involved is such that the certainty equivalent that type-$H$ agents assign to the lottery is no greater than their own deterministic consumption. This prevents any misrepresentation.

The next proposition shows that, when type $L$ is at least as risk averse as type $H$, lotteries are not useful. Let $A_i : \mathbb{R}_+ \to \mathbb{R}_+$ denote the index of absolute risk aversion for type $i$; that is, $A_i(c) = -\frac{U_i''(c)}{U_i'(c)}$ for $c \in \mathbb{R}_+$.

**Proposition 2.5** *If $A_L(c) \geq A_H(c)$ for all $c \in \mathbf{R}_+$, then $x_L^*$ is degenerate and assigns lower consumption in state* 1 *than in state* 2.

### 2.5.2 The incentives of type-$L$ agents

An analysis analogous to the one in the previous section allows us to characterize incentive efficient allocations in which the incentive constraint of type $L$ binds. In this case, incentive efficient allocations provide full insurance to type $L$ and over insurance to type $H$. Full insurance to a type-$L$ agent increases her utility as well as her incentive to tell the truth. The optimal way to induce type $L$ to tell the truth is to provide over insurance to type $H$. Intuitively, an over insured position is less attractive for type-$L$ agents than for type-$H$ agents since the former are less likely to be in the low endowment state. If type $L$ is sufficiently more risk averse than type $H$, the assignment to type $H$ may be random, since type $L$ is more reluctant to accept random assignments than type $H$. By contrast, if type $H$ is at least as risk averse as type $L$, optimal assignments to type $H$ are deterministic. The next proposition summarizes the properties of incentive efficient allocations when the incentive constraint of type $L$ binds.

**Proposition 2.6** *For any $\gamma_L \in (0, \bar{\gamma}_L)$, incentive efficient allocations satisfy the following.*

*(i) $x_L^*$ is degenerate and provides full insurance; and*

*(ii) If utilities are type-invariant, then $x_H^*$ is degenerate and assigns higher consumption in state 1 than in state 2. More generally, if utilities are not type invariant, and $A_H(c) \geq A_L(c)$ for all $c \in \mathbb{R}_+$, then $x_H^*$ is degenerate and assigns higher consumption in state 1 than in state 2.*

# 3 Moral Hazard

## 3.1 The Economy

Consider an exchange economy with two goods, namely time and a single consumption good, and a measure one of ex ante identical agents. Each agent faces an idiosyncratic shock leading to two possible states, $s = 1, 2$. In state $s$, the agent is endowed with $\omega_s$ units of consumption where $0 < \omega_1 < \omega_2$. Prior to the realization of the shock, the agent is endowed with one unit of time which he allocates between leisure and effort in preventing the realization of state 1. Each agent can choose to exert either high or low effort, with the set of effort levels denoted by $E = \{e_L, e_H\}$, where $0 < e_L < e_H < 1$. Exerting high rather than low effort reduces the probability that the agent will end up in state 1. The probability of state 1 with high and low effort will be denoted by $\theta_H$ and $\theta_L$, respectively, so that $0 < \theta_H < \theta_L < 1$. Agents have von Neumann-Morgerstern preferences as defined by the utility function $u : E \times \mathbb{R}_+ \to \mathbb{R}$. The utility of consumption $c$ under effort $e_i$ is given by $U_i(c) = u(e_i, c)$, where $U_i$ is assumed twice continuously differentiable, strictly increas ing, and strictly concave with $\lim_{c \to 0} U_i'(c) = \infty$ and $\lim_{c \to \infty} U_i'(c) = 0$. Since effort is costly, we assume that there is some positive constant $d$ such that $U_L(c) - U_H(c) > d$ for all $c \in \mathbb{R}_+$.

Idiosyncratic shocks are independent and render no aggregate uncertainty. The ex-post average endowment of the agents who provide effort $e_i$ is then given by $\bar{\omega}_i = \theta_i \omega_1 + (1 - \theta_i)\omega_2$. The structure of uncertainty is common knowledge and the realization of the endowment shocks is observable. State-contingent net trades are perfectly verifiable and fully enforceable. However, effort is private information.

## 3.2 Allocations

In this section, we define feasible and incentive compatible allocations. Then we define incentive efficient allocations.

Let $Z \subset \mathbb{R}^2_+$ denote the net trade set of an agent. For any $z = (z_1, z_2) \in Z$, the expected net trade of an agent with effort $e_i$ is

$$r_i(z_1, z_2) = \theta_i z_1 + (1 - \theta_i) z_2,$$

and his expected utility is given by

$$EU_i(z_1, z_2) = \theta_i U_i(\omega_1 + z_1) + (1 - \theta_i) U_i(\omega_2 + z_2).$$

An allocation in this economy specifies an effort level and a net trade for each agent. Both specifications are allowed to be random and are given as follows. First, the agent receives a lottery which prescribes an effort level. After the agent chooses his effort and conditional on the prescription received, a second lottery specifies a net trade. It is useful for our purposes to describe an allocation as a pair of measures $(x_L, x_H) \in M_c(Z) \times M_c(Z)$ such that

$$\langle \mathcal{I}, x_L + x_H \rangle = 1, \quad x_i \geq 0, \quad i = L, H.^{10} \tag{3.28}$$

Here, $||x_i|| = \langle \mathcal{I}, x_i \rangle$ is the probability that effort $e_i$ is specified in the first lottery, and the equality in (3.28) is an adding-up condition.[11] In addition, $\frac{1}{||x_i||} x_i$ is a probability measure which describes the random net trade assigned conditional on specification $e_i$ (i.e., the second lottery). Note that the uncertainty involved in an allocation is resolved in two steps. In the first step, the agent may be uncertain about the effort that he will be asked to provide. This occurs when both $||x_L||$ and $||x_H||$ are positive. In the second step, the agent finds out his effort specification, but he may be uncertain about his contingent consumption plan. This occurs when $\frac{1}{||x_i||} x_i$ is a non-degenerate measure. Allowing for random effort is natural since the consumption set, $E \times \mathbb{R}_+$, displays indivisibilities.[12] As we shall see, the role for random net trade assignments arises from the unobservability of effort.

From the perspective of the entire economy, $||x_i||$ is the fraction of agents who are assigned $e_i$, and $\frac{1}{||x_i||} x_i$ is the distribution of their net trades. The ex-post aggregate net trade of these agents is given by $\langle r_i, x_i \rangle = \int_Z r_i dx_i$ (provided the agents conform to their specification). An allocation is *feasible* if the aggregate net trade is non-positive:

$$\langle r_L, x_L \rangle + \langle r_H, x_H \rangle \quad \leq \quad 0. \tag{3.29}$$

[11]Here, $||x_i|| = \langle \mathcal{I}, x_i \rangle = x_i(Z)$ is the total variation of $x_i$.

[12]It is well known that lotteries play a role in when consumption sets are non-convex. See Shell and Wright [26].

When effort $e_i$ is specified but $e_j$ is the actual effort provided, the agent's expected utility is $\frac{1}{||x_i||}\langle EU_j, x_i\rangle$. An allocation is *incentive compatible* if it is not in the interest of the agents to deviate from their specifications:

$$\langle EU_i, x_i\rangle \;\geq\; \langle EU_j, x_i\rangle, \quad j \neq i, \quad i = L, H. \tag{3.30}$$

An *incentive efficient* allocation is a feasible and incentive compatible allocation that maximizes the ex-ante expected utility of the agents.

## 3.3 The Primal and Dual Problems

An incentive efficient allocation can be determined as a solution to a planning problem, more precisely a dual LSIP problem. The problem is to choose an allocation $(x_L, x_H) \in M_c(Z) \times M_c(Z)$ that solves

$$(D) \qquad \sup \quad \langle EU_L, x_L\rangle + \langle EU_H, x_H\rangle$$

s.t.

$$\begin{aligned}
\langle \mathcal{I}, x_L + x_H\rangle &= 1, && (3.31)\\
-\langle EU_L, x_L\rangle + \langle EU_H, x_L\rangle &\leq 0, && (3.32)\\
\langle EU_L, x_H\rangle - \langle EU_H, x_H\rangle &\leq 0, && (3.33)\\
\langle r_L, x_L\rangle + \langle r_H, x_H\rangle &\leq 0, && (3.34)\\
x_L, x_H &\geq 0. && (3.35)
\end{aligned}$$

The primal LSIP problem consists of finding a quadruple $(\alpha, \beta_L, \beta_H, q) \in \mathbb{R}^4$ that solves

$$(P) \qquad \inf \quad \alpha$$

s.t.

$$\begin{aligned}
\alpha &\geq EU_L(z) - \beta_L[EU_H(z) - EU_L(z)] - qr_L(z) \quad \forall z \in Z, && (3.36)\\
\alpha &\geq EU_H(z) - \beta_H[EU_L(z) - EU_H(z)] - qr_H(z) \quad \forall z \in Z, && (3.37)\\
&\beta_L, \beta_H, q \geq 0, && (3.38)
\end{aligned}$$

where $\alpha$, $(\beta_L, \beta_H)$, and $q$ are the shadow prices of the adding-up constraint (3.31), the incentive constraints (3.32)-(**??**), and the resource constraint (3.34), respectively.

In the Appendix, we show that problems $(P)$ and $(D)$ have optimal solutions and that their optimal values coincide. We also show that, in characterizing incentive efficient allocations, there is no loss of generality in restricting attention to measures with finite support. These results are analogous to Theorems 2.2 and 2.1 and Proposition 2.1 in Section 2.3.

## 3.4 Incentive Efficiency

In this section, we characterize incentive efficient allocations. Since the aggregate endowment is constant, under full information it is optimal that each agent consumes with certainty the average endowment in the economy. However, since effort is costly and cannot be publicly observed, an agent who is fully insured will shirk to low effort when high effort is specified. For this reason, allocations which specify high effort with positive probability can only provide partial insurance. On the other hand, agents must be subject to the minimum risk possible that is compatible with their incentives to conform to a high-effort specification. So the incentive constraint (**??**) binds with $\beta_H > 0$. Implementing a low-effort specification is trivial. Since the incentive constraint (3.32) does not bind, we may let $\beta_L = 0$.

The first constraint system in problem $(P)$ is given by

$$\alpha \geq v_L(z_L; q) \quad \forall z_L \in Z, \tag{3.39}$$

where the function

$$v_L(z_L; q) = EU_L(z_L) - qr_L(z_L) \tag{3.40}$$

represents the *net contribution to social welfare with low effort*. The first term in (3.40) is the contribution to welfare when agents are assigned effort $e_L$ and net trade $z_L$. The second term is the value of the associated aggregate net trade when the shadow price of the good is $q$. Note that, since there are no welfare effects of incentives, the net social contribution with low effort is the same both under full and private information.

The second constraint system is given by

$$\alpha \geq v_H(z_H; \beta_H, q) \quad \forall z_H \in Z, \tag{3.41}$$

where the function

$$v_H(z_H; \beta_H, q) = EU_H(z_H) - qr_H(z_H) - \beta_H[EU_L(z_H) - EU_H(z_H)] \tag{3.42}$$

represents the *net contribution to social welfare with high effort.* The first and second terms in (3.42) are the contribution to welfare when agents are assigned effort $e_H$ and net trade $z_H$, and the value of the associated aggregate net trade, respectively. The third term is the welfare effect of incentives. If the net trade assigned is such that the agent has incentives to deviate to $e_L$, the term is negative and reflects a welfare *cost* which is proportional to the utility gain in the deviation. If the net trade assigned is such that the agent wants to conform to $e_H$, the term is positive and reflects a *benefit* which is proportional to the utility loss that a deviation would imply. The direct (i.e. full information) net contribution of the assignment is thus adjusted upward (downward) when it gives the right (wrong) incentives. The total cost of an assignment under high effort is given by the resource cost net of the w elfare effect of incentives: $qr_H(z_H) + \beta_H[EU_L(z_H) - EU_H(z_H)]$. Note that the cost function is *not* convex.

The maximal net contributions with high and low effort at prices $\beta_H$ and $q$ are:

$$v_L^*(q) \equiv \sup_{z_L \in Z} v_L(z_L; q), \tag{3.43}$$

$$v_H^*(\beta_H, q) \equiv \sup_{z_H \in Z} v_H(z_H; \beta_H, q). \tag{3.44}$$

It is thus possible to write conditions (3.39)-(3.41) as

$$\alpha \geq v^*(\beta_H, q),$$

where $v^*(\beta_H, q)$ is the *largest* of the two maximal net social contributions:

$$v^*(\beta_H, q) \equiv \max\{v_L^*(q), v_H^*(\beta_H, q)\}. \tag{3.45}$$

Because the objective of the primal problem is to minimize $\alpha$, the optimal prices $\beta_H^*$ and $q^*$ are determined by minimizing $v^*(\beta_H, q)$:

$$(\beta_H^*, q^*) \in \arg\left\{ \min_{\beta_H, q \geq 0} \left\{v^*(\beta_H, q)\right\}\right\},$$

while optimal net contribution is given by $\alpha^* = v^*(\beta_H^*, q^*)$.

We now turn to the characterization of incentive efficient allocations. According to the complementary slackness theorem, if $(\alpha, \beta_H, q)$ and $(x_L, x_H)$ are feasible for problems $(P)$ and $(D)$, respectively, then they are optimal if and only if:

$$\beta_H(\langle EU_L - EU_H, x_H\rangle) = 0, \tag{3.46}$$

$$q(\langle r_L, x_L\rangle + \langle r_H, x_H\rangle) = 0, \tag{3.47}$$

$$\alpha = v^*(\beta_H, q) = v_i(z_i; \beta_H, q) \quad \text{if } x_i(z_i) > 0,\ i = L, H. \tag{3.48}$$

We have already noted that $\beta_H^* > 0$, and it can be checked that $q^* > 0$. Thus, an incentive efficient allocation $(x_L^*, x_H^*)$ has the following three properties. First, when $e_H$ is assigned, the agent is indifferent between exerting effort $(e_H)$ and shirking $(e_L)$. Second, the aggregate net trade is zero. Third, $x_i^*$ *puts weight only on net trades which achieve the optimal net contribution.* That is, $x_L^*(z_L)$ is positive provided: (i) $z_L$ maximizes $v_L(\cdot\,; q^*)$, and (ii) $v_L(q^*) = v^*(\beta_H, q)$. Similarly, $x_H^*(z_H)$ is positive provided: (i) $z_H$ maximizes $v_H(\cdot\,; \beta_H^*, q^*)$, and (ii) $v_H(\beta_H, q^*) = v^*(\beta_H, q)$. The following proposition follows from the third property and is a direct result of the strict concavity of $v_L(\cdot\,; q^*)$ and first-order conditions.

**Proposition 3.1** *If* $||x_L^*|| > 0$ *then* $x_L^*$ *is degenerate and provides full insurance.*

Since a low effort assignment does not generate incentive effects, an agent who is assigned $e_L$ should be fully insured. Under a high effort specification, however, an increase in insurance may raise the incentives to shirk (increasing the shadow cost of the assignment). Since the shadow cost function is not a convex, net trade assignments under high effort may even be random. If utility is separable in effort and consumption, however, optimal assignments under high effort are deterministic.

**Proposition 3.2** *Suppose that utility is separable in consumption and effort. If* $||x_H^*|| > 0$ *then* $x_H^*$ *is degenerate and assigns lower consumption in state* 1 *than in state* 2.

The planner would like to increase as much as possible the utility of agents who are specified high effort without adversely affecting their incentives. The planner may increase the net contribution with high effort by exploiting differences in preferences with high and low effort. Partial insurance makes high effort relatively more attractive because it raises the probability of being in the high consumption state. Random net trade assignments exploit differences in preferences for risk. When utility is separable, there are not such differences and random net trade assignments are not optimal. Consider, however, the extreme case where an agent is risk neutral when his effort is high and risk averse when it is low. Then it is easy to find a random allocation which is incentive compatible and first best efficient. In this allocation, agents who are specified high effort are assigned a random net trade which yields the first best expected consumption and involves sufficient risk for the agent not to have incentives to shirk to low effort.

The next proposition shows that, if risk aversion does not decrease with effort, random net trade assignments are not optimal.[13] Let $A_i : \mathbb{R}_+ \to \mathbb{R}_+$ be the index of

[13] Arnott and Stiglitz [3] derive this result through a different argument.

absolute risk aversion of an agent with effort $e_i$; that is, $A_i(c) = -\frac{U_i''(c)}{U_i'(c)}$ for $c \in \mathbb{R}_+$.

**Proposition 3.3** *Suppose that $A_H(c) \geq A_L(c)$ for all $c \in \mathbb{R}_+$. If $||x_L^*|| > 0$ then $x_H^*$ is degenerate and assigns lower consumption in state 1 than in state 2.*

*Remark.* The conditions which characterize incentive efficient allocations with adverse selection and moral hazard are very similar (conditions (2.25)-(2.27) and (3.46)-(3.48), respectively). Their main differences relate to the definition of the net contribution functions in each model, and thereby in the third condition of the characterization. The net contributions differ in the terms which describe the welfare effects of incentives. With adverse selection, the assignments of both types generate incentive effects. In particular, the assignment of the type with the highest average consumption generates external costs because it affects the truth-telling incentives of the other type. With moral hazard, only the assignments of those agents who are specified high effort generate incentive effects. In this case, it is the incentives of *these* agents (not the incentives of *others*) that are affected.[14] A second important difference is that, while in the adverse selection model the fraction of each type is exogenously given, in the moral hazard model the fraction of people who provide each effort level is endogenous. As a result, whereas in the first model the net social contributions of the two types are defined independently, in the second model the net social contributions with high and low effort are inter-related. In particular, the optimal maximal net contribution is the largest of the maximal contributions with high and low effort at the optimal prices.

The remainder of this section focuses on optimal effort assignments. To avoid trivial solutions, we assume throughout that it is not incentive efficient to assign low effort with probability one and to provide the agents with full insurance.[15] Below we give conditions under which the optimal effort is random.

Consider the most preferred allocation which is feasible and incentive compatible, and specifies high effort with probability one. There may exists a feasible and incentive compatible allocation which specifies low effort with positive probability and is strictly preferred by the agents. When a fraction of the population provides low effort, the aggregate endowment decreases. Yet, if this fraction is taken to be small, the

[14]The source of this difference is the different form of the incentive constraints in the two models. Whereas with adverse selection both incentive constraints depend on assignments of the two types $x_L$ and $x_H$, with moral hazard the (relevant) incentive constraint depends only on the net trade assignment under high effort $x_H$.

[15]Assume there exists a feasible and incentive compatible allocation $(x_L, x_H)$ such that $x_L = 0$ and $\langle EU_H, x_H \rangle > U_L(\bar{\omega}_L)$.

loss in endowment is also small. Starting from the original high effort allocation, it is feasible to give an arbitrarily high fixed consumption to a sufficiently small fraction of agents and allow these agents to provide low effort without decreasing the average consumption of others by too much. The cost of introducing this lottery is that, due to a reduction in expected consumption, agents are slightly worse off if high effort is specified. The benefit is that, ex ante, every agent has a positive probability of being fully insured, exerting less effort and receiving a highly subsidized consumption level.[16] Since marginal utility of consumption decreases to zero, the cost of giving up a small amount of expected consumption becomes negligible as agents get wealthier. However, the disutility of effort is strictly positive. Proposition 3.4 asserts that, under the natural assumption that marginal utility of consumption does not increase with effort (decrease with leisure) and as long as the loss in expected endowment from switching from high to low effort, $\bar{\omega}_H - \bar{\omega}_L$, is bounded, agents are willing to give up some expected consumption to participate in the lottery provided they are sufficiently wealthy.

**Proposition 3.4** *Suppose that*

*(i)* $U'_H(c) \leq U'_L(c)$ *for all* $c \in \mathbb{R}_+$*, and*

*(ii)* $(\theta_L - \theta_H)(\omega_2 - \omega_1) < M$ *for some constant* $M$*, so* $(\bar{\omega}_H - \bar{\omega}_L)$ *is bounded.*

*Then there exists a threshold* $\hat{\omega}_H$ *such that, if* $\bar{\omega}_H \geq \hat{\omega}_H$ *then* $||x^*_H|| < 1$ *.*

In recent work, Bennardo and Chiappori [4] study a moral hazard environment where the optimal effort is typically random.[17] In their model, consumption and leisure are assumed to be complementary goods, so that the marginal utility of consumption decreases with effort, and the cost of effort increases with consumption. An important result in that paper is that, if the marginal utility of consumption decreases fast enough with effort then there may be a limit to the amount of expected consumption that the agent can receive while still being willing to provide high effort. Put differently, a really wealthy individual may have no incentive to provide high effort. When the aggregate endowment exceeds a threshold level, part of the available resources cannot be consumed if high effort is specified with probability one. In this case, the best high-effort allocation implies an strictly negative aggregate net trade and cannot be incentive efficient. In terms of our former discussion, what is

[16] That is, of "winning the lottery" and reducing effort.

[17] Most models in the partial equilibrium literature consider only deterministic effort prescriptions. See, however, the general equilibrium model of Prescott and Townsend [23].

special about this case is that the lottery described in the previous paragraph can be implemented at no cost (provided that the probability of a low effort specification is sufficiently low) as there are resources available for free! The following proposition establishes the result formally.

**Proposition 3.5** *Suppose that*

*(i)* $U'_H(c) \leq U'_L(c)$ *for all* $c \in \mathbb{R}_+$*, and*

*(ii)* $\lim_{c\to\infty} \frac{U'_H(c)}{U'_L(c)} = 0$.

*Then there exists a threshold* $\breve{\omega}_H$ *such that, if* $\bar{\omega}_H \geq \breve{\omega}_H$ *then any allocation with* $||x_H|| = 1$ *satisfies the resource constraint (3.34) with strict inequality. In this case,* $||x^*_H|| < 1$.

Proposition 3.5 is a similar, slightly stronger result than the result in Bennardo and Chiappori [4, Lemma 3.6]. It shows that the key assumption behind the Bennardo-Chiappori result is that marginal utility of consumption ought to decrease faster with high than with low effort (Assumption (ii)). The main difference between Propositions 3.4 and 3.5 is that random effort prescriptions may be optimal in economies where agents have relatively low endowment provided that the ratio $\frac{U'_H(c)}{U'_L(c)}$ goes to zero sufficiently fast.

# Appendix A

## A.1 The Linear Semi-Infinite Programming Problems

In this section, we set up the primal LSIP problem and derive its dual. Following Charnes, Cooper and Kortanek [8], we define the restricted dual problem, so-called dual problem in Haar's sense. The LSIP

problems in Sections 2 and 3 obtain as particular cases of the problems in this section by applying the definitions in Table I.

### A.1.1 The Primal Problem

Let $1 \leq m \leq n$ and $\mathbb{R}^n$ be equipped with the Euclidean norm and partially ordered by means of the cone

$$K^n_m = \{\, y \in \mathbb{R}^n : \, y_j \geq 0, \, j = 1, ..., m\}.$$

Let $\omega \in \mathbb{R}^2_+$ and define $Z = \{ z \in \mathbb{R}^2 : z \geq -\omega \}$. Let $C(Z)$ denote the vector space of continuous real-valued functions on $Z$, endowed with the topology of uniform convergence on compact sets and partially ordered by means of the cone

$$C_+(Z) = \{ f \in C(Z) : f(z) \geq 0 \quad \forall z \in Z \}.$$

The *primal* problem is to find $y \in \mathbb{R}^n$ to solve

$$\begin{array}{lll} (P) & \inf & c \cdot y \\ & \text{s.t.} & Ay \geq b, \\ & & y \in K^n_m, \end{array}$$

where $c \in \mathbb{R}^n$, $b = (b_L, b_H) \in C(Z) \times C(Z)$, and $A : \mathbb{R}^n \to C(Z) \times C(Z)$ is a continuous linear mapping. Problem $(P)$ is linear and has $n$ unknowns and infinitely many constraints. Denote its optimal value by $\nu(P)$.

### A.1.2 The Dual Problem

Let $M_c(Z)$ denote the space of signed Borel measures on $Z$ which have compact support and are finite on compact sets. This space is the topological dual space of $C(Z)$ (Hewitt [16]).

Let $C(Z) \times C(Z)$ be paired in duality with $M_c(Z) \times M_c(Z)$. The reflexive space $\mathbb{R}^n$ is paired with itself. The two pairings are endowed with their natural bilinear forms (to highlight the dimensionality of the spaces in the pairing we use the dot product and bracket notation for finite and infinite dimensions, respectively):

$$\begin{array}{ll} \langle f, x \rangle = \displaystyle\int_Z f_L dx_L + \int_Z f_H dx_H, & f = (f_L, f_H) \in C(Z) \times C(Z), \\ & x = (x_L, x_H) \in M_c(Z) \times M_c(Z); \\ y \cdot z = \displaystyle\sum_{j=1}^n y_j z_j, & y \in \mathbb{R}^n, \ z \in \mathbb{R}^n. \end{array}$$

The adjoint of $A$, $A^* : M_c(Z) \times M_c(Z) \to \mathbb{R}^n$, is defined by the relation

$$y \cdot (A^* x) = \langle Ay, x \rangle, \qquad \text{for all } y \in K^n_m, \ x \in M_{c+}(Z) \times M_{c+}(Z). \tag{A.1}$$

We may write $Ay = \sum_{j=1}^n y_j f_j$, where $f_j = (f_{jL}, f_{jH}) \in C(Z) \times C(Z)$ for $j = 1, \cdots, n$. Then (A.1) can be expressed as

$$y \cdot (A^* x) = \sum_{j=1}^n y_j \langle f_j, x \rangle, \qquad \text{for all } y \in K^n_m, \ x \in M_{c+}(Z) \times M_{c+}(Z). \tag{A.2}$$

Write $A^*x \leq c$ as

$$\sum_{j=1}^{n} y_j(\langle f_j, x\rangle - c_j) \leq 0, \qquad \text{for all } y \in K_m^n.$$

The *dual* problem is to find $x \in M_c(Z) \times M_c(Z)$ to solve

$$\begin{array}{lll} (D) & \sup & \langle b, x\rangle \\ & \text{s.t.} & \langle f_j, x\rangle \leq c_j, \quad j = 1, ..., m, \\ & & \langle f_j, x\rangle = c_j, \quad j = m+1, ..., n, \\ & & x \geq 0. \end{array}$$

Problem $(D)$ is a linear programming problem with infinitely many unknowns and $n$ constraints. Denote its optimal value by $\nu(D)$. By the weak duality theorem (Krabs [19, Theorem I.3.1]), $\nu(D) \leq \nu(P)$.

### A.1.3 The Dual Problem in Haar's Sense

Let $\mathbb{R}^{(Z)}$ denote the vector space of all functions $\lambda_i : Z \to \mathbb{R}$ which vanish outside a *finite* subset of $Z$. For any $\lambda_i \in \mathbb{R}^{(Z)}$, we define the supporting set of $\lambda_i$ as

$$\operatorname{supp} \lambda_i = \{z_i \in Z : \ \lambda_i(z_i) \neq 0\}.$$

Let $C(Z) \times C(Z)$ be paired in duality with $\mathbb{R}^{(Z)} \times \mathbb{R}^{(Z)}$, with bilinear form

$$\begin{aligned} \langle f, \lambda\rangle \ &= \sum_{z_L \in \operatorname{supp} \lambda_L} f_L(z_L)\lambda_L(z_L) + \sum_{z_H \in \operatorname{supp} \lambda_H} f_H(z_H)\lambda_H(z_H), \\ & f = (f_L, f_H) \in C(Z) \times C(Z), \quad \lambda = (\lambda_L, \lambda_H) \in \mathbb{R}^{(Z)} \times \mathbb{R}^{(Z)}. \end{aligned}$$

The *dual problem in Haar's sense* is to find $\lambda \in \mathbb{R}^{(Z)} \times \mathbb{R}^{(Z)}$ to solve

$$\begin{array}{lll} (D_F) & \sup & \langle b, \lambda\rangle \\ & \text{s.t.} & \langle f_j, \lambda\rangle \leq c_j, \quad j = 1, ..., m, \\ & & \langle f_j, \lambda\rangle = c_j, \quad j = m+1, ..., n, \\ & & \lambda \geq 0. \end{array}$$

Problem $(D_F)$ is also a linear programming problem with infinitely many unknowns and $n$ constraints. Denote its optimal value by $\nu(D_F)$. To see the relation between problems $(D)$ and $(D_F)$ denote the set of finitely supported Borel measures on $Z$ by $M_F$. Also, denote the Dirac measure at $z \in Z$ by $\delta_z$ (i.e., for any Borel set $B \subset Z$, $\delta_z(B) = 1$ if $z \in B$ and $\delta_z(B) = 0$ otherwise). Any pair $\lambda = (\lambda_L, \lambda_H) \in$

| | Adverse Selection | Moral Hazard |
|---|---|---|
| $(n, m)$ | $(5, 3)$ | $(4, 3)$ |
| $y$ | $(\beta_L, \beta_H, q, \alpha_L, \alpha_H)$ | $(\beta_L, \beta_H, q, \alpha)$ |
| $c$ | $(0, 0, 0, 1, 1)$ | $(0, 0, 0, 1)$ |
| $b = (b_L, b_H)$ | $(\gamma_L EU_L, (1-\gamma_L)EU_H)$ | $(EU_L, EU_H)$ |
| $f_1 = (f_{1L}, f_{1H})$ | $(-EU_L, EU_L)$ | $(-EU_L + EU_H, 0)$ |
| $f_2 = (f_{2L}, f_{2H})$ | $(EU_H, -EU_H)$ | $(0, EU_L - EU_H)$ |
| $f_3 = (f_{3L}, f_{3H})$ | $(\xi_L r_L, (1-\xi_L) r_H)$ | $(r_L, r_H)$ |
| $f_4 = (f_{4L}, f_{4H})$ | $(\mathcal{I}, 0)$ | $(\mathcal{I}, \mathcal{I})$ |
| $f_5 = (f_{5L}, f_{5H})$ | $(0, \mathcal{I})$ | — |

Table I: Adverse Selection and Moral Hazard

$\mathbb{R}^{(Z)} \times \mathbb{R}^{(Z)}$ corresponds to a pair of finitely supported measures $x = (x_L, x_H)$ where $x_i = \sum_{z_i \in \operatorname{supp} \lambda_i} \lambda_i(z_i)\delta_{z_i}$ for $i = L, H$. Thus, the space $\mathbb{R}^{(Z)} \times \mathbb{R}^{(Z)}$ is isomorphic to $M_F \times M_F$. This implies that problem $(D_F)$ is equivalent to problem $(D)$ when dual variables are restricted to lie in the subset $M_F \times M_F$ of $M_c(Z) \times M_c(Z)$. Thus, $\nu(D_F) \leq \nu(D)$.

## A.2 Proofs of Theorems 2.2 and 2.1 and Proposition 2.1

We begin with three preliminary Lemmas. To prove Lemma A.1 we appeal to the properties of the expected utility $EU_i$ and the expected net trade functions $r_i$ defined in Table I. In particular, we use the continuity and strict concavity of $EU_i$, the fact that marginal utility of consumption decreases asymptotically to zero, and the linearity of $r_i$. The proof also uses the fact that at most one incentive constraint binds in problem $(D)$.

**Lemma A.1** *There exists a compact subset $T \subset Z$ such that, if all the constraints which are associated with elements $z \in Z|T$ are eliminated from problem $(P)$ then the set of optimal solutions does not change.*

*Proof.* Let $Y$ denote the set of feasible solutions in problem $(P)$. That is, $y \in Y$ if and only if $y \in K^n_m$ and

$$0 \;\geq\; h_i(z_i, y) \;\equiv\; b_i(z_i) - \sum_{j=1}^{n} y_j f_{ji}(z_i) \quad \text{for all } z_i \in Z, \quad i = L, H. \tag{A.3}$$

(See Table I). Note that $Y$ is a closed convex subset of $\mathbb{R}^n$. We establish the Lemma through a sequence of claims.

*Claim 1: $Y$ is non-empty.*
*Proof:* This follows straightforwardly from (A.3) given the strict concavity of $b_i$ and the linearity of $f_{mi}$ for $i = L, H$ (see Table I).

*Claim 2: If $y \in Y$ then $y \geq 0$. Further, there exist constants $M_j$, $j = 1, \ldots, n$, such that any optimal solution to problem $(P)$ lies in the set*

$$M = \{y \in Y : \quad y_j \leq M_j, \quad j = 1, \ldots, n\}.$$

*Proof:* By definition, any $y \in Y$ satisfies $y_j \geq 0$ for $j = 1, \ldots, m$. Now, $f_{mi}(0) = 0$ and utilities can be normalized so $b_i(0) = 0$. Because $0 \in Z$, (A.3) implies that $y_j \geq 0$ for $j \geq m + 1$. The existence of $M_j$ for $j \geq m + 1$ follows from Claim 1 and the primal o bjective function. By the weak duality theorem and since autarky is a feasible solution for problem $(D)$,

$$b_L(0) + b_H(0) \leq \nu(D) \leq \nu(P) \leq \sum_{j=m+1}^{n} M_j.$$

Finally, since at most one incentive constraint binds in problem $(D)$, it can be verified using (A.3) that at an optimal solution $y_j$ is bounded above for $1 \leq j \leq m$.

*Claim 3: There is some $\epsilon > 0$ such that $y_m > \epsilon$ for all $y \in M$.*
*Proof:* Assume not. Then there exists a sequence $\{y^k\}$ in $M$ such that $0 \leq y_m^k < \frac{1}{k}$ for all $k \in \mathbb{N}$. Since at most one incentive constraint binds, without loss of generality let $y_1 = 0$. Then, for some $i$ and any $y \in Y$,

$$0 \;\; \geq \;\; h_i(z_i, y) \; \geq \; b_i(z_i) - y_m f_{mi}(z_i) - y_n, \qquad \text{for all } z_i \in Z. \tag{A.4}$$

Since holds (A.4) for $y = y^k$, rearranging and taking limits gives

$$\lim_{k\to\infty} y_n^k \geq b_i(z_i) - \lim_{k\to\infty} y_m^k f_{mi}(z_i) \; = \; b_i(z_i), \quad \forall z_i \in Z.$$

Hence,

$$\lim_{k\to\infty} y_n^k \geq b_i(z_i), \quad \forall z_i \in Z.$$

When utility is unbounded, $\lim_{k\to\infty} y_{m+1}^k = \infty$, thereby contradicting Claim 2. When utility is bounded, $\lim_{z_i\to\infty} b_i(z_i) = B_i$. But $M_n$ can then always be found in $(0, B_i)$, leading to a similar contradiction.

*Claim 4: There is a $\bar{z}$ such that, for each $i = L, H$ and any $y \in M$, $\nabla h_i(z_i, y) << 0$ for all $z_i > \bar{z}$.*

*Proof:* Without loss of generality, take $i = L$. From Table I, $\nabla f_{jL} = 0$ for $j \geq m+1$. Also, $\nabla f_{mL}(z_L) = \bar{g}_L >> 0$. Then

$$\begin{aligned} \nabla h_L(z_L, y) &= \nabla b_L(z_L) - \sum_{j=1}^{m} y_j \nabla f_{jL}(z_L) \\ &= \nabla b_L(z_L) - \sum_{j=1}^{m-1} y_j (\nabla f_{jL}^{+}(z_L) - \nabla f_{jL}^{-}(z_L)) - y_m \bar{g}_L, \end{aligned}$$

where $\nabla f_{jL}^{+}$ and $\nabla f_{jL}^{-} \geq 0$ stand for the positive and negative parts of $\nabla f_{jL}$. This together with Claims 2 and 3 implies

$$\nabla h_L(z_L, y) \leq \nabla b_L(z_L) + \sum_{j=1}^{m-1} M_j \nabla f_{jL}^{-}(z_L) - \epsilon \bar{g}_L, \qquad \text{for all } z_L \in Z.$$

But as marginal utility decreases asymptotically to zero:

$$\begin{aligned} \lim_{z_L \to +\infty} \nabla b_L(z_L) &= 0, \\ \lim_{z_L \to +\infty} \nabla f_{jL}(z_L) &= 0, \quad 1 \leq j \leq m-1, \end{aligned}$$

and this gives

$$\lim_{z_L \to +\infty} \nabla h_L(z_L, y) = -\epsilon \bar{g}_L << 0.$$

Since $h_L(\cdot, y)$ is a continuously differentiable, there exists a constant $\bar{z}_L$ such that $\nabla h_L(z_L, y) << 0$ for all $z_L > \bar{z}_L$. A similar derivation gives $\bar{z}_H$. Setting $\bar{z} = \max\{\bar{z}_L, \bar{z}_H\}$ proves our claim.

*Claim 5: The set $T = [-\omega_1, \bar{z}] \times [-\omega_2, \bar{z}]$ satisfies Lemma* **??**.
*Proof:* Claim 5 is direct from Claim 4. This completes the proof of Lemma A.1. ■

Consider the LSIP problems which arise by replacing $Z$ by $T$ in problems $(P)$, $(D)$ and $(D_F)$. Denote these problems by $(P^T)$, $(D^T)$ and $(D_F^T)$, respectively. The proofs of Lemma A.2 and Lemma A.3 below appeal to some well-known results in LSIP theory. The proof of Lemma A.2 exploits also the strict concavity of $EU_i$ and the linearity of $r_i$.

**Lemma A.2** *The system of constraints in problem $(P^T)$ is canonically closed in the sense of Charnes, Cooper and Kortanek [9].*

*Proof.* First, since $T$ is compact and since for all $i$ and $j$ $b_i$ and $f_{ji}$ are continuous, the set

$$\{(f_1(t), f_2(t), \ldots, f_n(t), b(t)) : t \in T\}$$

is compact in $\mathbb{R}^{n+1}$.

Second, the Slater constraint qualification is satisfied. To see this, let $y_j^0 = 0$ for $1 \leq j \leq m-1$ and let $y_m^0 > 0$ be given. Since, $f_{mi}$ is linear and $b_i$ is strictly concave, there exist constants $a_L > 0$ and $a_H > 0$ and values for $y_j^0$ for $j \geq m+1$ such that,

$$\begin{aligned} 0 &\geq h_L(z_L, y^0) = b_L(z_L) - y_m^0 f_{mL}(z_L) - y_{m+1}^0 + a_L, \quad \text{for all } z_L \in Z, \\ 0 &\geq h_H(z_H, y^0) = b_H(z_H) - y_m^0 f_{mH}(z_H) - y_n^0 + a_H, \quad \text{for all } z_H \in Z. \end{aligned}$$

That is, $y^0$ is a Slater point. ■

**Lemma A.3** *Problem* $(D_F^T)$ *is solvable and* $\nu(D_F^T) = \nu(D^T) = \nu(P^T)$.

**Proof.** By weak duality of the pair $\{(P^T, D^T)\}$, and the definition of $(D_F^T)$,

$$\nu(D_F^T) \leq \nu(D^T) \leq \nu(P^T).$$

Given Lemma A.2, the inhomogeneous Haar theorem of Charnes, Cooper and Kortanek [8, Theorem 3] implies that the system of constraints in $(P_T)$ has the Farkas-Minkoswki property. Since $(P^T)$ and $(D_F^T)$ are consistent, the extended dual theorem of Charnes, Cooper and Kortanek [8, Theorem 4] implies then that $(D_F^T)$ is solvable and that $\nu(D_F^T) = \nu(P^T)$. ■

We are now ready to prove Theorems 2.1 and 2.2, and Proposition 2.1.

*Proof of Theorem 2.1.* By weak duality of the pair $\{(P), (D)\}$, and the definition of $(D_F)$,

$$\nu(D_F) \leq \nu(D) \leq \nu(P).$$

Also, since $\mathbb{R}^{(T)} \subset \mathbb{R}^{(Z)}$, it follows that $\nu(D_F^T) \leq \nu(D_F)$. By Lemma A.1, $\nu(P) = \nu(P^T)$. But then, Lemma A.3 implies that $\nu(D_F) = \nu(D) = \nu(P)$. ■

*Proof of Proposition 2.1.* The proof of Theorem 2.1 establishes that $\nu(D_F) = \nu(D)$. It also implies that $\nu(D_F^T) = \nu(D_F)$. Since by Lemma A.3, $(D_F^T)$ is solvable, so is $(D_F)$. ■

*Proof of Theorem 2.2.* The solvability of $(D)$ follows from Proposition 2.1. By Claims 1 and 2 in Lemma A.1, $Y$ is non-empty and may be assumed bounded. Since $Y$ is closed, problem $(P)$ maximizes a continuous function on a compact set, and so its value is attained. ■

## A.3 Proofs of Proposition 2.2 and 2.4 to 2.6, and 3.4 to 3.5

*Proof of Proposition 2.2.* Let $(x^*, x^*)$ denote the first best equal treatment allocation: $x^* = \delta_{z^*}$ with $z^* = (z_1^*, z_2^*)$ and $z_s^* = \bar{\omega} - \omega_s$ for $s = 1, 2$. Since $(x^*, x^*)$ is incentive compatible, it is also incentive efficient. Let $\bar{\gamma}_L$ and $\bar{\gamma}_H$ be the associated weights in problem $(D)$. Using first-order conditions,

$$v_i'(z^*; q^*) = \bar{\gamma}_i U_i'(\bar{\omega}) - \xi_i q^* = 0, \quad i = 1, 2.$$

Writing $\bar{\gamma}_H = 1 - \bar{\gamma}_L$ and rearranging gives

$$\bar{\gamma}_L = \Big(1 + \frac{(1 - \xi_L) U_L'(\bar{\omega})}{\xi_L U_H'(\bar{\omega})}\Big)^{-1}.$$

Any other incentive efficient allocation $(x_L^*, x_H^*)$ is such that either (i) one type is strictly better off, or (ii) both types are indifferent. Assume (i) and suppose, without loss of generality, that type $L$ is better off. Then, $\gamma_L > \bar{\gamma}_L$. Since $x^*$ provides full insurance, the expected consumption of type $L$ must exceed $\bar{\omega}$ and, by feasibility, that of type $H$ must be lower than $\bar{\omega}$. But then,

$$\langle U_L, x_H^* \rangle < \langle U_L, x^* \rangle < \langle U_L, x_L^* \rangle,$$

so the incentive constraint of type $L$ is does not bind ($\beta_L = 0$). Since the incentive constraint of type $H$ is satisfied, $x_L^*$ entails only partial insurance. This constraint must bind with $\beta_H > 0$; otherwise, the utility of type $L$ could be increased by reducing the risk in $x_L^*$ and maintaining the expected consumption.

Case (ii) is impossible. If each type $i$ is indifferent between $x_i^*$ and $x^*$ then $(x_L^*, x_H^*)$ must give both types an expected consumption of at least $\bar{\omega}$. But, by feasibility, the expected consumption of both types must equal $\bar{\omega}$. Since $(x_L^*, x_H^*)$ and $(x^*, x^*)$ are different, at least one type $i$ is not fully insured and strictly prefers $x^*$ to $x_i^*$, a contradiction. ■

*Proof of Proposition 2.4.* We first show that, when $U_i(\cdot) = U(\cdot)$ for $i = L, H$, the function $v_L(\cdot\,; \beta_H^*, q^*)$ is strictly concave. This function is additive across states,

$$v_L(z_L; \beta_H^*, q^*) = \sum_{s \in \{1,2\}} v_{Ls}(z_{Ls}; \beta_H^*, q^*),$$

where

$$\begin{aligned} v_{L1}(z_{L1}; \beta_H^*, q^*) &= \big(\gamma_L \theta_L - \beta_H^* \theta_H\big) U(w_1 + z_{L1}) - q^* \xi_L \theta_L z_{L1}, \\ v_{L2}(z_{L2}; \beta_H^*, q^*) &= \big(\gamma_L (1 - \theta_L) - \beta_H^* (1 - \theta_H)\big) U(w_2 + z_{L2}) - q^* \xi_L (1 - \theta_L) z_{L2}. \end{aligned}$$

Since $U'' < 0$, for $s = 1, 2$, the second derivative $v''_{Ls}$ never changes sign. Suppose $v''_{Ls} \geq 0$ for some $s$. Because $U' > 0$, then $v'_{Ls} < 0$. But then, by condition (2.27), the optimal assignment to type $L$ is deterministic and such that $w_s + z^*_{Ls} = 0$, which is impossible since $\lim_{c\to 0} U'_L(0) = \infty$. We conclude that $v''_{Ls} < 0$ for $s = 1, 2$.

Since $0 < \theta_L < \theta_H < 1$ and $U'' < 0$, the first-order conditions imply that the maximum of $v_L(\cdot\,; \beta^*_H, q^*)$ satisfies $\omega_1 + z_{L1} < \omega_2 + z_{L2}$. ■

*Proof of Proposition 2.5.* We may write

$$v_L(z_L; \beta^*_H, q^*) = \sum_{s\in\{1,2\}} v_{Ls}(z_{Ls}; \beta^*_H, q^*),$$

where

$$\begin{aligned} v_{L1}(z_{L1}; \beta^*_H, q^*) &= \gamma_L\theta_L U_L(w_1 + z_{L1}) - \beta^*_H\theta_H U_H(w_1 + z_{L1}) - q^*\xi_L\theta_L z_{L1}, \\ v_{L2}(z_{L2}; \beta^*_H, q^*) &= \gamma_L(1-\theta_L)U_L(w_2 + z_{L2}) - \beta^*_H(1-\theta_H)U_H(w_2 + z_{L2}) \\ &\quad -q^*\xi_L(1-\theta_L)z_{L2}. \end{aligned}$$

We first show that, if $A_L(c) \geq A_H(c)$ for all $c \in \mathbb{R}_+$ then $v_L(\cdot\,; \beta^*_H, q^*)$ is strictly concave. Write $v'_{L1} = (g_1 + g_2)g_3$ where

$$g_1 = \gamma_L\theta_L\tfrac{U'_L}{U'_H}, \quad g_2 = -\big(\beta^*_H\theta_H + \tfrac{q^*\xi_L\theta_L}{U'_H}\big), \quad g_3 = U'_H.$$

Clearly, $g'_2, g'_3 < 0$. Further,

$$\frac{1}{\gamma_L\theta_L}{g_1}' = \Big(\frac{U''_L U'_H - U'_L U''_H}{(U'_H)^2}\Big) = \left(\frac{\Big(\frac{U''_L U'_H}{U'_L U''_H} - 1\Big)U'_L U''_H}{(U'_H)^2}\right) = \left(\frac{\Big(\frac{A_L}{A_H} - 1\Big)U'_L U''_H}{(U'_H)^2}\right).$$

So $A_L \geq A_H$ implies ${g_1}' \leq 0$ and hence $v''_{L1} < 0$. Finally, because $0 < \theta_L < \theta_H < 1$, $v''_{L1} < 0$ implies $v''_{L2} < 0$, which proves our claim.

Now ${g_1}' \leq 0$ is equivalent to $\frac{U'_L}{U'_H}$ being non-increasing. Since $0 < \theta_L < \theta_H < 1$ and $U'' < 0$, the first-order conditions imply that the maximum of $v_L(\cdot\,; \beta^*_H, q^*)$ satisfies $\omega_1 + z_{L1} < \omega_2 + z_{L2}$. ■

*Proof of Proposition 2.6.* For $\gamma_L \in (0, \bar{\gamma}_L)$, the net contributions are:

$$\begin{aligned} v_L(z_L; \beta_L, q) &= \gamma_L EU_L(z_L) - q\xi_L r_L(z_L) + \beta_L EU_L(z_L), \\ v_H(z_H; \beta_L, q) &= (1-\gamma_L)EU_H(z_H) - q(1-\xi_L)r_H(z_H) - \beta_L EU_L(z_L). \end{aligned}$$

Similar arguments to those in the proofs of Propositions 2.3, 2.4 and 2.5 prove (i) and (ii). ■

*Proof of Proposition 3.2.* Define $U(\cdot) = U_H(\cdot)$, so $U_L(\cdot) = U(\cdot) + d$. Then

$$v_H(z_H; \beta_H^*, q^*) = \sum_{s\in\{1,2\}} v_{Hs}(z_{Hs}; \beta_H^*, q^*),$$

where

$$\begin{aligned} v_{H1}(z_{H1}; \beta_H^*, q^*) &= \big((1+\beta_H^*)\theta_H - \beta_H^*\theta_L\big)U(w_1 + z_{H1}) - q^*\theta_H z_{H1} - \beta_H^*\theta_L d, \\ v_{H2}(z_{H2}; \beta_H^*, q^*) &= \big((1+\beta_H^*)(1-\theta_H) - \beta_H^*(1-\theta_L)\big)U(w_2 + z_{H2}) - q^*(1-\theta_H)z_{H2} \\ &\quad -\beta_H^*(1-\theta_L)d. \end{aligned}$$

Analogous arguments to those in the proof of Proposition 2.4 show that $v_H(\cdot\,; \beta_H^*, q^*)$ is strictly concave and that its maximum is characterized by partial insurance. ■

The proof of Proposition 3.3 is analogous to that of proposition 2.5 and is omitted.

*Proof of Proposition 3.4.* Take an arbitrary endowment sequence $\{\omega^k\} \subset \mathbb{R}_+^2$ such that $\lim_{k\to\infty} \bar{\omega}_H^k = \infty$ and $\bar{\omega}_H^k - \bar{\omega}_L^k \le N$ for some constant $N$ (with $\bar{\omega}_i^k$ denoting the average endowment with effort $e_i$ when $\omega = \omega^k$). Let $(\alpha^k, \beta_H^k, q^k)$ and $(x_L^k, x_H^k)$ be optimal primal and dual solutions for $\omega = \omega^k$.

Suppose that $||x_H^k|| = 1$ for all $k$. Since $q^k > 0$, by condition (3.47), the support of $x_H^k$ becomes unbounded as $k$ increases. Since $\beta_H^k > 0$, by condition (3.46), there is a sequence $\{z_H^k\}$ where $z_H^k = (z_{H1}^k, z_{H2}^k)$ such that $x_H^k(z_H^k) > 0$ and $\lim_{k\to\infty} z_{H2}^k = \infty$. Write the first-order condition associated to (3.48) for $s = 2$ as

$$\left(\frac{1-\theta_L}{1-\theta_H} - \frac{U_H'(w_2^k + z_{H2}^k)}{U_L'(w_2^k + z_{H2}^k)}\right)\beta_H^k = \frac{U_H'(w_2^k + z_{H2}^k)}{U_L'(w_2^k + z_{H2}^k)} - \frac{q^k}{U_L'(w_2^k + z_{H2}^k)} \tag{A.5}$$

Since $0 \le \theta_H \le \theta_L \le 1$, $U_H' \le U_L'$ and $\beta_H^k > 0$, the right-hand side of (A.5) is positive. The first term in the left-hand side of (A.5) is bounded above by one. But as $\lim_{c\to\infty} U_L'(c) = 0$, for (A.5) to hold, $\lim_{k\to\infty} q^k = 0$.

Let $v_L^k(q^k)$ and $v_H^k(\beta_H^k, q^k)$ denote the maximal net contributions with $e_L$ and $e_H$ for $\omega = \omega^k$. By condition (3.48) and equality of the primal and dual optimal values,

$$\alpha^k = v_H^k(\beta_H^k, q^k) = \langle EU_H, x_H^k\rangle. \tag{A.6}$$

Let $\bar{c}_H^k$ denote the certainty equivalent associated to $x_H^k$, so $U_H(\bar{c}_H^k) = \langle EU_H, x_H^k\rangle$. Since $U_H'' < 0$, then $\bar{c}_H^k < \bar{\omega}_H^k$. Applying (3.43) for $z_{Ls}^k = \bar{c}_H^k - w_s^k$, $s = 1, 2$, gives

$$v_L^k(q^k) \ge U_L(\bar{c}_H^k) - q^k(\bar{c}_H^k - \bar{\omega}_L^k) > U_L(\bar{c}_H^k) - q^k(\bar{\omega}_H^k - \bar{\omega}_L^k). \tag{A.7}$$

Since $\lim_{k\to\infty} q^k = 0$ and $\lim_{k\to\infty}(\bar{\omega}_H^k - \bar{\omega}_L^k) \le M$, (A.7) implies that for any $\epsilon > 0$ there is $K$ such that $U_L(\bar{c}_H^k) - v_L^k(q^k) \le \epsilon$ for all $k \ge K$. Fix $\epsilon \le d$.

Because $U''_H < 0$ and $U_L(\cdot) - U_H(\cdot) = d$, (A.6) implies

$$\alpha^k = \langle EU_H, x_H^k \rangle < U_H(\bar{c}_H^k) = U_L(\bar{c}_H^k) - d, \tag{A.8}$$

and, since $||x_L^k|| = 0$, by condition (3.48),

$$\alpha^k > v_L^k(q^k). \tag{A.9}$$

(A.8) and (A.9) imply that $U_L(\bar{c}_H^k) - v_L^k(q^k) > d$, a contradiction if $k \geq K$. We conclude that $||x_H^k|| < 1$ for all $k \geq K$. ■

*Proof of Proposition 3.5.* Suppose we restrict the allocations to satisfy $x_L = 0$. Take any endowment sequence $\{\omega^k\} \subset \mathbb{R}_+^2$ with $\lim_{k\to\infty} \bar{\omega}_H^k = \infty$. Let $(\hat{\beta}_H^k, \hat{q}^k)$ and $\hat{x}_H^k$ be optimal primal and dual solutions to the restricted planner's problem for $\omega = \omega_k$. The complementary slackness conditions in this case are obtained by letting $x_L = 0$ in (3.46)-(3.48).

Suppose, in contrast to what we want to show, that $\langle r_H, \hat{x}_H^k \rangle = 0$ for all $k$. Since (3.46) holds, there is a sequence $\{z_H^k\}$ where $z_H^k = (z_{H1}^k, z_{H2}^k)$ such that $\hat{x}_H^k(z_H^k) > 0$, $\lim_{k\to\infty} z_{H2}^k = \infty$, and $(\omega_{H2} + z_{H2}^k - \omega_{H1} - z_{H1}^k) \geq \epsilon_1$ for some $\epsilon_1 > 0$ and all $k$. Write the first-order conditions associated to (3.48) as:

$$\left(\frac{\theta_L}{\theta_H} - \frac{U'_H(w_1^k + z_{H1}^k)}{U'_L(w_1^k + z_{H1}^k)}\right)\hat{\beta}_H^k = \frac{U'_H(w_1^k + z_{H1}^k)}{U'_L(w_1^k + z_{H1}^k)} - \frac{\hat{q}^k}{U'_L(w_1^k + z_{H1}^k)}, \tag{A.10}$$

$$\left(\frac{1-\theta_L}{1-\theta_H} - \frac{U'_H(w_2^k + z_{H2}^k)}{U'_L(w_2^k + z_{H2}^k)}\right)\hat{\beta}_H^k = \frac{U'_H(w_2^k + z_{H2}^k)}{U'_L(w_2^k + z_{H2}^k)} - \frac{\hat{q}^k}{U'_L(w_2^k + z_{H2}^k)}. \tag{A.11}$$

Since $\lim_{c\to\infty} \frac{U'_H(c)}{U'_L(c)} = 0$, taking limits in (A.11) yields

$$\left(\frac{1-\theta_L}{1-\theta_H}\right) \lim_{k\to\infty} \hat{\beta}_H^k = -\lim_{k\to\infty} \frac{\hat{q}^k}{U'_L(w_2^k + z_{H2}^k)},$$

which implies that $\lim_{k\to\infty} \hat{\beta}_H^k = \lim_{k\to\infty} \hat{q}^k = 0$.

For sufficiently large $k$, the right-hand side of (A.11) is positive, so $\hat{q}^k < U'_H(w_2^k + z_{H2}^k)$. Since $U''_L < 0$ and $\omega_{H2} + z_{H2}^k > \omega_{H1} - z_{H1}^k$, (A.10) implies

$$\left(\frac{\theta_L}{\theta_H} - \frac{U'_H(w_1^k + z_{H1}^k)}{U'_L(w_1^k + z_{H1}^k)}\right)\hat{\beta}_H^k > \frac{U'_H(w_1^k + z_{H1}^k)}{U'_L(w_1^k + z_{H1}^k)} - \frac{U'_H(w_2^k + z_{H2}^k)}{U'_L(w_2^k + z_{H2}^k)}. \tag{A.12}$$

But as $U_i$ is continuously differentiable and $(\omega_{H2} + z_{H2}^k - \omega_{H1} - z_{H1}^k) \geq \epsilon_1$, taking limits and rearranging in (A.12) gives

$$\lim_{k\to\infty} \hat{\beta}_H^k > \left(\frac{\theta_L}{\theta_H}\right)^{-1} \lim_{k\to\infty} \left(\frac{U'_H(w_1^k + z_{H1}^k)}{U'_L(w_1^k + z_{H1}^k)} - \frac{U'_H(w_2^k + z_{H2}^k)}{U'_L(w_2^k + z_{H2}^k)}\right) \geq \left(\frac{\theta_L}{\theta_H}\right)^{-1} \epsilon_2,$$

for some constant $\epsilon_2 > 0$, a contradiction. ■

# References


[1] N. I. Al-Najjar, Decomposition and characterization of risk with a continuum of random variables, *Econometrica* **63** (1995), 1195–1224.

[2] R. Arnott, B.C. Greenwald and J.E. Stiglitz, Information and economic efficiency, *Info. Econ. Pol.* **6** (1994), 77–88.

[3] R. Arnott and J.E. Stiglitz, Randomization with asymmetric information, *RAND J. Econ.* **19** (1988), 344–362.

[4] A. Bennardo and P.A. Chiappori, "Bertrand and Walras equilibria under moral hazard," mimeo (2000).

[5] A. Bisin and P. Gottardi, Competitive equilibria with asymmetric information, *J. Econ. Theory* **87** (1999), 1–48.

[6] A. Bisin and P. Gottardi, "Decentralizing Incentive Efficient Allocations of Economies with Adverse Selection", mimeo (2000).

[7] A. Bisin and D. Guatoli, "Inefficiency of competitive equilibrium with asymmetric information and financial intermediaries," CEPR Discussion Paper 1987 (1997).

[8] A. Charnes, W.W. Cooper, and K. Kortanek, Duality in semi-infinite programs and some works of Haar and Carathédory, *Manage. Sci.* **9** (1963), 209-228.

[9] A. Charnes, W.W. Cooper, and K. Kortanek, On representations of semi-infinite programs which have no duality gaps, *Manage. Sci.* **2** (1965), 113–121.

[10] H. L. Cole, Comment: general competitive analysis in an economy with asymmetric information, *Int. Econ. Rev.* **30** (1989), 249–253.

[11] P. Dubey, J. Geanakoplos, and M. Shubik, "Default and Efficiency in a General Equilibrium Model with Incomplete Markets," mimeo (1995).

[12] M.A. Goberna and M.A. López, "Linear Semi-Infinite Optimization," Wiley, Chichester, 1998.

[13] B.C. Greenwald and J.E. Stiglitz, Externalities in economies with imperfect information and incomplete markets, *Quart. J. Econ.* **101** (1986), 229–264.

[14] N.E. Gretsky, J.M. Ostroy, and W.R. Zame, Perfect competition in the continuous assignment model, *J. Econ. Theory* **88** (1999), 60-118.

[15] P.J. Hammond and M.B. Lisboa, "Monte Carlo integration and an exact law of large numbers for a continuum of independent random variables," mimeo (1998).

[16] E. Hewitt, Linear functionals on spaces of continuous functions, *Fundamentae Mathematicae* **37** (1959), 161–189.

[17] K. Judge, The law of large numbers with a continuum of iid random variables, *J. Econ. Theory* **35** (1985), 19-25.

[18] T.J. Kehoe, D.K. Levine, and E. Prescott, "Lotteries, sunspots and incentive constraints," mimeo (2000).

[19] W. Krabs, "Optimization and Approximation," Wiley, New York, 1979.

[20] L. Makowski and J. M. Ostroy, "Perfect competition via Linear Programming," mimeo (1996).

[21] R.B. Myerson, Optimal coordinating mechanisms in generalized principal-agent problems, *J. Math. Econ.* **10** (1982), 67–81.

[22] R.B. Myerson, Cooperative games with incomplete information, *Int. J. Game Theory* **13** (1984), 69–96.

[23] E. Prescott and R. Townsend, Pareto optima and competitive equilibria with adverse selection and moral hazard, *Econometrica* **52** (1984), 21–45.

[24] E. Prescott and R. Townsend, General competitive analysis in an economy with private information, *Int. Econ. Rev.* **25** (1984), 1–20.

[25] M. Rothschild and J.E. Stiglitz, Equilibrium in competitive insurance markets: an essay on the economics of imperfect information, *Quart. J. Econ.* **90** (1976), 629–649.

[26] K. Shell and R. Wright, Indivisibilities, lotteries and sunspot equilibria, *Econ. Theory* **3** (1993), 1–17. Income,"

[27] Y. Sun, A theory of hyperfinite processes: the complete removal of individual uncertainty via exact LLN, *J. Math. Econ.* **29** (1998), 419–503.

[28] C. Wilson, A model of insurance markets with incomplete information, *J. Econ. Theory* **16** (1977), 176–207.