Allocative and Productive Efficiency in REE with Asymmetric Information

by

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Abstract

We characterize the divergence between informational and economic efficiency in a rational expectations competitive market with asymmetric information about the costs of production. We find that prices may contain too much or too little information with respect to incentive efficient allocations depending on whether the main role of the price is, respectively, the traditional as index of scarcity or informational. Only when REE degenerate to Cournot equilibria the market solution does not show allocative inefficiency. With multidimensional uncertainty we find that the REE price does not have in general the incentive efficient information mix: It pays to sacrifice allocative efficiency at the REE to improve productive efficiency.

Keywords: Informational Externalities, Supply Function Equilibria, Rational Expectations, Asymmetric Information, Mechanism Design, Market Efficiency.

1 Introduction

In this paper we perform a welfare analysis of rational expectations equilibria (REE) in a competitive production economy with asymmetric information taking into account incentive constraints. We characterize precisely the divergence between informational and economic efficiency. Our paper is not the first to explore such divergence. Pesendorfer and Swinkels (1998) and Shin (1995) explore the tension between allocative and informational efficiency in some auction contexts. We provide here a complete characterization of both allocative and productive efficiency in a simple economy in which prices convey information about costs.

We take a mechanism design approach and examine the efficiency properties of REE in the class of Bayesian incentive compatible allocations. More specifically, we look at REE which are implementable in supply functions. Indeed, REE which are not implementable (that is, for which there is no game in which the REE emerges as equilibrium) may be seen as an artificial construct.[1]

Since the seminal work of Lucas (1972) (and Green (1973)), REE have been used pervasively in every field in economics. Applications in markets with asymmetric information are prominent in this respect (see Grossman (1981) for an introduction). The paucity of work on the welfare properties of REE in asymmetric information environments is therefore somewhat surprising (with the exception of the work by Laffont (1985)).[2]

Grossman (1981) shows that fully revealing REE (FRREE) are (ex post) Pareto optimal (and ex post, obviously, there is no need to consider incentive constraints because information has been revealed). Laffont (1985) shows however that FR-

[1]See Blume and Easley (1990), Palfrey and Srivastava (1986), and Postlewaite and Schmeidler (1986).

[2]Some headway in the welfare analysis of competitive equilibria with asymmetric information has been made by Prescott and Townsend (1984), Gale (1996), and Bisin and Gottardi (1999).

REE need not be incentive efficient in an interim sense because of the well-known Hirschleifer (1971) effect: REE may reveal too much information and eliminate valuable insurance opportunities. However, Laffont provides the following positive result (and conjectures that the property should be general) in a quasilinear linear-normal world, in which ex ante and ex post optimality coincide: linear REE are Pareto optimal in the class of linear Bayesian incentive compatible mechanisms (LBICM) which face the same communication constraints as the market (that is, that use the same pieces of aggregate information as the market). This can be considered to be the best possible case for the market to attain efficiency. We consider in this paper a quasilinear world which generalizes the case considered by Laffont(1985) and show that even in the best possible case for the market linear REE are not, in general, incentive efficient. We characterize precisely why this is so and examine the potential misalignment of informational and economic (allocative and productive) efficiency.

The essential ingredients of our model are as follows: we consider a partial equilibrium model where a continuum of risk neutral firms compete in a homogenous product market with potentially random demand. Costs of firms are strictly convex and subject to shocks of type $k = 1,2$. Shocks of type 1 affect all the firms in the same way while shocks of type 2 affect each firm differently. Furthermore, each type of shock has a common θ_k and an idiosyncratic s_k^i component. Both are correlated and the latter constitute private information to the firms. Idiosyncratic shocks provide thus information about the common ones. Firms may have asymmetric costs ex ante. It is assumed that firms compete in supply schedules. This provides a natural way to implement REE. In our paper REE will be just Bayesian equilibria of the supply function game. This is, for example, like in Kyle (1989) but without strategic interaction because of our continuum assumption.[3] The market price there-

[3]Grossman (1981) and Klemperer and Meyer (1989) consider supply function equilibria under

fore potentially reveals information about the common components of costs. For tractability reasons the specification we use is of the linear-normal variety, yielding unique linear Bayesian supply function equilibria (LBSFE). Our parametrization is rich enough to encompass the cases of FRREE, partially revealing (nonnoisy) RE, noisy REE (all of them implementable as BSFE), as well as displaying a FRREE which is not implementable.[4] Our model generalizes, in particular, the quasilinear model of Laffont (1985) and displays all the relevant welfare tradeoffs (within a quasilinear utility model). It is worth to recall that with quasilinear utility ex post Pareto optimality implies ex ante Pareto optimality.

The welfare analysis is conducted in the class of mechanisms which are linear and share the same communication constraints as the market, defining the class of LBICM. We say that an allocation rule is incentive efficient if it maximizes expected total surplus in the class of LBICM. Our LBSFE is a member of this class but except in very particular circumstances is not optimal. The basic reason is an informational externality: Firms do not take into account that their actions influence the informational content of the price (about costs) and therefore the decisions of other firms. An optimal mechanism will take into account the informational externality with the only constraint of incentive compatibility. The information externality is given by the expected average impact on total surplus of a change in production for each firm as a consequence of a change in public information. The welfare impact is just the average marginal effect on expected profits per firm because firms are price takers.

Our analysis characterizes and decomposes the informational externality present at the REE into a total output effect (TOE) and a distribution of output effect (DOE). The second is due to ex ante cost asymmetries among firms.

certainty and uncertainty, respectively, but with complete information. See also Wilson (1979).

[4]See Ausubel (1990) for a partially revealing non-noisy REE.

When $TOE = 0$ average production is at the first best efficient level. The total output effect is the only relevant effect if firms face ex ante symmetric cost functions. In this case and with random demand the responsiveness to private information at the REE is insufficient or excessive depending on whether the firms use (in equilibrium) downward or upward sloping supply functions. The fact that firms may use downward sloping supply functions should not be surprising given the double nature of prices, allocational and informational, at the REE. The price is as usual an index of scarcity and guides competitive supply: A larger price will tend to increase supply. However, a larger price may also contain news that the common component of costs is high and will therefore tend to depress supply. When the informational role of the price dominates then supply is downward sloping and the REE price is not informative enough. It pays to make it more informative by increasing the weights firms put on private signals which can be done in an incentive compatible way. On the contrary when the allocational role of the price dominates then supply is upward sloping and the REE price is too informative. It pays then to make it less informative by decreasing the weights firms put on private signals. For the boundary case in which firms do not respond to public information (because the two roles of price exactly balance each other) the REE is incentive efficient. This corresponds in fact with a Cournot market, in which firms do not condition on market price. In this case the Bayesian Cournot equilibrium, with a continuum of firms, is team optimal. That is, it maximizes expected total surplus under the constraint that firms use decentralized production strategies (Vives (1988)). We see that in general informational and allocative efficiency are not aligned.

A second effect is the distribution of output effect (DOE). This is due to the ex ante asymmetry in costs. It is the only relevant effect when demand is not random because then in our model price equals average marginal cost (with deterministic demand at the REE the price reveals $\theta_1 + \theta_2$, which determines average marginal costs). In this case average output at the REE is first best optimal but the distribution of output across firms is inefficient in general. An incentive efficient allocation will typically distort allocative efficiency to improve productive efficiency. At the REE firms put the same weight on average on signals independent of whether the shock has the same or a differential impact on costs. In a world without incentive constraints expected total surplus could be increased at the REE solution by making firms more (less) responsive to signals about the shock which has a differential (the same) impact on costs. Whenever the precisions of both types of signals is different it is possible to increase in an incentive compatible way the relative average weight to signals of type 2. Only when the precisions of both types of signals are the same incentive compatibility dictates that the weights should be the same and the REE is incentive efficient. This is the (knife-edge) case considered by Laffont (1985).

Among the papers that deal with information externalities it is worth pointing out the work by Stein (1987), Rob (1987, 1991) and Creane(1996). In all these papers an inefficiency arising from an information externality in a competitive market is characterized.

The paper is organized at follows. Section 2 describes the model. In Section 3 the competitive rational expectations equilibrium is derived as a linear Bayesian supply function equilibrium. Section 4 describes the restrictions imposed by the class of Linear Bayesian Incentive Compatible Mechanisms (LBICM) to which the REE will be compared. This class has the same communication constraints as the market. Section 5 analyzes the welfare properties of the REE solution in relation to the incentive efficient solution. Finally, in section 6 we conclude. The Appendix gathers some proofs.

2 The Model

A continuum of firms, indexed in the unit interval $i \in [0,1]$, compete in a homogenous product market facing a linear downward sloping inverse demand: $P(x) = 1+\lambda-x$, where $x = \int_0^1 x^i di$ is the aggregate output (and in our continuum economy also per capita output).[5] The demand intercept λ is random and normally distributed with zero mean and finite variance σ_λ^2, we write $\lambda \sim N(0, \sigma_\lambda^2)$. Firm i produces according to a strictly increasing and convex cost function:

$$C^i(x^i) = [\gamma s_1^i + \theta_1 + \alpha^i(\gamma s_2^i + \theta_2)]x^i + \beta(x^i)^2/2 \tag{1}$$

where $x^i$is the output of the firm and $\beta > 0$ and $\gamma \geq 0$. Costs are affected by the unobservable random parameters θ_1 and θ_2, as well as by the signals that the firm receives about them, s_1^i and s_2^i, respectively. Signals are of the type $s_k^i = \theta_k + \varepsilon_k^i$, where $\theta_k \sim N(\mu_k, \sigma_{\theta_k}^2)$ and $\varepsilon_k^i \sim N(0, \sigma_{\varepsilon_k}^2)$, $k = 1, 2$ for all i. The random variables $\theta_1, \theta_2, \varepsilon_1^i$, ε_2^i, ε_1^j and ε_2^j are mutually independent for any i and j. This means in particular that error terms are uncorrelated across firms. The parameter $\gamma \geq 0$ determines the sensitivity of firms' costs to their private signals.

We can think that the random variables θ_1 and θ_2 are industry specific cost parameters and therefore common to all firms while s_1^i and s_2^i are firm specific components of the costs which depends on the private signals received. For example, high skill labor contracts are, in general, directly negotiable between the employer and the employees. In contrast, unskilled labor contracts are often negotiated industry wide by unions. In a given industry skilled and unskilled labor costs are correlated and therefore, fixing wages for high skill workers of two types (s_1^i, s_2^i) provides a signal about the outcome of the industry wide union negotiation about unskilled labor wages (θ_1, θ_2).

[5]Without loss of generality and to simplify notation we set the slope of demand equal to one.

Firms also differ in their costs by the known constant $\alpha^i \in (0,2)$ where $\int_0^1 \alpha^i di = 1$. Let $\int_0^1 (\alpha^i - 1)^2 di = \sigma_\alpha^2$. Firms' costs are differentially affected by the term $(\gamma s_2^i + \theta_2)$. For example, firms with a lower α^i might be more efficient in using type 2 labor.

We make the convention that error terms cancel in the aggregate: $\int_0^1 \epsilon_1^i di = \int_0^1 \epsilon_2^i di = 0$ (almost surely, a.s. for short). The aggregation of all individual signals will reveal the underlying uncertainty: $\int_0^1 s_k^i di = \int_0^1 \theta_k di + \int_0^1 \epsilon_k^i di = \theta_k$, $k = 1, 2$.[6]

We are interested in the study of rational expectations equilibria under asymmetric information. A (competitive) rational expectations equilibrium is a price function $p(\theta_1, \theta_2)$ and productions $x^i, i \in [0,1]$ such that every firm i maximizes its expected profit ($E[\pi^i \mid \Omega^i]$) conditional on its information $\Omega^i = (s_1^i, s_2^i, p)$, where $\pi^i = p\, x^i - C(x^i)$, knowing the functional relationship $p(\theta_1, \theta_2)$ as well as the underlying distributions of the random variables.

There are well-known problems with the competitive REE concept (see, for example, Grossman and Stiglitz (1980), Hellwig (1980), Kyle (1989)). In this paper we will restrict attention to REE which are the outcome of a well specified game. That is, that are implementable. The natural way to implement REE is to consider competition in supply functions (by analogy to Wilson (1979) or Kyle (1989) in which traders choose demand functions). The strategy of firm i is a supply function contingent on its private information: $x^i(s_1^i, s_2^i, p)$. The market clearing price is

[6] More precisely, we will make the convention that the Strong Law of Large Numbers (SLLN) holds for the continuum economy. Suppose that $(q_i)_{i \in [0,1]}$ is a process of independent random variables with $E[q_i] = 0$ for all i and that variances ($Var q_i$) are uniformly bounded. Define $\int_0^1 q_i di = 0$ (a.s.). The convention is used taking as given the usual linearity properties of integrals. Note that the variances of the error terms are indeed uniformly bounded: $Var \epsilon_k^i = \sigma_{\varepsilon_k}^2$, k =1, 2. For a discussion of the issues involved in the convention see, for example, Judd (1985) or Vives (1988) for an application in a Cournot market.

then determined by the intersection of aggregate supply and demand. A REE is associated to a Bayesian Nash equilibrium of the game in supply functions. We will restrict attention to linear Bayesian Supply Function equilibria (LBSFE).

The reason why we have chosen the present model specification, apart from its tractability, is that the model is parsimonious in displaying a full variety of types of rational expectations equilibria (REE) as well as highlighting the potential problems with the concept. More concretely, if demand is non random ($\sigma_\lambda^2 = 0$ with $\lambda = 0$) the model encompasses the following cases: (i) when firms have ex ante symmetric cost functions ($\alpha^i = 1$ for all i) and signals do not affect costs ($\gamma = 0$) we can define a fully revealing REE (FRREE) which is not implementable; however, (ii) under symmetry if signals do affect costs ($\gamma > 0$) then there is a FRREE which is implementable as a LBSFE; (iii) with asymmetric cost functions there is a (non noisy) partially revealing REE which is implementable as a LBSFE if $\gamma > 0$ (when $\sigma_{\theta_1}^2 = \sigma_{\theta_2}^2$ and $\sigma_{\epsilon_1}^2 = \sigma_{\epsilon_2}^2$ we have an equivalent of the quasilinear model of Laffont (1985)) and (iv) there is no linear REE if $\gamma = 0$. The following table summarizes the cases:

$\sigma_\lambda^2 = 0$	$\gamma = 0$	$\gamma > 0$
$\alpha^i = 1 \forall i$	FRREE (not implementable)	FRREE
$\alpha^i \in (0, 2)$	non-existence of linear REE	partially revealing REE

Table 1: REE cases with non random demand

If demand is random ($\sigma_\lambda^2 > 0$) then there is a noisy REE which is implementable as a LBSFE for any $\gamma \geq 0$. The results are presented in the next section.

The timing will be the following: At $t = 0$ random variables θ_1, θ_2 and λ are drawn but not observed. At $t = 1$ firms observe their own private signals s_1^i and s_2^i and submit supply functions. Finally, the market clears and pay-offs are collected at $t = 2$.

3 The Rational Expectations Equilibrium

In this section we characterize the Linear Bayesian Equilibria of the game in which firms use supply functions contingent on their private information. That is, we restrict attention to equilibria in which strategies are linear in the information firms have. The strategy of firm i is a supply function contingent on its private information: $x^i(s_1^i, s_2^i, p)$. The market clearing price is then determined by the intersection of aggregate supply and demand.[7]

Firm i solves the problem $\max_{x^i} E[\pi^i \mid \Omega^i]$, where $\pi^i = p\ x^i - C(x^i)$ and $\Omega^i = (s_1^i, s_2^i, p)$. In particular, firms condition on their private signals s_1^i and s_2^i which in turn implies that the market clearing price will be a function of the aggregation of private signals or equivalently, according to our convention on the average error terms of the signals, of θ_1 and θ_2. Since the distribution of random variables and the underlying model are common knowledge, firms can infer how aggregate private information enters the pricing function and use this information in the estimation of the underlying cost uncertainty.

In order to characterize the (linear) REE we conjecture that firms use strategies of the following form: $x^i = \Phi(p) - a_1^i s_1^i - a_2^i s_2^i$, where $\Phi(p)$ is linear. Aggregate output, according to our convention on the average error terms of the signals, is then given by $x = \int_0^1 x^j dj = \Phi(p) - a_1\theta_1 - a_2\theta_2$ where $a_k = \int_0^1 a_k^i di, k = 1, 2.$[8] Using the inverse demand function $p = 1 + \lambda - x$ it is then easy to see that the random variable $z = \lambda + a_1\theta_1 + a_2\theta_2$ is informationally equivalent to the price. Note that z (and the price) will provide in general a noisy signal of the unknown parameters

[7] We can assume that the market shuts down if there is no market clearing price and that if there are many the one that maximizes volume is chosen.

[8] To apply the convention requires that the coefficients a_1^i and a_2^i be uniformly bounded in i. This is the case in equilibrium. We will drop the supersript of a variable or parameter when we average it over the population of firms: $y = \int_0^1 y^i di$, for example.

θ_1 and θ_2 because λ is random. We can write the information available to firm i as $\Omega^i = (s_1^i, s_2^i, z)$. Let us posit strategies of the form $x^i = b^i - a_1^i s_1^i - a_2^i s_2^i + c^i z$. Aggregate output and price are then given by $x = b - a_1\theta_1 - a_2\theta_2 + cz$ and $p = (1-b) + (1-c)z$, respectively (where $b = \int b^i$ and $c = \int c^i$) .[9]

The optimal (interior) production of firm i is determined by the first order condition (FOC) $\partial E[\pi^i \mid \Omega^i]/\partial x^i = p - E[MC^i(x^i) \mid \Omega^i] = 0$.[10] The supply function for firm i is given by: $X^i(s_1^i, s_2^i, p) = \beta^{-1}(p - \gamma(s_1^i + \alpha^i s_2^i) - E[\theta_1 + \alpha^i\theta_2 \mid \Omega^i])$. Using properties of the normal distribution one can calculate conditional expectations and solve for the equilibrium. The proposition states the result. In the analysis that follows we work mostly with precisions and let $\tau_y = (\sigma_y^2)^{-1}$ denote the precision of the normal random variable y. To ease notation we set the means of the cost parameters equal to zero: $\mu_1 = \mu_2 = 0$.[11] We consider first the case $\sigma_\lambda^2 > 0$ and then the case $\sigma_\lambda^2 = 0$.

Proposition 1 *Let $\sigma_\lambda^2 > 0$, then there is a unique Linear Bayesian Supply Function Equilibrium (LBSFE). Firm i uses the following strategy:*

$$x^i = b^i - a_1^i s_1^i - a_2^i s_2^i + c^i z$$

where:

$$a_1^i = \gamma/\beta + \left(\tau_{\varepsilon_1}[(\tau_{\theta_2} + \tau_{\varepsilon_2}) + \tau_\lambda a_2(a_2 - \alpha^i a_1)]\right)/\beta\Delta \tag{2}$$

$$a_2^i = \alpha^i\gamma/\beta + \left(\tau_{\varepsilon_2}[\alpha^i(\tau_{\theta_1} + \tau_{\varepsilon_1}) + \tau_\lambda a_1(\alpha^i a_1 - a_2)]\right)/\beta\Delta \tag{3}$$

[9]In equilibrium we will have that $c < 1$.

[10]The marginal costs for firm i are given by $MC^i = \left[\gamma s_{1i} + \theta_1 + \alpha^i(\gamma s_{2i} + \theta_2)\right] + \beta x^i$. The optimization problem is strictly concave given strict convexity of the cost function.

[11]We will not worry here about outputs (and prices) becoming negative because of our normality assumption. The probability of occurrence of these events can be controlled by an appropriate choice of parameters. Alternatively, we could work with pairs of prior and likelihoods which admit a bounded support and maintain the crucial property of linear conditional expectations which yields a tractable model (see Vives (1988)).

$$c^i = c + c_2(\alpha^i - 1) \tag{4}$$

$$b^i = \frac{1}{\beta + 1}$$

$$\Delta = (\tau_{\theta_1} + \tau_{\varepsilon_1})(\tau_{\theta_2} + \tau_{\varepsilon_2}) + \tau_\lambda[(a_1)^2(\tau_{\theta_2} + \tau_{\varepsilon_2}) + (a_2)^2(\tau_{\theta_1} + \tau_{\varepsilon_1})]$$

$$c = \int c^i = \frac{1}{\beta+1} - \frac{\tau_\lambda[a_1(\tau_{\theta_2} + \tau_{\varepsilon_2}) + a_2(\tau_{\theta_1} + \tau_{\varepsilon_1})]}{\Delta(\beta+1)}$$

$$c_2 = -\frac{\tau_\lambda a_2(\tau_{\theta_1} + \tau_{\varepsilon_1})}{\beta\Delta}$$

and a_1 and a_2 are the unique solution to the cubic equations $k, h = 1, 2, k \neq h$:

$$a_k = \int a_k^i = \gamma/\beta + \left(\tau_{\varepsilon_k}[(\tau_{\theta_h} + \tau_{\varepsilon_h}) + \tau_\lambda a_h(a_h - a_k)]\right)/\beta\Delta$$

Proof. See Appendix

Corollary 2 *In equilibrium,*

$$a_1 > \frac{\gamma}{\beta}\ ,\ a_2 > \frac{\gamma}{\beta} \text{ and } c < 1$$

Furthermore, $a_1 \gtreqless a_2$ if and only if $(\tau_{\varepsilon_1}/\tau_{\theta_1}) \gtreqless (\tau_{\varepsilon_2}/\tau_{\theta_2})$.

Proof. Taking the difference $a_1 - a_2$ in the equations that determine the equilibrium values it follows that $a_1 - a_2 = 0$ if and only if $\tau_{\varepsilon_1}\tau_{\theta_2} = \tau_{\theta_1}\tau_{\varepsilon_2}$. Let now $\tau_{\varepsilon_1}(\tau_{\theta_2} + \tau_{\varepsilon_2}) > \tau_{\varepsilon_2}(\tau_{\theta_1} + \tau_{\varepsilon_1})$. It follows that $a_1 > a_2$ from which $a_2 > \gamma/\beta$ and a fortiori $a_1 > \gamma/\beta$. If $\tau_{\varepsilon_1}(\tau_{\theta_2} + \tau_{\varepsilon_2}) < \tau_{\varepsilon_2}(\tau_{\theta_1} + \tau_{\varepsilon_1})$ we get that $a_2 - a_1 > 0$. It is immediate that $c < 1$. ■

Production strategies for the firms are asymmetric due to differences in costs and therefore, coefficients a_1^i, a_2^i and c^i depend on the cost parameter α^i. As can be seen from expressions (2), (3) and (4) firms with costs with higher sensitivity to θ_2 (higher values of α^i) tend to put a higher weight on signal s_2^i and lower weights on s_1^i and z.

The individual weights a_1^i and a_2^i on private signals are the sum of two components. The first component (γ/β and $\alpha^i\gamma/\beta$, respectively) comes from the fact that costs are signal-sensitive. The second component are the signal to noise ratios in the estimation of θ_1 and θ_2 with information $\Omega^i = (s_1^i, s_2^i, z)$, respectively. As usual in REE models the price ($p = (1-b)+(1-c)z$) serves a dual role as index of scarcity and as conveyor of information. Indeed, a high price has a direct effect to increase the competitive supply of a firm but also conveys news that costs are high. This can be seen clearly in the expression for c, the average response to public information z. The direct effect is $\frac{1}{\beta+1}$ while the information effect is $-\frac{\tau_\lambda[a_1(\tau_{\theta_2}+\tau_{\varepsilon_2})+a_2(\tau_{\theta_1}+\tau_{\varepsilon_1})]}{\Delta(\beta+1)}$. The parameter c can be positive or negative depending on which effect dominates. As τ_λ ranges from 0 to ∞, c decreases from $\frac{1}{\beta+1}$ to $\frac{1}{\beta+1}(1-\frac{\beta}{\gamma})$. The less noise in demand the larger the information component (in absolute value) and c is reduced.[12] When the price contains no information about costs ($\tau_\lambda = 0$) there is no information effect. When the public information is not very noisy (τ_λ is large enough) and costs are not very sensitive to private signals in relation to the slope of marginal costs ($\gamma < \beta$), the information effect dominates and $c < 0$. Then the aggregate supply is decreasing in the price. [13] In the particular case where the scarcity and informational effects balance, firms set a zero weight ($c^i = 0$) on public information. In this case firms do not condition on the price and the model reduces to the Cournot model where firms compete in quantities. However, here not reacting to the price (public

[12]It is easy to see also that the less noise in demand the more firms rely on public information and the less on private signals to learn about costs. For example, if $a_1 = a_2 = a$, then a decreases from $\gamma/\beta + \frac{\tau_{\varepsilon_i}}{\beta(\tau_{\theta_i}+\tau_{\varepsilon_i})}$ to γ/β as τ_λ ranges from 0 to ∞.

[13]At the individual firm level the slope of the supply function is governed by $c^i = c + c_2(\alpha^i - 1)$, where $c_2 < 0$ is the signal to noise ratio assigned to public information z in the estimation of θ_2 with information Ω^i. Since firms are asymmetrically affected by the unknown parameter θ_2 , they adjust the average response c by $c_2(\alpha^i - 1)$ to account for their cost specificity. Even when $c > 0$ firms with high sensitivity to θ_2 (with α^i high) may set $c^i < 0$.

information) is optimal when supply functions are allowed.

Firms take public information z as given and use it to form probabilistic beliefs about the underlying uncertain cost parameters θ_1 and θ_2. This in turn determines coefficients a_1^i, a_2^i and c^i for private and public information, respectively. At the same time, the informativeness of public information z depends on the (average) coefficients a_1 and a_2. In the REE firms behave as information takers and thus from the viewpoint of an individual firm public information is perceived as exogenous. This lies at the root of the informational externality present at the REE. Firms do not take into account their impact on public information and therefore on other firms.

Three particular cases deserve attention. The first is when signals are perfectly informative ($\tau_{\varepsilon_1} = \tau_{\varepsilon_2} = \infty$) and we are back to a full-information competitive equilibrium. As we know this is Pareto optimal. We then have that $a_1 = a_2 = (1+\gamma)/\beta$ and $c = 1/(1+\beta)$. The second is when signals are uninformative about the common cost parameters θ_1 and θ_2 ($\tau_{\varepsilon_1} = \tau_{\varepsilon_2} = 0$) but still firms rely on them because they affect costs directly ($\gamma > 0$). Then the LBSFE reduces to the following: $a_1^i = \gamma/\beta$, $a_2^i = \alpha^i\gamma/\beta$, $c^i = c - c_2(\alpha^i - 1)$, $c_2 = -\frac{\tau_\lambda a \tau_{\theta_1}}{\beta\Delta}$, $\Delta = \tau_{\theta_2}\tau_{\theta_1} + \tau_\lambda a^2(\tau_{\theta_2} + \tau_{\theta_1})$ with average values $a_1 = a_2 = a = \gamma/\beta$, and $c = \frac{1}{\beta+1} - \frac{\tau_\lambda a(\tau_{\theta_2}+\tau_{\theta_1})}{(\beta+1)\Delta}$. In both cases no informational externality arises because the weights assigned to private signals do not depend on the informativeness of public information.[14] The third case is when demand is not noisy ($\sigma_\lambda^2 = 0$ or $\tau_\lambda = \infty$):

Proposition 3 *Let $\sigma_\lambda^2 = 0$. Then if $\gamma = 0$ there is no LBSFE. If $\gamma > 0$ the (unique)*

[14]For $\tau_{\varepsilon_1} = \tau_{\varepsilon_2} = \infty$ firms are perfectly informed about random costs and public information does not add additional information. For $\tau_{\varepsilon_1} = \tau_{\varepsilon_2} = 0$, firms condition on private signals only because costs are signal sensitive. No weight is given to private information in the estimation of random costs.

LBSFE is given by:

$$a_1^i = \gamma/\beta + \frac{\tau_{\varepsilon_1}(1-\alpha^i)}{\beta(\tau_{\theta_1}+\tau_{\varepsilon_1}+\tau_{\theta_2}+\tau_{\varepsilon_2})} \tag{5}$$

$$a_2^i = \alpha^i\gamma/\beta + \frac{\tau_{\varepsilon_2}(\alpha^i-1)}{\beta(\tau_{\theta_1}+\tau_{\varepsilon_1}+\tau_{\theta_2}+\tau_{\varepsilon_2})} \tag{6}$$

$$c^i = c + c_2(\alpha^i - 1) \tag{7}$$

$$b^i = \frac{1}{\beta+1}$$

$$c = \frac{1}{\beta+1}\frac{\gamma-\beta}{\gamma}, c_2 = -\frac{(\tau_{\theta_1}+\tau_{\varepsilon_1})}{\gamma(\tau_{\theta_1}+\tau_{\varepsilon_1}+\tau_{\theta_2}+\tau_{\varepsilon_2})}$$

and $a_1 = a_2 = \gamma/\beta$.

Proof. When $\gamma > 0$ the result follows directly from Proposition 1 letting τ_λ tend to ∞. When $\gamma = 0$, the equations determining a linear equilibrium are inconsistent. ■

Recall that the costs of firm i are: $C^i(x^i) = [\gamma(s_1^i + \alpha^i s_2^i) + \theta_1 + \alpha^i\theta_2]x^i + \beta(x^i)^2/2$. When demand is not noisy and $a_1 = a_2 = \gamma/\beta > 0$, $z = \frac{\gamma}{\beta}(\theta_1+\theta_2)$, and the price reveals $\theta_1 + \theta_2$. This means that price equals average marginal cost, $p = MC$ $\left(= \int_0^1 MC^i di\right)$and therefore average production is at its full information first best level. Then if $\alpha^i = 1$ public information together with $s_1^i + s_2^i$ is fully revealing for the costs of firm i. This is an instance of a FRREE which is implementable. The equilibrium will be ex post Pareto optimal. If $\alpha^i \neq 1$ then $\Omega^i = (s_1^i, s_2^i, z)$ is partially revealing for the costs of firm i. This provides an instance of non noisy partially revealing REE.

When firms have ex ante symmetric cost functions ($\alpha^i = 1$ for all i) and signals do not affect costs ($\gamma = 0$), only $\theta_1 + \theta_2$ matters and we can define a fully revealing REE (FRREE). Indeed, this is just the competitive equilibrium of a full information market in which the firms know $\theta_1 + \theta_2$ (Grossman (1981)). This is given by $p = (\beta + (\theta_1 + \theta_2))/(1+\beta)$ with individual supply $x^i = \beta^{-1}(p - (\theta_1 + \theta_2))$. However, this REE is not implementable. That is, there is no game which has as equilibrium the REE. Indeed, for the price to be fully revealing it is needed that the supply of a firm be sensitive to the signals but then there is no reason for firms to rely on their signals![15] In terms of our model the nonexistence of a LBSFE when signals do not affect costs directly ($\gamma = 0$) is easy to understand. In this case equilibrium would call for $a_1 = a_2 = 0$ but then prices can not reveal any information. Furthermore, if prices do not reveal any information then firms have an incentive to rely on their signals and this makes the price informative.

It is worth summarizing the cases in which there is no information externality: When public information is pure noise ($\tau_\lambda = 0$), when signals are uninformative ($\tau_{\varepsilon_1} = \tau_{\varepsilon_2} = 0$), in the full information case ($\tau_{\varepsilon_1} = \tau_{\varepsilon_2} = \infty$), and when the equilibrium is fully revealing ($\tau_\lambda = \infty$, $\gamma > 0$ and $\alpha^i = 1$ for all i). In the last two cases there is no welfare loss at the LBSFE with respect to the first best because the market outcome replicates the full information competitive equilibrium. Otherwise there will be a welfare loss.

4 Linear Bayesian Incentive Compatible Mechanisms

Our objective is to analyze the welfare properties of REE (our LBSFE). We will do so taking into account incentive constraints in the market with private information as in Holmström and Myerson (1983) and Laffont (1985). We will study the performance of the REE in the class of linear incentive compatible Bayesian mechanisms (LICBM). This is the class of mechanisms which are:

1. linear in private and public information,

[15] Another way to put it is to realize that at the described FRREE the price is not measurable in the supplies of the firms. Anderson and Sonnenschein (1982) insist on defining REE requiring that prices be measurable in the demands of agents.

2. incentive compatible,

3. restricted to have the same communication constraints as the market, and

4. implementable in Bayesian Nash equilibrium.

We will say that an allocation is incentive efficient if it maximizes expected total surplus (ETS) in the class of LBICM.[16]

A LBICM has the following properties parallel to the REE. First, it is linear in private and public information as the linear REE. Second, since the mechanism has to infer private signals from firms it has to take incentive compatibility constraints into account. Third, the mechanism is bound to use the same communication constraints as in the competitive case. Finally, the game induced by the mechanism has a Bayesian Nash equilibrium which implements the desired allocation.

To compare the performance of REE in this class is to consider the best possible case for the market. Indeed, Laffont (1985) shows that incentive efficiency breaks down as soon as non-linear mechanisms are considered. Furthermore, if the mechanisms considered were not bound by the same communication constraints as the market then it would be very easy to improve upon it. For example, by pooling the private information of firms one could recover the true values of θ_1 and θ_2 and, therefore, replicate the full information outcome.

According to the revelation principle we can restrict attention to direct mechanisms in which the strategy space of a firm is just his type space (the space of private signals). Firms submit their signals to the center and then the center derives the statistic $z = (\lambda + a_1\theta_1 + a_2\theta_2)$, which is constructed in the same way as in the REE. Thus the mechanism is restricted to use the same communication constraints as the market. The center assigns productions to the firms according to the rules:

[16] In our world of quasilinear utility ex post efficiency implies ex ante efficiency.

$x^i = b^i - a_1^i s_1^i - a_2^i s_2^i + c^i z$.[17]

The aggregation over individual weights a_1^i and a_2^i determines the informativeness of statistic z about θ_1 and θ_2. By choosing individual weights the center has the possibility to influence information revelation through z.

Suppose the center assigns production $\widehat{x}^i = b^i - a_1^i \widehat{s}_1^i - a_2^i \widehat{s}_2^i + c^i \widehat{z}$, where $\widehat{z} = (\lambda + \int a_1^i \widehat{s}_1^i di + \int a_2^i \widehat{s}_2^i di)$, depending on announcements $\widehat{s}_1^i$ and $\widehat{s}_2^i$ of the firms. Firms submit signals $\widehat{s}_1^i$ and $\widehat{s}_2^i$ (not necessarily the true ones) that maximize expected profits conditional on the true signals observed (s_1^i, s_2^i): $\max_{\widehat{s}_1^i, \widehat{s}_2^i} E\left[p\widehat{x}^i - C(\widehat{x}^i) \mid s_1^i, s_2^i\right]$, where $p = (1-b) + (1-c)\widehat{z}$. This leads to the following incentive compatibility constraints (ICC):

Proposition 4 *At a the LBICM truth telling requires that for $x^i = b^i - a_1^i s_1^i - a_2^i s_2^i + c^i z$:*

$$E[(p - MC^i(x^i)) \mid s_1^i, s_2^i] = 0 \quad \forall s_1^i, \forall s_2^i, \forall i \tag{8}$$

which given normality implies

$$\begin{aligned} E[(p - MC^i(x^i))] &= 0 \quad \forall i & (9) \\ COV[(p - MC^i(x^i)), s_1^i] &= 0 \quad \forall i & (10) \\ COV[(p - MC^i(x^i)), s_2^i] &= 0 \quad \forall i & (11) \end{aligned}$$

Proof. See Appendix

The Proposition shows that to achieve incentive compatibility it is necessary to eliminate the covariation between signals and the margin $(p - MC^i(x^i))$ (10 and 11). Observe that[18]

$$E[(p - MC^i(x^i)) \mid s_1^i] = \frac{COV[(p - MC^i(x^i)), s_1^i]}{Var(s_1^i)} s^i \tag{12}$$

[17]We will use the same notation for the coefficients in the production rules of the LBICM and in the REE. No confusion should arise from this.

[18]Recall that $E[s_1^i] = 0$ by assumption.

If $COV[(p - MC^i), s_1^i] \neq 0$, then, given signal s_1^i, expected price is not equal to expected marginal cost and firm i would gain by announcing a different signal $\hat{s}_1^i \neq s_1^i$ and inducing $E[(p - MC^i(\widehat{x}^i)) \mid s_1^i] = 0$. The following corollary derives the implications for the coefficients of linear production rules at a LBICM.

Corollary 5 *Let* $t_k = \frac{\tau_{\epsilon_k}}{(\tau_{\epsilon_k} + \tau_{\theta_k})}$, $k = 1, 2$. *At a LBICM the following ICC on the coefficients of the production rules have to hold:*

$$
\begin{aligned}
a_1^i &= f_1(c^i, c) \equiv a_1(1 + t_1(c^i - c)) \\
a_2^i &= f_2(\alpha^i, c^i, c) \equiv a_2(\alpha^i + t_2\beta^{-1}\left((1 - c(1+\beta))(\alpha^i - 1) + \beta(c^i - c)\right) \\
b^i &= \frac{1}{(1+\beta)}
\end{aligned}
$$

and

$$
a_k = h_k(c) \equiv \frac{\gamma + t_k}{[t_k(1 - c(1+\beta)) + \beta]}, \; k = 1, 2.
$$

Proof. See Appendix.

An LBICM puts restrictions on three coefficients while there remains one free parameter (c^i). Note that the level of noise in demand does not affect the ICC. The weights assigned to private information depend on both c^i and the average value c. To keep incentives the center needs to impose production rules such as to eliminate the covariation between signals and $(p - MC^i)$ which in turn gives the restrictions in a_1^i and a_2^i. For example, the requirement $COV[(p - MC^i), s_1^i] = 0$ yields $a_1^i = f_1(c^i, c)$.[19] For a given c, a_1^i is increasing in c^i. A higher weight c^i given to z must be compensated with a higher a_1^i, otherwise the firm would misreport. The reason is that a higher c^i means that less weight is given to public information in the estimation of costs (because the coefficient for public information enters by

[19] And similarily $COV[(p - MC^i), s_2^i] = 0$ defines coefficient $a_2^i = f_2(\alpha^i, c^i, c)$. In the following we limit the analysis to coefficient a_1^i. Qualitatively it is the same for a_2^i.

convention in the production schedule recommended by the mechanism as $+\ c^i$ while for signal s_k^i enters as $-\ a_k^i$) and this must be compensated by a higher weight on private signals. This holds in the aggregate also and a_1 (as well as a_2) are increasing in c. It is worth to remark that incentive compatibility requires that $a_1 = a_2$ whenever signals of the two types are of the same precision: $t_1 = t_2$

It should be clear that the REE is incentive compatible. Indeed, incentive compatibility requires that the FOC at the REE $E\{(p - MC^i(x^i)) \mid s_1^i, s_2^i, p\} = 0$, hold on average given the private signals of the firms: $E[E\{(p - MC^i(x^i)) \mid s_1^i, s_2^i, p\} \mid s_1^i, s_2^i] = 0$. Another way to put it is to realize that maximizing expected profits subject to ICC yields the REE allocation. The following corollary states the result, which will be useful later.

Corollary 6 *The solution to*

$$
\begin{aligned}
& \max_{\{c^i\}} E[px^i - C(x^i)] \\
s.t. \quad &: \quad x^i = b^i - a_1^i s_1^i - a_2^i s_2^i + c^i z, \\
z \quad &= \quad (\lambda + a_1\theta_1 + a_2\theta_2) \\
a_1^i \quad &= \quad f_1(c^i, c),\ a_2^i = f_2(\alpha^i, c^i, c),\ \text{and } b^i = \frac{1}{(1+\beta)}
\end{aligned}
$$

results in the following FOC:

$$
\frac{\partial E[\pi^i]}{\partial c^i} = E\left[(p - MC^i)\left(-\frac{\partial a_1^i}{\partial c^i} s_1^i - \frac{\partial a_2^i}{\partial c^i} s_2^i + z\right)\right] = E\left[(p - MC^i)z\right] = 0 \quad \text{for all } i \tag{13}
$$

which determines coefficient c^i, and therefore a_1^i, a_2^i, as in the REE.

Indeed, from ICC we know that coefficients a_1^i and a_2^i are chosen so that $E\left[(p - MC^i)s_1^i\right] = E\left[(p - MC^i)s_2^i\right] = 0$. Then the optimality condition in the REE reduces to $E\left[(p - MC^i)z\right] =$

0 which determines the remaining coefficient c^i.[20] Allocations at a LBICM make efficient private use of the signals (yielding $E\left[(p - MC^i)s_1^i\right] = E\left[(p - MC^i)s_2^i\right] = 0$). To this the REE adds the privately efficient use of public information (which yields $E\left[(p - MC^i)z\right] = 0$).

5 Welfare Analysis

An incentive efficient allocation maximizes expected total surplus in the class of LBICM. The characterization of incentive efficient allocations boils down to solving an optimal control problem with integral objective.

Incentive compatibility constraints put restrictions on a_1^i, a_2^i and b^i as a function of coefficient c^i and its average $c = \int_0^1 c^i di$ (see Corollary 5). Coefficient c^i and its average c determine production x^i for firm i, which in turn determines total production $x = \int x^i di$ and total costs of production through the aggregation over firms. An incentive efficient allocation will maximize expected total surplus using as control c^i, $i \in (0,1)$:[21]

$$\max_{\{c^i\}} E\left[\int_0^x P(q)dq - \int_0^1 C^i(x^i)di\right] \tag{14}$$

$$\begin{aligned} s.t. \quad x^i &= b^i - a_1^i s_1^i - a_2^i s_2^i + c^i z,\ x = \int_0^1 x^i di \qquad (15) \\ z &= (\lambda + a_1\theta_1 + a_2\theta_2), \text{and} \qquad (16) \\ & ICC : \begin{cases} a_1^i = f_1(c^i, c) \\ a_2^i = f_2(\alpha^i, c^i, c) \\ b^i = \frac{1}{(1+\beta)}. \end{cases} \end{aligned}$$

The problem can be reformulated in the following way:

[20]The FOC is sufficient because $\left[\frac{\partial^2 E[\pi^i]}{\partial (c^i)^2}\right]_{\frac{\partial E[\mu^i]}{\partial c^i}=0} < 0$.

[21]The reader should not get confused with notation: while c^i and c refer to coefficients determining production in the mechanism, capital C refers to costs.

$$\max_{\{c^i\}} \int_0^1 \phi\left(i, c^i, c\right) di \tag{17}$$

where

$$\begin{aligned}
\phi\left(i, c^i, c\right) &= G(c) - F\left(i, c^i, c\right) \text{ with} \\
G(c) &= E\left\{\int_0^x P(q) dq\right\} = E\left\{(1+\lambda - x/2)\, x\right\} \text{ and } F\left(i, c^i, c\right) = EC^i(x^i), \text{ where} \\
x &= \frac{1}{(1+\beta)} - h_1(c)\theta_1 - h_2(c)\theta_2 + cz, \\
x^i &= \frac{1}{(1+\beta)} - f_1(c^i, c)s_1^i - f_2(\alpha^i, c^i, c)s_2^i + c^i z, \text{ and} \\
z &= (\lambda + h_1(c)\theta_1 + h_2(c)\theta_2).
\end{aligned}$$

The function $G(c)$ gives the expected gross surplus resulting from total production $x = \int_0^1 x^i di$ and the function $F(i, c^i, c)$ describes the expected cost of production for firm i and has to be integrated over all firms i to obtain the total expected costs of production. The following proposition characterizes incentive efficient allocations.

Proposition 7 *Incentive efficient allocations have to fulfill the following system of First Order Necessary Conditions (FONC):*

$$E\left[p\frac{\partial x}{\partial c} - \int_0^1 MC^j\left(\frac{\partial x^j}{\partial c}\right) dj - MC^i\left(\frac{\partial x^i}{\partial c^i}\right)\right] = 0 \quad \textit{for all } i. \tag{18}$$

Proof. According to Lemma 14 in the Appendix we obtain the following FONC from the solution to (17):

$$\begin{aligned}
\frac{\partial \phi\left(i, c^i, c\right)}{\partial c^i} + \int_0^1 \frac{\partial \phi\left(j, c^j, c\right)}{\partial c} dj &= 0 \text{ for all } i \text{ or} \\
-\frac{\partial F(i, c^i, c)}{\partial c^i} + \int_0^1 (G'(c) - \frac{\partial F(j, c^j, c)}{\partial c}) dj &= 0 \quad \text{for all } i
\end{aligned}$$

and the result follows from:

$$
\begin{aligned}
\frac{\partial F(i,c^i,c)}{\partial c^i} &= E\left\{\frac{\partial C(x^i)}{\partial c^i}\right\} = E\left\{MC^i(x^i)\left(\frac{\partial x^i}{\partial c^i}\right)\right\} \\
\frac{\partial F(j,c^j,c)}{\partial c} &= E\left\{\frac{\partial C(x^j)}{\partial c}\right\} = E\left\{MC^j(x^j)\left(\frac{\partial x^j}{\partial c}\right)\right\} \\
G'(c) &= p\frac{\partial x}{\partial c}. \blacksquare
\end{aligned}
$$

The FONC just say that the effect on the costs of firm i of a marginal change in c^i must be equated for all the firms and must equal the effect on net surplus (gross surplus minus costs) of changing the average parameter c. The FONC in (18) can be expressed as the sum of two parts: a term as in the FONC of the REE and an informational externality term IE. A necessary condition for the REE to be incentive efficient is that the informational externality be zero when evaluated at the REE.

Proposition 8 *The FONC for an incentive efficient allocation can be decomposed as follows:*

$$
\begin{aligned}
& E\left[p\frac{\partial x}{\partial c} - \int_0^1 MC^j\left(\frac{\partial x^j}{\partial c}\right)dj - MC^i\left(\frac{\partial x^i}{\partial c^i}\right)\right] \\
&= E\left[\left(p - MC^i\right)\left(\frac{\partial x^i}{\partial c^i}\right) + \int_0^1 \left(p - MC^j\right)\left(\frac{\partial x^j}{\partial c}\right)dj\right] \\
&= \underbrace{E\left\{\left(p - MC^i\right)z\right\}}_{=REE} + \underbrace{E\left\{\int_0^1 [(p - MC^j)c^j\frac{\partial z}{\partial c}]dj\right\}}_{=IE} = 0 \quad \textit{for all } i
\end{aligned} \tag{19}
$$

Proof. For the first equality we make use of the decomposition of the derivatives $\partial a_1/\partial c = \int_0^1 \left(\partial a_1^i/\partial c + \partial a_1^i/\partial c^i\right) di$ and $\partial a_2/\partial c = \int_0^1 \left(\partial a_2^i/\partial c + \partial a_2^i/\partial c^i\right) di$ and plug those expressions into the derivatives $\partial x^i/\partial c^i$ and $\partial x/\partial c$ and rearrange to obtain $E\left[p\frac{\partial x}{\partial c}\right] = E\left[p\frac{\partial x^i}{\partial c^i} + \int_0^1 p\frac{\partial x^j}{\partial c}dj\right]$. (See Lemma 13 in the Appendix for a complete proof.) For the second equality, note that the first part is the FOC of the REE as shown in Corollary 6. For the second part we have $E\left[(p-MC^j)\left(\frac{\partial x^j}{\partial c}\right)\right]=E\left[(p-MC^j)\left(-\frac{\partial a_1^j}{\partial c}s_1^j-\frac{\partial a_2^j}{\partial c}s_2^j+c^j\frac{\partial z}{\partial c}\right)\right]$. We obtain $E\left[(p-MC^j)\left(\frac{\partial a_1^j}{\partial c}s_1^j\right)\right]$ $=E\left[(p-MC^j)\left(\frac{\partial a_2^j}{\partial c}s_2^j\right)\right]=0$ because from ICC: $E\left[(p-MC^j)s_1^j\right]=0$ and $E\left[(p-MC^j)s_2^j\right]=0$ for all j. This leads to the result. ■

The first order necessary condition has two parts. First, changing weight c^i results in a marginal effect on expected profits of firm i given by: $E\left[(p-MC^i)\left(\frac{\partial x^i}{\partial c^i}\right)\right]=E\left[(p-MC^i)\,z\right]$. This part is the same as when maximizing expected profits with respect to c^i and coincides therefore, with the optimality condition of the REE, $E\left[(p-MC^i)\,z\right]=0$, as shown in Corollary 6. This condition is just marginal cost pricing in expected terms in the competitive environment with private information. The incentive efficient solution also takes into account the effect of changes in the average value $c=\int c^i$ (and hence a_1 and a_2 through ICC), which affects production x^i of firms. The marginal impact of a change in c on expected total surplus is given by $IE=E\left[\int_0^1(p-MC^j)\left(\frac{\partial x^j}{\partial c}\right)dj\right]$. An increase in c changes production of firm j by $\frac{\partial x^j}{\partial c}$ which has social value of $E\left[(p-MC^j)\left(\frac{\partial x^j}{\partial c}\right)\right]$. Since firms are price takers, the social value of that change in production is just the marginal impact on expected profits per firm, which depends on the covariation of $\frac{\partial x^j}{\partial c}$ with $(p-MC^j)$. Averaging this effect over all firms gives the IE. Of course, this effect is not taken into account at the REE, because there firms take the informativeness of public information as given.

The FONC in (19) hold for all firms and when taking the average of the equation we can derive an expression for the optimal weight c^i assigned to public information for firm i as shown in the following corollary.

Corollary 9 *The optimal individual weight c^i on public information is given by:*

$$c^i=c+c_2(\alpha^i-1) \tag{20}$$

$$c_2=-\frac{\tau_\lambda a_2(\tau_{\varepsilon_1}+\tau_{\theta_1})}{\beta\Delta}$$

$$\Delta = \left[(\tau_{\theta_1} + \tau_{\varepsilon_1}) (\tau_{\theta_2} + \tau_{\varepsilon_2}) + \tau_\lambda \left[(a_1)^2 (\tau_{\theta_2} + \tau_{\varepsilon_2}) + (a_2)^2 (\tau_{\theta_1} + \tau_{\varepsilon_1}) \right] \right]$$

and therefore, individual coefficients a_1^i and a_2^i can be written as

$$a_1^i = a_1 + \frac{\tau_{\varepsilon_1} a_1}{(\tau_{\theta_1} + \tau_{\varepsilon_1})} c_2 \left(\alpha^i - 1 \right) \tag{21}$$

$$a_2^i = \alpha^i a_2 + \frac{\tau_{\varepsilon_2} a_2}{\beta (\tau_{\theta_2} + \tau_{\varepsilon_2})} \left((1 - c(1+\beta) + \beta c_2 \right) \left(\alpha^i - 1 \right) \tag{22}$$

Proof. *See Appendix.*

Observe that the functional form for the optimal c^i coincides with the optimal c^i in (4) for the LBSFE. However, the optimal value c^i will be different from its REE value, depending on the optimal average coefficient c which can be found by averaging (19) over firms. This will determine parameter c and from ICC also parameters a_1 and a_2 and therefore, average production $x(c)$. Once c is determined we can go back to (20), (21) and (22) to obtain individual coefficients c^i, a_1^i and a_2^i which determine individual productions x^i. It is clear then that average production is at the incentive efficient level if and only if individual productions are also incentive efficient (for almost all firms).

The externality term $IE = E\left\{\int [(p - MC^j) c^j \frac{\partial z}{\partial c}]\right\}$ vanishes in three cases: (1) with marginal cost pricing ($p = MC^j$) ; (2) for $\partial z / \partial c = 0$, that is when coefficient c does not affect public information z, and (3) for $c^j = 0$, that is when firms do not condition on public information.

(1) Marginal cost pricing ($p = MC^j$) prevails if private signals are perfectly informative ($\tau_{\epsilon_1} = \tau_{\epsilon_2} = \infty$) about cost uncertainty (the θ parameters). In this case firms do not rely on public information in the estimation of costs (since they are already fully informed) and consequently assign first best weights to private signals ($a_1^j = (\gamma + 1)/\beta$; $a_2^j = \alpha^j (\gamma + 1)/\beta$). Then, production x^j will be such that $p = MC^j$ under certainty because marginal costs are not random.

(2) The coefficient c does not affect public information z if private signals are pure noise . Indeed, for $\tau_{\epsilon_1} = \tau_{\epsilon_2} = 0$ we have from ICC that $\partial a_1/\partial c = \partial a_2/\partial c = 0$ and therefore also $\partial z/\partial c = 0$. The reason is that private signals are only used to account for the signal-sensitivity of costs and not for estimation purposes since signal precisions are zero. As a consequence, the REE and the incentive efficient solutions coincide and optimal weights are given by $a_1 = a_2 = \gamma/\beta$.

(3) If firms' production x^j does not depend on public information ($c^j = 0$), then making z more or less revealing can not affect production and therefore, expected profits of firms. In fact, for $c^j = 0$ for all j the model reduces to a Cournot Model with private information. According to the corollary this case arises when $c = 0$ and $\alpha^j = 1$ for all j . [22]

From the above corollary we see immediately that the informational externality term involves two sources of misallocation. First, a misallocation of total production, or Total Output Effect (TOE) and second, a misallocation of individual production across firms, or Distribution of Output Effect (DOE). The DOE only appears when cost functions are ex ante asymmetric.

Proposition 10 *Let $MC = \int MC^j dj$. Then*

$$IE = \underbrace{E\left[(p - MC)\left(\frac{\partial z}{\partial c}\right)\right]c}_{TOE} + \underbrace{E\left[\int_0^1 (p - MC^j)\left(\frac{\partial z}{\partial c}\right)(\alpha^j - 1)c_2 dj\right]}_{DOE} \tag{23}$$

Proof. Follows directly from equation (19) using $c^i = c + c_2(\alpha^i - 1)$. ■

We will characterize the two effects separately and derive solutions for two particular cases: first, when costs are ex ante symmetric ($\sigma_\alpha^2 = 0$) and second, when

[22] This is consistent with Vives (1988) where it is shown that a Cournot market with private information and a continuum of firms would solve a team problem with expected total surplus as its objective function. In our terminology, the competitive Cournot market is restricted efficient in the class of LBICM.

demand is not random ($\sigma_\lambda^2 = 0$). The former case corresponds to a situation where $DOE = 0$ while the latter one corresponds to $TOE = 0$.

5.1 Total Output Effect

The following proposition derives the sign of the information externality for the TOE when evaluated at the REE solution:

Proposition 11 *If demand is noisy, $\sigma_\lambda^2 > 0$, then sign $TOE = sign(-c^{REE})$. If $\sigma_\lambda^2 = 0$, then $TOE = 0$.*

Proof. We have that $TOE = E\left[(p - MC)\left(\frac{\partial z}{\partial c}\right)\right]c = E\left[(p - MC)\left(\frac{\partial a_1}{\partial c}\theta_1 + \frac{\partial a_2}{\partial c}\theta_2\right)\right]c$. Derivatives $\partial a_1/\partial c$ and $\partial a_2/\partial c$ are strictly positive from ICC. Also from ICC we obtain the following restrictions: $E\left[(p - MC^i)s_1^i\right] = E\left[(p - MC^i)s_2^i\right] = 0$, which can be used to show that: $E\left[(p - MC)\theta_k\right] = \int E\left[(p - MC^j)\theta_k\right]dj = \int E\left[MC^j\varepsilon_k^j\right]dj = \left[(\gamma - \beta a_k)\sigma_{\varepsilon_k}^2\right] < 0$ because using the equilibrium characterization in Corollary 2 we have that $a_k^{REE} > \frac{\gamma}{\beta}$ (for $\sigma_\lambda^2 > 0$). In summary, if $\sigma_\lambda^2 > 0$ we find therefore:

$$TOE = \left[\underbrace{E\left[(p - MC)\theta_1\right]}_{<0}\underbrace{\frac{\partial a_1}{\partial c}}_{>0} + \underbrace{E\left[(p - MC)\theta_2\right]}_{<0}\underbrace{\frac{\partial a_2}{\partial c}}_{>0}\right]c^{REE} \tag{24}$$

If $\sigma_\lambda^2 = 0$, then we know that at the REE, $p = MC$ because $E\left[(p - MC)\theta_k\right] = \int E\left[MC^j\varepsilon_k^j\right] = 0$ since $a_k^{REE} = \frac{\gamma}{\beta}$ (for $k = 1, 2$) and $TOE = 0$. ■

The conclusion is that when costs are ex ante symmetric incentive efficiency requires to increase (decrease) c when c^{REE} is negative (positive).[23] For $c^{REE} < 0$ the informational role of the price dominates and the price reveals too little information. In this case an increase in c means that more weight should be given to private signals

[23]With symmetric costs it is easy to check that the FONC for an incentive efficient allocation are also sufficient.

(a_1 and a_2 increase) and public information becomes therefore more revealing. On the contrary, when the price is mainly an index of scarcity, $c^{REE} > 0$, then the price reveals too much information and c should be reduced. Only in the knife-edge (Cournot) case where $c^{REE} = 0$ the REE is incentive efficient.

Example 1 *Consider the classical case in which signals do not affect costs,* $\gamma = 0$, *demand is noisy,* $\sigma_\lambda^2 > 0$, *costs are symmetric,* $\alpha^i = 1$ *for all* i *and there is only one uncertain cost parameter,* $\tau_{\theta_2} = \infty$ *(assume* $\theta_2 = 0$*). Then* $a_1^{REE} = \tau_{\epsilon_1} / \left(\beta\left(\tau_{\epsilon_1} + \tau_p\right)\right)$ *where* $\tau_p = \tau_{\theta_1} + \tau_\lambda a_1^2$ *is the precision (informativeness) of the price. The parameter* a_1 *is the unique solution to the cubic equation:* $\beta\tau_\lambda a_1^3 + \beta\left(\tau_{\epsilon_1} + \tau_{\theta_1}\right) a_1 - \tau_{\epsilon_1} = 0$. *The optimal weight* a_1 *is the unique positive solution*[24] *to the fourth degree equation:* $\beta\tau_\lambda a_1^3 \left\{E_1\right\} + \beta\left(\tau_{\epsilon_1} + \tau_{\theta_1}\right) a_1 - \tau_{\epsilon_1} = 0$ *with* $E_1 = \left\{a_1 \frac{\beta(\tau_{\epsilon_1} + \tau_{\theta_1}) + \tau_{\epsilon_1}}{\tau_{\epsilon_1}}\right\}$. *Only when* $E_1 = 1$ *or equivalently* $a_1 = \tau_{\varepsilon_1} / \left(\beta(\tau_{\varepsilon_1} + \tau_{\theta_1}) + \tau_{\varepsilon_1}\right)$ *the REE is incentive efficient. From ICC the weight c on public information is given by* $c = \frac{a_1\left(\beta(\tau_{\epsilon_1} + \tau_{\theta_1}) + \tau_{\epsilon_1}\right) - \tau_{\epsilon_1}}{\tau_{\epsilon_1}(1+\beta)a_1}$, *which equals 0 for* $E_1 = 1$. *When* $c^{REE} > 0$, *c should be reduced and when* $c^{REE} < 0$ *it should be increased. Qualitatively the same result holds for the team solution where ETS is maximized under the only constraint that firms use decentralized production strategies, that is*

$$\max_{a,b,c} E[TS] = \max_{a,b,c} E\left[(1 + \lambda - x/2)\, x - \int C(x^i) di\right]$$

subject to $x^i = b - as^i + cz$, *with* $z = \lambda + a\theta$. *Taking derivatives and evaluating the expressions at the solution to the REE we find the following distortion TE (Team*

[24]Negative solutions can be ruled out easily.

Externality):[25]

$$TE = c\underbrace{E\left[\left(p - MC^i\right)\theta_1\right]}_{<0 \text{ (by ICC of REE)}}$$

Therefore, the solutions coincide for $c^{REE} = 0$ (as in the LBICM). For $c^{REE} < 0$ the distortion is positive and c should be increased (as in the LBICM) while the contrary is true for $c^{REE} > 0$. The ICC do not affect the direction of change for c only the magnitude.

5.2 Distribution of Output Effect

Proposition 12 *Evaluating DOE at the REE yields:*

$$\begin{aligned} DOE &= E\left[\int_0^1 \left((p - MC^j)c_2(\alpha^j - 1)dj\right)\left(\frac{\partial a_1}{\partial c}\theta_1 + \frac{\partial a_2}{\partial c}\theta_2\right)\right] \\ &= \underbrace{\frac{(-c_2)\sigma_\alpha^2\tau_\lambda}{\Delta}}_{>0}\left[\underbrace{\frac{\partial a_2}{\partial c}\left((a_1)^2 + \frac{(\tau_{\varepsilon_1} + \tau_{\theta_1})}{\tau_\lambda}\right)}_{>0} - \underbrace{\frac{\partial a_1}{\partial c}}_{>0} a_1 a_2\right] \end{aligned} \tag{25}$$

Proof. First, we substitute for $\partial z/\partial c = (\partial a_1/\partial c)\,\theta_1 + (\partial a_2/\partial c)\,\theta_2$ into DOE from (23). Taking expectations and integrating over firms gives the result. Signs are obtained when evaluating the expression at optimal REE values ($a_1 > 0$, $a_2 > 0$) (see Appendix for a complete proof). ■

[25] Derivatives are given by $\frac{\partial x}{\partial a} = -s^i + c\theta$, $\frac{\partial x}{\partial b} = 1$ and $\frac{\partial x}{\partial c} = z$ which yields the following (sufficient) FOC's:

$$\begin{aligned} \frac{\partial E[TS]}{\partial a} &= E\left[\left(p - MC^i\right)\left(-s^i + c\theta\right)\right] = 0 \\ \frac{\partial E[TS]}{\partial b} &= E\left[\left(p - MC^i\right)\right] = 0 \\ \frac{\partial E[TS]}{\partial c} &= E\left[\left(p - MC^i\right)z\right] = 0 \end{aligned}$$

Evaluating the team solution at the REE where $E\left[\left(p - MC^i\right)\right] = E\left[\left(p - MC^i\right)z\right] = E\left[\left(p - MC^i\right)\left(-s^i\right)\right] = 0$, we get the team externality.

The DOE is the sum of two parts: first, a positive effect on welfare of increasing c making public information more revealing (by $\frac{\partial a_2}{\partial c}$) about θ_2 and second, a negative effect of making public information more revealing (by $\frac{\partial a_1}{\partial c}$) about θ_1. The sign of DOE depends on which of the two effects dominates. It is clear that changes in opposite direction for a_1 and a_2 would be optimal: A welfare improvement could be achieved by decreasing revelation of θ_1 (through a decrease in a_1) and increasing revelation of θ_2 (through an increase in a_2). Starting from the REE it pays to increase the average responsiveness to the signal about the shock θ_2 which has an asymmetric impact on the costs of firms and to decrease it to the signal about the shock θ_1. However, due to the constraints imposed by the LBICM public information can only become more (or less) revealing about θ_1 and θ_2 jointly and changes in opposite direction are not possible.

When noise in demand vanishes ($\sigma_\lambda^2 = 0$) we can transform and evaluate the DOE at the solution to the REE as follows :

$$\begin{aligned}
DOE &= \int_0^1 c_2(\alpha^j - 1)\left\{\underbrace{E\left[(p - MC^j)\theta_1\right]}_{=(\alpha^j-1)/\Delta}\frac{\partial a_1}{\partial c} + \underbrace{E\left[(p - MC^j)\theta_2\right]}_{=(1-\alpha^j)/\Delta}\frac{\partial a_2}{\partial c}\right\} dj && (26)\\
&= \frac{-c_2\sigma_\alpha^2}{\Delta}\left[\frac{\partial(a_2 - a_1)}{\partial c}\right] \text{ and } sign\left[\frac{\partial(a_2 - a_1)}{\partial c}\right] = sign\left(\tau_{\varepsilon_2}\tau_{\theta_1} - \tau_{\varepsilon_1}\tau_{\theta_2}\right), && (27)\\
\text{where } \Delta &= (\tau_{\theta_2} + \tau_{\varepsilon_2}) + (\tau_{\theta_1} + \tau_{\varepsilon_1}).
\end{aligned}$$

If demand is not noisy then λ is a constant, $a_1^{REE} = a_2^{REE} = \gamma/\beta$ and public information becomes $z = a(\theta_1 + \theta_2)$. At the REE, z is now fully revealing with respect to average production $x = b - a(\theta_1 + \theta_2) + cz$. Then, $E\left[(p - MC)\theta_1\right] = 0$ and $E\left[(p - MC)\theta_2\right] = 0$, the total output effect vanishes ($TOE = 0$) and first best average production is obtained. Furthermore, whenever $\sigma_\alpha^2 > 0$ it can be shown that the incentive efficient solution and the REE coincide if and only if $\tau_{\varepsilon_2}\tau_{\theta_1} - \tau_{\varepsilon_1}\tau_{\theta_2} = 0$

. This is precisely the case considered by Laffont (1985).[26] The result is that if $\tau_{\varepsilon_2}\tau_{\theta_1} - \tau_{\varepsilon_1}\tau_{\theta_2} \neq 0$ then $a_2 - a_1$ should be increased. If $\tau_{\varepsilon_2}\tau_{\theta_1} - \tau_{\varepsilon_1}\tau_{\theta_2} > (<)\ 0$, $a_2 - a_1$ should be increased by increasing (decreasing) c from the REE value. Thus it pays to distort allocative efficiency at the REE to improve productive efficiency.

The REE always puts the same weight on the two types of signals, $a_1^{REE} = a_2^{REE}$, when $\sigma_\lambda^2 = 0$ because say that $a_2 > a_1$, then the public statistic $z = (a_1\theta_1 + a_2\theta_2)$ is more precise about θ_2 than about θ_1 and, on average, firms would rely more on signal s_1^i than s_2^i. However, this would imply $a_1 > a_2$, a contradiction. This is only optimal when the two types of signals are of the same precision, $\tau_{\varepsilon_2}/\tau_{\theta_2} = \tau_{\varepsilon_1}/\tau_{\theta_1}$, because then incentive compatibility requires that $a_2 = a_1$. We have seen that in principle it always pays to increase $a_2 - a_1$ to increase the relative revelation of θ_2. This can be achieved in an incentive compatible way by increasing (decreasing) c from the REE value when $(\tau_{\varepsilon_2}/\tau_{\theta_2}) - (\tau_{\varepsilon_1}/\tau_{\theta_1}) > (<)\ 0$.

6 Conclusion

We have studied the allocative and productive efficiency properties of REE in a simple production economy with asymmetric information about costs. We have insisted in that REE be implementable in Bayesian equilibrium and have considered competition in supply schedules. Our model is rich enough to encompass all relevant cases of REE: fully or partially revealing, with noise (in demand) or without. We have looked at efficiency in the class of linear Bayesian incentive compatible mechanisms which have the same communication constraints as the market. This is the most favorable situation for the efficiency of market allocations. Here are the results.

[26] When $\tau_{\varepsilon_2}\tau_{\theta_1} - \tau_{\varepsilon_1}\tau_{\theta_2} = 0$ Laffont (1985) noted that increasing $a_2 - a_1$ was welfare improving but that incentive compatibility required that $a_2 = a_1$.

The first result is that, except in very particular cases, REE are not incentive efficient. The source of the inefficiency is an informational externality. Firms are competitive (price-takers) but when responding to their private signals they do not take into account the fact that they modify the information content of the public signal (the market price). Furthermore, at the REE prices may contain too little or too much information (and this has nothing to do with the Hirshleifer effect, the potential destruction of insurance opportunities since firms are risk neutral). That is, informational efficiency need not be aligned with allocative efficiency.

We concentrate attention on two polar cases. In the first firms are ex ante symmetric and demand is random. In the second firms are ex ante asymmetric and demand is nonrandom. When firms are ex ante symmetric and demand is random allocative efficiency is distorted at REE except in the (degenerate) case in which the market turns into Cournot competition. That is, except if firms use strategies which are not contingent on the price in which case there is no information externality. Otherwise, if the informational role of the price prevails at an incentive efficient solution more weight should be given to private signals while if the traditional role of the price as index of scarcity prevails then less weight should be given to private signals. In the first instance the REE price reveals too little information and in the second too much. When firms are ex ante asymmetric and demand is nonrandom then the total output at the REE is first best optimal but the REE allocation is not incentive efficient except in the case in which the precision of the two types of signals are identical (which is the case considered by Laffont (1985)). Otherwise it always pays to distort allocative efficiency to improve productive efficiency. The REE price does not contain enough information on the random cost parameter which affects costs asymmetrically. In conclusion, with multidimensional uncertainty (and of higher dimension than the price system) the information content of the REE has the wrong composition.

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7 Appendix

Proof. of Proposition (1)

In order to characterize LBSFE we follow a standard procedure. We posit linear strategies for the firms in terms of (s_1^i, s_2^i, z), derive a linear relationship between prices and the random variables $(\lambda, \theta_1, \theta_2)$ and work through the optimization and updating problems of the firms to obtain revised strategies. Identifying coefficients of the initial and revised linear strategies we obtain:

$$\begin{aligned}
a_1^i &= \gamma/\beta + \left(\tau_{\varepsilon_1}[(\tau_{\theta_2}+\tau_{\varepsilon_2}) + \tau_\lambda a_2(a_2 - \alpha^i a_1)]\right)/\beta\Delta \\
a_2^i &= \alpha^i\gamma/\beta + \left(\tau_{\varepsilon_2}[\alpha^i(\tau_{\theta_1}+\tau_{\varepsilon_1}) + \tau_\lambda a_1(\alpha^i a_1 - a_2)]\right)/\beta\Delta \\
c^i &= (1-c)/\beta - \frac{\tau_\lambda[a_1(\tau_{\theta 2}+\tau_{\varepsilon 2}) + \alpha^i a_2(\tau_{\theta 1}+\tau_{\varepsilon 1})]}{\beta\Delta} \\
b^i &= \frac{1}{\beta}(1-b) - \frac{1}{\beta}E[\theta_1+\alpha^i\theta_2] + \left((1-c)\beta^{-1} - c^i\right)E[z] + \left(a_1^i - \gamma/\beta\right)E[s_1^i] + \left(a_2^i - \alpha^i\gamma/\beta\right)E[\cdot \\
\Delta &= (\tau_{\theta_1}+\tau_{\varepsilon_1})(\tau_{\theta_2}+\tau_{\varepsilon_2}) + \tau_\lambda[(a_1)^2(\tau_{\theta_2}+\tau_{\varepsilon_2}) + (a_2)^2(\tau_{\theta_1}+\tau_{\varepsilon_1})]
\end{aligned}$$

The expressions depend on average coefficients a_1 and a_2 which are defined (through aggregation) by the following cubic equations:

$$\begin{aligned}
a_1 &= \gamma/\beta + (\tau_{\varepsilon_1}[(\tau_{\theta_2}+\tau_{\varepsilon_2}) + \tau_\lambda a_2(a_2 - a_1)])/\beta\Delta \\
a_2 &= \gamma/\beta + (\tau_{\varepsilon_2}[(\tau_{\theta_1}+\tau_{\varepsilon_1}) + \tau_\lambda a_1(a_1 - a_2)])/\beta\Delta.
\end{aligned}$$

To proof existence and uniqueness of the REE allocation we make use of a result which we derive in Corollary (6). There we show that the REE allocation can be equivalently obtained by maximizing expected profits subject to incentive compatibility constraints given by (see Corollary (5)) $a_1(c) = \frac{[\gamma(\tau_{\theta_1}+\tau_{\varepsilon_1})+\tau_{\varepsilon_1}]}{[\tau_{\varepsilon_1}(1-c(1+\beta))+\beta(\tau_{\theta_1}+\tau_{\varepsilon_1})]}$ and $a_2(c) = \frac{[\gamma(\tau_{\theta_2}+\tau_{\varepsilon_2})+\tau_{\varepsilon_2}]}{\tau_{\varepsilon_2}(1-c(1+\beta))+\beta(\tau_{\theta_2}+\tau_{\varepsilon_2})}$. The optimal allocation is then determined by the following FOC (see Corollary (6)):

$$\frac{\partial E[\pi^i]}{\partial c^i} = E\left[\left(p - MC^i\right)z\right] = 0$$

Averaging over firms $\int E\left[(p - MC^i)\, z\right] di = 0$ the optimal solution is defined by a function $f(c) = 0$ given by:

$$f(c) = c - \frac{1}{(1+\beta)} - \frac{\tau_\lambda a_1(c)}{\tau_{\varepsilon_1}(1+\beta)}(\gamma - \beta a_1(c)) - \frac{\tau_\lambda a_2(c)}{\tau_{\varepsilon_2}(1+\beta)}(\gamma - \beta a_2(c)) = 0$$

Taking limits for $a_1(c)$ and $a_2(c)$ yields: $\lim_{c\to\infty} a_1(c) = \lim_{c\to-\infty} a_1(c) = 0$ and $\lim_{c\to\infty} a_2(c) = \lim_{c\to-\infty} a_2(c) = 0$. Therefore we have

$$\begin{aligned} \lim_{c\to\infty} f(c) &= \infty \\ \lim_{c\to-\infty} f(c) &= -\infty \end{aligned}$$

Furthermore,

$$\begin{aligned} f'(c) &= 1 - \left[\frac{\tau_\lambda \overbrace{a_1'(c)}^{>0}}{\tau_{\varepsilon_1}(1+\beta)} \overbrace{(\gamma - \beta a_1(c))}^{<0} - \frac{\tau_\lambda \overbrace{a_1(c)}^{>0}}{\tau_{\varepsilon_1}(1+\beta)} \beta \overbrace{a_1'(c)}^{>0} \right] \\ &\quad - \left[\frac{\tau_\lambda \overbrace{a_2'(c)}^{>0}}{\tau_{\varepsilon_2}(1+\beta)} \overbrace{(\gamma - \beta a_2(c))}^{<0} - \frac{\tau_\lambda \overbrace{a_2(c)}^{>0}}{\tau_{\varepsilon_2}(1+\beta)} \beta \overbrace{a_2'(c)}^{>0} \right] \end{aligned}$$

and $\partial a_1/\partial c > 0$, $\partial a_2/\partial c > 0$, $a_1 > \gamma/\beta$ and $a_2 > \gamma/\beta$ and therefore $f'(c)\mid_{f(c)=0} > 0$. In summary, there exists a unique solution solving $f(c) = 0$ which determines the remaining coefficients a_1^i, a_2^i and c^i as given in the Proposition. Coefficient b^i is given by the following expression:

$$\begin{aligned} b^i &= \frac{1}{1+\beta} + \frac{\tau_\lambda [a_1(\tau_{\theta_2} + \tau_{\varepsilon_2}) + a_2(\tau_{\theta_1} + \tau_{\varepsilon_1})]}{\beta \Delta} E[z] \\ &\quad + \left(a_1^i - \frac{\gamma}{\beta} - \frac{1}{\beta}\right) E\left[s_1^i\right] + \left(a_2^i - \frac{\alpha^i \gamma}{\beta} - \frac{\alpha^i}{\beta}\right) E\left[s_2^i\right] \end{aligned}$$

and without loss of generality we assume that unconditional means are zero, that is $\overline{\lambda} = \mu_1 = \mu_2 = 0$, and therefore, $b^i = b = 1/(1+\beta)$. ■

Proof. of Proposition (4) and Corollary (5)

Given the mechanism described above and truth telling behavior by $j \neq i$ the optimal announcements $\widehat{s}_1^i$ and $\widehat{s}_2^i$ of firm i are derived from the following maximization program:

$$\max_{\hat{s}_1^i, \hat{s}_2^i} E\left[p\widehat{x}^i - C(\widehat{x}^i) \mid s_1^i, s_2^i\right]$$

where $\widehat{x}^i$ is the production recommended by the center when receiving messages $\hat{s}_1^i, \hat{s}_2^i$. The FOC's. yield:

$$E\left[p - MC^i(\widehat{x}^i) \mid s_1^i, s_2^i\right] \frac{\partial \widehat{x}^i}{\partial \hat{s}_1^i} = 0 \qquad \forall i \tag{28}$$

$$E\left[p - MC^i(\widehat{x}^i) \mid s_1^i, s_2^i\right] \frac{\partial \widehat{x}^i}{\partial \hat{s}_2^i} = 0 \qquad \forall i \tag{29}$$

Unless the weights a_1^i and a_2^i on private signals are zero the following condition has to be satisfied: $E[(p - MC^i(\widehat{x}^i)) \mid s_1^i, s_2^i] = 0$ for almost all s_1^i, s_2^i. For truth telling to be optimal this has to hold for $s_1^i = \widehat{s}_1^i$ and $s_2^i = \widehat{s}_2^i$. In fact, there is a linear combination of announcements $\widehat{\omega} = (a_1^i \widehat{s}_1^i + a_2^i \widehat{s}_2^i)$ that maximize expected profits and truth telling is one of them.Given normality, we obtain:

$$E[(p - MC^i) \mid s_1^i, s_2^i] = E[(p - MC^i)] + \frac{COV[(p - MC^i), s_1^i]}{Var(s_1^i)}\left(s_1^i - E[s_1^i]\right)$$
$$+ \frac{COV[(p - MC^i), s_2^i]}{Var(s_2^i)}\left(s_2^i - E[s_2^i]\right) = 0 \qquad \forall s_1^i, \forall s_2^i$$

Since the equation has to hold for all possible signals s_1^i and s_2^i, $Var(s_1^i) > 0$ and $Var(s_2^i) > 0$ we have $E[(p - MC^i)] = COV[(p - MC^i), s_1^i] = COV[(p - MC^i), s_2^i] = 0$. Solving these equations yields the restrictions on b, a_1^i and a_2^i as stated in the Corollary. Substituting for $p = (1 - b) + (1 - c)z$, $MC^i(x^i) = [\gamma s_1^i + \theta_1 + \alpha^i(\gamma s_2^i + \theta_2)] + \beta(b^i - a_1^i s_1^i - a_2^i s_2^i + c^i z)$ and $z = (\lambda + a_1\theta_1 + a_2\theta_2)$ we obtain the following:

$$E[(p - MC^i(x^i))] = 0 \Longrightarrow \left(1 - b - \beta b^i\right) = 0 \Longrightarrow b^i = b = \frac{1}{1 + \beta}.$$

$$
\begin{aligned}
& COV[(p - MC^i(x^i)), s_1^i] = 0 \\
\Longrightarrow \quad & \frac{1}{\tau_{\theta_1}\tau_{\varepsilon_1}}\left[(1-c)a_1\tau_{\varepsilon_1} - \gamma\left(\tau_{\theta_1}+\tau_{\varepsilon_1}\right) - \tau_{\varepsilon_1} + \beta a_1^i\left(\tau_{\theta_1}+\tau_{\varepsilon_1}\right) - \beta c^i a_1\tau_{\varepsilon_1}\right] = 0 \\
\Longrightarrow \quad & a_1^i = \frac{\left[\gamma\left(\tau_{\theta_1}+\tau_{\varepsilon_1}\right)+\tau_{\varepsilon_1}\right] - (1-c)a_1\tau_{\varepsilon_1} + \beta c^i a_1\tau_{\varepsilon_1}}{\beta\left(\tau_{\theta_1}+\tau_{\varepsilon_1}\right)}.
\end{aligned}
$$

Integrating over all firms yields average coefficient a_1 and plugging back into the expression for a_1^i gives the result:

$$
\begin{aligned}
a_1 &= \frac{\left[\gamma\left(\tau_{\theta_1}+\tau_{\varepsilon_1}\right)+\tau_{\varepsilon_1}\right]}{\left[\tau_{\varepsilon_1}(1-c(1+\beta))+\beta\left(\tau_{\theta_1}+\tau_{\varepsilon_1}\right)\right]} \\
a_1^i &= \frac{\left[\gamma\left(\tau_{\theta_1}+\tau_{\varepsilon_1}\right)+\tau_{\varepsilon_1}\right]\left(\tau_{\varepsilon_1}(c^i-c)+\left(\tau_{\theta_1}+\tau_{\varepsilon_1}\right)\right)}{\left[\tau_{\varepsilon_1}(1-c(1+\beta))+\beta\left(\tau_{\theta_1}+\tau_{\varepsilon_1}\right)\right]}.
\end{aligned}
$$

The third restriction yields the following:

$$
\begin{aligned}
& COV[(p - MC^i(x^i)), s_2^i] = 0 \\
\Longrightarrow \quad & \frac{1}{\tau_{\theta_2}\tau_{\varepsilon 2}}\left[(1-c)a_2\tau_{\varepsilon_2} - \gamma\alpha^i\left(\tau_{\theta_2}+\tau_{\varepsilon_2}\right) - \alpha^i\tau_{\varepsilon_2} + \beta a_2^i\left(\tau_{\theta_2}+\tau_{\varepsilon_2}\right) - \beta c^i a_2\tau_{\varepsilon_2}\right] = 0 \\
\Longrightarrow \quad & a_2^i = \frac{\left[\alpha^i\left[\gamma\left(\tau_{\theta_2}+\tau_{\varepsilon_2}\right)+\tau_{\varepsilon_2}\right] - (1-c)a_2\tau_{\varepsilon_2} + \beta c^i a_2\tau_{\varepsilon_2}\right]}{\beta\left(\tau_{\theta_2}+\tau_{\varepsilon_2}\right)}
\end{aligned}
$$

and integrating over all a_2^i determines coefficient a_2 which when plugging back into the expression for a_1^i gives the following values:

$$
\begin{aligned}
a_2 &= \frac{\left[\gamma\left(\tau_{\theta_2}+\tau_{\varepsilon_2}\right)+\tau_{\varepsilon_2}\right]}{\tau_{\varepsilon_2}(1-c(1+\beta))+\beta\left(\tau_{\theta_2}+\tau_{\varepsilon_2}\right)} \\
a_2^i &= \frac{\left[\gamma\left(\tau_{\theta_2}+\tau_{\varepsilon_2}\right)+\tau_{\varepsilon_2}\right]\left[(\alpha^i-1)\tau_{\varepsilon_2}(1-c)+\alpha^i\beta\left(\tau_{\theta_2}+\tau_{\varepsilon_2}\right)+\beta\tau_{\varepsilon_2}(c^i-\alpha^i c)\right]}{\beta\left(\tau_{\theta_2}+\tau_{\varepsilon_2}\right)\tau_{\varepsilon_2}(1-c(1+\beta))+\beta\left(\tau_{\theta_2}+\tau_{\varepsilon_2}\right)},
\end{aligned}
$$

which are the expressions in the text. ■

Lemma 13 $E\left[p\frac{\partial x}{\partial c}\right] = E\left[p\frac{\partial x^i}{\partial c^i} + \int_0^1 p\frac{\partial x^j}{\partial c}dj\right]$

The Lemma relies on two properties of the derivatives:

$$
\frac{\partial a_1}{\partial c} = \int_0^1\left(\frac{\partial a_1^j}{\partial c}+\frac{\partial a_1^j}{\partial c^j}\right)dj = \left(\int_0^1\frac{\partial a_1^j}{\partial c}dj\right)+\frac{\partial a_1^i}{\partial c^i} \tag{30}
$$

$$
\frac{\partial a_2}{\partial c} = \int_0^1\left(\frac{\partial a_2^j}{\partial c}+\frac{\partial a_2^j}{\partial c^j}\right)dj = \left(\int_0^1\frac{\partial a_2^j}{\partial c}dj\right)+\frac{\partial a_2^i}{\partial c^i}. \tag{31}
$$

First, differentiating coefficients a_1 and a_2 gives:

$$\begin{aligned}\frac{\partial a_1}{\partial c} &= \frac{[\gamma(\tau_{\varepsilon_1}+\tau_{\theta_1})+\tau_{\varepsilon_1}]\tau_{\varepsilon_1}(1+\beta)}{[\tau_{\varepsilon_1}(1-c(1+\beta))+\beta(\tau_{\varepsilon_1}+\tau_{\theta_1})]^2} \\ \frac{\partial a_2}{\partial c} &= \frac{[\gamma(\tau_{\varepsilon_2}+\tau_{\theta_2})+\tau_{\varepsilon_2}]\tau_{\varepsilon_2}(1+\beta)}{[\tau_{\varepsilon_2}(1-c(1+\beta))+\beta(\tau_{\varepsilon_2}+\tau_{\theta_2})]^2},\end{aligned} \tag{32}$$

further, we have:

$$\begin{aligned}\frac{\partial a_1^j}{\partial c} &= \frac{[\gamma(\tau_{\varepsilon_1}+\tau_{\theta_1})+\tau_{\varepsilon_1}]\tau_{\varepsilon_1}\left(c^j\tau_{\varepsilon_1}(1+\beta)+\tau_{\theta_1}\right)}{(\tau_{\varepsilon_1}+\tau_{\theta_1})\left[\tau_{\varepsilon_1}(1-c(1+\beta))+\beta(\tau_{\varepsilon_1}+\tau_{\theta_1})\right]^2} \\ \frac{\partial a_2^j}{\partial c} &= \frac{[\gamma(\tau_{\varepsilon_2}+\tau_{\theta_2})+\tau_{\varepsilon_2}]\tau_{\varepsilon_2}\left(c^j\tau_{\varepsilon_2}(1+\beta)+\tau_{\theta_2}\right)}{(\tau_{\varepsilon_2}+\tau_{\theta_2})\left[\tau_{\varepsilon_2}(1-c(1+\beta))+\beta(\tau_{\varepsilon_2}+\tau_{\theta_2})\right]^2},\end{aligned} \tag{33}$$

and finally

$$\begin{aligned}\frac{\partial a_1^i}{\partial c^i} &= \frac{[\gamma(\tau_{\varepsilon_1}+\tau_{\theta_1})+\tau_{\varepsilon_1}]\tau_{\varepsilon_1}}{(\tau_{\varepsilon_1}+\tau_{\theta_1})\left[\tau_{\varepsilon_1}(1-c(1+\beta))+\beta(\tau_{\varepsilon_1}+\tau_{\theta_1})\right]} \\ \frac{\partial a_2^i}{\partial c^i} &= \frac{[\gamma(\tau_{\varepsilon_2}+\tau_{\theta_2})+\tau_{\varepsilon_2}]\tau_{\varepsilon_2}}{(\tau_{\varepsilon_2}+\tau_{\theta_2})\left[\tau_{\varepsilon_2}(1-c(1+\beta))+\beta(\tau_{\varepsilon_2}+\tau_{\theta_2})\right]}.\end{aligned} \tag{34}$$

Integrating over the sum of $\frac{\partial a_1^j}{\partial c}$ and $\frac{\partial a_1^j}{\partial c^i}$ and noting that $\frac{\partial a_1^i}{\partial c^i} = \frac{\partial a_1^j}{\partial c^j}$ and $\frac{\partial a_2^i}{\partial c^i} = \frac{\partial a_2^j}{\partial c^j}$ are constants and independent of i, j we obtain

$$\begin{aligned}&\int_0^1 \left(\frac{\partial a_1^j}{\partial c} + \frac{\partial a_1^j}{\partial c^j}\right) dj = \int_0^1 \frac{\partial a_1^j}{\partial c} dj + \frac{\partial a_1^i}{\partial c^i} \\ =\;& \frac{[\gamma(\tau_{\varepsilon_1}+\tau_{\theta_1})+\tau_{\varepsilon_1}]\tau_{\varepsilon_1}}{(\tau_{\varepsilon_1}+\tau_{\theta_1})\left[\tau_{\varepsilon_1}(1-c(1+\beta))+\beta(\tau_{\varepsilon_1}+\tau_{\theta_1})\right]}\left[\int_0^1 \frac{\left(c^j\tau_{\varepsilon_1}(1+\beta)+\tau_{\theta_1}\right)}{\left[\tau_{\varepsilon_1}(1-c(1+\beta))+\beta(\tau_{\varepsilon_1}+\tau_{\theta_1})\right]} dj + 1\right] \\ =\;& \frac{[\gamma(\tau_{\varepsilon_1}+\tau_{\theta_1})+\tau_{\varepsilon_1}]\tau_{\varepsilon_1}\left(\left(c\tau_{\varepsilon_1}(1+\beta)+\tau_{\theta_1}\right)+\left[\tau_{\varepsilon_1}(1-c(1+\beta))+\beta(\tau_{\varepsilon_1}+\tau_{\theta_1})\right]\right)}{(\tau_{\varepsilon_1}+\tau_{\theta_1})\left[\tau_{\varepsilon_1}(1-c(1+\beta))+\beta(\tau_{\varepsilon_1}+\tau_{\theta_1})\right]^2} \\ =\;& \frac{[\gamma(\tau_{\varepsilon_1}+\tau_{\theta_1})+\tau_{\varepsilon_1}]\tau_{\varepsilon_1}}{\left[\tau_{\varepsilon_1}(1-c(1+\beta))+\beta(\tau_{\varepsilon_1}+\tau_{\theta_1})\right]^2}(1+\beta) \\ =\;& \frac{\partial a_1}{\partial c}.\end{aligned}$$

Equivalently, using expressions (33) and (34) it can be shown that $\int_0^1 \left(\frac{\partial a_2^j}{\partial c} + \frac{\partial a_2^j}{\partial c^j}\right) dj = \frac{\partial a_2}{\partial c}$. Using these results we can do the following transformation:

$$E\left[p\frac{\partial x}{\partial c}\right] = E\left[p\left(-\frac{\partial a_1}{\partial c}\theta_1 - \frac{\partial a_2}{\partial c}\theta_2 + z + c\frac{\partial z}{\partial c}\right)\right]$$

$$
\begin{aligned}
&= E\left[p\left(-\left[\left(\int_0^1 \frac{\partial a_1^j}{\partial c}dj\right)+\frac{\partial a_1^i}{\partial c^i}\right]\theta_1-\left[\left(\int_0^1 \frac{\partial a_2^j}{\partial c}dj\right)+\frac{\partial a_2^i}{\partial c^i}\right]\theta_2+z+c\frac{\partial z}{\partial c}\right)\right] \\
&= E\left[p\left(-\frac{\partial a_1^i}{\partial c^i}\theta_1-\frac{\partial a_2^i}{\partial c^i}\theta_2+z\right)+\int_0^1 p\left(-\frac{\partial a_1^j}{\partial c}\theta_1-\frac{\partial a_2^j}{\partial c}\theta_2+c^j\frac{\partial z}{\partial c}\right)dj\right] \\
&= E\left[p\left(-\frac{\partial a_1^i}{\partial c^i}s_1^i-\frac{\partial a_2^i}{\partial c^i}s_2^i+z\right)+\int_0^1 p\left(-\frac{\partial a_1^j}{\partial c}s_1^j-\frac{\partial a_2^j}{\partial c}s_2^j+c^j\frac{\partial z}{\partial c}\right)dj\right] \\
&= E\left[p\frac{\partial x^i}{\partial c^i}+\int_0^1 p\frac{\partial x^j}{\partial c}dj\right].
\end{aligned}
$$

For the first equality we wrote out the derivative $\partial x/\partial c$. In the second we make use of the decomposition of the derivative as shown in (30) and (31). For the third equality we first group derivatives depending on i and j, respectively. Further, we can take the integral outside the derivatives. For the fourth equality note that $E\left[p\left(-\frac{\partial a_1^i}{\partial c^i}s_1^i-\frac{\partial a_2^i}{\partial c^i}s_2^i+z\right)\right] = E\left[p\left(-\frac{\partial a_1^i}{\partial c^i}\theta_1-\frac{\partial a_2^i}{\partial c^i}\theta_2+z\right)\right]$ because errors are not correlated with the price: $E\left[p\varepsilon_1^i\right]=E\left[p\varepsilon_2^i\right]=0\ \forall i$. Finally, the last equality follows by replacing expressions with derivatives $\frac{\partial x^i}{\partial c^i}$ and $\frac{\partial x^j}{\partial c}$, respectively. ■

Proof. of Corollary (9)

Averaging the individual FONC in (19) we have

$$
\int_0^1 \left(FONC^i\right)di = E\left[\int_0^1 \left[\left(p-MC^i\right)z\right]di+\int_0^1 \left(p-MC^j\right)\left(\frac{\partial x^j}{\partial c}\right)dj\right]=0
$$

which together with condition (19) implies $E\left[\left(p-MC^i\right)z\right]-E\left[\int\left(p-MC^j\right)zdj\right]=0$. Taking expectations and using coefficients from ICC we can show:

$$
\begin{aligned}
& E\left[\left(p-MC^i\right)z\right]-E\left[\int\left(p-MC^j\right)zdj\right] \\
&= \left[\frac{1}{\tau_\lambda}\left(1-c-c^i\beta\right)+\frac{a_1}{\tau_{\varepsilon_1}}\left(\gamma-\beta a_1^i\right)+\frac{a_2}{\tau_{\varepsilon_2}}\left(\alpha^i\gamma-\beta a_2^i\right)\right]-\left[\frac{1}{\tau_\lambda}\left(1-c-c\beta\right)+\frac{a_1}{\tau_{\varepsilon_1}}\left(\gamma-\beta a_1\right.\right. \\
&= \frac{\beta}{\tau_\lambda}\left[\left(c-c^i\right)\right]+\frac{\beta\left(a_1\right)^2}{\left(\tau_{\theta_1}+\tau_{\varepsilon_1}\right)}\left(c-c^i\right)+\frac{\beta\left(a_2\right)^2}{\left(\tau_{\theta_2}+\tau_{\varepsilon_2}\right)}\left(c-c^i\right)-\left(\alpha^i-1\right)\left[\frac{a_2}{\left(\tau_{\theta_2}+\tau_{\varepsilon_2}\right)}\right]=0
\end{aligned}
$$

$$\implies \quad c^i = \frac{\left[\frac{\beta}{\tau_\lambda} + \frac{\beta(a_1)^2}{(\tau_{\theta_1}+\tau_{\varepsilon_1})} + \frac{\beta(a_2)^2}{(\tau_{\theta_2}+\tau_{\varepsilon_2})}\right] c - (\alpha^i - 1)\left[\frac{a_2}{(\tau_{\theta_2}+\tau_{\varepsilon_2})}\right]}{\left[\frac{\beta}{\tau_\lambda} + \frac{\beta(a_1)^2}{(\tau_{\theta_1}+\tau_{\varepsilon_1})} + \frac{\beta(a_2)^2}{(\tau_{\theta_2}+\tau_{\varepsilon_2})}\right]}$$

$$\implies \quad c^i = c - (\alpha^i - 1)\frac{a_2\tau_\lambda\left(\tau_{\theta_1}+\tau_{\varepsilon_1}\right)}{\beta\Delta} = c + c_2(\alpha^i - 1),$$

where for the first equality we take expectations: $E\left[(p - MC)\lambda\right] = (1 - c(1+\beta))\, E\left[\lambda^2\right] = \left[1 - c(1+\beta)\right]/\tau_\lambda$, $\int E\left[(p - MC^i)\theta_1\right] = \int \left(\gamma - \beta a_1^i\right)/\tau_{\varepsilon_1} = \left(\gamma - \beta a_1\right)/\tau_{\varepsilon_1}$ and $\int E\left[(p - MC^i)\theta_2\right]$ $\int \left(\alpha^j\gamma - \beta a_2^i\right)/\tau_{\varepsilon_2} = \left(\alpha^j\gamma - \beta a_2\right)/\tau_{\varepsilon_2}$. For the second equality we use expressions for $a_1^i = a_1 + \frac{\tau_{\varepsilon_1}a_1}{(\tau_{\theta_1}+\tau_{\varepsilon_1})}\left(c^i - c\right)$ and $a_2^i = \alpha^i a_2 + \frac{\tau_{\varepsilon_2}a_2}{\beta(\tau_{\theta_2}+\tau_{\varepsilon_2})}\left((1 - c(1+\beta))(\alpha^i - 1) + \beta(c^i - c)\right)$ from ICC and group terms depending on a_1 and a_2, respectively. The result follows then by rearranging terms. ■

Proof. of Proposition (12)

Take DOE from Proposition (10). Taking expectations as before ($E\left[(p - MC)\lambda\right] = \left[1 - c(1+\beta)\right]/\tau_\lambda$, $\int E\left[(p - MC^i)\theta_1\right] = \left(\gamma - \beta a_1\right)/\tau_{\varepsilon_1}$ and $\int E\left[(p - MC^i)\theta_2\right] = \left(\gamma - \beta a_2\right)/\tau_{\varepsilon_2}$), we can do the following transformation:

$$\begin{aligned}
& E\left[\int_0^1 (p - MC^j)\left(\frac{\partial a_1}{\partial c}\theta_1 + \frac{\partial a_2}{\partial c}\theta_2\right)(\alpha^j - 1)c_2 dj\right] \\
= \;& \int\left[\frac{\partial a_1}{\partial c}\frac{\left(\gamma - \beta a_1^j\right)}{\tau_{\varepsilon_1}} + \frac{\partial a_2}{\partial c}\frac{\left(\alpha^j\gamma - \beta a_2^j\right)}{\tau_{\varepsilon_2}}\right](\alpha^j - 1)c_2 dj \\
= \;& -c_2\sigma_\alpha^2\left[\frac{\partial a_1}{\partial c}\left(\frac{a_1\beta c_2}{(\tau_{\theta_1}+\tau_{\varepsilon_1})}\right) - \frac{\partial a_2}{\partial c}\left(\frac{a_2\beta c_2}{(\tau_{\theta_2}+\tau_{\varepsilon_2})} + \frac{1}{(\tau_{\theta_2}+\tau_{\varepsilon_2})}\right)\right] \\
= \;& \frac{-c_2\sigma_\alpha^2\tau_\lambda}{\Delta}\left[\frac{\partial a_2}{\partial c}\left(\frac{\Delta}{\tau_\lambda\left(\tau_{\theta_2}+\tau_{\varepsilon_2}\right)} - \frac{(a_2)^2\left(\tau_{\varepsilon_1}+\tau_{\theta_1}\right)}{(\tau_{\theta_2}+\tau_{\varepsilon_2})}\right) - \frac{\partial a_1}{\partial c}\left(a_1 a_2\right)\right] \\
= \;& \frac{-c_2\sigma_\alpha^2\tau_\lambda}{\Delta}\left[\frac{\partial a_2}{\partial c}\left(\frac{\left(\tau_{\varepsilon_1}+\tau_{\theta_1}\right) + \tau_\lambda\left(a_1\right)^2}{\tau_\lambda}\right) - \frac{\partial a_1}{\partial c}\left(a_1 a_2\right)\right] \\
= \;& \frac{-c_2\sigma_\alpha^2\tau_\lambda}{\Delta}\left[a_1\left(\frac{\partial a_2}{\partial c}a_1 - \frac{\partial a_1}{\partial c}a_2\right) + \frac{\partial a_2}{\partial c}\frac{\left(\tau_{\varepsilon_1}+\tau_{\theta_1}\right)}{\tau_\lambda}\right] \\
= \;& \frac{-c_2\sigma_\alpha^2\tau_\lambda}{\Delta}\left[\frac{a_1\left(a_1 a_2\right)^2(1+\beta)\beta}{\left[\gamma(\tau_{\varepsilon_1}+\tau_{\theta_1}) + \tau_{\varepsilon_1}\right]\left[\gamma(\tau_{\varepsilon_2}+\tau_{\theta_2}) + \tau_{\varepsilon_2}\right]}\left(\tau_{\varepsilon_2}\tau_{\theta_1} - \tau_{\varepsilon_1}\tau_{\theta_2}\right) + \frac{\partial a_2}{\partial c}\frac{\left(\tau_{\varepsilon_1}+\tau_{\theta_1}\right)}{\tau_\lambda}\right].
\end{aligned}$$

For the first equality we have taken expectations. For the second equality we use coefficients a_1^j and a_2^j from Corollary (9) to show:

$$\begin{aligned}\int_0^1 \left(\gamma - \beta a_1^j\right)(\alpha^j - 1)c_2 dj &= \left[-\frac{\tau_{\varepsilon_1}\beta a_1}{(\tau_{\theta_1} + \tau_{\varepsilon_1})}(c_2)^2 \sigma_\alpha^2\right] \text{and} \\ \int_0^1 \left(\alpha^j\gamma - \beta a_2^j\right)(\alpha^j - 1)c_2 dj &= \left[-\frac{\tau_{\varepsilon_2}\beta a_2}{(\tau_{\theta_2} + \tau_{\varepsilon_2})}(c_2)^2 \sigma_\alpha^2 - \frac{\tau_{\varepsilon_2}}{(\tau_{\theta_2} + \tau_{\varepsilon_2})} c_2 \sigma_\alpha^2\right].\end{aligned}$$

For the third equality we substitute for $c_2 = -(\tau_\lambda a_2(\tau_{\varepsilon_1} + \tau_{\theta_1}))/\beta\Delta$ and simplify in the forth and the fifth equality. Finally, using $\frac{\partial a_2}{\partial c} = (a_2)^2 \frac{\tau_{\varepsilon_2}(1+\beta)}{[\gamma(\tau_{\varepsilon_2}+\tau_{\theta_2})+\tau_{\varepsilon_2}]}$ and $\frac{\partial a_1}{\partial c} = (a_1)^2 \frac{\tau_{\varepsilon_1}(1+\beta)}{[\gamma(\tau_{\varepsilon_1}+\tau_{\theta_1})+\tau_{\varepsilon_1}]}$ gives the expression as stated in the Proposition. ■

Lemma 14 *Let ϕ be continuously differentiable. The FONC to the optimal control problem*

$$\max_{\{c^i\}_{i\in[0,1]}} \int_0^1 \phi\left(i, c^i, c\right) di, \textit{with } c = \int_0^1 c^i di \tag{35}$$

are given by $\frac{\partial \phi\left(i,c^i,c\right)}{\partial c^i} + \int_0^1 \frac{\partial \phi\left(j,c^j,c\right)}{\partial c} dj = 0 \quad \forall i.$

Proof. Suppose that $\{c^i\}_{i\in[0,1]}$ is an optimal solution and consider the perturbed solution $\widetilde{c}^i = c^i + \epsilon\, q^i$ and $\widetilde{c} = \int_0^1 \widetilde{c}^i di = c + \epsilon \int_0^1 q^i di$. Let $V(\epsilon) = \int_0^1 \phi\left(i, \widetilde{c}^i, \widetilde{c}\right) di$, with $\widetilde{c} = \int_0^1 \widetilde{c}^i di$. Since c^i is optimal by assumption, a necessary condition is that $V'(0) = 0$.

$V'(0) = \int_0^1 \left\{\frac{\partial \phi}{\partial c^i}\left(i, c^i, c\right) q^i + \frac{\partial \phi}{\partial c}\left(i, c^i, c\right)\left[\int_0^1 q^i di\right]\right\} di =$
$\int_0^1 \frac{\partial \phi}{\partial c^i}\left(i, c^i, c\right) q^i di + \left[\int_0^1 q^i di\right]\left[\int_0^1 \frac{\partial \phi}{\partial c}\left(i, c^i, c\right) di\right] =$
$\int_0^1 \left\{\frac{\partial \phi}{\partial c^i}\left(i, c^i, c\right) + \left[\int_0^1 \frac{\partial \phi}{\partial c}\left(i, c^i, c\right) di\right]\right\} q^i di = 0.$

Given that the assignment q^i is arbitrary we conclude that to satisfy the above equation, the term inside the bracket has to be equal to zero for all i :

$$\frac{\partial \phi}{\partial c^i}\left(i, c^i, c\right) + \left[\int_0^1 \frac{\partial \phi}{\partial c}\left(i, c^i, c\right) di\right] = 0 \qquad \forall i$$

■