THE CHI-COMPROMISE VALUE FOR NON–TRANSFERABLE UTILITY GAMES

by

Gustavo Bergantiños[+*] and Jordi Massó[+†]

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Abstract: We introduce and study a compromise value for non-transferable utility games: the Chi-compromise value. It is closely related to the Compromise value introduced by Borm, Keiding, McLean, Oortwijn, and Tijs (1992) and to the MC-value introduced by Otten, Borm, Peleg, and Tijs (1998). The main difference being that the maximal aspiration a player may have in the game is his maximal (among all coalitions) marginal contribution. We show that it is well defined on the class of essential and non-level games.

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[*]Facultade de Ciencias Económicas. Universidade de Vigo. Campus Universitario de Lagoas Marcosende. 36271, Vigo (Pontevedra). Spain. E-mail: gbergant@uvigo.es

[†]Departament d'Economia i d'Història Econòmica and CODE. Universitat Autònoma de Barcelona. 08193, Bellaterra (Barcelona), Spain. E-mail: jordi.masso@uab.es

1 Introduction

The purpose of this paper is to introduce a new compromise value for non-transferable utility games (NTU-games): the Chi-compromise value. As with all compromise values it chooses as the solution of the game the efficient vector lying in the segment between the vectors of maximal and minimal utilities that each player may expect to obtain; that is, it is a compromise between their maximum and minimum aspirations. For pure bargaining problems (that is, situations where all agreements have to be unanimous) the Kalai-Smorodinsky solution (Kalai and Smorodinsky (1975)) is based on a compromise of this type. When partial agreements are possible and utility is transferable across players (that is, TU-games) we defined (Bergantiños and Massó (1996)) a compromise value called the Chi-value. Our proposal here extends these two particular solutions to general problems where players may reach partial agreements and utility is not necessarily transferable (that is, NTU-games).

We propose as the maximum aspiration for a player in a game his maximal (among all coalitions) marginal contribution and as the minimum aspiration the maximum remainder he can obtain by going with a coalition of players and offering them their maximum aspirations. In non-level NTU-games our proposed vectors of aspirations have the following three properties: (1) Giving players their maximum aspirations will always exhaust all possible gains from cooperation. (2) The vector of maximum aspirations is component-wise larger than the vector of minimal aspirations. (3) The minimum aspiration obtained in this rather indirect way coincides with the vector of individually rational payoffs. We find this last property interesting because it means that we have as a result that the minimum aspiration for each player in a game coincides with what they can obtain without any cooperation. It seems to us that this property may also be a good indication that the proposed maximum aspiration is meaningful.

The paper is organized as follows. After a preliminary section which gives the main notation and concepts comes Section 3. This contains the definition of the Chi-compromise value; Propositions 1, 2, and 3 which establish that properties (1), (2), and (3) above hold for non-level NTU-games; the demonstration that the Chi-compromise value exists for all non-level and essential NTU-games; and finally, a number of examples which illustrate the concept. Section 4 provides two characterizations of the Chi-compromise value using the following axioms: Pareto optimality, covariance, symmetry, and restricted monotonicity (or strong symmetry instead of symmetry and restricted monotonicity). Section 5 proposes a different compromise value based on applying our Chi-value for TU-games to the characteristic function obtained by the classical $\lambda-$transfer approach. Section 6 concludes by suggesting (as a generalization of Moulin (1984)'s implementation of the Kalai-Smorodinsky solution for pure bargaining problems) a non-cooperative extensive form game whose subgame perfect equilibrium payoffs coincide with the Chi-compromise value. We also compare, briefly, our value with other well-known NTU-values.

2 Preliminaries

Players are the elements of a finite set $N = \{1, ..., n\}$ where $n \geq 2$. A non-empty subset of players is called a *coalition* . We denote by s the number of players of coalition S and, abusing notation, by i the set $\{i\}$.

A (cooperative) game with *non-transferable utility* (NTU-game) is an ordered pair (N, V) where $N = \{1, ..., n\}$ is the set of *players* and V is a mapping, called the *characteristic function*, which assigns to each non-empty coalition S a non-empty subset of $\mathbb{R}^s$. The set $V(S)$ is interpreted as the collection of payoffs or utilities that members of S can reach by cooperating among themselves. We will concentrate only on games with non-transferable utility having the standard properties that for each coalition S, the set $V(S)$ is closed, non-empty, and comprehensive (*i.e.*, $x \in V(S)$ and $y \leq x$ imply $y \in V(S)$).[1] Also, the set $V(S) \cap \mathbb{R}^s_+$ is bounded and non-empty (where $\mathbb{R}^s_+ = \{x \in \mathbb{R}^s \mid x \geq 0\}$). This last requirement is a payoff normalization and it implies that $0 \in V(S)$ for each coalition. For each player $i \in N$ there exists a payoff $w_i \geq 0$, called the *individually rational payoff*, such that $V(i) = \{x \in \mathbb{R} \mid x \leq w_i\}$. We denote by $\mathbf{V}_n$ the class of games with non-transferable utility with n players.

We will often use the following properties of games with non-transferable utility.

[1] Given $x, y \in \mathbb{R}^k$, $y \leq x$ means $y_i \leq x_i$ for all $i = 1, ..., k$ while $y < x$ means $y_i < x_i$ for all $i = 1, ..., k$. Given $x \in \mathbb{R}^n$ and a coalition S, denote by x_S the restriction of x to the coordinates corresponding to the members of S; *i.e.*, $x_S = (x_i)_{i \in S}$.

Definition 1. *A game* (N,V) *is* **non-level** *if for each coalition* S *we have that for all* $x,y \in V(S) \cap I\!R^s_+$ *such that* $y \geq x \geq w_S$ *and* $x \neq y$ *there exists* $z \in V(S)$ *with the property that* $z > x$.

Definition 2. *A game* (N,V) *is* **essential** *if* $w \in V(N)$.

We denote by $\mathbf{C}_n$ the subclass of non-level and essential games with non-transferable utility.

A *solution* on a subclass of games $\mathbf{G}_n \subseteq \mathbf{V}_n$ is a function $\varphi : \mathbf{G}_n \to I\!R^n$ which assigns a vector $\varphi(N,V) \in V(N)$ to each $(N,V) \in \mathbf{G}_n$.

We will consider, and use as references, two special subclasses of games. A game (N,V) has *transferable utility* if there is a real-valued function v such that $V(S) = \left\{x \in I\!R^s \mid \sum_{i \in S} x_i \leq v(S)\right\}$; namely, each coalition S can achieve a maximum level of utility $v(S)$ which can be distributed amongst its members in all possible ways. We denote by $\mathbf{v}_n$ the subclass of games with transferable utility with n players. A generic game with transferable utility will be denoted by (N,v). A game (N,V) is a *bargaining game* if $w \in V(N)$ and $V(S) = \{x \in I\!R^s \mid x \leq w_S\}$ for every coalition $S \neq N$; namely, gains from cooperation come only from unanimous agreements. We denote by $\mathbf{B}_n$ the subclass of bargaining games with n players. A generic bargaining game will be denoted by (w,B), where B stands for the set $V(N)$ and w represents the disagreement point.

We are specially interested in extending two compromise solutions of these subclasses to games with non-transferable utility. The first one is the Kalai-Smorodinsky solution (Kalai and Smorodinsky (1975)) on bargaining games which represents an efficient compromise between the maximal aspiration of each player, compatible with individual rationality of the others, and the disagreement point. Formally, given $(w,B) \in \mathbf{B}_n$ define the *Kalai-Smorodinsky solution*, denoted by $KS(w,B)$, as follows: for all $i \in N$

$$KS_i(w,B) = \lambda M_i^{KS}(w,B) + (1-\lambda)w_i,$$

where $M_i^{KS}(w,B) = \max\left\{x_i \in I\!R \mid \left(x_i, x_{N\setminus i}\right) \in B \text{ and } \left(x_i, x_{N\setminus i}\right) \geq w\right\}$ and $\lambda \in [0,1]$ is such that $KS(w,B) \in P(B)$, where $P(B)$ denotes the Pareto frontier of B.[2]

[2]In general, given a set $A \subseteq \boldsymbol{R}^k$, the *Pareto frontier of* A is the set $P(A) = \{x \in A | \nexists y \in A$ with the property that $y \geq x, y \neq x\}$ and the *weak Pareto frontier of* A is the set $WP(A) = \{x \in A | \nexists y \in A$ with the property that $y > x\}$. Given a set A and a vector y we say that y is *undominated for* A if $\nexists x \in A$ such that $x \geq y$ and $x \neq y$. Obviously, if $y \in V(S) \setminus P(V(S))$ then y is dominated for $V(S)$.

The second one is the χ-value (Bergantiños and Massó (1996)) on the subclass of games with transferable utility. It is also based on selecting an efficient compromise between maximal and minimal aspirations of players. In this case, the maximal aspiration of a player is his largest marginal contribution while his minimal aspiration is the largest remainder he can obtain after conceding to the other players their maximal aspiration. Formally, let (N, v) be a game with transferable utility. For each $i \in N$, define player i's maximum aspiration in the game as

$$M_i^{\chi}(N, v) = \max_{S \subseteq N, i \in S} \{v(S) - v(S \setminus i)\}.$$

Given the vector $M^{\chi}(N, v)$ define player i's minimum aspiration in the game as

$$m_i^{\chi}(N, v) = \max_{S \subseteq N, i \in S} \left\{ v(S) - \sum_{j \in S \setminus i} M_j^{\chi}(N, v) \right\}.$$

Define the *χ-value* on $\mathbf{v}_n$, denoted by $\chi(N, v)$, as the unique efficient vector in the lineal segment having as extreme points $m^{\chi}(N, v)$ and $M^{\chi}(N, v)$; that is,

$$\chi(N, v) = \gamma M^{\chi}(N, v) + (1 - \gamma) m^{\chi}(N, v),$$

where $\gamma \in [0, 1]$ is such that $\sum_{i \in N} \chi_i(N, v) = v(N)$. Bergantiños and Massó (1996) showed that the χ-value exists in the class of essential games.

3 The Chi-compromise value

In this section we define and study a compromise value for NTU-games. Let (N, V) be a game in $\mathbf{V}_n$. For each $i \in N$ define player i's *maximum aspiration* in the game as

$$M_i^{\chi}(N, V) = \max_{S \subseteq N, i \in S} \left\{ t \in I\!R \mid (t, x) \in V(S) \cap I\!R_+^s, x \in P(V(S \setminus i)) \right\}.$$

Notice that $M_i^{\chi}(N, V) \geq w_i$ (take $S = \{i\}$ and $t = w_i$). We also have that $M_i^{\chi}(N, V) < +\infty$ because $V(S) \cap I\!R_+^s$ is compact and $P(V(S \setminus i))$ is closed. Therefore, $M_i^{\chi}(N, V)$ is well defined for all (N, V) in $\mathbf{V}_n$.

Given the vector $M^\chi(N,V)$ define player i's *minimal aspiration* in the game as

$$m_i^\chi(N,V) = \max_{S\subseteq N, i\in S} \left\{ t \in \mathbb{R} \mid \left(t, M_{S\setminus i}^\chi(N,V)\right) \in V(S) \right\}.$$

Of course $m_i^\chi(N,V) \geq w_i$ (again take $S=\{i\}$ and $t=w_i$). Notice that for each S containing i, the projection of $V(S)$ on i's coordinate is closed and bounded above. Therefore the maximum defining $m_i^\chi(N,V)$ does exist for all (N,V) in $\mathbf{V}_n$.

From now on, and when this does not lead to confusion, we will omit the reference to the game (N,V) to denote the aspiration vectors m^χ and M^χ.

The following propositions state that the three important properties of the vectors of aspirations already explained in the Introduction hold for non-level games. Proposition 1 says that, for every coalition S, the vector of maximum aspirations is undominated for $V(S)$.

Proposition 1. *Let (N,V) be a non-level NTU-game. Then, for all $S \subseteq N$ we have that*

$$M_S^\chi \notin V(S) \setminus P(V(S)).$$

Proof: If S has only one player the result holds. Suppose it is true when S has at most $p-1$ players; we will show that the statement holds in the case of coalitions with p players.

In order to get a contradiction assume that S has p players and $M_S^\chi \in V(S) \setminus P(V(S))$. Then, there exists $y_S \in V(S)$ such that $y_S \geq M_S^\chi$ and $i \in S$ with $y_i > M_i^\chi$. As $M_{S\setminus i}^\chi \notin V(S\setminus i) \setminus P(V(S\setminus i))$ (by the induction hypothesis) and (N,V) is non-level we can find $x_{S\setminus i} \in P(V(S\setminus i))$ such that $x_{S\setminus i} \leq M_{S\setminus i}^\chi$. Then, by comprehensiveness, $\left(y_i, x_{S\setminus i}\right) \in V(S)$ and therefore $M_i^\chi \geq y_i > M_i^\chi$. ■

Proposition 2 says that for non-level games the maximum aspiration is larger or equal to the minimum aspiration.

Proposition 2. *Let (N,V) be a non-level NTU-game. Then,*

$$m^\chi \leq M^\chi.$$

Proof: Let $i \in N$ be an arbitrary player and let $t \in \mathbb{R}$ be such that there exists a coalition $S \subseteq N$ containing i such that $\left(t, M_{S\setminus i}^\chi\right) \in V(S)$. Since

$w \leq m^{\chi}$ and $w \leq M^{\chi}$ we may restrict attention only to t's such that $0 \leq w_i \leq t$. By Proposition 1 we must be able to find $x \in P(V(S \setminus i))$ such that $0 \leq x \leq M^{\chi}_{S \setminus i}$. Therefore, by comprehensiveness of the game,

$$(t, x) \leq \left(t, M^{\chi}_{S \setminus i}\right) \in V(S).$$

Then, we have $(t, x) \in V(S) \cap I\!R^s_+$. Hence,

$$\begin{aligned} m^{\chi}_i &= \max_{S \subseteq N, i \in S} \left\{ t \in I\!R \mid \left(t, M^{\chi}_{S \setminus i}\right) \in V(S) \right\} \\ &\leq \max_{S \subseteq N, i \in S} \left\{ t \in I\!R \mid (t, x) \in V(S) \cap I\!R^s_+ \text{ and } x \in P(V(S \setminus i)) \right\} \\ &= M^{\chi}_i. \quad \blacksquare \end{aligned}$$

Proposition 3 below shows that, for non-level NTU-games, the vector of minimal aspirations coincides, as it should, with the vector of individually rational payoffs. But, again, notice that m^{χ} is obtained endogenously as the maximum reminder after giving to other players in the coalition their maximal aspirations. We interpret this property as an indication that our definition of maximal aspiration is sensible.

Proposition 3. *Let (N, V) be a non-level NTU-game. Then,*

$$m^{\chi} = w.$$

Proof: From the definition of m^{χ}_i it follows that $m^{\chi}_i \geq w_i$ just by taking $S = \{i\}$. To see that $m^{\chi}_i \leq w_i$ it will be sufficient to show that $t \leq w_i$ for all $t \in I\!R$ and all $S \subseteq N$ such that $i \in S$ and $\left(t, M^{\chi}_{S \setminus i}\right) \in V(S)$. The proof is by induction on the number of players in the coalition S.

Assume that $S = \{i, j\}$. If $\left(t, M^{\chi}_j\right) \in V(\{i, j\})$ and $t > w_i$ then, by comprehensiveness of the game, $\left(x, M^{\chi}_j\right) \in V(\{i, j\}) \cap I\!R^2_+$ for all $x \leq t$, which is impossible by non-levelness of the game and the definition of M^{χ}_j.

Assuming that the result is true if S contains $p \geq 2$ players (the induction hypothesis), we will show that it is true for all coalitions with $p+1$ players. Let $S = \{i_1, ..., i_p, i\}$ be any set with $p+1$ players containing i and assume that $\left(t, M^{\chi}_{S \setminus i}\right) \in V(S) \cap I\!R^s_+$. If the following implication is true

$$\left[\left(t, M^{\chi}_{S \setminus i}\right) \in V(S)\right] \Rightarrow \left[\left(t, M^{\chi}_{i_1}, ..., M^{\chi}_{i_{p-1}}\right) \in V(S \setminus i_p)\right]$$

then, $t \leq w_i$ would follow by the induction hypothesis. Therefore, to get a contradiction, assume that $\left(t, M_{i_1}^{\chi}, ..., M_{i_{p-1}}^{\chi}\right) \notin V(S \setminus i_p)$. Then, there exists a vector $x \in P(V(S \setminus i_p))$ such that $x < \left(t, M_{i_1}^{\chi}, ..., M_{i_{p-1}}^{\chi}\right)$. Therefore, $\left(x, M_{i_p}^{\chi}\right) \leq \left(t, M_{S\setminus i}^{\chi}\right) \in V(S)$ implying, by non-levelness of the game, that we can find a vector $y \in V(S)$ with the property that $y > \left(x, M_{i_p}^{\chi}\right)$. Therefore, $y_{i_p} > M_{i_p}^{\chi}$ which contradicts the definition of $M_{i_p}^{\chi}$. ■

Example 1 below shows that the conclusion of Proposition 3 does not hold for level NTU-games.

Example 1. Let (N, V) be the NTU-game where $N = \{1, 2\}$, $w_1 = w_2 = 0$, and $V(N) = \mathit{comp}(\mathit{conv}(\{(1,1), (2,0)\}))$.[3] The vector of maximum aspirations is $M^{\chi}(N, V) = (2, 1)$ and the vector of minimum aspirations is $m^{\chi}(N, V) = (1, 0)$ which for player 1 is strictly larger than $w_1 = 0$.

We can now define the Chi-compromise value for NTU-games as well as state the most important result of the paper which identifies a large class of games (non-level and essential) in which the Chi-compromise value does exist.

Definition 3. *The* **Chi-compromise value** *on* $\mathbf{V}_n$*, denoted by* $\chi(N, V)$*, is the unique efficient vector in the lineal segment having as extreme points* $m^{\chi}(N, V)$ *and* $M^{\chi}(N, V)$*; that is,*

$$\chi(N, V) = \gamma M^{\chi}(N, V) + (1 - \gamma) m^{\chi}(N, V),$$

where $\gamma \in [0, 1]$ *is such that* $\chi(N, V) \in P(V(N))$.

Theorem 1. *For all* $(N, V) \in \mathbf{C}_n$ *there exists* $\chi(N, V)$.

Proof: It follows by combining Propositions 1, 2, and 3 and the essentiality of the game. ■

Remark 1. It is straightforward to show that the Chi-compromise value coincides with the Kalai-Smorodinsky solution in bargaining problems and with the χ-value in TU-games.

[3]In general, if $A \subseteq \boldsymbol{R}^n$, $\mathit{comp}(A)$ denotes the comprehensive hull of A (*i.e.*, the smallest comprehensive set containing A) and $\mathit{conv}(A)$ the convex hull of A.

Now, we compare more specifically our value with two prominent compromise values in the literature: the Compromise value of Borm *et al.* (1992) and the MC-value of Otten *et al.* (1998).

Given an NTU-game (N,V), the Compromise value is defined as the unique vector on the lineal segment between $M^C(N,V)$ and $m^C(N,V)$ which lies in $V(N)$ and is closest to $M^C(N,V)$, where for any $i \in N$

$$M_i^C(N,V) = \sup\left\{ t \in \mathbb{R} \mid \begin{array}{l} (t,x) \in V(N), x \notin V(N\backslash i) \backslash WP(V(N\backslash i)), \\ \text{and } x \geq w_{N\backslash i} \end{array} \right\}$$

and

$$m_i^C(N,V) = \max_{S \subseteq N, i \in S} \left\{ t \in \mathbb{R} \mid \begin{array}{l} \exists x \in \mathbb{R}^{s-1}, (t,x) \in V(S), \\ \text{and } x > M_{S\backslash i}^C(N,V) \end{array} \right\}.$$

The Compromise value exists for the class of compromise admissible NTU-games, defined as,

$$\mathbf{CA}_n = \left\{ (N,V) \in \mathbf{V}_n \mid \begin{array}{l} m^C(N,V) \leq M^C(N,V), m^C(N,V) \in V(N), \\ \text{and } M^C(N,V) \notin V(N) \backslash WP(V(N)) \end{array} \right\}.$$

Borm *et al.* (1992) proved that for any $(N,V) \in \mathbf{V}_n$ and any $i \in N$, $m_i^C(N,V) \geq w_i$. Suppose that (N,V) is non-level and hence $P(V(S)) = WP(V(S))$ for all $S \subseteq N$. Then, $m_i^C(N,V) \geq m_i^\chi(N,V)$. If $(t,x) \in V(N)$, $x \notin V(N\backslash i) \backslash WP(V(N\backslash i))$, and $x \geq w_{N\backslash i}$, by non-levelness, we can find $x' \in P(V(N\backslash i))$ such that $x' \leq x$ and hence $(t,x') \in V(N) \cap \mathbb{R}_+^{n-1}$. Now, it is easy to conclude that $M_i^C(N,V) \leq M_i^\chi(N,V)$. Then, in the class of non-level NTU-games, $\mathbf{CA}_n \subset \mathbf{C}_n$; that is, if the Compromise value exists then the Chi-compromise value also exists.

Note that if in the definition of M_i^χ we change $x \in P(V(S\backslash i))$ to $x \in WP(V(S\backslash i))$ (denote this alternative maximum aspiration by $\overline{M}_i^\chi$) then it is straightforward to check that $\overline{M}_i^\chi(N,V) \geq M_i^C(N,V)$ for all NTU-games. Therefore, the corresponding Chi-compromise value using the $\overline{M}^\chi$ vector as maximum aspirations is defined whenever the Compromise value exists. However, it seems to us that it is more appropriate to obtain the maximum aspiration of a player i in a coalition S as the remainder assuming that the members of coalition $S\backslash i$ exhaust all their possible gains of cooperation by reaching Pareto (and not weakly Pareto) agreements.

The MC-value of Otten *et al.* (1998) is defined as the efficient outcome lying on the lineal segment between the vector of individually rational payoffs and a vector of maximum aspiration obtained by giving to each player the *sum* of *all* his marginal contributions in all possible orderings of the set of players. Since in many cases each component of this upper value vector may be unfeasible it seems difficult to justify it as a vector of maximal aspirations. Otten *et al.* (1998) showed that the MC-value is well defined in the class of monotonic, zero-normalized NTU-games, which is unrelated to the class of non-level and essential NTU-games.

We end this section by calculating the Chi-compromise value in three well-known examples of NTU-games and comparing it with other proposed values.

Example 2 (Roth, 1980). Let (N, V) be a NTU-game such that $N = \{1, 2, 3\}$,

$$\begin{aligned} V(\{i\}) &= \{x_i \in I\!R \mid x_i \leq 0\}, \text{ for } i \in N, \\ V(\{1,2\}) &= \{(x_1, x_2) \in I\!R^2 \mid (x_1, x_2) \leq (0.5, 0.5)\}, \\ V(\{1,3\}) &= \{(x_1, x_3) \in I\!R^2 \mid (x_1, x_3) \leq (0.25, 0.75)\}, \\ V(\{2,3\}) &= \{(x_2, x_3) \in I\!R^2 \mid (x_2, x_3) \leq (0.25, 0.75)\}, \end{aligned}$$

and

$$V(N) = \{x \in I\!R^3 \mid \exists y \in conv\{(0.5, 0.5, 0), (0.25, 0, 0.75), (0, 0.25, 0.75)\}, x \leq y\}.$$

For this example the Shapley-NTU value (Aumann (1985)) is $(0.333, 0.333, 0.333)$, the Harsanyi-NTU value (Harsanyi (1963)) is $(0.416, 0.416, 0.166)$, the Consistent value (Maschler and Owen (1989, 1992)) is $(0.25, 0.25, 0.5)$, the MC-value coincides with the Shapley-NTU value, and the Compromise value is $(0.5, 0.5, 0)$.

Although the game does not satisfy non-levelness we can compute the Chi-compromise value, which coincides with $(0.5, 0.5, 0)$, the unique Core outcome.

Example 3 (Shafer, 1980).[4] Consider the following exchange economy with three agents and two commodities. The initial commodity bundles of agents

[4] We present the modification of Shafer (1980)'s example as it was used in Hart and Kurz (1983).

1, 2, and 3 are

$$\omega^1 = (1-\epsilon, 0), \omega^2 = (0, 1-\epsilon), \text{ and } \omega^3 = (\epsilon, \epsilon),$$

where $0 \leq \epsilon \leq \frac{1}{5}$, and their respective utility functions, $u_i : I\!R^2_+ \to I\!R$, are given by

$$u_1(y,z) = u_2(y,z) = \min\{y,z\}, \text{ and } u_3(y,z) = \frac{1}{2}(y+z).$$

Following Shapley and Shubik (1969) the corresponding NTU-game (N,V) is given by:

$$\begin{aligned}
V(\{i\}) &= \{x_i \in I\!R \mid x_i \leq 0\}, \text{ for } i = 1,2, \\
V(\{3\}) &= \{x_3 \in I\!R \mid x_3 \leq \epsilon\}, \\
V(\{1,2\}) &= \left\{(x_1,x_2) \in I\!R^2 \mid (x_1,x_2) \leq (1-\epsilon, 1-\epsilon), x_1 + x_2 \leq 1-\epsilon\right\}, \\
V(\{1,3\}) &= \left\{(x_1,x_3) \in I\!R^2 \mid (x_1,x_3) \leq \left(\epsilon, \frac{1+\epsilon}{2}\right), x_1 + x_3 \leq \frac{1+\epsilon}{2}\right\}, \\
V(\{2,3\}) &= \left\{(x_2,x_3) \in I\!R^2 \mid (x_2,x_3) \leq \left(\epsilon, \frac{1+\epsilon}{2}\right), x_2 + x_3 \leq \frac{1+\epsilon}{2}\right\},
\end{aligned}$$

and

$$V(N) = \{x \in I\!R^3 \mid (x_1,x_2,x_3) \leq (1,1,1), x_1 + x_2 + x_3 \leq 1\}.$$

In this game the Shapley-NTU value is $(\frac{5-5\epsilon}{12}, \frac{5-5\epsilon}{12}, \frac{1+5\epsilon}{6})$, the Harsanyi-NTU value is $(\frac{3-5\epsilon}{6}, \frac{3-5\epsilon}{6}, \frac{5\epsilon}{3})$, the MC-value coincides with the Shapley-NTU value, and the Compromise value is $(\frac{1-\epsilon}{2}, \frac{1-\epsilon}{2}, \epsilon)$.

The Chi-compromise value is $(\frac{2-2\epsilon}{5-5\epsilon}, \frac{2-2\epsilon}{5-5\epsilon}, \frac{1-\epsilon}{5-5\epsilon})$.

Example 4 (Owen, 1972). Let (N,V) be an NTU-game such that $N = \{1,2,3\}$;

$$\begin{aligned}
V(\{i\}) &= \{x_i \in I\!R \mid x_i \leq 0\}, \text{ for } i \in N, \\
V(\{1,2\}) &= \{(x_1,x_2) \in I\!R^2 \mid x_1 + 4x_2 \leq 100, x_1 \leq 100, x_2 \leq 25\}, \\
V(\{1,3\}) &= \{(x_1,x_3) \in I\!R^2 \mid x_1 \leq 0, x_3 \leq 0\}, \\
V(\{2,3\}) &= \{(x_2,x_3) \in I\!R^2 \mid x_2 \leq 0, x_3 \leq 0\},
\end{aligned}$$

and

$$V(N) = \{x \in I\!R^3 \mid \sum_{i \in N} x_i \leq 100; \forall i \in N, x_i \leq 100; \forall i,j \in N, x_i + x_j \leq 100\}.$$

In this example the Shapley-NTU value is $(50, 50, 0)$, the Harsanyi-NTU value is $(40, 40, 20)$, the Consistent value is $(50, 37.5, 12.5)$, the MC-value is $(50, 33.33, 16.67)$, and the Compromise value is $(36.36, 36.36, 27.27)$.

The Chi–compromise value is $(36.36, 36.36, 27.27)$.

4 Characterizations of the Chi-compromise value

In this section we study several properties of the Chi-compromise value. Moreover two characterizations of this value are provided.

Proposition 4. *The Chi-compromise value satisfies the following properties:*

Pareto optimality. $\chi(N,V) \in P(V(N))$ *for all* $(N,V) \in \mathbf{C}_n$.

Covariance. *Given* $(N,V),(N,W) \in \mathbf{C}_n$ *such that for all* $S \subseteq N$, $W(S) = \alpha_S * V(S) + \beta_S$ *(where* $\alpha_S * V(S) = \left\{(\alpha_i x_i)_{i\in S} \mid x_S \in V(S)\right\}$, $\alpha \in \mathbb{R}^n, \alpha > 0$ *and* $\beta \in \mathbb{R}^n$*) we have that* $\chi(N,W) = \alpha * \chi(N,V) + \beta$.

Symmetry. *If* $i,j \in N$ *are symmetric players in the game* $(N,V) \in \mathbf{C}_n$ *then* $\chi_i(N,V) = \chi_j(N,V)$. *Players* i *and* j *are called* symmetric *in a game* (N,V) *if for all* $S \subseteq N \setminus \{i,j\}$ *and all* $x \in V(S \cup i)$ *there exists* $y \in V(S \cup j)$ *defined by* $y_j = x_i$ *and* $y_S = x_S$.

Strong symmetry. *If* $w_i = w_j$ *and* $M_i^\chi(N,V) = M_j^\chi(N,V)$ *then* $\chi_i(N,V) = \chi_j(N,V)$.

Restricted Monotonicity. *If* $(N,V),(N,V') \in \mathbf{C}_n$ *are such that* $V(N) \subseteq V'(N)$, $w = w'$, *and* $M^\chi(N,V) = M^\chi(N,V')$ *then* $\chi(N,V) \leq \chi(N,V')$.

Proof: It is straightforward to check that the Chi-compromise value satisfies these five properties. ■

Theorem 2. *The Chi-compromise value is the unique solution on* $\mathbf{C}_n$ *satisfying Pareto optimality, covariance, symmetry, and restricted monotonicity.*

Proof: We have just established in Proposition 4 that the Chi-compromise value satisfies the four properties.

Now, we prove uniqueness. Suppose F is another solution satisfying the four properties. By covariance it suffices to prove that $\chi(N,V) = F(N,V)$ when, for all $i \in N$, $w_i = 0$ and $M_i^\chi(N,V) = 1$.

Clearly, for all $i \in N$, the vector $c^i \in I\!R^n$ defined by $c^i_j = \chi_j(N,V) + \epsilon$ if $i = j$ and $c^i_j = 0$ if $j \neq i$ belongs to $V(N)$ for ϵ sufficiently small. The non-levelness ensures that ϵ is strictly positive. Note that for all $i \in N$, $\chi_i(N,V) \leq 1$.

Let (N,W) be such that for all $i \in N$

$$W(\{i\}) = \{x \in I\!R \mid x \leq 0\},$$

for all $S \subset N$ such that $2 \leq s \leq n-1$

$$W(S) = comp\left\{x_S \in I\!R^S \mid \forall i \in S, 0 \leq x_i \leq 1, \text{ and } \sum_{i \in S} x_i \leq 1\right\},$$

and

$$W(N) = comp\left(conv\left(\{c^i \in I\!R^n \mid i \in N\} \cup \chi(N,V)\right)\right) \cap V(N).$$

Then $(N,W) \in \mathbf{C}_n$, $M^\chi_i(N,W) = 1$ for all $i \in N$, and $\chi(N,V) = \chi(N,W)$. By symmetry for all $i,j \in N$, $F_i(N,W) = F_j(N,W)$. Note that even though $W(N)$ is not necessarily a symmetric set, (N,W) is a symmetric game. Therefore by Pareto optimality, $F(N,W) = \chi(N,W)$. By restricted monotonicity $F(N,W) \leq F(N,V)$, which implies $\chi(N,V) \leq F(N,V)$. But since χ satisfies Pareto optimality we can conclude that $\chi(N,V) = F(N,V)$. ∎

Theorem 3. *The Chi-compromise value is the unique solution on* $\mathbf{C}_n$ *satisfying Pareto optimality, covariance, and strong symmetry.*

Proof: Proposition 4 establishes that the Chi-compromise value satisfies these properties.

Now we prove uniqueness. Suppose F is another solution satisfying these properties. By covariance it suffices to prove that $\chi(N,V) = F(N,V)$ when, for all $i \in N$, $w_i = 0$ and $M^\chi_i(N,V) = 1$.

By strong symmetry, for all $i,j \in N$, $F_i(N,V) = F_j(N,V)$ and $\chi_i(N,V) = \chi_j(N,V)$. By Pareto optimality, $F(N,V) = \chi(N,V)$. ∎

Note that all axioms used in both characterizations are independent. The egalitarian solution defined by Kalai and Samet (1985) satisfies all five properties except covariance. The solution f^1 defined as $f^1(N,V) = w$ for all $(N,V) \in \mathbf{C}_n$ satisfies all properties except Pareto optimality. The solution f^2

defined as the Shapley value when (N,V) is an essential TU-game and the Chi-compromise value in the rest of the class $\mathbf{C}_n$ satisfies all properties except strong symmetry and restricted monotonicity. The solution f^3 defined as $f_i^3(N,V) = w_i$ for $i \neq 1$ and $f_1^3(N,V) = \max\left\{t \in \mathbb{R} \mid \left(t, w_{N\setminus 1}\right) \in V(N)\right\}$, satisfies all properties except symmetry.

These axiomatic characterizations can be extended in the following way. Theorem 2 is also true for the class of NTU-games for which the Chi-compromise value exists and the condition of non-levelness is satisfied only for the set $V(N) \cap \mathbb{R}_+^n$. Theorem 3 is also true for the class of NTU-games where the Chi-compromise value exists.

Moreover, notice that in both characterizations the sets $V(S)$ need not be convex. While this is also possible in the characterization of the MC-value it is not the case in the characterization of the Compromise value where the set $V(N)$ has to be convex.

5 The Lambda-transfer Chi-value

Shapley (1969) defined the family of λ-transfer TU-games corresponding to an NTU-game. Using this family of games, and their corresponding Shapley values, he defined the NTU-Shapley value. We proceed in the same way using our χ-value for TU-games instead of the Shapley value.

Define $\Delta_+^n = \left\{\lambda \in \mathbb{R}_+^n \mid \sum_{i\in N} \lambda_i = 1\right\}$ as the n-dimensional unit simplex. Given a NTU-game (N,V) we say that the vector $\lambda \in \Delta_+^n$ is *feasible* if $\sup\left\{\sum_{i\in S} \lambda_i x_i \mid x \in V(S)\right\} < \infty$ for all $S \subseteq N$. For each feasible vector $\lambda \in \Delta_+^n$ we define the TU-game $\left(N, v^\lambda\right)$ by associating with each coalition $S \subseteq N$ the number $v^\lambda(S) = \sup\left\{\sum_{i\in S} \lambda_i x_i \mid x \in V(S)\right\}$.

Definition 4. *The* **Lambda-transfer Chi-value** *on* $\mathbf{V}_n$*, denoted by* $\chi^\Lambda(N,V)$*, is the set*

$$\chi^\Lambda(N,V) = \left\{x \in V(N) \mid \lambda * x \geq \chi\left(N, v^\lambda\right) \textit{ for some } \lambda \in \Delta_+^n \textit{feasible}\right\}.$$

Before stating a result establishing sufficient conditions under which the Lambda-transfer Chi-value set is non-empty we need to define two standard properties of NTU-games.

Definition 5. *A NTU-game* (N,V) *is* **compactly generated** *if for all* $S \subseteq N$ *there exists a compact set* $K_S \subset \mathbb{R}^s$ *with the property that* $V(S) =$

$\{x \in \mathbb{R}^s \mid x \leq y \textit{for some } y \in K_S\}$. *A NTU-game* (N, V) *is* **convex** *if for all* $S \subseteq N$ *the set* $V(S)$ *is convex.*

Theorem 4. *Let* (N, V) *be an essential, compactly generated, and convex NTU-game. Then,* $\chi^\Lambda(N, V) \neq \emptyset$.

Proof: First, we will show that if the NTU-game (N, V) is essential then for any $\lambda \in \Delta_+^n$ feasible the TU-game (N, v^λ) is essential as well. Consider any $i \in N$. By definition $v^\lambda(i) = \lambda_i w_i$. Moreover, by the essentiality of (N, V),

$$\begin{aligned} v^\lambda(N) &= \sup\left\{\sum_{i \in N} \lambda_i x_i \mid x \in V(N)\right\} \\ &\geq \sum_{i \in N} \lambda_i w_i \\ &= \sum_{i \in N} v^\lambda(\{i\}), \end{aligned}$$

which means that the TU-game (N, v^λ) is essential.

The non-emptiness of the set $\chi^\Lambda(N, V)$ follows using a fixed-point argument similar to that of Shapley (1969). ∎

The game of Example 2 illustrates the fact that, in general, the Chi-compromise value and the Lambda-transfer Chi-value may be different. After a simple, but very tedious computation, it is possible to see that $\chi^\Lambda(N, V) = (0.33, 0.33, 0.33)$ while $\chi(N, V) = (0.5, 0.5, 0)$.

6 Concluding remarks

Following the Nash program, there is a long tradition of justifying axiomatic bargaining solutions by means of equilibria of a non-cooperative game associated to the original bargaining problem. Moulin (1984) exhibits an extensive-form game whose subgame perfect equilibria induce the Kalai-Smorodinsky solution. Here, and following the procedure used by Hart and Mas-Colell (1996) to obtain the Consistent value by extending the non-cooperative implementation of the Nash bargaining solution (which also coincides with the Shapley value for TU-games) to NTU-games, we extend Moulin's implementation of the Kalai-Smorodinsky solution for bargaining problems to NTU-games as follows:

- *Round* 0. Each player i makes a bid p_i where $0 < p_i \leq 1$ and the players are renumbered in decreasing order of their bids, $p_1 \geq p_2 \geq ... \geq p_n$ (players with tied bids are ordered randomly among themselves).

- *Round* 1. Player 1 proposes a payoff vector $x = (x_1, ..., x_n)$ for the approval of player n, who can either accept or reject it. If he accepts it the game proceeds to round 2.

  In Moulin's implementation, if player n rejects the initial offer he must make a counteroffer to the rest of the players, who can reject or accept it. If somebody rejects it the disagreement point is enforced. In our model player n, who rejected the initial offer, can make a proposal to some smaller coalition. This modification of Moulin's implementation is motivated because in NTU-games partial agreements are also possible.

- *Rounds* $2, ..., n-1$ are similar to round 1 but now players $n-1, ..., 2$ (instead of player n) have to accept or reject the offer of player 1.

By induction on the number of players, it is possible to show that the associated payoffs of all subgame-perfect equilibria of this extensive form coincide with the Chi-compromise value of the NTU-game.

Before finishing this paper we would like to briefly compare our proposal with other NTU-values. As with all compromise values it is easier to compute than the Shapley, Harsanyi, and the Consistent values. However, the Shapley and Harsanyi values have nice characterizations, while those of all compromise values including ours are *ad hoc* (in the sense that the vectors of maximum and minimum aspirations are used in the definitions of some of the key axioms); on the contrast, to our knowledge the Consistent value has yet to be fully characterized (Maschler and Owen (1989) characterize it for the class of hyperplane games). Except for the Compromise value, whose existence is guaranteed only for games with non-empty cores (a proper subclass of compromise admissible NTU-games), the existence of all other NTU-values is guaranteed for classes of games which are relatively larger than this and unrelated to each other. Finally, to our knowledge, only the Consistent value (Hard and Mas-Colell (1996)) and our Chi-compromise value have been shown to be implementable by extensive-form games.

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