Self-Selection Consistent Functions*

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Abstract

This paper studies collective choice rules whose outcomes consist of a collection of simultaneous decisions, each one of which is the only concern of some group of individuals in society. The need for such rules arises in different contexts, including the establishment of jurisdictions, the location of multiple public facilities, or the election of representative committees. We define a notion of allocation consistency requiring that each partial aspect of the global decision taken by society as a whole should be ratified by the group of agents who are directly concerned with this particular aspect. We investigate the possibility of designing envy-free allocation consistent rules, we also explore whether such rules may also respect the Condorcet criterion.

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1. Introduction.

Collective choices often involve multiple simultaneous decisions, whose particular aspects may affect different agents to different degrees. If new borders are drawn in a region of the world, I am mainly affected by what my country will look like, although I may also care about the whole map of the region. If a committee is chosen to negotiate on behalf of my union, I am especially interested on those delegates that I am acquainted with, and/or who will more closely represent my interest.

In this paper we concentrate on polar cases, where each agent is solely concerned with one of the components of the global decision, and congestion effects are ignored. For example, several hospitals may be built simultaneously, but if each agent is only allowed to use one of them (and congestion levels are similar), then he will essentially evaluate the overall decision in terms of the particular hospital he is assigned to. Under these circumstances, we discuss the merits of different social choice procedures to determine (1) what set of objects should be chosen, and (2) which agents should benefit from each of the objects. Each procedure is defined to give answers to both questions, for each possible set of agents and for any number of objects to be chosen.

Since we consider that agents are assigned to specific objects, and that they only care about them, an interesting question arises regarding the overall consistency of the collective procedure. Once a decision is taken, all agents who share the same object emerge naturally as a meaningful group. All those citizens of a new nation after border redrawing, all the trade union members whose opinion will be channeled by a given representative, all users of a new hospital are concerned about the same aspects of the global decision. What if they challenge the global decision by suggesting that, as far as they are concerned, the particular object that they have been assigned to should be changed for another one? What if all people who, given the public decision to build hospitals H_1, H_2, are assigned to H_1, then demand that H_1' be built instead? What if, after talking to one delegation member, the agents he is supposed to represent meet and vote in favor of substituting him for somebody else? In all these cases, there would be some inconsistency between a global decision which turns a group into the major beneficiary of one of its aspects, and the partial decision that these same concerned agents would suggest, regarding this particular aspect. This would be a definite problem if one of these groups is not only interested in changing the partial decision, but is actually entitled to do so, because the rule assigns a different single decision to the group when only one has to be chosen. Social choice procedures which avoid these problems will be called *allocation consistent*.

Many authors have been concerned about the connections among different decisions taken by societies when their members or their resources vary. Different conditions have been imposed requiring that the changes in the social decision associated with changes in the membership of society, or with changes in the set of possible outcomes, respect some notion of consistency (see for example Thomson (1998) for a survey on consistency).

Our concern can also be viewed as one of consistency, but we must qualify the analogy. We want to emphasize the fact that our consistency requirement refers to the connections between global choices and their particular aspects: on this account, our focus is restricted, since we only consider models where this distinction makes sense. A second difference is, we believe, in favor of our notion. We do not look at exogenous changes in the membership of society, which may or may not be reasonably expected. We concentrate on the connections between global decisions, taken by the society at large, and their partial components, as viewed by those agents in the very same society who are affected and concerned by those partial aspects of the decision. For those problems where the structure of the global decision is naturally decomposable, and agents are particularly concerned with only parts of the global picture, we find our notion of allocation consistency to be particularly attractive.

In this paper, we focus on rules which are allocation consistent, but also satisfy a condition of *no-envy.* In our context, no-envy is a normative property requiring that the assignment of agents to objects should be compatible with the will of agents. Allocation rules which are allocation consistent and envy-free are called *self-selection consistent,* to emphasize the idea that the concerned agents self-select themselves to play this role, through their voluntary identification with one of the projects, hospitals, representatives or nations.

In order to discuss these general issues in a specific context, we concentrate on a particular setup. Formally, we study the problem of choosing k objects on the real line, when the preferences of agents over single objects are single peaked.

We first prove that self-selection consistent procedures can be defined. In fact, we present a systematic procedure to create them, starting from simple rules which only determine a single choice for each group. We show that if the simple rule satisfies a condition of participation, then a self-selection consistent rule can be defined for any set of agents and any number of partial decisions. Moreover, this general rule will recommend the same partial decisions that would be made, under the initial simple rule, by each of the groups that self selects themselves as interested in each one of them, if the group alone was called to make a one-element decision. In this sense, the definition of the general rule can be interpreted as an extension of the simple one, and our result is that we can extend any simple rule satisfying participation to create self-selection consistent general rules. We also offer a partial converse of the preceding result. Start from a self selection consistent general rule and consider its restriction to the cases where groups are only called to make one simple decision. If the general rule satisfies efficiency and an additional condition of simplicity, then its associated simple rule must satisfy participation. Finally, we investigate the extent to which self-selection consistent rules can be made compatible with the Condorcet principle, a requirement which is often considered a must for a collective choice rule to be well behaved. We show that there is no problem in accomodating the Condorcet principle for simple decisions, involving the choice of one object only. But we also prove that no general rule can be self-selection consistent and respect the Condorcet principle, when the rule is a function defined for all sets of voters and any number of simultaneous decisions. This unwelcome impossibility result must be qualified: it could be overcome if we allowed our rules to be somewhat undecisive, by considering correspondences instead of functions.

Setups where several points are chosen on a line and agents cluster around these points arise naturally in different parts of the econommic literature and they admit several interpretations. They provide the basic model for the analysis of local public goods and jurisdictional questions (see Alesina and Spolaore (1997), Greenberg and Weber (1993), Jehiel and Schotchmer (1997), Konishi et al.,(1998), Milchtaich and Winter (1998), Tiebout (1956)). Our specific model is particularly explicit about the connections between the global decision of the whole group and the partial decisions of its different subgroups; our main focus is on allocation consistency. This is done at some cost. We explicitly rule out congestion effects, which are important in many contexts. We also take the number of objects to be chosen as an exogenous parameter (in contrast with models where the number of jurisdictions is an endogenous variable). These two restrictive features of our model are borrowed from a series of recent papers by Miyagawa (1997). His model is very similar to ours, but we have expanded it to encompass the possibility of a variable electorate to choose a variable number of objects. This extension allows us to stress the issue of consistency and the endogenous character of the groups that share each single object. Even if our models are similar, Miyagawa's analysis and conclusions are very different from ours. His choice of axioms leads him to characterize different rules which tend to select rather extreme outcomes. Moreover, his formal analysis often stops at the case where only two objects are chosen. In contrast, our analysis applies to any fixed number of partial choices.

The rest of the paper is organized as follows. In Section 2, we present our model in detail. Section 3 presents the main results of the paper regarding existence and characterization of self-selection consistent rules. Section 4 studies the existence of self-selection consistent rules respecting the Condorcet criterion.

2. The Model.

We consider problems that involve any finite set of agents. Agents are identified with elements in $\mathbb{N}$, the set of natural numbers. Let $\mathfrak{S}$ be the class of all finite subsets of $\mathbb{N}$. Elements of $\mathfrak{S}$, denoted as $S, S', ...$, stand for particular societies, whose cardinality is denoted by $|S|$, $|S'|$, etc.

We now describe the decisions that societies can face. These are determined by the number and the position of relevant locations, and by the sets of agents who are allocated to each location.

A natural number $k \in \mathbb{N}$ will stand for the number of locations. Then, given $S \in \mathfrak{S}$ and $k \in \mathbb{N}$, an $S/k-$*decision* is a $k-$tuple of pairs $d = (x_h, S_h)_{h=1}^{k}$, where $x_h \in \mathbb{R}$, and $(S_1, .., S_k)$ is a partition of S. We interpret each x_h as a location and S_h as the set of agents who is assigned to the location $x_{h.}$ Notice that elements in the partition may be empty. This will be the case, necessarily, if $k > |S|$. We call $d_L = (x_1, .., x_k)$ the vector of *locations*, and $d_A = (S_1, .., S_k)$ the vector of *assignments.*

Let $D(S,k)$ be the set of $S/k-$decisions, $D(k) = \bigcup_{S \in \mathfrak{S}} D(S,k)$ the set of $k-$decisions, and $D = \bigcup\limits_{k \in \mathbb{N}} D(k)$ the set of decisions. For each agent $j \in N$, the set of $k-$decisions which concern j is $D_j(k) = \bigcup_{\{S \in \mathfrak{S} | j \in S\}} D(S,k)$ and the set of decisions that concern j is $D_j = \bigcup\limits_{k \in \mathbb{N}} D_j(k)$.

Agents are assumed to have complete, reflexive, transitive preferences over decisions which concern them. That is, *agent i's preferences* are defined on D_i, and thus, rank any pair of S/k and $S'/k'-$decisions provided that $i \in S \cap S'$. Denote by $\succcurlyeq_i$ the preferences of agent i on D_i.

We shall assume all along that preferences are *singleton-based.* Informally, this means that agents' rankings of decisions only depend on the location they are assigned to, not on the rest of locations or on the assignment of other agents to locations. This assumption is compatible with our interpretation that agents can only use the good provided at one location, and that this is a public good subject to no congestion. Formally, a preference $\succcurlyeq_i$ on D_i is singleton-based if there is a preference $\bar{\succcurlyeq}_i$ on $\mathbb{R}$ such that for all d, $d' \in D_i$, $d \succcurlyeq_i d'$ if and only if $x(i,d) \bar{\succcurlyeq}_i x(i,d')$, where $x(i,d)$ denotes the location to which agent i is assigned under the decision d.

In all that follows, we shall assume that for all $i \in N$, $\succcurlyeq_i$ is singleton-based, and in addition, that the order $\bar{\succcurlyeq}_i$ is *single-peaked.* That is: for each $\bar{\succcurlyeq}_i$, there is an alternative $p(\ i)$ which is the unique best element for $\bar{\succcurlyeq}_i$; moreover, for all x, y, if $p(\ i) \geq x > y$, then $x \ \bar{\succ}_i y$, and if $y > x \geq p(\ i)$, then $x \ \bar{\succ}_i y$. Abusing notation we will use the same symbol $\succcurlyeq_i$ for both orders.

Given $S \in \mathfrak{S}$, *preference profiles for* S are $|S|$-tuples of preferences, and we denote them by $P_S, P'_S, ...$

We denote by $\mathcal{P}$ the set of all preferences described above, and by $\mathcal{P}^S$ the set of preference profiles for S satisfying those requirements.

A *collective choice rule* will select a $k-$decision, for each given k, on the basis of the preferences of agents in S, for any $S \in \mathfrak{S}$. Formally,

Definition 1. *A collective choice rule is a function* $\varphi : \bigcup_{S \in \mathfrak{S}} \mathcal{P}^S \times \mathbb{N} \rightarrow D$ *such that, for all* $S \in \mathfrak{S}$, $P_S \in \mathcal{P}^S$ *and* $k \in \mathbb{N}$, $\varphi(P_S, k) \in D(S, k)$.

We now propose two natural and attractive properties that a collective choice rule may or may not satisfy. The first one, efficiency, is well known. The second one, allocation consistency, is proposed here for the first time.

First, we formulate the condition of Pareto efficiency.

Definition 2. *An* $S/k-$*decision* d *is efficient if there is no* $S/k-$*decision* d' *such that* $d' \succcurlyeq_i d$ *for every agent* $i \in S$ *and* $d' \succ_j d$ *for some* $j \in S$.

We propose our notion of *allocation consistency* for collective choice rules.

Definition 3. *A collective choice rule φ is allocation consistent if for all $S \in \mathfrak{S}$, $P_S \in \mathcal{P}^S$ and $k \in \mathbb{N}$, if $((x_1, S_1), ..., (x_k, S_k)) = \varphi(P_S, k)$, then $(x_h, S_h) = \varphi(P_{S_h}, 1)$ for all h such that $S_h \neq \emptyset$.*[1]

Next we give some examples of allocation consistent choice rules.

Example 1. *Consider the natural order. For any given finite set of agents $S \in \mathfrak{S}$, consider the restricted order on this set. Given $S \in \mathfrak{S}$, $P_S \in \mathcal{P}^S$ and $k \in \mathbb{N}$, let $\varphi(P_S, k) = (x_h, S_h)_{h=1}^k$ be such that $x_h = p(h)$ for all $h \in \{1, .., k\}$, $S_h = \{h\}$ for all $h \in \{1, .., k-1\}$, and S_k is the set of the remaining agents, that is, $S_k = S \setminus \cup_{h=1}^{k-1} S_h$. Clearly, this rule is allocation consistent.*

Example 2. *Consider the natural order. For any given finite set of agents $S \in \mathfrak{S}$, consider the restricted order on this set. Given $S \in \mathfrak{S}$, $P_S \in \mathcal{P}^S$ and $k \in \mathbb{N}$, let $\varphi(P_S, k) = (x_h, S_h)_{h=1}^k$ be such that $x_h = p(h)$, $S_h = \{i \in S \mid x_h \succcurlyeq_i x_t$ for all $t \in \{h+1, .., k\}$, and $x_h \succ_i x_l$ for all $l \in \{1, .., h-1\}\}$ for all $h \in \{1, .., k\}$. This rule is also allocation consistent.*

The difference between the above two examples lies on the assignment of the agents. In the second example the assignment of agents to objects is compatible with the will of agents. They are assigned to the location they most prefer, so the assignment does not generate *envy.*

We now proceed to define the notion of *no-envy,* and to study it compatibility with other desirable futures of the collective choice rules.

[1] Since for $k = 1$ there is a unique assignment of the agents (all of them together), we will often abuse notation and write $x_h = \varphi(P_{S_h}, 1)$ when $(x_h, S_h) = \varphi(P_{S_h}, 1)$.

Definition 4. *An $S/k-$decision $d \in D(S,k)$ is envy-free if for all $i \in S$, $x(i,d) \succcurlyeq_i x_h$ for all $x_h \in d_L$.*

Remark 1. *Notice that any efficient $S/k-$decision is envy-free.*

Definition 5. *A collective choice rule φ is envy-free if for all $S \in \mathfrak{S}$, $P_S \in \mathcal{P}^S$ and $k \in \mathbb{N}$, $\varphi(P_S,k)$ selects an envy-free $S/k-$decision.*

We now present the definition which appears in the title of the paper.

Definition 6. *A collective choice rule is self-selection consistent if it is envy-free and satisfies allocation consistency. That is, for all $S \in \mathfrak{S}$, $P_S \in \mathcal{P}^S$ and $k \in \mathbb{N}$, $\varphi(P_S,k) \in \{d = (x_h, S_h)_{h=1}^k \in D(S,k) \mid d$ is envy-free and $x_h = \varphi(P_{S_h}, 1)$ for all h s.t. $S_h \neq \emptyset\}$.*

We emphasize the conjunction of the two properties which give rise to self-selection consistency, because the groups of agents whose partial decisions must match with the global decision are self selected as the set of people who would attach themselves to each location, out of a voluntary choice.

3. MAIN RESULTS

Examples in the preceding Section tell us that there exist self-selection consistent rules (Example 2), and also that some allocation consistent rules may fail to be self-selection consistent (Example 1) because they might generate envy.

In order to understand the structure of self-selection consistent rules, it is crucial to relate them to simple functions, of the form $f : \bigcup_{S \in \mathfrak{S}} \mathcal{P}^S \to \mathbb{R}$, which can be interpreted to determine a single location for each preference profile.

We can show that any such function f satisfying efficiency and a condition which we call *participation* can be used to define a self-selection consistent rule φ.

This is the object of Proposition 1, which thus answers the question of existence of self-selection consistent rules in the positive and by way of a constructive proof. In fact, the function φ constructed with the use of f will be such that, for all $S \in \mathfrak{S}$ and all $P_S \in \mathcal{P}^S$, $f(P_S) = \varphi(P_S, 1)$.

We now define *participation* and state the announced result.

Definition 7. *A function $f : \bigcup_{S \in \mathfrak{S}} \mathcal{P}^S \to \mathbb{R}$ satisfies participation if for all $S \in \mathfrak{S}$, all $i \notin S$, and for all $(P_S, P_{\{i\}}) \in \mathcal{P}^{S \cup \{i\}}$, $f(P_S, P_{\{i\}}) \succcurlyeq_i f(P_S)$.*

Notice that many functions defined on single-peaked preferences satisfy participation: selections from the median of the peaks, the mean of the peaks, the minimum or the maximum peak, etc...

Proposition 1. *Let $f : \bigcup_{S \in \mathfrak{S}} \mathcal{P}^S \to \mathbb{R}$ be a function which satisfies participation and is Pareto efficient. Then, for all $S \in \mathfrak{S}$, $P_S \in \mathcal{P}^S$ and $k \in \mathbb{N}$, any function φ which selects $\varphi(P_S, k)$ from the set $\{d = (x_h, S_h)_{h=1}^k \in D(S, k) \mid d$ is envy-free and $x_h \in f(P_{S_h})$ for all h s.t. $S_h \neq \emptyset\}$ and such that $\varphi(P_S, 1) = f(P_S)$ is a self-selection consistent choice rule.*

Given the construction, all we have to show is that the functions described are well defined, which is equivalent to prove that the set $\{d = (x_h, S_h)_{h=1}^k \in D(S, k) \mid$ d is envy-free and $x_h \in f(P_{S_h})$ for all h s.t. $S_h \neq \emptyset\}$ is not empty for all $S \in \mathfrak{S}$, $P_S \in \mathcal{P}^S$ and $k \in \mathbb{N}$. We will prove that constructively, by describing a procedure which always converges and finds decisions which belong to the desirable set.

The following Lemmas are useful to show that our construction works.

Lemma 1. *Let $f : \bigcup_{S \in \mathfrak{S}} \mathcal{P}^S \to \mathbb{R}$ satisfy participation. Given $S, S' \in \mathfrak{S}$ with $S \cap S' = \emptyset$, let $P_S \in \mathcal{P}^S$, $P_{S'} \in \mathcal{P}^{S'}$ and $y \in \mathbb{R}$ be such that for all $i \in S'$, $f(P_S) \leq p(i) < y$, and $f(P_S) \succcurlyeq_i y$. Then $f(P_S, P_{S'}) \succcurlyeq_i y$ for all $i \in S'$. Furthermore, $f(P_S) \leq f(P_S, P_{S'}) < y$.*

In words, Lemma 1 states that if a group of agents S', whose peaks are to the right of the location chosen by f for the group S, joins the group S, then the choice of f for the larger group will be to the right of the choice of f for S.

Proof. Let $x = f(P_S)$. Since for all $i \in S'$, $x \leq p(i) < y$, and $x \succcurlyeq_i y$, any location in $[x, y)$ will be preferred to y by any agent in S'. Let's see that $f(P_S, P_{S'}) \in [x, y)$. For each agent $i \in S'$, let $x_i \in \mathbb{R}$ be such that agent i is indifferent between x and x_i. Notice that for all $i \in S'$, $x_i < y$. Then, let us order the agents in S' by increasing order of their x_i, and suppose, without loss of generality, that $S' = \{1, 2, .., m\}$. Take the first agent, and consider the set of agents $S \cup \{1\}$. Since f satisfies participation, $f(P_S, P_{\{1\}}) \in [x, x_1]$. Let $z_1 = f(P_S, P_{\{1\}})$. Notice that $z_1 \in [x, x_2]$. Let $z_2 = f(P_S, P_{\{1,2\}})$. Because of participation $z_2 \succcurlyeq_2 z_1$, and therefore $z_2 \in [x, x_2]$. If we keep adding agents of S' to S in this order, we will finally get that $z_m = f(P_S, P_{S'}) \in [x, x_m]$. ■

Lemma 2. *Let $f : \bigcup_{S \in \mathfrak{S}} \mathcal{P}^S \to \mathbb{R}$ be Pareto efficient and satisfy participation. Given $S \in \mathfrak{S}$, $P_S \in \mathcal{P}^S$, and $y \in \mathbb{R}$, $y < f(P_S)$, let $S' \subset S$ be such that for all $i \in S'$ $y \succcurlyeq_i f(P_S)$. Then, $f(P_S) \leq f(P_{S \setminus S'})$.*

In a similar vein than Lemma 1, Lemma 2 now states that, if a subset S' of S is removed from S, and all agents in S' had their peaks to the left of $f(P_S)$, then $f(P_{S \setminus S'})$ will move to the right of $f(P_S)$.

Proof. Let $x = f(P_S)$. For each $i \in S'$, let $x_i \in \mathbb{R}$ be such that agent i is indifferent between x and x_i. Notice that for all $i \in S'$, $x_i \leq y < x$. Let us order the agents in S' by increasing order of x_i, and suppose, without loss of generality, that $S' = \{1, 2, .., m\}$. Notice that $[x_m, x] \subseteq [x_{m-1}, x] \subseteq ... \subseteq [x_1, x]$. Let $z_1 = f(P_{S \setminus \{1\}})$. By participation, $x \succcurlyeq_1 z_1$, which implies that $z_1 \notin (x_1, x)$. By efficiency, $z_1 \geq x$. Notice that all the agents in $S' \setminus \{1\}$ prefer y to z_1. Let $z_2 = f(P_{S \setminus \{1,2\}})$. By participation, $z_1 \succcurlyeq_2 z_2$, which implies that $z_2 \notin (x_2, x)$. Then, by efficiency,

$z_2 \geq x$. Following this process, we will finally get that, because participation, agent m prefers z_{m-1} to $z_m = f(P_{S \setminus S'})$, which implies that $z_m \notin (x_m, x)$, and this in turn implies that $z_m \geq x$ by efficiency. ■

We can now complete the proof of Proposition 1.

Proof of Proposition 1. Let $S \in \mathfrak{S}$, $P_S \in \mathcal{P}^S$ and $k \in \mathbb{N}$. If there are at most k different peaks, we are done. Hence, suppose that there are at least k different peaks. Let us order the agents by increasing order of their peaks. Let $S_h^1 = \{i \in S \mid p(i) = p(h)\}$ for all $h \in \{1,..,k-1\}$, and $S_k^1 = \{i \in S \mid p(i) > p(k-1)\}$. Let $x^1 = (x_h^1)_{h=1}^k$ be such that $x_h^1 = f(P_{S_h^1})$, for $h \in \{1,..,k\}$. Notice that, since f is efficient, $x_h^1 = p(h)$ for all $h \in \{1,..,k-1\}$. If the decision $(x_h^1, S_h^1)_{h=1}^k$ is envy-free, we are done. Otherwise, let $S_{k-1} = S_{k-1}^1 \cup \{i \in S_k^1 \mid x_{k-1}^1 \succcurlyeq_i x_k^1\}$. By Lemma 1, $x_{k-1}^1 \leq f(P_{S_{k-1}}) < x_k^1$. Notice that we can have agents in S_k^1 which prefer x_k^1 to x_{k-1}^1, but they have their peaks between x_{k-1}^1 and $f(P_{S_{k-1}})$, which implies that they prefer $f(P_{S_{k-1}})$ to x_k^1. Let $S_k' \subset S_k^1$ be the set of those agents. For each $i \in S_k'$, let z_i be such that agent i is indifferent between x_k^1 and z_i. Let us order the agents in S_k' by increasing order of z_i. Then $[z_{i+1}, x_k^1] \subseteq [z_i, x_k^1]$. Take the first agent in this set. By participation $f(P_{S_{k-1}\cup\{1\}}) \succcurlyeq_1 f(P_{S_{k-1}})$, and since $f(P_{S_{k-1}}) \succcurlyeq_1 x_k^1$, $f(P_{S_{k-1}\cup\{1\}}) \in [z_1, x_k^1]$. If $f(P_{S_{k-1}\cup\{1\}}) \in [z_2, x_k^1]$, add agent 2 to $S_{k-1} \cup \{1\}$, and again by participation $f(P_{S_{k-1}\cup\{1,2\}}) \succcurlyeq_2 f(P_{S_{k-1}\cup\{1\}})$, which will imply that $f(P_{S_{k-1}\cup\{1,2\}}) \in [z_2, x_k^1]$. We keep adding agents from S_k' in the above defined order whenever $f(P_{S_{k-1}\cup\{1,2,..,i\}}) \in [z_{i+1}, x_2^1]$. Let $S_k'^1$ be this subset of agents. Then, for all $i \in S_k' \setminus S_k'^1$, $f(P_{S_{k-1}\cup S_k'^1}) \notin [z_i, x_k^1]$. Notice that all agents in $S_k' \setminus S_k'^1$ have their peaks between $f(P_{S_{k-1}\cup S_k'^1})$ and x_k^1. Once this process is completed, consider the following sets of agents: $S_{k-1}^2 = S_{k-1} \cup S_k'^1$, $S_k^2 = S_k^1 \setminus \{\{i \in S_k^1 \mid x_{k-1}^1 \succcurlyeq_i x_k^1\} \cup S_k'^1\}$, and $S_h^2 = S_h^1$ for all $h \in \{1,..,k-2\}$. Let $x_h^2 = f(P_{S_h^2})$ for all $h \in \{1,..,k\}$. We have shown that $x_{k-1}^2 \geq x_{k-1}^1$, and by

Lemma 2, $x_k^2 \geq x_k^1$. If the decision $(x_h^2, S_h^2)_{h=1}^k$ is envy-free, we are done. If it is not we repeat the process. Notice that, if in step j we do not get an envy-free decision, it is because some agents from S_h^j prefer what agents in S_{h-1}^j have gotten. Thus, if this is the case, in each step we add agents from S_h^j to S_{h-1}^j in the way as described above. The process will end in a finite number of steps because there are a finite number of agents, because by Lemma 1 and Lemma 2 at each step $x_h^j \geq x_h^{j-1}$ for all $h \in \{1, ..k\}$ and furthermore, $S_1^{j-1} \subseteq S_1^j$ and $S_k^j \subseteq S_k^{j-1}$. ■

We have seen how any function $f : \bigcup_{S \in \mathfrak{S}} \mathcal{P}^S \rightarrow \mathbb{R}$ satisfying efficiency and participation can be used to induce a self-selection consistent choice rule φ. We now turn to the opposite question. Given a rule φ, its restriction to $k = 1$ is indeed a function of the above type. What can we say about its properties, if we know that φ is a self-selection consistent choice rule? We can not give a complete answer to this question, but we can provide a definite answer if, in addition, φ satisfies two conditions (continuity and peaks only) which actually only refer to $\varphi(., 1)$.

We first express these two conditions and then state the result.

Definition 8. *A rule* $f : \bigcup_{S \in \mathfrak{S}} \mathcal{P}^S \rightarrow \mathbb{R}$ *is peaks-only if for all* $S \in \mathfrak{S}$, *and* $P_S, P'_S \in \mathcal{P}^S$ *such that* $p(i) = p'(i)$ *for all* $i \in S$, *then* $f(P_S) = f(P'_S)$.

Definition 9. *A rule* $f : \bigcup_{S \in \mathfrak{S}} \mathcal{P}^S \rightarrow \mathbb{R}$ *satisfies continuity if given* $S, S' \in \mathfrak{S}$, $P_S \in \mathcal{P}^S$, $P_{S'} \in \mathcal{P}^{S'}$ *such that* $f(P_S) = x$, *and* $f(P_{S'}) = y$, *there is* $n \in \mathbb{N}$ *such that* $f(P_S, ...^n...., P_S, P_{S'}) = x$.

Continuity means that if x is chosen by a group S and y by a group S', we can replicate the group S sufficiently many times such that the union of all these replicas together with S' choose x.[2]

Proposition 2. *Let* $\varphi : \bigcup_{S \in \mathfrak{S}} \mathcal{P}^S \times \mathbb{N} \to D$ *be an efficient self-selection consistent choice rule such that* $\varphi(.,1)$ *is peaks-only and satisfies continuity. Then,* $\varphi(.,1)$ *satisfies participation.*

Proof. Suppose that φ does not satisfy participation. Then there is a group of agents $S \in \mathfrak{S}$, an agent $i \notin S$, and a preference profile $(P_S, P_{\{i\}}) \in \mathcal{P}^{S \cup \{i\}}$ such that $\varphi(P_S, 1) \succ_i \varphi(P_S, P_{\{i\}}, 1)$. Let us order the agents in $S \cup \{i\}$ by increasing order of their peaks. Suppose, without loss of generality that $S \cup \{i\} = \{1, 2, .., m\}$ and $\varphi(P_S, 1) < \varphi(P_S, P_{\{i\}}, 1)$. Let $x \in \mathbb{R}$ be such that $x < \varphi(P_S, 1)$ and every agent in S prefers anything in the interval $[p(1), p(m)]$ to x. Let $P'_{\{i\}} \in \mathcal{P}^{\{i\}}$ be such that $p'(i) = p(i)$, and $\varphi(P_S, 1) \succ'_i x \succ'_i \varphi(P_S, P'_{\{i\}}, 1)$. Since $\varphi(.,1)$ is peaks-only, $\varphi(P_S, P'_{\{i\}}, 1) = \varphi(P_S, P_{\{i\}}, 1)$. Consider an agent j different from i and not in S with $P_{\{j\}} \in \mathcal{P}^{\{j\}}$ such that $p(j) = x$. By efficiency, $\varphi(P_{\{j\}}, 1) = x$. Let's consider as many replicas of j as necessary, say n, to guarantee that for any possible subgroup $S' \subseteq S \cup \{i\}$, $\varphi(P_{S'}, P_{\{j\}}, \overset{n}{\ldots}, P_{\{j\}}, 1) = x$. By continuity this is always possible. Let $((x_1, S_1), (x_2, S_2)) = \varphi(P_S, P'_{\{i\}}, P_{\{j\}}, \overset{n}{\ldots}, P_{\{j\}}, 2)$. By efficiency $x_1 \neq x_2$. Let's see that all the replicas of j are in the same group. Suppose this is not the case, then, by no-envy x_1 should be indifferent to x_2 for all agents identical to agent j. Since $x_1 \neq x_2$, $x_1 < x_2$, or $x_2 < x_1$. Suppose that $x_1 < x_2$, and therefore, $x_1 < x < x_2$. By allocation consistency, $x_1 = \varphi(P_{S_1}, 1)$. But the preference profile is constructed in such a way that for all agent $k \in S_1$, $p(k) \geq x$, and so, by efficiency $\varphi(P_{S_1}, 1) \in [x, \max_{k \in S_1} p(k)]$, which contradicts the assumption that $x_1 < x$. Consequently, all replicas of j should be in the same

[2] This definition is borrowed from the social choice literature. See Moulin (1988)

group, say S_1, and since φ satisfies allocation consistency, $x_1 = x$. Let's now see that in S_1 there is no agent except for the replicas of agent j. By efficiency, $\varphi(P_{S_2}, 1) \in [p(1), p(m)]$. But any point in this interval is preferred for all agents in S to x. Thus, by no-envy, $S_1 \cap S = \emptyset$. Suppose that $i \in S_1$, then $\varphi(P_{S_2}, 1) = \varphi(P_S, 1)$, but then agent i will envy the agents in S. Therefore $S_2 = S \cup \{i\}$, and S_1 is just the set of replicas of j. By allocation consistency, $x_2 = \varphi(P_S, P'_{\{i\}}, 1)$, but them the decision $((x_1, S_1), (x_2, S_2))$ is not envy-free, because agent i prefers x to x_2, which contradicts the assumption that φ is self-selection consistent. Therefore $\varphi(., 1)$ satisfies participation. ■

4. Self–Selection Consistent rules and the Condorcet Principle.

We now want to investigate whether it is possible to design self selection consistent collective choice rules which respect the Condorcet criterion. For a given preference profile, each set S of agents and each given k, an $S/k-$decision is a Condorcet winner if no other $S/k-$decision would be prefered to it by a majority of agents. The Condorcet criterion requires that, whenever Condorcet winners exist in the set of S/k-decisions one of them should be chosen. Because our rules must be defined for all profiles, all S's and all k's, the Condorcet criterion imposes a multiplicity of requirenments at each profile. We provide a negative result, establishing that no self-selection consistent collective choice function can satisfy the Condorcet criterion (for all S and k). Our negative result can be somewhat qualified in two directions. One by observing that it is not hard to find self selection consistent rules which respect the Condorcet criterion for all $S's$, when $k = 1$. Another qualification is that the negative result comes from our strict requirement that our collective choice rules should be functions. In a companion paper, we study the added flexibility to be gained by considering multivalued social choice rules.

We now present the formal definitions and results.

Definition 10. *An $S/k-$decision $d \in D(S,k)$ is a Condorcet winner for S if*

$$\left|\{i \in S \mid d \succ_i d'\}\right| \geq \left|\{i \in S \mid d' \succ_i d\}\right| \text{ for all } d' \in D(S,k)$$

Given $S \in \mathfrak{S}$, $P_S \in \mathcal{P}^S$ and $k \in \mathbb{N}$, let $CW(P_S,k)$ be the set of $S/k-$decisions that are Condorcet winners for S.

Notice that any $S/k-$decision that is a Condorcet winner for S is an efficient decision.

Definition 11. *A collective choice function φ respects the Condorcet criterion if for all $S \in \mathfrak{S}$, $P_S \in \mathcal{P}^S$ and $k \in \mathbb{N}$ such that $CW(P_S,k) \neq \emptyset$, $\varphi(P_S,k) \in CW(P_S,k)$.*

We first show that, whenever a Condorcet winner exists for $k > 1$, each of its components is a Condorcet winner for its corresponding group.

Proposition 3. *Given $S \in \mathfrak{S}$, $P_S \in \mathcal{P}^S$ and $k \in \mathbb{N}$, if an $S/k-$decision $d = ((x_h, S_h))_{h=1}^k \in CW(P_S,k)$, then $d_h = (x_h, S_h) \in CW(P_{S_h}, 1)$ for all h such that $S_h \neq \emptyset$.*

Proof. Suppose that there is an h such that $d_h = (x_h, S_h)$ is not a Condorcet winner for S_h. Then there is a $d'_h = (x'_h, S_h)$, such that

$$\left|\{i \in S_h \mid d'_h \succ_i d_h\}\right| > \left|\{i \in S_h \mid d_h \succ_i d'_h\}\right|$$

Let $S_{h1} = \{i \in S_h \mid d'_h \succ_i d_h\}$, $S_{h2} = \{i \in S_h \mid d_h \succ_i d'_h\}$, and $d' = (d'_h, d_{-h})$. Then, $\{i \in S \mid d' \succ_i d\} = S_{h1}$, and $\{i \in S \mid d \succ_i d'\} = S_{h2}$. But then,

$|\{i \in S \mid d' \succ_i d\}| > |\{i \in S \mid d \succ_i d'\}|$, which contradicts the assumption that d is a Condorcet winner for S. ■

Rules satisfying the condition of Proposition 3 would partially respect the Condorcet criterion. The reader can chek that it is not hard to find self-selection consistent rules which respect the Condorcet principle to that extend. Unfortunately, our last result shows that a complete respect of the principle will be incompatible with self selection consistency.

Proposition 4. *No self-selection consistent rule respects the Condorcet criterion.*

Proof. It will be sufficient to prove that a violation of the Condorcet criterion must arise for any function at some specific profiles and some $S's$ and k's. Let $S = \{1, 2, ..., 11, 12\}$, and $P_S = (\succcurlyeq_i)_{i=1}^{12}$ be such that for all i, $\succcurlyeq_i$ is euclidean on $\mathbb{R}$ with the following peaks: $p(1) = 0$, $p(2) = 2$, $p(3) = 6$, $p(4) = 8$, $p(5) = 9$, $p(6) = 11$, $p(7) = 15$, $p(8) = 17$, $p(9) = 18$, $p(10) = 20$, $p(11) = 24$, $p(12) = 26$. Consider first the set of agents $S' = \{1, 2, .., 8\}$. First, notice that no efficient $S'/2-$decision $d = (x_h, S'_h)_{h=1}^2$, such that $0 \leq x_1 \leq 2$ and $9 \leq x_2 \leq 11$ can be a Condorcet winner. If $x_2 < 11$, then the decision $((8, \{1, 2, 3, 4, 5\}), (11, \{6, 7, 8\}))$ defeats d, and if $x_2 = 11$, the decision $((9, \{1, 2, 3, 4, 5, 6\}), (15, \{7, 8\}))$ is the one which defeats d . Second, notice that no efficient $S'/2-$decision $d = (x_h, S'_h)_{h=1}^2$, such that $6 \leq x_1 \leq 8$ and $15 \leq x_2 \leq 17$ can be a Condorcet winner. If $x_1 > 6$, then the decision $((6, \{1, 2, 3\}), (9, \{4, 5, 6, 7, 8\}))$ defeats d. If $x_1 = 6$, then the decision $((2, \{1, 2\}, (8, \{3, 4, 5, 6, 7, 8\}))$ beats d. Third, notice that no efficient $S'/2-$decision $d = (x_h, S'_h)_{h=1}^2$, such that $2 \leq x_1 < 5$ and $11 < x_2 \leq 15$ can be a Condorcet winner. If $x_1 = 2$ and $x_2 = 15$, then the decision $((9, \{1, 2, 3, 4, 5, 6\}, (17, \{7, 8\}))$ defeats d. If $x_1 > 2$, the decision $((2, \{1, 2\}),$ $(7, \{3, 4, 5, 6, 7, 8\}))$ defeats d. If $x_2 < 15$, then the decision $((7, \{1, 2, 3, 4, 5, 6\}),$

$(15, \{7, 8\}))$ defeats d. Fourth, notice that no efficient $S'/2-$decision $d = (x_h, S'_h)_{h=1}^2$, such that $2 \leq x_1 < 6$ and $12 < x_2 \leq 15$ can be a Condorcet winner. Since we have seen that $x_1 = 2$ and $x_2 = 15$ can be defeated, let's suppose that $x_1 > 2$. Then the decision $((2, \{1, 2, 3\}), (10, \{4, 5, 6, 7, 8\}))$ defeats d. Alternatively, if $x_2 < 15$, the decision $((10, \{1, 2, 3, 4, 5, 6\}), (15, \{7, 8\}))$ defeats d. Because of Proposition 3 and given that none of the decisions described above can be Condorcet winners, the set of possible candidates is reduced to the set of decisions $d = (x_h, S'_h)_{h=1}^2$ such that $5 \leq x_1 \leq 6$, $S'_1 = \{1, 2, 3, 4\}$ and $11 \leq x_2 \leq 12$, $S'_2 = \{5, 6, 7, 8\}$. In particular notice that the decision $((6, \{1, 2, 3, 4\}), (11, \{5, 6, 7, 8\}))$ is one, but not the only among the Condorcet winners in our set. Now, a self-selection consistent rule which respects the Condorcet criterion should select a Condorcet winner among those identified above, and any such choice should be such that $11 \leq \varphi(P_{S'_2}, 1) \leq 12$. In a completely symmetric way, consider now the set of agents $S'' = \{5, 6, ..., 12\}$. The set of $S''/2-$decisions which are candidates to Condorcet winners is the set of decisions $d = (x_h, S'_h)_{h=1}^2$ such that $14 \leq x_1 \leq 15$, $S''_1 = \{5, 6, 7, 8\}$ and $20 \leq x_2 \leq 21$, $S''_2 = \{9, 10, 11, 12\}$. Thus, a self-selection consistent rule which respects the Condorcet criterion should select a Condorcet winner from that set, and it should be such that $14 \leq \varphi(P_{S''_1}, 1) \leq 15$. But, the set S''_1 is exactly the set S'_2. Therefore $\varphi(P_{S'_2}, 1)$ should be equal to $\varphi(P_{S''_1}, 1)$ which is incompatible with the restriction that $\varphi(P_{S'_2}, 1) \in [11, 12]$ and $\varphi(P_{S''_1}, 1) \in [14, 15]$. Consequently, there is no self-selection consistent rule which respects the Condorcet criterion. ■

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