

# Candidate Stability and Voting Correspondences

Carmelo Rodríguez-Álvarez<sup>1</sup>

Dept. Economia i d'Història Econòmica

Edifici B. Universitat Autònoma de Barcelona

08193 Bellaterra (Barcelona)

carmelo@idea.uab.es

July 2001

<sup>1</sup>Tel.: +34 93 581 18 11. Fax: +34 93 581 20 12. I would like to thank Salvador Barberà, Carmen Beviá, Bhaskar Dutta, Matt Jackson, Jordi Massó and David Pérez-Castrillo for their helpful comments and suggestions. Financial support by Research Grant 1998 FI 00022 from Comisionat per Universitat i Recerca, Generalitat de Catalunya is also gratefully acknowledged.

## **Abstract**

We study the incentives of candidates to enter or to exit elections in order to strategically affect the outcome of a voting correspondence. We extend the results of Dutta, Jackson and Le Breton [9], who only considered single-valued voting procedures by admitting that the outcomes of voting may consist of sets of candidates. We show that, if candidates form their preferences over sets according to Expected Utility Theory and Bayesian updating, every unanimous and non dictatorial voting correspondence violates candidate stability. When candidates are restricted to use even chance prior distributions, only dictatorial or bidictatorial rules are unanimous and candidate stable. We also analyze the implications of using other extension criteria to define candidate stability that open the door to positive results.

*JEL* Classification Number: D71.

Keywords: Voting Correspondence, Candidate Stability, Extension Criterion.

Running Title: Candidate Stability.

# 1 Introduction

The Social Choice literature has addressed the study of strategic manipulation of collective choice processes. Much work has been devoted to analyze the incentives of the voters to misrepresent their preferences. But, there are other ways in which the participants in an election can affect social choices. The set of candidates (or possible alternatives) is often taken as exogenous and independent of any individual decision. Nevertheless, to enter an election implies an explicit decision by the candidates, and candidates can influence the result of the choice procedure by leaving the fray, even when they have no chance of winning. For instance, we can think of a presidential election in which there are three candidates, one of them, conservative, with the support of 40% of the voters, and two of them, leftist, with the support of 35%, and 25% respectively. If the voting rule is plurality and voters cast their ballot according to their true preferences, the conservative candidate wins, but if one of the leftist candidates quits, the other leftist candidate will be the one chosen. If the leftist candidates prefer a leftist president to a conservative one, they will have incentives to exit the poll and transfer their votes to the other leftist candidate.

In this paper, we examine a two stage process. In the first stage, candidates decide whether they will stay in the election. In the second stage, a voting correspondence is implemented to select from the candidates who do not withdraw. The second stage voting mechanism is only assumed to respect unanimity. We focus on the strategic incentives in the first stage. Thus, we study the possibility of constructing voting correspondences such that, at any preference profile, each candidate prefers to run in the election rather than to exit. In other terms, staying as a candidate when the rest of candidates stay is always a Nash Equilibrium strategy for all the candidates. This condition is called "candidate stability".

The above problem was first addressed by Dutta, Jackson and Le Breton [9]<sup>1</sup> for

---

<sup>1</sup>Hereinafter, DJL.

single-valued voting procedures. That is, rules that choose a single candidate for each set of candidates entering the ballot and each profile of voters' preferences. Then, the voting stage of the voting procedure is a social choice function. In DJL's framework, only dictatorial rules are candidate stable and unanimous when candidates are not allowed to vote.<sup>2</sup> The same result holds when candidates can vote, but then stronger versions of candidate stability and unanimity are introduced.

In this work, a more general setting is adopted. We do not restrict the voting correspondence to be single-valued: the second stage of the process is modeled as a social choice correspondence. We suppose that for each set of candidates at stake and each preference profile, there is a group of candidates who may be finally elected. Candidates consider the voting correspondence as a first screening device which selects a set of candidates. Candidates know this set, but they do not have enough information to infer which candidate is going to be finally selected. Hence, candidates interpret this first selection as the outcome of the entry game.

A possible interpretation considers environments in which the voting mechanism played by the voters admits multiple equilibria. We can think of situations in which the candidates know the set of equilibria of the voting mechanism at any preference profile, but they cannot deduce which are the strategies actually played by the voters and the equilibrium that eventually arises. In these cases, the candidates cannot focus on a unique equilibrium of the voting stage. Thus, the candidates consider the set of possible equilibria as the result of their decisions about running the election.

When the result of the voting correspondence is multi-valued, the study of the candidates' strategic concerns becomes problematic. Individuals are naturally endowed with preferences over candidates, but such preferences do not contain enough information to compare sets of candidates. Thus, we have to provide candidates with preferences over

---

<sup>2</sup>Weymark [24] proposes an alternative proof for this result. The importance of this assumption will become clear in footnote 7 and in Appendix 1.

sets that are consistent with their original preferences over singletons. There is a vast literature on the extension of preferences over alternatives to the power set of alternatives, and many solutions have been proposed.<sup>3</sup> The suitability of each proposal depends on the specific context. According to the two stage process we analyze, in which sets are associated to uncertainty in the final resolution of the election, we propose and explore different extensions. Some of them are based on Expected Utility Theory and Bayesian updating, while assuming distinct degrees of information. Others rely upon extreme attitudes towards risk (leximin, maximin and maximax). These extensions have been applied to the study of strategy-proofness of social choice correspondences (see Pattanaik [18] and [19], Gärdenfors [12], Barberà, Dutta and Sen [2],<sup>4</sup> and Campbell and Kelly [5]).

We show that the results of DJL still hold in multi-valued environments if candidates use the most sophisticated extension criterion presented by BDS. Nevertheless, this is not longer true when other extension principles are employed. In these cases, candidate stability becomes less stringent and positive results are obtained. For instance, bidictatorial rules are candidate stable when the weakest extension proposed by BDS is used.<sup>5</sup> Moreover, the voting correspondence that selects the Pareto correspondence is candidate stable according to the leximin extension.

The incentives of the candidates and the endogeneity of feasible sets in general collective decision environments is a relatively new issue in the social choice literature. Up to our knowledge, DJL and Weymark [24] are the first works dealing with it. When this paper was already complete we became aware of a very related one due to Eraslan and McLennan [11]. These authors independently investigate candidate stability for voting correspondences. However, they do not provide a formal model of the role of candidates' preferences when formulating their condition of candidate stability, and they restrict at-

---

<sup>3</sup>See Barberà, Bossert and Pattanaik [3] for a recent survey on the topic.

<sup>4</sup>Henceforth, BDS.

<sup>5</sup>A voting correspondence is bidictatorial if it chooses the union of the best candidates of two fixed voters.

tention to only one of the possible scenarios considered here.

In the literature of mechanism design, the works of Postlewaite [20], Hurwicz, Maskin and Postlewaite [13] and Sönmez [23] are naturally related with ours. The first two papers consider strategic withholding of endowments in exchange settings, while the last one concentrates on matching models.

Finally, we can mention the works of Majumdar [14] and Dutta and Pattanaik [10]. They show that sponsors may indulge in strategic behavior by not proposing their most preferred alternatives under some circumstances. More recently, Osborne and Slivinski [17] and Besley and Coate [4] analyzed strategic candidacy in the context of a representative democracy. They study the number of candidates who enter an election and the pattern of entry when the winner is elected by plurality.

The remainder of the paper is structured as follows. We devote section 2 to outline the setting and provide definitions and notation; and section 3 to develop the implications of candidate stability under the BDS extensions of preferences when candidates are not allowed to vote. In section 4, we refer to the possibilities that arise when another extension criteria are employed. Finally, we include in Appendix 1 the analysis of the case in which candidates can vote and the proofs of the main Theorems. We prove some intermediate results in Appendix 2.

## 2 Definitions

### 2.1 Candidates, Voters and Preferences

Let  $\mathcal{N}$  be a society formed by a finite set of voters  $\mathcal{V}$ , and a finite set of candidates  $\mathcal{C}$ ,  $\mathcal{N} = \mathcal{V} \cup \mathcal{C}$ . We consider the case in which  $\#\mathcal{C} \geq 3$ ,<sup>6</sup> (since the strategic concerns with  $\#\mathcal{C} \leq 2$  are trivial). We are going to focus on the case in which there is no overlap between the sets of voters and candidates. It could be an approximation to the situation in which

---

<sup>6</sup> $\#A$  refers to the cardinality of the set  $A$ .

the set of candidates is relatively small with respect to the set of voters, and hence, their capability to influence in the social choice is negligible. Moreover, in this scenario, we can isolate the incentives of the candidates to participate in an election, regardless of their concerns as voters.<sup>7</sup>

Each individual  $i \in \mathcal{N}$  is endowed with a linear order  $P_i$  on  $\mathcal{C} \cup \emptyset$ ,<sup>8</sup> where the empty set refers to the situation in which no candidate is elected. We assume that for all  $i$ , any candidate is preferred to the empty set. Preferences of voters over candidates are unrestricted, but each candidate considers herself as the best alternative. These assumptions will allow us to obtain complete preferences over the possible outcomes of voting stage.

We denote by  $P_i$  a preference order of individual  $i$  and by  $P$  a preference profile.  $\mathcal{P}_i^r$  refers to the set of admissible preferences for some  $i \in \mathcal{N}$  and then,  $\mathcal{P}^r$  to the set of preference profiles for which for all  $a \in \mathcal{C}$ , for any  $b \in \mathcal{C} \setminus a$ ,  $aP_a b$ . For each  $A \subseteq \mathcal{C}$ ,  $P|_A$  denotes the restriction of  $P$  to the set  $A$ . *top*  $(A, P_i)$  and *bottom*  $(A, P_i)$  are, respectively, the best and worst element of  $A$  according to  $P_i$ . We say  $B \subseteq \mathcal{C}$  is a restricted top-set of profile  $P$  if for all  $i \in \mathcal{N}$ , for all  $b \in B$ ,  $c \in (\mathcal{C} \setminus i) \setminus B$ ,  $bP_i c$ , and we write  $P \in \mathcal{P}^r(B)$ . Finally,  $B$  is a restricted top set of  $P$  at profile  $\bar{P}$ , if  $B$  is a restricted top-set of  $P$  and  $P|_{\mathcal{C} \setminus B} = \bar{P}|_{\mathcal{C} \setminus B}$ , we denote the set of profiles for which  $B$  is a top-set at profile  $\bar{P}$  as  $\mathcal{P}^r(B, \bar{P})$ .<sup>9</sup>

---

<sup>7</sup>When candidates can be voters, the analysis of stability becomes more complicated, because candidates preferences are assumed to favor their own election. As stronger conditions are required and their interpretation is not clear in terms of candidates' incentives, this case is left to Appendix 1.

<sup>8</sup>A linear order is a complete, antisymmetric and transitive binary relation. Eraslan and McLennan[11] allow the agents to have strict or weak preferences (weak orders) on the sets of candidates. This assumption does not imply substantial changes on the final results.

<sup>9</sup>Abusing notation, for any set  $I \subseteq \mathcal{N}$ ,  $P_I$  refers to the restriction of the profile  $P$  to the members of  $I$ ,  $\mathcal{P}_I^r$ , is obviously defined as the set of admissible preferences profiles for  $I$ . Analogously, when for any  $I \subseteq \mathcal{N}$  the set  $B \subseteq \mathcal{C}$  is a restricted top-set for  $I$  at profile  $P$ , we write  $P_I \in \mathcal{P}_I^r(B)$ .

## 2.2 Voting Correspondences

The object of interest of this article is an aggregation rule, a voting correspondence  $v$ , that for each configuration of the ballot and each preference profile selects a set of the candidates at contest. Let  $2^{\mathcal{C}} \setminus \emptyset$  denote the set of all non-empty subsets of  $\mathcal{C}$ .

**Definition 1** *A voting correspondence is a mapping  $v : 2^{\mathcal{C}} \setminus \emptyset \times \mathcal{P}^r \rightarrow 2^{\mathcal{C}} \setminus \emptyset$  such that for all  $A \in 2^{\mathcal{C}} \setminus \emptyset$  and for all  $P \in \mathcal{P}^r$ :*

- i)  $v(A, P) \subseteq A$ , for all  $P \in \mathcal{P}^r$  and  $A \subseteq \mathcal{C}$ ;*
- ii)  $v(A, P) = v(A, P')$  for all  $P' \in \mathcal{P}^r$  such that,  $P_i = P'_i$  for all  $i \in \mathcal{V}$ ;*
- iii)  $v(A, P) = v(A, P')$  for all  $P' \in \mathcal{P}^r$  such that,  $P|_A = P'|_A$ .*

Item *i)* makes reference to the fact that candidates cannot be selected if they are not at stake.

In *ii)* we state formally the fact that only the preferences of voters affect the outcome of the voting correspondence. When  $\mathcal{V} \cap \mathcal{C} = \emptyset$ , the candidates' preferences do not affect the result of the social decision process.

Finally, *iii)* is in the spirit of Arrow's Independence of Irrelevant Alternatives, only the preferences of voters over the candidates at stake matter.

Notice that a voting correspondence is a collection of social choice correspondences, one for each configuration of the agenda.

## 2.3 Unanimity

In this work we make complete abstraction of the mechanism used in the voting stage. We only assume unanimity, which is uncontroversial: when all the voters agree on who is the best candidate, this is elected.

**Definition 2** *The voting correspondence  $v$  satisfies unanimity iff whenever there is  $b \in B \subseteq \mathcal{C}$ , with  $b = \text{top}(B, P_i)$  for all  $i \in \mathcal{V}$ , then  $v(B, P) = b$ .*



## 2.4 Candidate Stability

A voting correspondence  $v$  is candidate stable if and only if for all  $a \in \mathcal{C}$  and for all  $P \in \mathcal{P}^r$ ,  $a$  does not prefer  $v(\mathcal{C} \setminus a, P)$  to  $v(\mathcal{C}, P)$ . Equivalently, a voting correspondence is candidate stable if and only if the entry of all candidates constitutes a Nash equilibrium for any configuration of the preferences.

Candidate stability implies that the set of candidates can be treated as exogenous (since no candidate will have incentives to leave the poll). The existence of other entry equilibria  $A \subset \mathcal{C}$  is not contemplated, since it leads immediately to impossibility results. Notice that if  $v$  is unanimous no  $A \subset \mathcal{C}$  can be a Nash equilibrium for all configurations of preferences, given that any  $b \in \mathcal{C} \setminus A$  will have incentives to enter when voters unanimously prefer her to the remaining candidates.

Candidate stability is not a condition on the entry-exit game form, but on the family of games which are generated from the game form once the preferences of agents on outcomes (and their pay-offs) are specified. Since the result of the stay-quit decision by the candidates is not necessarily a singleton, pay-off functions cannot be trivially derived from candidates' preferences over candidates. Thus, candidates have to be endowed with some criterion in order to extend their preferences from alternatives to sets of alternatives.<sup>10</sup> Moreover, the extension principles must generate complete preferences on the power set of candidates if we want the associated pay-off functions to be fully defined. If the preferences over sets were not complete, we could not construct the associated games, and we could not talk about the existence of a Nash equilibrium in the entry stage.

The choice of the extension principle is crucial for our analysis since the different criteria lead to different pay-off functions and different normal games for the same entry-exit game form. We delay the formulation of the extension principles to the following sections for clarity of exposition, but we include the generic definition of candidate stability

---

<sup>10</sup>Extension criteria must respect the original preferences over singleton sets, and then DJL's framework is a special case of ours.

according to any given extension  $\mathcal{E}$ .

**Definition 3** *Given  $a \in \mathcal{C}$ , let  $\mathcal{D}_a^\mathcal{E}$ , denote the set of preferences over sets that can be generated from some  $P_a \in \mathcal{P}_a^r$  according to extension principle  $\mathcal{E}$ . The voting correspondence is candidate stable according to extension of preferences  $\mathcal{E}$  iff for all  $a \in \mathcal{C}$ , for all  $P \in \mathcal{P}^r$  and for all  $\succsim_a^\mathcal{E} \in \mathcal{D}_a^\mathcal{E}$ ,  $v(\mathcal{C}, P) \succsim_a^\mathcal{E} v(\mathcal{C} \setminus a, P)$ .*

### 3 Candidate Stability and Conditional Expected Utility Maximaxers

We begin by introducing the extension criteria proposed by BDS for the study of strategy-proof social choice correspondences. Candidates are assumed to interpret the possible social choice outcomes as the basis of lotteries and to have von Neumann-Morgenstern preferences over lotteries. If a set is chosen, candidates attribute some probabilities of choice to the elements of this. We present two scenarios, which differ in the description of the way how individuals will assess the probabilities associated to sets. In the first one, BDS1, each candidate is assumed to subjectively assess one probability distribution over  $\mathcal{C}$  and to evaluate each subset according to its conditional utility. In the second scenario, BDS2, the initial distribution of probability is restricted to be uniform, all candidates associate each set with an even chance lottery over all its components. We state this formally.

A utility function is a mapping  $u_i : \mathcal{C} \rightarrow \mathbb{R}$ . We say that  $u_i$  is consistent with  $P_i$  or represents  $P_i$  if for any  $a, b \in \mathcal{C}$ ,  $aP_ib$  iff  $u_i(a) > u_i(b)$ . An individual assessment  $\lambda_i$  is a function  $\lambda_i : \mathcal{C} \rightarrow (0, 1)$ , such that  $\sum_{a \in \mathcal{C}} \lambda_i(a) = 1$ .

**Definition 4 (BDS1)** *For any  $X, Y \subseteq \mathcal{C}$ ,  $X$  is at least as good as  $Y$  according to BDS1 for candidate  $a$  with preferences  $P_a$  represented by the utility function  $u_a$ , and with prob-*

ability assessment  $\lambda_a (X \succsim_a^1 Y)$  iff:

$$\sum_{x \in X} \frac{\lambda_a(x) u_a(x)}{\left(\sum_{x \in X} \lambda_a(x)\right)} \geq \sum_{y \in Y} \frac{\lambda_a(y) u_a(y)}{\left(\sum_{y \in Y} \lambda_a(y)\right)}.$$

For any  $a \in \mathcal{C}$ , let  $\mathcal{D}_a^1$  denote the set of all orderings consistent with the BDS1 criterion obtained from some  $P_a \in \mathcal{P}_a^r$ .<sup>11</sup>

**Definition 5 (BDS 2)** For any  $X, Y \subseteq \mathcal{C}$ ,  $X$  is at least as good as  $Y$  according to BDS2 for candidate  $a$  with preferences  $P_a$  and consistent utility function  $u_a (X \succsim_i^2 Y)$  iff:

$$\frac{1}{\#X} \sum_{x \in X} u_a(x) \geq \frac{1}{\#Y} \sum_{y \in Y} u_a(y).$$

For any  $a \in \mathcal{C}$ , let  $\mathcal{D}_a^2$  denote the set of all BDS2 orderings obtained from some  $P_a \in \mathcal{P}_a^r$ .

**Remark 1**  $\mathcal{D}_a^2 \subset \mathcal{D}_a^1$ .

Before proceeding to state the restrictions that candidate stability implies in both environments, we begin with a lemma on the existence of various admissible preference orderings in  $\mathcal{D}_a^1$  and  $\mathcal{D}_a^2$ .

**Lemma 1 i)** Let  $a \in \mathcal{C}$ ,  $X, Y \subset \mathcal{C}$  be such that  $a \notin (X \cup Y)$ , then there is  $\succsim_a^2 \in \mathcal{D}_a^2$ ,  
 $X \succ_a^2 Y$ .

**ii)** Let  $a \in \mathcal{C}$ ,  $X, Y \subset \mathcal{C}$  be such that  $a \in X$ ,  $a \notin Y$  and  $X = a \cup Y$ ; then for all  $\succsim_a^1 \in \mathcal{D}_a^1$ ,  
 $X \succ_a^1 Y$ .

**iii)** Let  $a \in \mathcal{C}$ ,  $X, Y \subseteq \mathcal{C}$  be such that  $X \neq Y$ ,  $\#X \neq 1$ ,  $a \in X$ ,  $X \neq (Y \cup a)$ , then there  
is  $\succsim_a^1 \in \mathcal{D}_a^1$ ,  $Y \succ_a^1 X$ .

**iv)** Let  $a, b \in \mathcal{C}$ ,  $X, Y \subseteq \mathcal{C}$ , be such that  $X \neq Y$ ,  $a \in X$ ,  $b \in (Y \cup \emptyset)$  and  $(X \setminus a) = (Y \setminus b)$   
then for all  $\succsim_a^2 \in \mathcal{D}_a^2$ ,  $X \succ_a^2 Y$ .

---

<sup>11</sup>For any  $a \in \mathcal{C}$  and  $\succsim \in \mathcal{D}_a^1$ , the strict term of  $\succsim$  is defined in the usual way. For any two sets  $X, Y \subseteq \mathcal{C}$ ,  $X$  is preferred to  $Y$  according to  $\succsim$  iff  $X \succ Y$ , but not  $Y \succ X$ .

v) Let  $a \in \mathcal{C}$ ,  $X, Y \subseteq \mathcal{C}$  be such that  $X \neq Y$ ,  $\#X \neq 1$ ,  $a \in X$ , and for any  $b \in (Y \cup \emptyset)$ ,  $(X \setminus a) \neq (Y \setminus b)$ , then there is  $\succsim_a^2 \in \mathcal{D}_a^2$ ,  $Y \succ_a^2 X$ .

The proof of this Lemma appears in Appendix 2. Lemma 1 presents the main features of the admissible preferences over sets. Items i), iii) and v) imply that the preferences of the candidates over a significant amount of sets are free, independently of the candidates' self preference. Moreover, item iv) says that the BDS2 criterion generates responsive preferences, for any two sets that differ in only one candidate, the set containing the more desirable candidate is always preferred.

Now, we can state the implications of candidate stability in both BDS scenarios when candidates do not vote.

**Lemma 2** *Let  $\mathcal{V} \cap \mathcal{C} = \emptyset$ , then the voting correspondence  $v$  is candidate stable according to BDS1 iff for all  $P \in \mathcal{P}^r$  and  $a \in \mathcal{C}$ :*

- i) *If  $a \in v(\mathcal{C}, P)$  and  $v(\mathcal{C}, P)$  is not a singleton, then  $v(\mathcal{C}, P) = v(\mathcal{C} \setminus a, P) \cup a$ .*
- ii) *If  $a \notin v(\mathcal{C}, P)$ ,  $v(\mathcal{C}, P) = v(\mathcal{C} \setminus a, P)$ .*

**Proof.** Assume there are  $P \in \mathcal{P}^r$  and  $a \in \mathcal{C}$  such that  $a \notin v(\mathcal{C}, P)$ ,  $v(\mathcal{C}, P) \neq v(\mathcal{C} \setminus a, P)$ . As  $a \notin v(\mathcal{C}, P) \cup v(\mathcal{C} \setminus a, P)$ , we can apply i) of Lemma 1 and then we know there is  $\succsim_a^1 \in \mathcal{D}_a^1$  for which  $v(\mathcal{C} \setminus a, P) \succ_a^1 v(\mathcal{C}, P)$ , contradicting candidate stability. If there are  $P \in \mathcal{P}^r$  and  $a \in \mathcal{C}$  such that  $a \in v(\mathcal{C}, P)$ ,  $a \neq v(\mathcal{C}, P)$ ,  $v(\mathcal{C}, P) \neq v(\mathcal{C} \setminus a, P) \cup a$ . By the item iii) of the previous lemma there is  $\succsim_a^1 \in \mathcal{D}_a^1$  such that  $v(\mathcal{C} \setminus a, P) \succ_a^1 v(\mathcal{C}, P)$ , contradicting candidate stability.

If for all  $P \in \mathcal{P}^r$  the voting correspondence satisfies the conditions of the lemma, then it is BDS1 candidate stable. Candidate stability holds trivially whenever  $a \notin v(\mathcal{C}, P)$ , and  $v(\mathcal{C}, P) = v(\mathcal{C} \setminus a, P)$ , or  $a = v(\mathcal{C}, P)$ . Finally, in the remaining possibilities, if for all  $P$  and for all  $a \in v(\mathcal{C}, P)$ ,  $v(\mathcal{C}, P) = v(\mathcal{C} \setminus a, P) \cup a$ , then, by item ii) of the previous lemma  $v(\mathcal{C}, P) \succ_a^1 v(\mathcal{C} \setminus a, P)$ , for all  $\succsim_a^1 \in \mathcal{D}_a^1$ . ■

**Lemma 3** *Let  $\mathcal{V} \cap \mathcal{C} = \emptyset$ . Then the voting correspondence  $v$  is candidate stable under BDS2 iff for all  $P \in \mathcal{P}^r$  and  $a \in \mathcal{C}$ :*

- i) If  $a \in v(\mathcal{C}, P)$  and  $v(\mathcal{C}, P)$  is not a singleton either:*
  - $v(\mathcal{C}, P) = v(\mathcal{C} \setminus a, P) \cup a$  or,
  - there is  $b \in \mathcal{C} \setminus a$  such that  $v(\mathcal{C} \setminus a, P) = (v(\mathcal{C}, P) \setminus a) \cup b$ .
- ii) If  $a \notin v(\mathcal{C}, P)$ ,  $v(\mathcal{C}, P) = v(\mathcal{C} \setminus a, P)$ .*

**Proof.** Assume  $v$  is candidate stable but does not satisfy the conditions of the Lemma. From the arguments in the previous lemma, we know that the only possibility is that there are  $P \in \mathcal{P}^r$ ,  $a, b \in \mathcal{C}$  such that  $v(\mathcal{C} \setminus a, P) = (v(\mathcal{C}, P) \setminus a) \cup b$ . From item v) of Lemma 1 we know that there is at least one  $\succsim_a^2 \in \mathcal{D}_a^2$ , such that  $v(\mathcal{C} \setminus a, P) \succ_a^2 v(\mathcal{C}, P)$ , contradicting BDS2 candidate stability.

For sufficiency we only have to check the case in which for some  $P \in \mathcal{P}^r$  there are  $a, b \in \mathcal{C}$  such that  $a \in v(\mathcal{C}, P)$  and  $v(\mathcal{C} \setminus a, P) = (v(\mathcal{C}, P) \setminus a) \cup b$ . Item iv) of Lemma 1 implies that for all  $\succsim_a^2 \in \mathcal{D}_a^2$ , if there is some  $b \in \mathcal{C} \setminus a$ , such that  $(v(\mathcal{C}, P) \setminus a) \cup b = v(\mathcal{C} \setminus a, P)$ ,  $v(\mathcal{C}, P) \succ_a^2 v(\mathcal{C} \setminus a, P)$ .

■

The constraints that candidate stability implies on  $v$  in both scenarios are quite similar. Notice that when  $v(\mathcal{C}, P)$  is a singleton, the elected candidate never has an incentive to leave the stake, since she considers herself as the best possible set of outcomes of the second stage voting correspondence. Moreover, non elected candidates cannot change the outcomes of the voting stage if they withdraw. We highlight these facts in the following remark.

**Remark 2** *If  $\mathcal{V} \cap \mathcal{C} = \emptyset$  and  $v$  is candidate stable according to BDS1 or BDS2, then for all  $P \in \mathcal{P}^r$ , and  $a \in \mathcal{C}$ ,  $v(\mathcal{C}, P) = v(\mathcal{C} \setminus a, P)$  if  $a \notin v(\mathcal{C}, P)$  and  $v(\mathcal{C}, P) \subseteq v(\mathcal{C} \setminus a, P) \cup a$ , if  $a \in v(\mathcal{C}, P)$ .*

The following theorem shows that almost any unanimous voting correspondence is subject to the incentives of the candidates to leave the ballot when they conform their preferences over sets according to the BDS1 or BDS2 extensions.

A voting correspondence  $v$  is dictatorial if there exists a voter  $i \in \mathcal{V}$  such that  $v(\mathcal{C}, P) = \text{top}(\mathcal{C}, P_i)$  and  $v(\mathcal{C} \setminus a, P) = \text{top}(\mathcal{C} \setminus a, P_i)$  for all  $P \in \mathcal{P}^r$  and for all  $a \in \mathcal{C}$ . A voting correspondence is bidictatorial if there are two voters  $i, j \in \mathcal{V}$  such that  $v(\mathcal{C}, P) = \text{top}(\mathcal{C}, P_i) \cup \text{top}(\mathcal{C}, P_j)$  and  $v(\mathcal{C} \setminus a, P) = \text{top}(\mathcal{C} \setminus a, P_i) \cup \text{top}(\mathcal{C} \setminus a, P_j)$  for all  $P \in \mathcal{P}^r$  and for all  $a \in \mathcal{C}$ .<sup>12</sup>

**Theorem 1** *Let  $\mathcal{V} \cap \mathcal{C} = \emptyset$  and  $v$  be a unanimous voting correspondence:*

- i)  $v$  is candidate stable according to BDS1 if and only if  $v$  is dictatorial.*
- ii) Let  $\#\mathcal{C} \geq 4$ ;  $v$  is candidate stable according to BDS2 if and only if  $v$  is dictatorial or bidictatorial.*

In the same vein than DJL, we obtain the result as a corollary to a more general result, Theorem 3, that appears in Appendix 1. Theorem 3 allows for an overlap between the candidates' and voters' sets at the cost of employing stability notions that are not directly derived from the candidates' incentives to withdraw the election. The proofs of Theorems 1 and 3 are provided in Appendix 1 and we just sketch them here. We first focus on a restricted domain in which three candidates are a restricted top-set, and voters also agree in their preferences over the remaining candidates. In this case, we can prove that if  $v$  is candidate stable and unanimous we can construct a group decision rule that always generates quasitransitive orderings on  $\mathcal{C}$ ,<sup>13</sup> and satisfies Unrestricted Domain, Pareto and Arrow's Independence of Irrelevant Alternatives. Thus, by a well known result by Mas-Colell and Sonnenschein [15] the group decision rule is oligarchical and there is

---

<sup>12</sup>This restricted definition of dictatorial rules is due to the fact that we only care about the stability of the whole set of candidates.

<sup>13</sup>A binary relation is quasitransitive if its strict component is transitive.

a set of voters (vetoers or oligarchs) holding veto power. When we restrict our attention to candidate stability under BDS1, we can prove immediately that there can be only one vetoer, a dictator, although this is not true according to BDS2. Finally, we extend the results from the restricted domain to arbitrary profiles of preferences, and if  $\#\mathcal{C} \geq 4$ , we attain the result for BDS2.<sup>14</sup>

When  $\#\mathcal{C} = 3$ , there exist many rules satisfying unanimity and BDS2 candidate stability. For instance, the Pareto correspondence, or the union of the top candidates generate BDS2 candidate stable voting correspondences, but many other examples exist. Although we cannot provide a full characterization, we can state that even in any case there is always a group of voters holding veto power.

**Example 1** *Let  $\mathcal{V} = \{1, \dots, \#\mathcal{V}\}$ ,  $\#\mathcal{C} = 3$ ,  $\mathcal{V} \cap \mathcal{C} = \emptyset$ , and  $k^* = \min\{i \in \mathcal{V}, \text{ s.t. } \text{top}(A, P_i) \neq \text{top}(A, P_1)\}$ . Let  $\tilde{v}$  be a voting correspondence such that for all  $A \subseteq \mathcal{C}$ , and for all  $P \in \mathcal{P}^r$ ,  $\tilde{v}(A, P) = \text{top}(A, P_1) \cup \text{top}(A, P_{k^*})$ .  $\tilde{v}$  is unanimous and candidate stable according to BDS2.*

Notice that, apart from the case  $\#\mathcal{C} = 3$ , Theorem 1 is in the line of the BDS results. BDS show that only dictatorial rules are unanimous and strategy-proof according to BDS1, while only dictatorial or bidictatorial rules are unanimous and BDS2 strategy-proof. In any case, it is clear that candidate stability and strategy-proofness are logically independent conditions, and quite different methods of proof are required.

---

<sup>14</sup>The theorem admits other proofs. For instance, following Weymark [24], we can show that if a voting correspondence is candidate stable according BDS1 or BDS2 and unanimous, it is Pareto Efficient when the agenda contains at least  $\#\mathcal{C} - 1$  candidates. Then, from the results of Denicolò [8] on nonbinary choice, we can develop the arguments that lead us to complete the proof. Denicolò's theorems are not concerned with candidate stability, but they make use of consistency conditions that can be derived from both BDS versions of candidate stability.

## 4 Discussion on the Extension Criterion

The results stated in the previous section depend dramatically on how candidates compare the sets of possible equilibria of the voting stage. Both BDS scenarios are plausible when candidates can make use of plenty of information and construct preferences over sets in a somehow sophisticated way. In this section, we assume that candidates cannot assess any probability distribution on the set of possible equilibria of the voting stage. In such situations, candidates may have extreme attitudes towards risk, consistent with the leximin, maximin or maximax extension criteria.<sup>15</sup> They may also be reluctant to express strict preference among sets which are not clearly better or worse than other. In this case the candidates may use other extensions like, for example, the Gärdenfors extension.

### Candidate Stability for Leximiners

**Definition 6** *Given  $a \in \mathcal{C}$ ,  $P_a \in \mathcal{P}_a^r$  and  $X \subseteq \mathcal{C}$ , define  $X_a(1)$  as the lowest ranking alternative of  $X$  according to  $P_a$ . Once  $X_a(t)$  is defined,  $X_a(t+1) = \emptyset$  if  $X = \{X_a(1), \dots, X_a(t)\}$ , and otherwise  $X_a(t+1) = (X \setminus \{X_a(1), \dots, X_a(t)\})_i(1)$ . Then, for any  $X, Y \subseteq \mathcal{C}$ , ( $X \neq Y$ ), and being  $t$  the smallest integer such that  $X_a(t) \neq Y_a(t)$ ;  $X \succ_a^{lex} Y$  if  $X_a(t) P_a Y_a(t)$  or  $Y_a(t) = \emptyset$ . For any  $a \in \mathcal{C}$ , let  $\mathcal{D}_a^{lex}$  denote the set of all preferences over sets consistent with leximin for some  $P_a \in \mathcal{P}_a^r$ .<sup>16</sup>*

Leximin reflects the way in which sophisticated pessimistic individuals can compare sets. If two sets have the same worst alternative, leximiners care about the next to the worst one, and if these are also the same, they take the following one and the procedure continues in the same fashion. Evidently, indifference is only obtained if the sets are identical.

---

<sup>15</sup>These extension principles were firstly proposed by Pattanaik [18] and [19].

<sup>16</sup>Analogously to leximin, we could define the leximax extension by reversing the way in which individuals order alternatives in each set, that is from the best alternative to the worst one.



**Lemma 4** *Let  $\mathcal{V} \cap \mathcal{C} = \emptyset$ . Then  $v$  is candidate stable according to leximin iff for all  $P \in \mathcal{P}^r$  and  $a \in \mathcal{C}$ ,  $v(\mathcal{C}, P) \subseteq v(\mathcal{C} \setminus a, P) \cup a$  if  $a \in v(\mathcal{C}, P)$ , and  $v(\mathcal{C}, P) = v(\mathcal{C} \setminus a, P)$ , if  $a \notin v(\mathcal{C}, P)$ .*

**Proof.** Assume  $b \in v(\mathcal{C}, P)$ ,  $b \notin v(\mathcal{C} \setminus a, P)$ , then there is  $P_a \in \mathcal{P}_a^r$ , with  $b = \text{bottom}(\mathcal{C}, P_a)$ , and  $\succsim_a^{\text{lex}} \in \mathcal{D}_a^{\text{lex}}$  such that  $v(\mathcal{C} \setminus a, P) \succ_a^{\text{lex}} v(\mathcal{C}, P)$ . On the other hand, assume  $a \notin v(\mathcal{C}, P)$  and there is some  $c \in v(\mathcal{C} \setminus a, P) \setminus v(\mathcal{C}, P)$ . Take  $P'_a \in \mathcal{P}_a^r$  such that for all  $c \in v(\mathcal{C} \setminus a, P) \setminus v(\mathcal{C}, P)$ ,  $c P'_a \max(v(\mathcal{C}, P), P_a)$ . Since, by candidate stability we know that  $v(\mathcal{C}, P) \subseteq v(\mathcal{C} \setminus a, P)$ , then for some  $\succsim_a^{\text{lex}} \in \mathcal{D}_a^{\text{lex}}$  and  $P \in \mathcal{P}^r$ ,  $v(\mathcal{C} \setminus a, P) \succ_a^{\text{lex}} v(\mathcal{C}, P)$ , contradicting candidate stability.

If  $a \notin v(\mathcal{C}, P)$  and  $v(\mathcal{C}, P) = v(\mathcal{C} \setminus a, P)$ , for all  $\succsim_a^{\text{lex}} \in \mathcal{D}_a^{\text{lex}}$ ,  $v(\mathcal{C}, P) \succsim_a^{\text{lex}} v(\mathcal{C} \setminus a, P)$ . If  $a \in v(\mathcal{C}, P)$  and  $v(\mathcal{C}, P) \subseteq v(\mathcal{C} \setminus a, P) \cup a$ , we know that being  $t$  the first integer for which  $[v(\mathcal{C}, P)]_a(t) \neq [v(\mathcal{C} \setminus a, P)]_a(t)$ ,  $a = [v(\mathcal{C}, P)]_a(t)$ , and for all  $P_a \in \mathcal{P}_a^r$ ,  $a P_a [v(\mathcal{C} \setminus a, P)]_a(t)$ , and therefore for all  $v(\mathcal{C}, P) \succsim_a^{\text{lex}} v(\mathcal{C} \setminus a, P)$ .

■

Candidate stability under leximin is slightly weaker than under BDS, but it includes the main features we use in the proofs of previous section, namely Remark 2. Hence, the arguments of the proof of Theorem 1 and 3 are also valid for the leximin case.

For any  $I \subseteq \mathcal{V}$ , for all  $P \in \mathcal{P}^r$  and  $A \subseteq \mathcal{C}$ ,  $\text{Pareto}(A, P_I) = \{a \in A, \text{ s.t. for no } b \in A, b P_i a \text{ for all } i \in I\}$ .

**Theorem 2** *Let  $\mathcal{V} \cap \mathcal{C} = \emptyset$  and  $v$  be a voting correspondence.*

- i) *If  $v$  is unanimous and candidate stable according to leximin then there is a subset of voters  $S \subseteq \mathcal{V}$ , such that for all  $a \in \mathcal{C}$ ,  $v(\mathcal{C}, P) \subseteq \text{Pareto}(\mathcal{C}, P_S)$ , and  $v(\mathcal{C} \setminus a, P) \subseteq \text{Pareto}(\mathcal{C} \setminus a, P_S)$ .*
- ii) *If there is  $S \subseteq \mathcal{V}$  such that for all  $P \in \mathcal{P}^r$  and for all  $a \in \mathcal{C}$ ,  $v(\mathcal{C}, P) = \text{Pareto}(\mathcal{C}, P_S)$ , and  $v(\mathcal{C} \setminus a, P) = \text{Pareto}(\mathcal{C} \setminus a, P_S)$ , then  $v$  is unanimous and candidate stable according to the leximin extension.*

**Proof.** Item i) is a direct corollary of Theorem 4 in Appendix 1.

It is clear that Pareto correspondences are unanimous. On the other hand, if  $a \notin \text{Pareto}(\mathcal{C}, P_S)$ , then all alternatives that are Pareto undominated when all candidates are at stake they are not dominated by any alternative in  $\mathcal{C} \setminus a$ ; and those that were dominated, are also dominated by some alternative in  $\mathcal{C} \setminus a$ . Hence  $\text{Pareto}(\mathcal{C}, P_S) = \text{Pareto}(\mathcal{C} \setminus a, P_S)$ . Finally, if  $a \in \text{Pareto}(\mathcal{C}, P_S)$  there cannot be any alternative  $b \in \text{Pareto}(\mathcal{C}, P_S) \setminus \text{Pareto}(\mathcal{C} \setminus a, P_S)$ , since if  $b$  is not dominated by any alternative in  $\mathcal{C}$ , it cannot be dominated by any alternative in  $\mathcal{C} \setminus a$ .

■

Theorem 2 shows that unanimous and leximin candidate stable voting correspondences endow some individuals with veto power. We want to remark that, besides of Pareto correspondences, many other rules satisfy the requirements of Theorem 2 independently of the size of the set of candidates. For instance, we can mention the union of the top alternatives of a group of vetoers, or other oligarchical rules as the one presented in Example 1. Moreover, since any unanimous and leximin candidate stable voting correspondence is efficient, Theorem 2 can be viewed as a positive result: many interesting voting correspondences do not provide incentives for candidates to withdraw if they use the leximin extension criterion.<sup>17</sup>

The remainder of this section is devoted to present the possibilities that arise when candidates compare sets according to less compelling extension criteria, like maximin, maximax or Gärdenfors Extension. In these scenarios, candidate stability is not very stringent, and many interesting rules are not affected by the incentives of the voters to quit the election. The arguments we use in the proof of the previous theorems do not apply to these new environments and we cannot obtain complete characterization results.

---

<sup>17</sup>Campbell and Kelly [5] prove that a social choice correspondence is unanimous and strategy-proof according to leximin iff it is the union of the best alternatives of an arbitrary group of voters at any preference profile.

The crucial point is that candidate stability no longer implies rationalizability of the voting procedure. Hence, we derive the specific implications of candidate stability in each context and provide some relevant examples of voting correspondences that are candidate stable in these new environments.

### Candidate Stability under the Gärdenfors Extension

**Definition 7** *The set  $X$  is preferred to the set  $Y$  by an individual  $a \in \mathcal{N}$  with preferences over candidates  $P_a \in \mathcal{P}_a^r$  according to the Gärdenfors Extension,  $X \succ_a^G Y$ , iff for some  $u_a$  representing  $P_a$ , for all probability assessment over the set of candidates  $\mathcal{C}$ ,  $\lambda$ ;*

$$\sum_{x \in X} \frac{\lambda(x) u_a(x)}{\left(\sum_{x \in X} \lambda(x)\right)} > \sum_{y \in Y} \frac{\lambda(y) u_a(y)}{\left(\sum_{y \in Y} \lambda(y)\right)}.$$

*Equivalently  $X \succ_a^G Y$  iff for all  $b \in X \setminus Y$ ,  $c \in X \cap Y$ ,  $d \in Y \setminus X$ ;  $bP_a c P_a d$ . For any  $X, Y \subseteq \mathcal{C}$ , if neither  $X \succ_a^G Y$ , nor  $Y \succ_a^G X$ ,  $X$  is said indifferent to  $Y$ ,  $X \sim_a^G Y$ .<sup>18</sup>*

The Gärdenfors Extension fits uncertain environments in which any distribution of probabilities among alternatives is plausible and candidates are reluctant to use a specific one. Only if for all possible assessments the conditional expected utility of  $X$  is higher than for  $Y$ , an individual prefers  $X$  to  $Y$ .

Let  $\mathcal{B}$  denote the set of complete binary relations on  $\mathcal{C}$ . For any  $P \in \mathcal{P}^r$  and any  $a, b \in \mathcal{C}$ , the majority relation,  $M_P$ , is defined as follows,  $aM_P b$  iff  $\#\{i \in \mathcal{V} \text{ s.t. } aP_i b\} \geq \#\{j \in \mathcal{V} \text{ s.t. } bP_j a\}$ . Obviously, the strict component of the majority relation,  $\bar{M}_P$ , is defined in the usual way. For any  $A \subseteq \mathcal{C}$  and for all  $P \in \mathcal{P}^r$ , the Top Cycle of the majority

---

<sup>18</sup>A proof of this statement can be found in Ching and Zhou [6]. They make use of degenerate lotteries, and the equivalence is stated for all  $u_a$  representing  $P_a$ , but one can easily see that a slightly modified argument applies to our case. As we have already mentioned in Section 2.3, we introduce a definition of the Gärdenfors extension which generates complete preference relations over sets to get a correct definition of candidate stability.

relation  $TC(A, P)$  is defined as the minimal (w.r.t. inclusion) set on  $A$  such that all its elements defeat by majority comparisons all the candidates in  $A$  outside it.<sup>19</sup>

**Proposition 1** *Assume candidates are not allowed to vote. Then:*

- i) A voting correspondence  $v$  is candidate stable according to the Gärdenfors extension iff for all  $a \in \mathcal{C} \setminus v(\mathcal{C}, P)$ ,  $v(\mathcal{C}, P) = v(\mathcal{C} \setminus a, P)$ .<sup>20</sup>*
- ii) The Top Cycle Correspondence is unanimous and candidate stable according to the Gärdenfors extension.*

**Proof.** i) Any candidate  $a \in v(\mathcal{C}, P)$  never has incentives to leave the ballot, since we can find a lottery for which she is better off by staying in rather than quitting. It is enough to take a prior distribution in which  $\lambda(a)$  is close enough to 1. On the other hand, if for some  $P \in \mathcal{P}^r$  there is  $a \notin v(\mathcal{C}, P)$  such that  $v(\mathcal{C}, P) \neq v(\mathcal{C} \setminus a, P)$ , then  $a$  would have incentives to withdraw if her preferences over candidates,  $P_a$ , are such that for all  $b \in v(\mathcal{C} \setminus a, P) \setminus v(\mathcal{C}, P)$ ,  $c \in v(\mathcal{C}, P) \cap v(\mathcal{C} \setminus a, P)$  and  $d \in v(\mathcal{C}, P) \setminus v(\mathcal{C} \setminus a, P)$ ,  $bP_a c P_a d$ .

ii) Unanimity of the Top Cycle correspondence is clear. To prove Gärdenfors candidate stability is also immediate. If a candidate  $a \notin TC(\mathcal{C}, P)$  leaves the stake all candidates within  $TC(\mathcal{C}, P)$  beat all candidates outside it. Since  $TC(\mathcal{C}, P)$  is minimal in  $\mathcal{C}$ , it is also minimal in  $\mathcal{C} \setminus a$ . Then for all  $P \in \mathcal{P}^r$  and for all  $a \notin TC(\mathcal{C}, P)$ ,  $TC(\mathcal{C}, P) = TC(\mathcal{C} \setminus a, P)$ .

■

We want to remark that Gärdenfors' candidate stability condition is equivalent to the DJL's formulation of candidate stability when referred to functions, but its implications are dramatically different in multi-valued environments. In the single-valued case, it

---

<sup>19</sup>Analogously, we could define the Top Cycle of any binary relation  $B \in \mathcal{B}$  in the same fashion. The results of the next Proposition are also valid for the Top Cycle of any binary relation  $B \in \mathcal{B}$ .

<sup>20</sup>We have to note that the candidate stability condition for the leximax extension can be equivalently formulated. Hence, the conclusions we reach in this section for the Gärdenfors extension apply also when individuals are leximaxers.

is equivalent to a regularity condition on choice functions that implies the transitive rationalizability of the voting procedure. However, the implications are much weaker when applied to multi-valued environments.<sup>21</sup>

### Candidate Stability under Maximin

The maximin criterion can be interpreted as the behavior of extreme risk averters who only care about the worst alternative in a set.  $X \succsim_a^{\min} Y$  according to the maximin criterion for candidate  $a$  with preferences  $P_a$  iff  $\text{bottom}(X, P_a) R_a \text{bottom}(Y, P_a)$ .

An effectivity function  $E$  is a mapping from the set of non empty-sets of voters,  $2^{\mathcal{V}} \setminus \emptyset$ , to  $2^{\mathcal{C}} \setminus \emptyset$  such that (i) for any  $A \in 2^{\mathcal{C}} \setminus \emptyset$ ,  $A \in E(\mathcal{V})$ ; and (ii), for any  $S \in 2^{\mathcal{V}} \setminus \emptyset$ ,  $\mathcal{C} \in E(S)$ . Effectivity functions specify the power of the coalitions of voters on subsets of candidates, regardless of their preferences. Given the effectivity function  $E$ , we say that the coalition  $S \subseteq \mathcal{V}$  has the power to force the social outcome to belong to the set  $A \subseteq \mathcal{C}$  iff  $A \in E(S)$ .

The coalition  $S \subseteq \mathcal{V}$  blocks candidate  $b$  at profile  $P \in \mathcal{P}^r$  if there is another alternative  $a \in \mathcal{C}$  such that for all  $i \in S$ ,  $a P_i b$ , and  $a \in E(S)$ . We can define the domination relation associated to  $E$  at profile  $P$ ,  $D_P \in \mathcal{B}$ , in such a way that for any two candidates  $a, b \in \mathcal{C}$ , we say  $a$  dominates  $b$ ,  $a D_P b$ , iff there is a coalition  $S \subseteq \mathcal{V}$  that blocks  $b$  using  $a$ . For any  $A \subseteq \mathcal{C}$  and  $P \in \mathcal{P}^r$ , the Core correspondence associated to the effectivity function  $E$ ,  $C_E(A, P)$ , selects the set of candidates in  $A$  which are not dominated by other candidate in  $A$  at profile  $P$ .<sup>22</sup>

The Pareto correspondence is a special case of Core correspondence. It corresponds to the situation in which only the whole set of voters has the power to block some candidate. Many other examples of core correspondences can be constructed by defining different effectivity functions and different distributions of the veto power among the coalitions

---

<sup>21</sup>See the preliminaries to the proof of Theorem 3 in Appendix 1, and Aizerman and Aleskerov [1].

<sup>22</sup>The Core correspondence at profile  $P$  is non-empty iff the domination relation generated by  $E$  is acyclical. See Moulin [16], Section 11.4.

of voters. For instance, special qualified majorities may be required for selecting some candidates.

**Proposition 2** *Let  $\mathcal{V} \cap \mathcal{C} = \emptyset$ . Then:*

*i)  $v$  is candidate stable according to maximin if and only if for all  $P \in \mathcal{P}^r$ , for all  $a \in \mathcal{C}$ ,  $v(\mathcal{C}, P) \subseteq v(\mathcal{C} \setminus a, P) \cup a$ .*

*ii) Given an effectivity function  $E$ , if for all  $P \in \mathcal{P}^r$  its associated Core correspondence,  $C_E$ , is non-empty, then it is unanimous and maximin candidate stable.*

**Proof.** i) Notice that if it is not the case, there is  $P \in \mathcal{P}_a^r$ , such that some  $b \in v(\mathcal{C}, P) \setminus v(\mathcal{C} \setminus a, P)$ ,  $b = \text{bottom}(\mathcal{C}, P_a)$ , contradicting candidate stability. On the other hand, if for all  $P \in \mathcal{P}^r$  and for all  $a \in \mathcal{C}$ ,  $v(\mathcal{C}, P) \subseteq v(\mathcal{C} \setminus a, P) \cup a$ , it cannot be the case that for some  $a \in \mathcal{C}$  and some  $P_a \in \mathcal{P}_a^r$ ,  $\text{bottom}(v(\mathcal{C} \setminus a, P)) \not\subseteq P_a \text{bottom}(v(\mathcal{C}, P))$ .

ii) Core correspondences are always unanimous. It is easy to check maximin candidate stability. If some candidate is not dominated by any other alternative within  $\mathcal{C}$ , it is not dominated by any candidate in  $\mathcal{C} \setminus a$ , for any  $a \in \mathcal{C}$ , and then for all  $a \in \mathcal{C}$  and all  $P \in \mathcal{P}^r$   $C_E(\mathcal{C}, P) \subseteq C_E(\mathcal{C} \setminus a, P)$ .

■

### Candidate Stability for Maximaxers

Finally, maximax extension is obviously defined as the way in which extremely optimistic people compare sets of candidates.  $X \succsim_a^{\text{max}} Y$  according to the maximax criterion for individual  $a$  with preferences  $P_a$  iff  $\text{top}(X, P_a) R_a \text{top}(Y, P_a)$ .

Let  $\#\mathcal{V}$  be odd. We say  $a$  is covered by  $b$  according to the majority relation  $M_P$  if  $b \bar{M}_P a$  and for all  $c$  such that  $a \bar{M}_P c$ , also  $b \bar{M}_P c$ . For all  $A \subseteq \mathcal{C}$ , and for all  $P \in \mathcal{P}^r$  define the Uncovered Set of the majority relation  $UC(A, P)$ , as the set of candidates in  $A$  who are not covered by other candidate in  $A$ .

**Proposition 3** *Let  $\mathcal{V} \cap \mathcal{C} = \emptyset$ . Then:*

- i) A voting correspondence  $v$  is candidate stable according to the maximax extension if and only if for all  $a \in \mathcal{C} \setminus v(\mathcal{C}, P)$ ;  $v(\mathcal{C} \setminus a, P) \subseteq v(\mathcal{C}, P)$ .*
- ii) If  $\#\mathcal{V}$  is odd, the Uncovered Set of the majority relation is unanimous and maximax candidate stable.*

**Proof.** i) A candidate within  $v(\mathcal{C}, P)$  never has incentives to leave the fray if she is a maximaxer. Moreover, if  $v(\mathcal{C} \setminus a, P) \subseteq v(\mathcal{C}, P)$ , for all  $a \notin v(\mathcal{C}, P)$ ,  $top(v(\mathcal{C} \setminus a, P), P_a) \subseteq top(v(\mathcal{C}, P), P_a)$ , and candidate  $a$  has no incentive to exit. On the other hand, if there are  $a, b \in \mathcal{C}$  and a preference profile  $P \in \mathcal{P}^r$  such that  $b \in v(\mathcal{C}, P) \setminus v(\mathcal{C} \setminus a, P)$ , if  $b = top(\mathcal{C} \setminus a, P_a)$ , candidate  $a$  has an incentive to leave the election.

ii) Unanimity holds, since the Uncovered Set is a subset of the Top Cycle. To see that the uncovered set is maximax candidate stable, assume there are  $P \in \mathcal{P}^r$ , and  $a, b \in \mathcal{C}$ , such that  $a \notin UC(\mathcal{C}, P)$ ,  $b \in UC(\mathcal{C} \setminus a, P) \setminus UC(\mathcal{C}, P)$ . Notice that if  $b \in \mathcal{C}$  is not covered by any  $c \in \mathcal{C} \setminus a$  but it is covered in  $\mathcal{C}$ , then  $b$  is covered by  $a$ . By the transitivity of the covering relation and as  $a \notin UC(\mathcal{C}, P)$ ,  $b$  also should be covered by some candidate in  $\mathcal{C} \setminus a$ , a contradiction.

■

Maximax candidate stability is slightly weaker than Gärdenfors' candidate stability. Hence, any Gärdenfors candidate stable voting correspondence is also maximax candidate stable, while the converse is not true. In fact, it is easy to see that the uncovered set of the majority rule is not candidate stable according to the Gärdenfors Extension.

**Example 2** *Let  $\#\mathcal{V} = 3$ ,  $\mathcal{C} = \{a, b, c, d, e\}$  and  $aP_1eP_1bP_1cP_1d$ ;  $bP_2cP_2dP_2aP_2e$  and  $cP_3eP_3dP_3aP_3b$ , then  $UC(\mathcal{C}, P) = \{a, b, c, e\}$ , whereas  $UC(\mathcal{C} \setminus d, P) = \{a, b, c\}$ . If the candidate  $d$ 's preferences over candidates are  $dP_4aP_4bP_4cP_4e$  and she uses the Gärdenfors criterion to compare sets of candidates,  $d$  has an incentive to quit.*

**Remark 3** *It is not difficult to check that scoring rules as the set of Borda Count winners do not satisfy candidate stability neither according to maximin, nor according to maximax. However, Condorcet consistent rules as the Top Cycle or the Uncovered Set are candidate stable according to the maximax criterion.*

We finish this section with a comment on the relation among the different implications of candidate stability in these new environments.

**Remark 4** *If candidates are not allowed to vote, a voting procedure  $v$  is maximin and Gärdenfors (or maximax) candidate stable if and only if  $v$  is candidate stable according to the leximin extension.*

This final remark implies that a core correspondence (that are unanimous and maximin candidate stable) is maximax candidate stable only if it is the Pareto correspondence.<sup>23</sup>

## 5 Appendices

### 5.1 Appendix 1: Overlap between Voters and Candidates

As we have already mentioned, we will prove Theorems 1 and 2 as corollaries to stronger theorems regarding the possibility in which candidates are also allowed to vote. Theorem 3 will be used to prove Theorem 1, and Theorem 4 will be used to prove Theorem 2. Firstly, we have to introduce new definitions of stability.

**Definition 8** *The voting correspondence  $v$  is strongly candidate stable according to BDS1*

---

<sup>23</sup>Demange [7] shows that many core correspondences, those generated from convex effectivity functions, are not manipulable according to maximax. This family of core correspondences strictly includes Pareto correspondences. Hence, we cannot state that candidate stability is generically weaker than strategy-proofness, as many maximax strategy-proof social choice correspondences are not maximax candidate stable.



iff for all  $P \in \mathcal{P}^r$  and  $a \in \mathcal{C}$ ,  $v(\mathcal{C}, P) = v(\mathcal{C} \setminus a, P) \cup a$  if  $v(\mathcal{C}, P)$  is not a singleton and  $a \in v(\mathcal{C}, P)$ , and  $v(\mathcal{C}, P) = v(\mathcal{C} \setminus a, P)$  if  $a \notin v(\mathcal{C}, P)$ .<sup>24</sup>

**Definition 9** *The voting correspondence  $v$  is strongly candidate stable according to BDS2 iff for all  $P \in \mathcal{P}^r$  and  $a \in \mathcal{C}$ , either  $v(\mathcal{C}, P) = v(\mathcal{C} \setminus a, P) \cup a$ ; or there is some  $b \in \mathcal{C} \setminus a$ , such that  $(v(\mathcal{C}, P) \setminus a) = (v(\mathcal{C} \setminus a, P) \setminus b)$ , if  $v(\mathcal{C}, P)$  is not a singleton and  $a \in v(\mathcal{C}, P)$ ; and  $v(\mathcal{C}, P) = v(\mathcal{C} \setminus a, P)$ , if  $a \notin v(\mathcal{C}, P)$ .*

**Definition 10** *The voting correspondence  $v$  is strongly candidate stable according to leximin iff for all  $P \in \mathcal{P}^r$  and  $a \in \mathcal{C}$ ,  $v(\mathcal{C}, P) \subseteq v(\mathcal{C} \setminus a, P) \cup a$  if  $a \in v(\mathcal{C}, P)$ , and  $v(\mathcal{C}, P) = v(\mathcal{C} \setminus a, P)$ , if  $a \notin v(\mathcal{C}, P)$ .*

**Remark 5** *If  $v$  is strongly candidate stable according to BDS1 or BDS2 then  $v$  is strongly candidate stable according to leximin.*

When candidates can vote unanimity has no bite. As candidates consider themselves as the best alternative, in fact, there is not any unanimous profile. Hence, we need to introduce a stronger definition of unanimity.

**Definition 11** *The voting correspondence  $v$  is strongly unanimous iff for all  $P \in \mathcal{P}^r$ , and all  $B \subseteq \mathcal{C}$ , whenever there is  $b \in B$ , such that  $b$  is a restricted top-set of  $B$  then  $v(B, P) = b$ .*

It is immediate to see that strong unanimity is equivalent to unanimity when candidates cannot vote.

Evidently, strong candidate stability under BDS1 (BDS2, leximin) implies candidate stability according to BDS1 (BDS2, leximin) when  $\mathcal{V} \cap \mathcal{C} = \emptyset$ . Nevertheless the strong versions of candidate stability do not take into account candidates' preferences and their

---

<sup>24</sup>Strong candidate stability according to BDS1 is equivalent to the definition of candidate stability used by Eraslan and McLennan [11].

incentives to quit the election. It is not difficult to find rules that are candidate stable according to both BDS extensions (or leximin) but they are not strongly candidate stable when candidates are also voters.

**Example 3** *This example resembles Example 3 in DJL. Let  $\mathcal{V} = \{1, a, b, c\}$  and  $\mathcal{C} = \{a, b, c\}$ . The voting correspondence  $v'$  is such that voter 1 is a dictator when only two candidates enter. When all the candidates are at stake, the best candidate according to the voter 1's preferences is always selected, but if there is a candidate who is a restricted top-set according to the candidates' preferences, she is also chosen.  $v'$  is strongly unanimous and candidate stable according to BDS1. Nevertheless,  $v'$  is not strongly candidate stable according to BDS1.*

**Theorem 3** *Let  $v$  be a strongly unanimous voting correspondence:*

- i)  $v$  is strongly candidate stable under BDS1, if and only if  $v$  is dictatorial and the dictator belongs to  $\mathcal{V} \setminus \mathcal{C}$ .*
- ii) Let  $\#\mathcal{C} \geq 4$ ;  $v$  is strongly candidate stable under BDS2 if and only if either  $v$  is dictatorial or  $v$  is bidictatorial, with dictators belonging to  $\mathcal{V} \setminus \mathcal{C}$ .*

It is evident that Theorem 1 is a corollary of Theorem 3, since strong unanimity and the strong versions of candidate stability are equivalent to unanimity and candidate stability when candidates cannot vote. Following DJL, we highlight the less general Theorem 1 because of it admits much better the interpretation in terms of candidates' incentives.<sup>25</sup>

---

<sup>25</sup>Eraslan and McLennan's [11] Theorem is the natural extension to weak preference orders of the necessity part of item i). They prove that when voters are allowed to express weak preferences, if  $v$  is strongly candidate stable according to BDS1 and strongly unanimous, then  $v$  is a serial dictatorship. That is, there is a group of voters such that the first one always selects her top candidates. Then, a second voter chooses her most preferred candidates among those selected by voter 1. The process continues in the same way until only one candidate remains or there are no voters. If more than one candidate survives, the chosen candidates are those most preferred under a given weak preference ordering.

### Proof of Theorem 3.

#### Preliminaries

The following definitions will become useful. A group decision rule  $F$  on  $\mathcal{P}^r$  is a function  $F_v : \mathcal{P}^r \rightarrow \mathcal{B}$ . For all  $P \in \mathcal{P}^r$ , the base binary relation associated to  $v$  at preferences  $P$ ,  $R_P \in \mathcal{B}$  is defined by  $aR_P b$  iff  $a \in v(\{a, b\}, P)$ ,<sup>26</sup> it is obviously complete and reflexive for all  $P \in \mathcal{P}^r$ . We define the group decision rule  $F_v$  by  $F_v(P) = R_P$  for all  $P \in \mathcal{P}^r$ . We say  $F_v$  is quasitransitive iff  $F_v(P) = R_P$  is a quasiorder for all  $P \in \mathcal{P}^r$ .  $F_v$  is unanimous iff for all  $a, b \in \mathcal{C}$  and for all  $P \in \mathcal{P}^r$  such that for all  $i \in \mathcal{V}$   $aP_i b$ , then  $a\bar{R}_P b$ .  $F_v$  satisfies Arrow's Independence of Irrelevant Alternatives iff for all  $a, b \in \mathcal{C}$  and for all  $P, P' \in \mathcal{P}^r$  such that for any  $i \in \mathcal{V}$   $aP_i b$  iff  $aP'_i b$  then  $aR_P b$  implies  $aR_{P'} b$ . Finally,  $F_v$  is oligarchical iff for all  $P \in \mathcal{P}^r$  there is a set of individuals (vetoers)  $S$ , such that for all  $a, b \in \mathcal{C}$ ,  $aR_P b$  iff there is  $j \in S$  with  $aP_j b$ .

We also include a set of regularity conditions for choice functions, and some related results that will become useful in the proof. Excellent compendia of the choice theory literature can be found in Sen [21] and [22], Moulin [16], and Aizerman and Aleskerov [1].

**Condition 1** *The voting correspondence  $v$  satisfies Chernoff iff for all  $P \in \mathcal{P}^r$ , for all  $A, B \subseteq \mathcal{C}$ ,  $v(A \cup B, P) \subseteq v(A, P) \cup B$ .*

**Condition 2** *The voting correspondence  $v$  satisfies Aizerman iff for all  $P \in \mathcal{P}^r$ ,  $A, B \subseteq \mathcal{C}$ ,  $v(B, P) \subseteq A \subseteq B$  implies  $v(A, P) \subseteq v(B, P)$ .*

**Condition 3** *The voting correspondence  $v$  satisfies Expansion iff for all  $P \in \mathcal{P}^r$ , for all  $A, B \subseteq \mathcal{C}$ ,  $v(A, P) \cup v(B, P) \subseteq v(A \cup B, P)$ .*

The voting correspondence  $v$  is rationalizable iff for all  $P \in \mathcal{P}^r$  and for all  $A \subseteq \mathcal{C}$ ,  $v(A, P) = \{a \in A; \text{for all } y \in A, xR_P y\}$ .

<sup>26</sup>We denote the strict component of  $R_P$  as  $\bar{R}_P$ .

**Proposition 4 (Sen [21])** *v satisfies Chernoff and Aizerman iff  $R_P$  is a quasiorder for all  $P \in \mathcal{P}^r$ .*

**Proposition 5 (Sen [21])** *v satisfies Chernoff, Aizerman and Expansion iff v is rationalizable by a quasiorder.*

The arguments of the proof of Theorem 3 proceed through a series of lemmata.

**Lemma 5** *Consider any  $\{a, b, c\} \in \mathcal{C}$ . If v satisfies strong candidate stability according BDS1 or BDS2 and strong unanimity, then  $v(\mathcal{C}, P) \subseteq \{a, b, c\}$  for all  $P \in \mathcal{P}^r(\{a, b, c\}, \bar{P})$ .*

**Proof.** Pick any  $\{a, b, c\} \subseteq \mathcal{C}$  and  $\bar{P} \in \mathcal{P}^r$ . Consider  $P' \in \mathcal{P}^r(\{a, b, c\}, \bar{P})$  such that  $\{a, b\}$  is a restricted top-set. Either  $a \notin v(\mathcal{C}, P')$ , or  $b \notin v(\mathcal{C}, P')$ , or  $\{a, b\} \subseteq v(\mathcal{C}, P')$ . Without loss of generality, assume  $b \notin v(\mathcal{C}, P')$ , then by strong candidate stability  $v(\mathcal{C}, P') = v(\mathcal{C} \setminus b, P')$ , and by strong unanimity  $v(\mathcal{C} \setminus b, P') = a$ , and also  $v(\mathcal{C}, P') = a$ . If  $\{a, b\} \subseteq v(\mathcal{C}, P')$  by strong unanimity  $v(\mathcal{C} \setminus a, P') = b$ , and  $v(\mathcal{C} \setminus b, P') = a$ , and then as  $v(\mathcal{C}, P') \subseteq v(\mathcal{C} \setminus x, P') \cup x$ ,  $v(\mathcal{C}, P') = \{a, b\}$ . As the choice of  $a$  and  $b$  was arbitrary, it follows that if  $\{x, y\} \subset \{a, b, c\}$  is a restricted top-set of  $P' \in \mathcal{P}^r(\{a, b, c\}, \bar{P})$ , then  $v(\mathcal{C}, P') \subseteq \{x, y\}$ .

Now, consider any  $P \in \mathcal{P}^r(\{a, b, c\}, \bar{P})$ . Suppose that there is some  $x \notin \{a, b, c\}$ ;  $x \in v(\mathcal{C}, P)$ . Take  $v(\mathcal{C} \setminus c, P)$ , by *iii*) in the definition of voting correspondence, it should be the case that  $v(\mathcal{C} \setminus c, P) = v(\mathcal{C} \setminus c, \hat{P})$ , where  $P|_{\mathcal{C} \setminus c} = \hat{P}|_{\mathcal{C} \setminus c}$ , but  $\{a, b\}$  is a restricted top-set for  $\hat{P}$ . By the arguments in the previous paragraph,  $v(\mathcal{C}, \hat{P}) \subseteq \{a, b\}$ , and by strong candidate stability,  $v(\mathcal{C}, \hat{P}) = v(\mathcal{C} \setminus c, \hat{P})$ . Hence,  $v(\mathcal{C} \setminus c, P) \subseteq \{a, b\}$ , and applying again strong candidate stability,  $v(\mathcal{C}, P) \subseteq \{a, b\} \cup \{c\}$ , contrary to our supposition.

Moreover, for any  $d \in \mathcal{C}$ ,  $v(\mathcal{C} \setminus d, P) \subseteq \{a, b, c\} \setminus d$ . Notice that if  $d \notin \{a, b, c\}$   $v(\mathcal{C} \setminus d, P) = v(\mathcal{C}, P)$ , and if  $d \in \{a, b, c\}$ ,  $v(\mathcal{C} \setminus d, P) = v(\mathcal{C} \setminus d, \tilde{P}) = v(\mathcal{C}, \tilde{P})$ , where  $\tilde{P} \in \mathcal{P}_{\{a, b, c\}}^r(\bar{P})$ , and  $P|_{\mathcal{C} \setminus d} = \tilde{P}|_{\mathcal{C} \setminus d}$ , and  $d = \text{bottom}(\{a, b, c\}, \tilde{P}_i)$  for all  $i \in \mathcal{V}$ .

■

At this point, we can define an ancillary restricted voting correspondence  $\hat{v} : 2^{\{a,b,c\}} \setminus \emptyset \times \mathcal{P}^r(\{a, b, c\}, \bar{P}) \longrightarrow 2^{\{a,b,c\}} \setminus \emptyset$ , in such a way that for all  $P \in \mathcal{P}^r(\{a, b, c\}, \bar{P})$  and for all  $x \in \{a, b, c\}$ ;  $\hat{v}(\{a, b, c\}, P) = v(\mathcal{C}, P)$ , and  $\hat{v}(\{a, b, c\} \setminus x, P) = v(\mathcal{C} \setminus x, P)$ . Then,  $\hat{v}$  is a well defined choice function in the restricted domain  $\mathcal{P}^r(\{a, b, c\}, \bar{P})$  and satisfies strong unanimity and strong candidate stability according to BDS1 and/or BDS2 if  $v$  does.

**Lemma 6** *Let  $\hat{v} : 2^{\{a,b,c\}} \setminus \emptyset \times \mathcal{P}^r(\{a, b, c\}, \bar{P}) \longrightarrow 2^{\{a,b,c\}} \setminus \emptyset$  be a strongly unanimous voting correspondence in the domain  $\mathcal{P}^r(\{a, b, c\}, \bar{P})$ .*

- i) If  $\hat{v}$  satisfies strong candidate stability according to BDS1, then  $\hat{v}$  is rationalizable and  $F_{\hat{v}}$  is quasitransitive.*
- ii) If  $\hat{v}$  satisfies strong candidate stability under BDS2, then  $F_{\hat{v}}$  is quasitransitive.*

**Proof.** This result holds from Propositions 4 and 5. We have to check that  $\hat{v}$  satisfies the necessary conditions for rationalizability and quasitransitivity of the base binary relation.

**”Chernoff”.** If  $A$  and  $B$  are singletons,  $A$  or  $B$  are  $\{a, b, c\}$ ,  $A \cup B$  is a duple Chernoff trivially holds. So suppose  $A \cup B = \{a, b, c\}$ ; if  $\#B = 1$ , then Chernoff holds since  $\hat{v}(A \cup B) \subseteq \hat{v}(A, P) \cup B = \{a, b, c\}$ , and by strong candidate stability,  $\hat{v}(A \cup B) \subseteq \hat{v}(B, P) \cup A$ . So the remaining case is that  $A$  and  $B$  are duples, then there is some  $d = A \cap B$ . Notice that by strong candidate stability  $\hat{v}(A \cup B, P) \subseteq \hat{v}(A \cup B \setminus [A \setminus B], P) \cup (A \setminus B)$  but it is also contained in  $\hat{v}(A \cup B \setminus [A \setminus B], P) \cup ([A \setminus B] \cup d) = \hat{v}(A, P) \cup B$ .

**”Aizerman”.** Clearly, if  $A$  is a singleton or  $\{a, b, c\}$  Aizerman holds, so w.l.o.g. take  $A = \{a, b\}$ ,  $B = \{a, b, c\}$ . If  $v(\{a, b, c\}, P) \subseteq \{a, b\}$ , then  $\hat{v}(\{a, b\}, P) = \hat{v}(\{a, b, c\}, P)$  by strong candidate.

**”Expansion”.** Assume  $v$  satisfies strong candidate stability according to BDS1 but it violates Expansion. Then there are  $A, B \in 2^{\{a,b,c\}} \setminus \emptyset$ . with some  $d \in \hat{v}(A, P) \cap \hat{v}(B, P)$ ,  $d \notin \hat{v}(A \cup B, P)$ . If either  $A$  or  $B$  is a singleton or  $\{a, b, c\}$  Expansion holds, so  $A$  and

$B$  should be duples and  $A \cup B = \{a, b, c\}$ . Without lost of generality, let  $A = \{a, b\}$ ,  $B = \{b, c\}$ ,  $b \in \hat{v}(\{a, b\}, P) \cap \hat{v}(\{b, c\}, P)$ ,  $b \notin \hat{v}(\{a, b, c\}, P)$ . By the previous lemma,  $\hat{v}(\{a, b, c\}, P)$  is either  $a$ , or  $c$  or  $\{a, c\}$ . If  $\hat{v}(\{a, b, c\}, P) = a$ , then  $\hat{v}(\{a, b\}, P) = a$ , hence  $\hat{v}(\{a, b, c\}, P) = \{a, c\}$ , contradicting strong candidate stability according to BDS1. ■

**Lemma 7** *Let  $\hat{v}$  be a strongly unanimous voting correspondence that generates a quasitransitive group decision rule  $F_{\hat{v}}$ . Then  $F_{\hat{v}}$  is oligarchical, with the vetoers belonging to  $\mathcal{V} \setminus \{a, b, c\}$ .*

**Proof.** Consider the group decision rule associated to  $\hat{v}$  when preferences of candidates  $a, b, c$  on  $\{a, b, c\}$  are such that,  $aP_a bP_a c$ ;  $bP_b cP_b a$ , and  $cP_c aP_c b$ . Then if for all  $i \in \mathcal{V} \setminus \{a, b, c\}$ ,  $P_i = P^1$ ,  $aP^1 bP^1 c$ ,  $F_{\hat{v}}(P^1_{-\{a,b,c\}}, P_{\{a,b,c\}}) = \bar{R}_{P^1}$  by strong unanimity  $a\bar{R}_{P^1} b$ , and  $b\bar{R}_{P^1} c$ , and as  $R_{P^1}$  is a quasiorder,  $a\bar{R}_{P^1} b\bar{R}_{P^1} c$ . By the same arguments if for all  $i \in \mathcal{V} \setminus \{a, b, c\}$ ,  $P_i = P^2$ ;  $bP^2 cP^2 a$ ,  $b\bar{R}_{P^2} c\bar{R}_{P^2} a$ , and if for all  $i \in \mathcal{V} \setminus \{a, b, c\}$ ,  $P_i = P^3$ ,  $cP^3 aP^3 b$ ;  $c\bar{R}_{P^3} a\bar{R}_{P^3} b$ . If for all  $i \in \mathcal{V} \setminus \{a, b, c\}$ ,  $P_i = P^4$ ,  $aP^4 cP^4 b$ , then by iii) in the definition of voting correspondence, as  $P^4|_{ac} = P^1|_{ac}$ , and  $P^4|_{bc} = P^3|_{bc}$ ,  $a\bar{R}_{P^4} c$ , and  $c\bar{R}_{P^4} b$ , and by quasitransitivity  $a\bar{R}_{P^4} c\bar{R}_{P^4} b$ . Similarly, if  $P_i = P^5$ ,  $bP^5 cP^5 a$ ;  $b\bar{R}_{P^5} a\bar{R}_{P^5} c$ , and if  $P_i = P^6$ ;  $cP^6 bP^6 a$ , then  $c\bar{R}_{P^6} b\bar{R}_{P^6} a$ .

Then,  $F_{\hat{v}}$  satisfies unanimity on  $P_{-\{a,b,c\}}$ , unrestricted domain, since  $P_{-\{a,b,c\}}$  over  $\{a, b, c\}$  are completely free, Arrow's Independence of Irrelevant Alternatives and quasitransitivity when  $P_{\{a,b,c\}} = (P_a, P_b, P_c)$ . Hence, we make use of a theorem by Mas-Colell and Sonnenschein [15], thus  $F_{\hat{v}}$  is oligarchical, and since the preferences of  $a, b$  and  $c$  are fixed, the vetoers belong to  $\mathcal{V} \setminus \{a, b, c\}$ .

Now, we have to extend this partial result to all possible configuration of the preferences of candidates  $a, b$  and  $c$ . If  $a, b$  and  $c$  are not voters, we are already done, since their preferences do not affect the voting correspondence, so we suppose that  $\{a, b, c\}$  are also voters. Suppose that  $aP'_a cP'_a b$  and denote by  $R'_P = F_{\hat{v}}(P_{-\{a,b,c\}}, P'_a, P_b, P_c)$  for all  $P_{-\{a,b,c\}} \in \mathcal{P}^r_{-\{a,b,c\}}$ . Then, if  $P_i = P^3$  for all  $i \in \mathcal{V} \setminus \{a, b, c\}$ ,  $(P^3_{-\{a,b,c\}}, P'_a, P_b, P_c)|_{ac} =$

$(P_{-\{a,b,c\}}^3, P_a, P_b, P_c) \mid_{ac}$  and  $(P_{-\{a,b,c\}}^3, P'_a, P_b, P_c) \mid_{ab} = (P_{-\{a,b,c\}}^3, P'_a, P_b, P_c) \mid_{ab}$ , thus  $c\bar{R}'_{P_3}a$  and  $a\bar{R}'_{P_3}b$ , and by quasitransitivity  $aR'_{P_3}cR'_{P_3}b$ . The same argument applies if for all  $i \in \mathcal{V} \setminus \{a, b, c\}$   $P_i = P^5$ , and then  $bR'_{P_5}cR'_{P_5}a$ . At this point, using AIIA implied by iii) in the definition of voting correspondence, we know that  $F_{\hat{\delta}}$  satisfies unanimity when  $P_{\{a,b,c\}} = (P'_a, P_b, P_c)$ , and again we know that the associated group decision rule is oligarchical with the vetoers in  $\mathcal{V} \setminus \{a, b, c\}$ . Now, we prove that they are the same vetoers that those for  $(P_a, P_b, P_c)$ . Assume the contrary, and w.l.o.g. suppose there is  $i \in \mathcal{V} \setminus \{a, b, c\}$ , such that  $i$  is a vetoer when  $P_{\{a,b,c\}} = (P'_a, P_b, P_c)$  but  $i$  is not when  $P_{\{a,b,c\}} = (P_a, P_b, P_c)$ . Assume that for all  $j \in \mathcal{V} \setminus \{a, b, c, i\}$ ,  $aP_jbP_jc$ , while  $bP_iaP_ic$ . Then  $\hat{v}(\{a, b\}, P_{-\{a,b,c\}}, P_a, P_b, P_c) = a$ , while  $b \in \hat{v}(\{a, b\}, P_{-\{a,b,c\}}, P'_a, P_b, P_c)$ , which is contradiction with the definition of voting correspondence since  $(P_{-\{a,b,c\}}, P_a, P_b, P_c) \mid_{ab} = (P_{-\{a,b,c\}}, P'_a, P_b, P_c) \mid_{ab}$ .

Iterative application of this argument for all possible  $P_{\{a, b, c\}} \in \mathcal{P}_{\{a, b, c\}}^r$  to obtain the desired result.

■

We now introduce the implications on  $v$  of the existence of an oligarchy  $S$  for  $F_{\hat{\delta}}$ .

In the BDS1 scenario,  $v$  does not admit more than one vetoer. Assume the contrary, let  $v$  be strongly candidate stable according to BDS1, and  $\#S \geq 2$ . Let  $P \in \mathcal{P}^r(\{a, b, c\}, \bar{P})$  be such that for some  $i \in S$ ,  $aP_ibP_ic$ , while for all  $j \in S \setminus i$ ,  $bP_jcP_ja$ . Then as  $v$  is rationalizable by an oligarchical binary relation,  $v(\mathcal{C}, P) = \text{Pareto}(\mathcal{C}, P_S) = v(\mathcal{C}, P) = \{a, b\}$ , and  $v(\mathcal{C} \setminus b, P) = \{a, c\}$ . But then  $v(\mathcal{C}, P) \neq v(\mathcal{C} \setminus b, P) \cup b$ , which contradicts strong candidate stability according to BDS1.

If  $v$  is strongly candidate stable according to BDS2, then we can prove that for all  $P \in \mathcal{P}^r(\{a, b, c\}, \bar{P})$ , and for all  $A = \{\mathcal{C}, \{\mathcal{C} \setminus a\}_{a \in \mathcal{C}}\}$ ,  $v(A, P) \subseteq \text{Pareto}(A, P_S)$  and whenever  $\#\text{Pareto}(A, P_S) \leq 2$ ,  $v(A, P) = \text{Pareto}(A, P_S)$ . Assume, that there is some alternative  $d \in v(\mathcal{C}, P)$ ,  $d \notin \text{Pareto}(\mathcal{C}, P_S)$ , then there is some candidate in  $\{a, b, c\}$ , w.l.o.g.  $a$ , which dominates  $d$ . Notice that  $d \notin \hat{v}(\{a, b\}, P) = v(\mathcal{C} \setminus c, P)$ , and then

$d \notin v(\mathcal{C}, P) \subseteq v(\mathcal{C} \setminus c, P) \cup c$ . The same argument applies to the cases in which for some  $e \in \mathcal{C}$ ,  $d \notin \text{Pareto}(\mathcal{C} \setminus e, P_S)$ , then  $d \notin v(\mathcal{C} \setminus e, P)$ . If  $\#\text{Pareto}(A, P_S) \leq 2$ , assume w.l.o.g. that  $c$  is dominated by  $a$ . Then, as  $c \notin v(\mathcal{C}, P)$ ,  $v(\mathcal{C} \setminus c, P) = v(\mathcal{C}, P)$  by strong candidate stability, and again  $v(\mathcal{C}, P) = \hat{v}(\{a, b\}, P) = \text{Pareto}(\mathcal{C}, P_S)$ . A similar reasoning implies that  $\#v(\mathcal{C}, P) = 1$  only if  $\#\text{Pareto}(\mathcal{C}, P_S) = 1$ .

Therefore, in the BDS1 case  $\#S = 1$ , there is a dictator in  $\mathcal{P}^r(\{a, b, c\}, \bar{P})$  as in DJL. However, we can not address directly to DJL to state that the dictator for  $\mathcal{P}^r(\{a, b, c\}, \bar{P})$  is a dictator for all  $P \in \mathcal{P}^r$ , since the proof of DJL depends crucially on the fact that  $v$  is always single-valued. Nevertheless, the arguments we use to extend the results for the BDS2 extension are also valid with minimal modifications for the BDS1 case.

**Lemma 8** *Consider any three distinct candidates,  $\{a, b, c\} \in \mathcal{C}$ . If  $v$  satisfies strong unanimity and strong candidate stability under BDS2, then  $v$  is oligarchical in  $\mathcal{P}^r(\{a, b, c\})$ .*

**Proof.** At this step we extend our previous result to those profiles for which  $\{a, b, c\}$  is top-set but are not uniform for the rest of alternatives. Take  $\bar{P}, \tilde{P}$  such that  $\{a, b, c\}$  is a top-set for both profiles, but there is  $d \in \mathcal{C} \setminus \{a, b, c\}$ ,  $\bar{P} \neq \tilde{P}$ ; but  $\bar{P}|_{\mathcal{C} \setminus d} = \tilde{P}|_{\mathcal{C} \setminus d}$ . We know, by the previous lemma, there are oligarchies for each profile  $\bar{S}, \tilde{S}$  respectively. Suppose to the contrary  $\bar{S} \neq \tilde{S}$ , then there is some vetoer for some profile but not for another, assume there is some  $j \in \bar{S}$ ,  $j \notin \tilde{S}$ . Suppose that, for all  $i \in \mathcal{V} \setminus j$ ,  $aP_i bP_i c$ , while  $bP_j aP_j c$ . Then,  $v(\mathcal{C}, \tilde{P}) = a$ , while  $b \in v(\mathcal{C}, \bar{P})$ , and  $c \notin v(\mathcal{C}, \bar{P})$ . Then, we know,  $v(\mathcal{C} \setminus d, \tilde{P}) = a$ , while  $b \in v(\mathcal{C} \setminus d, \bar{P})$ . But  $\bar{P}|_{\mathcal{C} \setminus d} = \tilde{P}|_{\mathcal{C} \setminus d}$ , which implies  $v(\mathcal{C} \setminus d, \bar{P}) = v(\mathcal{C} \setminus d, \tilde{P})$ , and contradicts our supposition about  $\bar{S} \neq \tilde{S}$ . We can repeat the argument as many times as necessary to prove the statement. ■

**Lemma 9** *If  $v$  satisfies strong candidate stability according to BDS2 and strong unanimity then  $v$  is oligarchical in  $\mathcal{P}^r(\{a, b, c\})$  for every  $\{a, b, c\} \subseteq \mathcal{C}$  (with the same vetoers on each of these domains).*



**Proof.** This follows directly if  $\#\mathcal{C} = 3$ . So suppose now that  $\#\mathcal{C} \geq 4$ . It is enough to consider some  $\{a, b, d\}$  distinct from  $\{a, b, c\}$  and show that the same vetoers "dictate" on  $\mathcal{P}^r(\{a, b, c\})$  and  $\mathcal{P}^r(\{a, b, d\})$ . By the previous lemma, we know there are an oligarchy  $S$  which dictates on  $\mathcal{P}^r(\{a, b, c\})$  and an oligarchy  $S'$  which dictates on  $\mathcal{P}^r(\{a, b, d\})$ . Suppose to the contrary,  $S \neq S'$ , and  $j \in S' \setminus S$ . Consider the profile  $\hat{P} \in \mathcal{P}^r(\{a, b, c, d\})$  such that  $\hat{P}_i \in \mathcal{P}_i^r(a)$  for all  $i \in \mathcal{V} \setminus j$  but  $b = \text{top}(\mathcal{C}, \hat{P}_j)$ ,  $a = \text{top}(\mathcal{C} \setminus b, \hat{P}_j)$  and  $d = \text{bottom}(\{a, b, c, d\}, \hat{P}_i)$  for all  $i \in \mathcal{V}$ . Then, as  $\hat{P} \in \mathcal{P}^r(\{a, b, c\})$ ,  $v(\mathcal{C}, \hat{P}) = a$ , and by strong candidate stability, we know also that  $v(\mathcal{C} \setminus c, \hat{P}) = a$ . Now, take a profile  $P^*$ , such that  $c$  and  $d$  are switched, hence  $\hat{P} \upharpoonright_{\mathcal{C} \setminus c} = P^* \upharpoonright_{\mathcal{C} \setminus c}$ , and  $\hat{P} \upharpoonright_{\mathcal{C} \setminus d} = P^* \upharpoonright_{\mathcal{C} \setminus d}$ , and  $P^* \in \mathcal{P}^r(\{a, b, d\})$ ,  $b \in v(\mathcal{C}, P^*)$ , and as  $c$  is Pareto dominated,  $c \notin v(\mathcal{C}, P^*)$ , by candidate stability  $b \in v(\mathcal{C} \setminus c, P^*)$ . However, this contradicts *iii*) in the definition of voting correspondence, since  $\hat{P} \upharpoonright_{\mathcal{C} \setminus c} = P^* \upharpoonright_{\mathcal{C} \setminus c}$ . We can repeat the argument for any set of three candidates.

■

Note that as the set of vetoers is the same for any  $\{a, b, c\}$  and the candidates cannot be vetoers, this implies that any strongly unanimous voting correspondence is not strongly candidate stable according to BDS1 or BDS2 if  $\mathcal{V} = \mathcal{C}$ .

**Lemma 10** *Let  $v$  be strongly candidate stable under BDS2 and strongly unanimous with  $S$  being the oligarchy for all  $\mathcal{P}^r(\{a, b, c\})$  domains, then for all  $P$  such that  $P_S \in \mathcal{P}_S^r(\{a, b, c\})$ , members of  $S$  still dictate.*

**Proof.** This lemma generalizes the previous one since we do not impose all voters to agree in a top-set. The preferences of the voters who are not members of the oligarchy do not matter when the vetoers agree in a triple as restricted top-set.

Take  $P' \in \mathcal{P}^r(\{a, b, c\})$ , and let  $S$  be the set of vetoers. Construct the profile  $P$  such that  $P_{-S} \upharpoonright_{\{a, b, c\}} = P'_{-S} \upharpoonright_{\{a, b, c\}}$ , and  $P_{-S} \upharpoonright_{\mathcal{C} \setminus \{a, b, c\}} = P'_{-S} \upharpoonright_{\mathcal{C} \setminus \{a, b, c\}}$ ,  $(P_S, P'_{-S}) \in \mathcal{P}^r(\{a, b, c\})$ . Find individual  $k \in \mathcal{V} \setminus S$ , and alternatives  $f = \text{bottom}(\{a, b, c\}, P'_k)$ ,  $e =$

$top(\mathcal{C} \setminus \{a, b, c\}, P'_k)$ . Let  $P''_k$  be such that  $P'_k \upharpoonright_{\mathcal{C} \setminus \{e, f\}} = P''_k \upharpoonright_{\mathcal{C} \setminus \{e, f\}}$ , but  $P_k \upharpoonright_{\{e, f\}} = P''_k \upharpoonright_{\{e, f\}}$ , that is,  $f$  and  $e$  are reversed from  $P''_k$ . Evidently,  $(P_S, P'_{-S, k}, P''_k) \notin \mathcal{P}^r(\{a, b, c\})$ , but  $(P_S, P'_{-S, k}, P''_k) \upharpoonright_{\mathcal{C} \setminus e} = (P_S, P'_{-S}) \upharpoonright_{\mathcal{C} \setminus e}$ , and  $(P_S, P'_{-S, k}, P''_k) \upharpoonright_{\mathcal{C} \setminus f} = (P_S, P'_{-S}) \upharpoonright_{\mathcal{C} \setminus f}$ . Then, as  $e \notin Pareto(\mathcal{C}, P_S)$ ,  $v(\mathcal{C}, (P_S, P'_{-S})) = v(\mathcal{C} \setminus e, (P_S, P'_{-S})) = v(\mathcal{C} \setminus e, (P_S, P'_{-S, k}, P''_k))$ . Moreover, as  $\#Pareto(\mathcal{C} \setminus f, P_S) \leq 2$ ; we know that,

$$v(\mathcal{C} \setminus f, (P_S, P'_{-S})) = v(\mathcal{C} \setminus f, (P_S, P'_{-S, k}, P''_k)) = Pareto(\mathcal{C} \setminus f, P_S),$$

and, by strong candidate stability,  $v(\mathcal{C}, (P_S, P'_{-S, k}, P''_k)) \subseteq v(\mathcal{C} \setminus f, (P_S, P'_{-S, k}, P''_k)) \cup f$ . Given that  $e \notin Pareto(\mathcal{C} \setminus f, P_S)$ ,  $e \notin v(\mathcal{C}, (P_S, P'_{-S, k}, P''_k))$ , this implies that:

$$v(\mathcal{C}, (P_S, P'_{-S, k}, P''_k)) = v(\mathcal{C} \setminus e, (P_S, P'_{-S, k}, P''_k)) = v(\mathcal{C}, (P_S, P'_{-S})).$$

Now, it remains to prove that  $v(\mathcal{C} \setminus d, (P_S, P'_{-S, k}, P''_k)) = v(\mathcal{C} \setminus d, (P_S, P'_{-S}))$  for all  $d \in \mathcal{C}$ . If  $d \notin Pareto(\mathcal{C}, P_S)$ ,  $d \notin v(\mathcal{C}, (P_S, P'_{-S, k}, P''_k)) = v(\mathcal{C}, (P_S, P'_{-S}))$ , then by strong candidate stability,  $v(\mathcal{C} \setminus d, (P_S, P'_{-S, k}, P''_k)) = v(\mathcal{C} \setminus d, (P_S, P'_{-S}))$ . So assume  $d \in Pareto(\mathcal{C}, P_S)$ , if  $d = f$  the conclusion holds by the arguments in the previous paragraph, so take  $d \neq f$ . Let  $P_S^* \upharpoonright_{\mathcal{C} \setminus d} = P_S \upharpoonright_{\mathcal{C} \setminus d}$  with  $d = bottom(\{a, b, c\}, P_i)$  for all  $i \in S$ . Notice that  $d \notin Pareto(\mathcal{C}, P_S^*)$ ,  $Pareto(\mathcal{C} \setminus d, P_S^*) = Pareto(\mathcal{C} \setminus d; P_S)$ , and  $\#Pareto(\mathcal{C} \setminus d; P_S) \leq 2$ . Using an already well known reasoning, as  $v(\mathcal{C} \setminus f, (P_S^*, P'_{-S, k}, P''_k)) = \{a, b, c\} \setminus \{d, f\}$ , then  $v(\mathcal{C}, (P_S^*, P'_{-S, k}, P''_k)) \subseteq (\{a, b, c\} \setminus \{d, f\}) \cup f$ . Hence,  $d \notin v(\mathcal{C}, (P_S^*, P'_{-S, k}, P''_k))$ , and also:

$$v(\mathcal{C}, (P_S^*, P'_{-S, k}, P''_k)) = v(\mathcal{C} \setminus d, (P_S^*, P'_{-S, k}, P''_k)) = Pareto(\mathcal{C}, P_S^*).$$

$$\text{And, } v(\mathcal{C} \setminus d, (P_S^*, P'_{-S, k}, P''_k)) = v(\mathcal{C} \setminus d, (P_S, P'_{-S, k}, P''_k)) = Pareto(\mathcal{C} \setminus d, P_S).$$

Finally, as  $v(\mathcal{C} \setminus d, (P_S, P'_{-S})) = Pareto(\mathcal{C} \setminus d, P_S)$ , we obtain that  $v(\mathcal{C} \setminus d, (P_S, P'_{-S})) = v(\mathcal{C} \setminus d, (P_S, P'_{-S, k}, P''_k))$ .

This argument can be repeated, with one such change at each stage for all  $j \in \mathcal{V} \setminus S$  and  $f \in \{a, b, c\}$  and  $e \notin \{a, b, c\}$  to complete the transition from  $P'_{-S}$  to  $P_{-S}$ .

■

Notice now, that if  $\#S = 1$ , as for any  $P \in \mathcal{P}^r$ ,  $P_S \in \mathcal{P}_S^r(\{a, b, c\})$  for some triple of candidates  $\{a, b, c\}$ , and for all  $A \subseteq \mathcal{C}$ ,  $\#Pareto(A, P_S) = 1$ , then  $v(\mathcal{C}, P) = top(\mathcal{C}, P_i) = Pareto(\mathcal{C}, P_i)$ , and for all  $d \in \mathcal{C}$ ,  $v(\mathcal{C} \setminus d, P) = top(\mathcal{C} \setminus d, P_i)$  for all  $P \in \mathcal{P}^r$ . So, we conclude the proof assuming that  $\#S \geq 2$ , and showing that if  $\#\mathcal{C} \geq 4$ , and  $v$  is not dictatorial then it is bidictatorial. We attain the final result by a series of steps.

**Step 1** For any  $\{a, b, c\} \subset \mathcal{C}$ , for all  $P_S \in \mathcal{P}_S^r(\{a, b, c\})$ ;  $\#v(\mathcal{C}, P) \leq 2$ .

Assume there is some  $P \in \mathcal{P}^r$ ,  $P_S \in \mathcal{P}_S^r(\{a, b, c\})$ , and  $v(\mathcal{C}, P) > 2$ , then  $v(\mathcal{C}, P) = \{a, b, c\}$ . We can assume w.l.o.g.  $a = top(\mathcal{C}, P_1)$ . Now, take  $P' \in \mathcal{P}^r(\{a, b, c, d\})$ , such that  $P' \upharpoonright_{\mathcal{C} \setminus d} = P \upharpoonright_{\mathcal{C} \setminus d}$ , and for all  $i \in \mathcal{V} \setminus 1$ ,  $d = top(\mathcal{C}, P'_i)$ , while  $d = top(\mathcal{C} \setminus a, P'_1)$ . Take now  $P^* \in \mathcal{P}^r(\{a, b, d\})$  such that  $P' \upharpoonright_{\mathcal{C} \setminus c} = P^* \upharpoonright_{\mathcal{C} \setminus c}$ , then  $v(\mathcal{C} \setminus c, P') = v(\mathcal{C} \setminus c, P^*) = \{a, d\} = Pareto(\mathcal{C} \setminus c, P^*)$ , since vetoers agree on a triple as top-set at  $P^*$ . Analogously, construct  $\bar{P} \in \mathcal{P}^r(\{a, c, d\})$  such that  $P' \upharpoonright_{\mathcal{C} \setminus b} = \bar{P} \upharpoonright_{\mathcal{C} \setminus b}$ , then  $v(\mathcal{C} \setminus b, P') = v(\mathcal{C} \setminus b, \bar{P}) = \{a, d\}$ . Hence  $v(\mathcal{C}, P') = \{a, d\}$ , since  $v(\mathcal{C}, P') \subseteq v(\mathcal{C} \setminus c, P') \cup c$ , and  $v(\mathcal{C}, P') \subseteq v(\mathcal{C} \setminus b, P') \cup b$ , and then as  $c \notin v(\mathcal{C}, P')$ ,  $v(\mathcal{C}, P') = v(\mathcal{C} \setminus c, P')$ . Notice now that, as  $P' \upharpoonright_{\mathcal{C} \setminus d} = P \upharpoonright_{\mathcal{C} \setminus d}$ ,  $v(\mathcal{C} \setminus d, P') = \{a, b, c\}$ , but this contradicts BDS2 strong candidate stability.

**Step 2**  $\#S = 2$ .

Assume the contrary, and take  $P \in \mathcal{P}^r$ ,  $P_S \in \mathcal{P}_S^r(\{a, b, c\})$ , such that w.l.o.g. there are vetoers  $1, 2, 3 \in S$  with  $top(\mathcal{C}, P_1) = a$ ,  $top(\mathcal{C}, P_2) = b$ ,  $top(\mathcal{C}, P_3) = c$ . As we know that  $\#v(\mathcal{C}, P) = 2$ , (it cannot be 1 since  $\#Pareto(\mathcal{C}, P_S) > 1$ ), at least one of the vetoers in  $S$  cannot include her top candidate, say 1, and then w.l.o.g.  $v(\mathcal{C}, P) = \{b, c\}$ . Now construct  $P' \in \mathcal{P}^r(\{a, b, c, d\})$ , such that  $P' \upharpoonright_{\mathcal{C} \setminus d} = P \upharpoonright_{\mathcal{C} \setminus d}$  and for all  $i \in \mathcal{V} \setminus 1$ ,  $d = top(\mathcal{C}, P'_i)$ , while  $d = top(\mathcal{C} \setminus a, P'_1)$ . Using the same argument we employed in the previous step, it holds that  $v(\mathcal{C}, P') = \{a, d\}$ , while as  $P' \upharpoonright_{\mathcal{C} \setminus d} = P \upharpoonright_{\mathcal{C} \setminus d}$ ,  $v(\mathcal{C} \setminus d, P') = \{b, c\}$ , another contradiction with strong candidate stability according to BDS2. Then, if  $v$  is not dictatorial, there should be at most two vetoers.

**Step 3** Let  $S = \{1, 2\}$ , then  $v(\mathcal{C}, P) = \text{top}(\mathcal{C}, P_1) \cup \text{top}(\mathcal{C}, P_2)$ ,  $v(\mathcal{C} \setminus x, P) = \text{top}(\mathcal{C} \setminus x, P_1) \cup \text{top}(\mathcal{C} \setminus x, P_2)$ , for all  $x \in \mathcal{C}$ , and for all  $P \in \mathcal{P}^r$ .

Firstly, we prove the statement for all  $P \in \mathcal{P}^r$ , such that  $P_S \in \mathcal{P}_S^r(\{a, b, c\})$ . Assume that  $v(\mathcal{C}, P) \neq \text{top}(\mathcal{C}, P_1) \cup \text{top}(\mathcal{C}, P_2)$ , by the restrictions imposed by the lemmata, it should be the case that  $\#Pareto(\mathcal{C}, P_S) = 3$  and w.l.o.g.  $aP_1bP_1c$ , and  $cP_2bP_2a$ ;  $v(\mathcal{C}, P) = \{a, b\}$ . We construct again  $P' \in \mathcal{P}^r(\{a, b, c, d\})$ , such that  $P' \upharpoonright_{\mathcal{C} \setminus d} = P \upharpoonright_{\mathcal{C} \setminus d}$  and for all  $i \in \mathcal{V} \setminus 2$ ,  $d = \text{top}(\mathcal{C}, P'_i)$ , while  $d = \text{top}(\mathcal{C} \setminus c, P'_i)$ . By an already familiar argument  $v(\mathcal{C}, P') = \{c, d\}$ , while as  $P' \upharpoonright_{\mathcal{C} \setminus d} = P \upharpoonright_{\mathcal{C} \setminus d}$ ,  $v(\mathcal{C} \setminus d, P') = \{a, b\}$ , which implies a new contradiction with strong candidate stability. Then, for all  $P \in \mathcal{P}^r$ , such that  $P_S \in \mathcal{P}_S^r(\{a, b, c\})$   $v(\mathcal{C}, P) = \text{top}(\mathcal{C}, P_1) \cup \text{top}(\mathcal{C}, P_2)$  and immediately, it also follows that for all  $d \in \mathcal{C}$ ,  $v(\mathcal{C} \setminus x, P) = \text{top}(\mathcal{C} \setminus d, P_1) \cup \text{top}(\mathcal{C} \setminus d, P_2)$ .

Finally, we extend the result to any arbitrary preference profile. Take any  $P \in \mathcal{P}^r(\{a, b, c, d\})$ , and assume  $a = \text{top}(\mathcal{C}, P_1)$ ,  $b = \text{top}(\mathcal{C}, P_2)$ .<sup>27</sup> Take now  $\hat{P} \in \mathcal{P}^r(\{a, b, c\})$ , such that  $P \upharpoonright_{\mathcal{C} \setminus d} = \hat{P} \upharpoonright_{\mathcal{C} \setminus d}$ , then  $v(\mathcal{C} \setminus d, P) = v(\mathcal{C} \setminus d, \hat{P}) = \{a, b\}$ , analogously, pick  $\tilde{P} \in \mathcal{P}^r(\{a, b, d\})$ ,  $P \upharpoonright_{\mathcal{C} \setminus c} = \tilde{P} \upharpoonright_{\mathcal{C} \setminus c}$ ,  $v(\mathcal{C} \setminus c, P) = v(\mathcal{C} \setminus c, \tilde{P}) = \{a, b\}$ , then, by strong candidate stability,  $v(\mathcal{C} \setminus c, P) \subseteq v(\mathcal{C} \setminus d, P) \cup d$ , then as  $c \notin v(\mathcal{C} \setminus d, P)$ ,  $c \notin v(\mathcal{C}, P)$ , and  $v(\mathcal{C}, P) = v(\mathcal{C} \setminus c, P) = \{a, b\} = \text{top}(\mathcal{C}, P_1) \cup \text{top}(\mathcal{C}, P_2)$ . Evidently, for all  $e \neq \{a, b\}$ , by strong candidate stability,  $v(\mathcal{C}, P) = v(\mathcal{C} \setminus e, P) = \{a, b\} = \text{top}(\mathcal{C} \setminus e, P_1) \cup \text{top}(\mathcal{C} \setminus e, P_2)$ . Finally, if  $e = a$  or  $e = b$ ,  $v(\mathcal{C} \setminus e, P) = v(\mathcal{C} \setminus e, P^*)$ , where  $P^* \in \mathcal{P}^r(\{a, b, c, d\} \setminus x)$  and  $P^* \upharpoonright_{\mathcal{C} \setminus e} = P \upharpoonright_{\mathcal{C} \setminus e}$  and thus as  $\text{top}(\mathcal{C} \setminus e, P_i) = \text{top}(\mathcal{C} \setminus e, P_i^*)$  for all  $i \in \mathcal{V}$ ,  $v(\mathcal{C} \setminus e, P) = \text{top}(\mathcal{C} \setminus e, P_1) \cup \text{top}(\mathcal{C} \setminus e, P_2)$ . We can repeat as many times as necessary the same argument to prove that the statement is valid for all  $P \in \mathcal{P}^r$ .

■

Finally, we present in Theorem 4 the implications of strong candidate stability according to leximin. Evidently, item i) of Theorem 2 is a direct corollary of the next result.

---

<sup>27</sup>The case in which  $\text{top}(\mathcal{C}, P_1) = \text{top}(\mathcal{C}, P_2)$  can be similarly handled and then it is omitted.

**Theorem 4** *Let  $v$  be a strongly unanimous voting correspondence. If  $v$  is strong candidate stable according to leximin then there is a subset of voters  $S \subseteq \mathcal{V} \setminus \mathcal{C}$ , such that for all  $a \in \mathcal{C}$ ,  $v(\mathcal{C}, P) \subseteq \text{Pareto}(\mathcal{C}, P_S)$ , and  $v(\mathcal{C} \setminus a, P) \subseteq \text{Pareto}(\mathcal{C} \setminus a, P_S)$ .*

**Proof.** By direct application of the lemmata of the proof of the previous theorem, we know there is an oligarchy  $S \subseteq \mathcal{V} \setminus \mathcal{C}$ , such that for all  $P \in \mathcal{P}^r$  with  $P_S \in \mathcal{P}_S^r(\{a, b, c\})$ , for some  $\{a, b, c\} \subseteq \mathcal{C}$ ; members of  $S$  dictate. An argument that mimics the proof of Step 3 in Theorem 3 yields the desired result. ■

## 5.2 Appendix 2: Proof of Lemma 1

i) As  $a$ 's preferences over the candidates within  $X \cup Y$  are free, we know there is  $P_a \in \mathcal{P}_a^r$ , such that for all  $b \in X \setminus Y$ ,  $c \in X \cap Y$  and  $d \in Y \setminus X$ ,  $bP_a cP_a d$ . Pick any  $u_a$  consistent with  $P$ , then for all assessment  $\lambda$ :

$$\begin{aligned} \frac{1}{\sum_{x \in X} \lambda(x)} \left( \sum_{b \in X \setminus Y} \lambda(b) u_a(b) + \sum_{c \in X \cap Y} \lambda(c) u_a(c) \right) &> \frac{1}{\left( \sum_{c \in X \cap Y} \lambda(c) \right)} \sum_{b \in X \cap Y} \lambda(c) u_a(c) > \\ &> \frac{1}{\sum_{y \in Y} \lambda(y)} \left( \sum_{c \in X \cap Y} \lambda(c) u_a(c) + \sum_{d \in Y \setminus X} \lambda(d) u_a(d) \right). \end{aligned}$$

Take now,  $\lambda$  such that for all  $a \in \mathcal{C}$ ,  $\lambda(a) = \frac{1}{\#\mathcal{C}}$ , and there is  $\succsim_a^2 \in \mathcal{D}_a^2$ , such that  $X \succ_a^2 Y$ .

ii) Assume the contrary, then there is  $P_a \in \mathcal{P}_a^r$ ,  $u_a$  and  $\lambda$  such that:

$$\sum_{y \in Y} \frac{\lambda(y) u_a(y)}{\left( \sum_{y \in Y} \lambda(y) \right)} \geq \frac{\lambda(a) u_a(a)}{\lambda(a) + \sum_{y \in Y} \lambda(y)} + \sum_{y \in Y} \frac{\lambda(y) u_a(y)}{\left( \lambda(x) + \sum_{y \in Y} \lambda(y) \right)},$$

and then,

$$\sum_{y \in Y} \frac{\lambda(x) \lambda(y) u_a(y)}{\left( \lambda(x) + \sum_{y \in Y} \lambda(y) \right) \left( \sum_{y \in Y} \lambda(y) \right)} \geq \frac{\lambda(x) u_a(x)}{\lambda(x) + \sum_{y \in Y} \lambda(y)}.$$

This implies that  $\sum_{y \in Y} \frac{\lambda(y)u_a(y)}{\left(\sum_{y \in Y} \lambda(y)\right)} \geq u_a(a)$ , which is a contradiction with  $u_a$  being consistent with some  $P_a \in \mathcal{P}_a^r$ .

iii) Let  $X, Y \subseteq \mathcal{C}$ ,  $X \neq Y$ ,  $a \in X$  be such that  $\#X > 1$ ,  $X \neq a \cup Y$ . Then either there is  $b \in X \setminus a$ ,  $b \notin Y$ ; or there is some  $c \in Y$ ,  $c \notin X$  (or both).

Assume we are in the first case, take  $P_a$  with  $b = \text{bottom}(\mathcal{C}, P_a)$  and  $\lambda$  such that  $\lambda(b) = 1 - \varepsilon$ , for some  $\varepsilon > 0$ ,  $\lambda(x) = \frac{\varepsilon}{\#\mathcal{C}-1}$  for all  $x \in \mathcal{C} \setminus b$ . Denote by  $x(k)$  the candidate in the  $k$ -th position according to  $P_a$  ( $k = 1, \dots, \#\mathcal{C}$ ). Construct now the  $P_a$  consistent utility function  $u_a$  in such a way that  $u_a(x(k)) = 1 - \frac{(k-1)}{\#\mathcal{C}}\varepsilon$  for  $k = 1, \dots, \#\mathcal{C} - 1$ , while  $u_a(x(\#\mathcal{C})) = u_a(b) = 0$ . Notice that, for some  $\varepsilon > 0$  small enough  $\sum_{y \in Y} \frac{\lambda(y)u_a(y)}{\left(\sum_{y \in Y} \lambda(y)\right)}$  is close to 1, while  $\sum_{x \in X} \frac{\lambda(x)u_a(x)}{\left(\sum_{x \in X} \lambda(x)\right)}$  is almost 0.

Assume now there is  $c \in Y$ ,  $c \notin X$ . Take  $\lambda'$  such that  $\lambda'(c) = 1 - \varepsilon'$  for some  $\varepsilon' > 0$ , and  $\lambda'(x) = \frac{\varepsilon'}{\#\mathcal{C}-1}$  for all  $x \in \mathcal{C} \setminus c$ , and  $P_a$  such that  $c = \text{top}(\mathcal{C} \setminus a, P_a)$ . Let  $u_x(x) = 1$ ,  $u_a(c) = 1 - \varepsilon'$ , and  $u_a(x(k)) = \frac{\varepsilon'}{k}$  for all  $k = 3, \dots, \#\mathcal{C}$ . Then, as  $\varepsilon'$  goes to 0,  $\sum_{y \in Y} \frac{\lambda(y)u_a(y)}{\left(\sum_{y \in Y} \lambda(y)\right)}$  goes to 1, while  $\sum_{x \in X} \frac{\lambda(x)u_a(x)}{\left(\sum_{x \in X} \lambda(x)\right)}$  is close to  $\frac{1}{\#X} < 1$ , since  $\#X > 1$ . Then, in any case we can find some  $\succsim_a^1 \in \mathcal{D}_a^1$ ; such that  $Y \succ_a^1 X$ .

iv) Notice that if  $b = \emptyset$ , the result holds by the direct application of item ii) since  $\mathcal{D}_a^2 \subset \mathcal{D}_a^1$ , so we suppose  $b \in Y$ , and  $(X \setminus a) = (Y \setminus b)$ . If  $X$  is a singleton the result holds trivially, so suppose that  $X$  (and then  $Y$ ) are not singletons. Then as  $\#X = \#Y$ , and  $X \setminus a = X \cap Y = Y \setminus b$ , it holds that  $\frac{1}{\#X} \left( u_a(a) + \sum_{x \in X \cap Y} u_a(x) \right) > \frac{1}{\#Y} \left( u_a(b) + \sum_{x \in X \cap Y} u_a(x) \right)$ , for all  $u_a$  consistent with some  $P_a \in \mathcal{P}_a^r$ .

v) Let  $a \in \mathcal{C}$ ,  $X, Y \subseteq \mathcal{C}$  be such that  $X \neq Y$ ,  $\#X \neq 1$ ,  $a \in X$ , and for any  $b \in (Y \cup \emptyset)$ ,  $(X \setminus a) \neq (Y \setminus b)$ , then either there is  $c \in X \setminus a$ ,  $c \notin Y$ , or  $(X \setminus a) \subset Y$  and there is a set of candidates  $\{b_1, \dots, b_n\}$ , such that  $b_k \notin X$  for all  $k$ , and either  $Y = X \cup \{b_1, \dots, b_n\}$ , ( $n \geq 1$ ), or  $Y = (X \setminus a) \cup \{b_1, \dots, b_n\}$ , ( $n \geq 2$ ). In the former case take  $P_a$  such that  $c$  is the worst

candidate for candidate  $a$ , and  $u_a$  such that  $u_a(c) < \left( \frac{\#X}{\#Y} \sum_{y \in Y} u_a(y) - \sum_{x \in X \setminus c} u_a(x) \right)$ . This is enough to get  $Y \succ_a^2 X$ . Thus, if  $(X \setminus a) \subset Y$  and there exist  $\{b_1, \dots, b_n\} \in Y \setminus X$ , take  $P_a$  such that  $b_l P_a y$  for all  $l = 1, \dots, n$  and for all  $y \in \mathcal{C} \setminus (\{b_1, \dots, b_n\} \cup a)$ . Construct  $u_a$  in such a way that for some  $\varepsilon > 0$  small enough  $u_a(x(k)) = 1 - \frac{k-1}{\#C} \varepsilon$  for all  $k = 1, \dots, n+1$ , and  $u_a(x(k)) = \frac{\varepsilon}{k}$  otherwise. Notice now, that for  $\varepsilon$  close to 0,  $\frac{1}{\#X} \left( \sum_{x \in X} u_a(x) \right)$  is close to  $\frac{1}{\#X}$  while  $\frac{1}{\#Y} \left( \sum_{y \in Y} u_a(y) \right)$  tends to  $\frac{n+1}{\#X+n}$  if  $a \in Y$  or to  $\frac{n}{\#X+n-1}$  if  $a \notin Y$ , that are larger than  $\frac{1}{\#X}$  if  $\#X > 1$ . Then, there is  $\succsim_a^2 \in \mathcal{D}_a^2$ , such that  $Y \succ_a^2 X$ .

## References

- [1] Aizerman M. and F. Aleskerov (1995). "Theory of Choice", Amsterdam, North Holland.
- [2] Barberà S., B. Dutta and A. Sen (1999). Strategy-Proof Social Choice Correspondences; forthcoming *Journal of Economic Theory*.
- [3] Barberà S. W. Bossert and P. Pattanaik (2001). Ranking Sets of Objects, in S. Barberà, P.J. Hammond and C. Seidl (eds.); Handbook of Utility Theory, Volume 2. Kluwert Academic Publishers, forthcoming.
- [4] Besley T. and S. Coate (1997). An Economic Model of Representative Democracy; *Quarterly Journal of Economics* 112, pp. 85-114.
- [5] Campbell D. and J.S. Kelly (2000). A Trade-off Result for Preference Revelation; *Journal of Mathematical Economics* 34, pp. 129-141.
- [6] Ching S. and L. Zhou (1997). Multivalued Strategy-Proof Social Choice Rules; forthcoming *Social Choice and Welfare*.
- [7] Demange G. (1987). Nonmanipulable Cores; *Econometrica* 55, pp. 1057-1074.

- [8] Denicolò V. (1987). Some Further Results on Nonbinary Social Choice; *Social Choice and Welfare* 4, pp. 277-285.
- [9] Dutta B., M.O. Jackson and M. Le Breton (2000). Strategic Candidacy and Voting Procedures; forthcoming *Econometrica*.
- [10] Dutta B. and P.K. Pattanaik (1978). On Strategic Manipulation of Issues in Group Decision Making; in P.K. Pattanaik, "Strategy and Group Choice", Amsterdam, New Holland.
- [11] Eraslan H., A. McLennan (2001). Strategic Candidacy and Multivalued Voting Procedures, mimeo, University of Minnesota.
- [12] Gärdenfors P.(1976). Manipulation of Social Choice Functions; *Journal of Economic Theory* 13, pp. 217-228.
- [13] Hurwicz L., E. Maskin and A. Postlewaite (1995). Feasible Nash Implementation of Social Choice Rules when the Designer does not Know Endowments or Production Sets; in: J. Ledyard (ed.); "Essays in the Honor of S. Reiter", pp. 367-433.
- [14] Majumdar P. (1956). Choice and Revealed Preference; *Econometrica* 24, pp. 71-73.
- [15] Mas-Colell A. and H. Sonnenschein (1972). General Possibility Theorems for Groups Decision. *Review of Economic Studies* 39, pp. 185-192.
- [16] Moulin H. (1988). "Axioms in Collective Decision Making"; Cambridge, Cambridge University Press.
- [17] Osborne M.J. and A. Slivinski (1996). A Model of Political Competition with Citizen Candidates; *Quarterly Journal of Economics* 111, pp. 65-96.
- [18] Pattanaik P.K. (1973). On the Stability of Sincere Voting Situations; *Journal of Economic Theory* 6, pp. 558-574.



- [19] Pattanaik P.K. (1978). "Strategy and Group Choice"; Amsterdam, New Holland.
- [20] Postlewaite (1979) Manipulation Via Endowments; *Review of Economic Studies* 46, pp. 255-262.
- [21] Sen A.K. (1971). Choice Functions and Revealed Preference; *Review of Economic Studies* 38, pp. 307-317.
- [22] Sen A.K. (1977). Social Choice: A Reexamination; *Econometrica* 45, pp. 53-90.
- [23] Sönmez T. (1997). Manipulation Via Capacities in Two-Sided Matching Models; *Journal of Economic Theory* 77, pp. 197-204.
- [24] Weymark J. (2000). Candidate Stability and No Binary Choice, mimeo, Vanderbilt University.