# MAXIMAL DOMAIN OF PREFERENCES

# IN THE DIVISION PROBLEM*

by

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## Abstract

The division problem consists of allocating an amount $M$ of a perfectly divisible good among a group of $n$ agents. Sprumont [13] showed that, given $M$, if agents have single-peaked preferences over their shares, the uniform allocation rule is the unique strategy-proof, efficient, and anonymous rule. Ching and Serizawa [9] extended this result by demonstrating that, when the rule depends not only on the preferences of the agents but also on the amount $M$ to be allocated, the domain of single-plateaued preferences is the maximal one under which there exists at least one rule satisfying the properties of strategy-proofness, efficiency, and symmetry. We characterize the maximal domain for each value of $M$ and $n$ and show that it is larger than the set of single-plateaued preferences. In addition, we show that their intersection, as $M$ varies from 0 to $\infty$, is precisely the domain of single-plateaued preferences, a result that implies that of Ching and Serizawa [9]..



## Resumen

El problema de la división consiste en distribuir una cantidad $M$ de un bien perfectamente divisible entre un grupo de $n$ agentes. Sprumont [13] demostró que, dado $M$, si los agentes tienen preferencias unimodales sobre las cantidades recibidas del bien, la regla de distribución uniforme es la única regla no manipulable, eficiente y anónima. Ching y Serizawa [9] ampliaron este resultado al demostrar que, cuando la regla depende no sólo de las preferencias de los agentes sino también de la cantidad $M$ a distribuir, el dominio de preferencias unimodales con plateau es el maximal para el cual existe al menos una regla que cumpla las propiedades de no manipulabilidad, eficiencia y simetria. Para cada $M$ y $n$ caracterizamos el dominio maximal y demostramos que es mayor que el conjunto de preferencias unimodales con plateau. Además, demostramos que su intersección, al variar $M$ desde 0 a $\infty$, es precisamente el dominio de preferencias unimodales con plateau, un resultado que implica el de Ching y Serizawa [9].

# 1 Introduction

The division problem consists of allocating an amount $M$ of a perfectly divisible good among a group of $n$ agents. A rule maps preference profiles of agents into $n$ shares of the amount $M$. Sprumont [13] shows that, given $M$, if agents have single-peaked preferences over their shares, the uniform allocation rule is the unique strategy-proof, efficient, and anonymous rule. This is a nice example of a large literature that, by restricting the domain of preferences, investigates the possibility of designing strategy-proof rules.[1] Moreover, in this case, single-peakedness does not only admit strategy-proof rules but also efficient ones.

In this paper we ask how much can we enlarge the set of single-peaked preferences and still allow for rules satisfying interesting properties. In particular, we characterize the maximal domain of preferences, including the set of single-peaked preferences, under which there exists at least one rule on this domain satisfying the properties of strategy-proofness, efficiency, and symmetry.

It turns out that this maximal domain depends crucially on both $M$ and $n$, since the egalitarian share $M/n$ plays, as a consequence of the symmetry requirement, a fundamental role in its description. In particular, our domain includes only preferences whose set of best shares is an interval. Additionally, the following requirement is satisfied. If the highest share in this interval is smaller than $M/n$, then the preference has to be "decreasing" between this highest share and $M/n$, although it may have "small" intervals of indifference ("small" because the sum of the extremes can not exceed $M$); moreover, the egalitarian share $M/n$ has to be at least as good as all larger shares, but all orderings are possible among them. Symmetrically, if the smallest of the best shares is bigger than $M/n$, then the preference has to be "increasing" between $M/n$ and this smallest share, although it may have "large" intervals of indifference ("large" because the sum of the extremes has to be larger than $M$);[2] moreover, the egalitarian share $M/n$ has to be at least as good as all smaller shares, but also all orderings are possible among them. Finally, if $M/n$ is itself one of the best shares, no additional requirement is imposed. Notice that the set of these preferences, given $M$ and $n$, is much larger than the single-plateaued domain studied by Moulin [10] and Berga [4] in a public good context, since single-plateaued preferences are strictly monotonic in both sides of the plateau.

Furthermore, and as a consequence of our main result, we find that the single-plateaued domain coincides with the intersection of all of our maximal domains, when $M$ varies from 0 to $\infty$. This also implies that, when the rule depends not only on the preferences of the agents but also on the amount $M$ to be allocated, the maximal domain coincides with the set of single-plateaued preferences as already shown by Ching and Serizawa [9]. Notice that in their setting, $M$ is treated as a variable of the problem rather than one of its data. We want to emphasize though, that in spite of their result, our analysis with a fixed amount $M$ is meaningful since there are many

[1] See Sprumont [14] and Barberà [1] for two comprehensive surveys of this literature as well as for two exhaustive bibliographies.

[2] See Example 1 at the end of Section 2 for an illustration of why efficiency imposes this condition on the intervals of indifference.

allocation problems where to assume the contrary is senseless.

Different papers have also identified maximal domains of preferences allowing for strategy-proof social choice functions in voting environments. Barberà, Sonneschein, and Zhou [3] show that the set of separable preferences is the largest domain preserving strategy-proofness of voting by committees without both, dummies and vetoers. Serizawa [12], Barberà, Massó, and Neme [2], Berga and Serizawa [6], and Berga [5] improve upon this result in several directions; for instance, by either looking at a more general voting model and/or by admitting larger classes of social choice functions. We want to emphasize that we do not claim that the domain identified here has economic relevance; rather, we understand our result as giving a precise and definite answer to an interesting and economically relevant question raised by all restricted domain literature; namely, how much can we enlarge the restricted domain and still be able to define on it strategy-proof rules?

Finally, it is worth mentioning that, in contrast to all the papers mentioned above, the rule that we exhibit when showing our maximality result is not "tops-only" in the sense that it does not depend exclusively on the $n$ sets of best shares. The efficiency requirement forces the rule to be sensible to intervals of indifference outside the "top".

The paper is organized as follows. Section 2 contains notation, definitions, and the statement of our result. This is proven in Section 3. Section 4 concludes by obtaining Ching and Serizawa [9] result as a corollary of our theorem and by relating our maximal domains with the "option" sets associated with strategy-proof, efficient, and symmetric rules.

# 2 Preliminaries, Definitions and the Theorem

*Agents* are the elements of a finite set $N = \{1,..,n\}$ where $n \geq 2$. They have to share the amount $M \in \Re_{++}$ of a perfectly divisible good. An *allocation* is a vector $(x_1, ..., x_n) \in \Re^n_+$ such that $\sum x_i = M$. We denote by $Z(M)$ the set of allocations. Every agent $i \in N$ has a *preference* ordering over the interval $[0, M]$ represented by a complete preorder $R_i$. Let $P_i$ be the strict preference relation associated with $R_i$ and let $I_i$ be its indifference relation. We assume that agents have continuous preferences in the sense that for each $x \in [0, M]$ the sets $\{y \in [0, M] \mid xR_iy\}$ and $\{y \in [0, M] \mid yR_ix\}$ are closed. We denote by $\mathcal{R}(M)$ the set of continuous preferences on $[0, M]$ and by $\mathcal{U}$ a generic subset of $\mathcal{R}(M)$. *Preference profiles* are $n$-tuples of continuous preferences on $[0, M]$ and they are denoted by $\mathbf{R} = (R_1, ..., R_n) \in \mathcal{R}(M)^n$. When we want to stress the role of agent $i$'s preference we will represent a preference profile by $(R_i, R_{-i})$.

A *rule on* $\mathcal{U} \subset \mathcal{R}(M)$ is a function $\Phi : \mathcal{U}^n \longrightarrow Z(M)$; that is, $\sum \Phi_i(\mathbf{R}) = M$ for all $\mathbf{R} \in \mathcal{U}^n$.

Rules on a set of preferences require each agent to report a preference on this set. A rule is strategy-proof if it is always in the best interest of agents to reveal their preferences truthfully. Formally,

**Definition 1** *A rule on* $\mathcal{U}$, $\Phi : \mathcal{U}^n \longrightarrow Z(M)$, *is strategy-proof if for all* $(R_1, ..., R_n) \in \mathcal{U}^n$, *all* $i \in N$, *and all* $R'_i \in \mathcal{U}$ *we have* $\Phi_i(R_i, R_{-i}) R_i \Phi_i(R'_i, R_{-i})$.

We are also interested in rules satisfying the following two properties.

**Definition 2** *A rule on $\mathcal{U}$, $\Phi : \mathcal{U}^n \longrightarrow Z(M)$, is efficient if for all $\mathbf{R} \in \mathcal{U}^n$ there is no $(z_1, ..., z_n) \in Z(M)$ such that for all $i \in N$, $z_i R_i \Phi_i(\mathbf{R})$, and for at least one $j \in N$ we have $z_j P_j \Phi_j(\mathbf{R})$.*

**Definition 3** *A rule on $\mathcal{U}$, $\Phi : \mathcal{U}^n \longrightarrow Z(M)$, is symmetric if for all $\mathbf{R} \in \mathcal{U}^n$ and all $i,j \in N$ such that $R_i = R_j$ we have $\Phi_i(\mathbf{R}) = \Phi_j(\mathbf{R})$.*[3]

We will consider different subsets of preferences, all of them related to single-peakedness. Before stating the definitions, we need the following notation. Given a preference $R_i \in \mathcal{R}(M)$ we denote the set of preferred shares according to $R_i$ as $p(R_i) = \{x \in [0, M] \mid xR_iy \text{ for all } y \in [0, M]\}$. Let $\underline{p}(R_i) = \inf p(R_i)$ and $\overline{p}(R_i) = \sup p(R_i)$. Abusing notation, we will also denote by $p(R_i)$ the unique element of the set $p(R_i)$ whenever $\underline{p}(R_i) = \overline{p}(R_i)$.

The first definition is the classical notion of single-peakedness. It requires that the preference $R_i$ has a *unique* maximal element $p(R_i)$ and at each of its sites the preference is monotonic and strict. Formally,

**Definition 4** *A preference $R_i \in \mathcal{R}(M)$ is single-peaked if $p(R_i)$ is a singleton and for all $x, y \in [0, M]$ we have $xP_iy$ whenever $y < x < p(R_i)$ or $p(R_i) < x < y$.*

Let $\mathcal{R}_s(M)$ be the set of single-peaked preferences on $[0, M]$. The following rule on $\mathcal{R}_s(M)$, the uniform allocation rule, has been extensively studied.

**Definition 5** *The uniform allocation rule on $\mathcal{R}_s(M)$, $\varphi : \mathcal{R}_s(M)^n \rightarrow Z(M)$, is defined as follows: for all $\mathbf{R} \in \mathcal{R}_s(M)^n$ and $i \in N$,*

$$\varphi_i(\mathbf{R}) = \begin{cases} \min\{p(R_i), \lambda(\mathbf{R})\} & \text{if } M \leq \sum p(R_j), \\ \max\{p(R_i), \lambda(\mathbf{R})\} & \text{if } M \geq \sum p(R_j), \end{cases}$$

*where $\lambda(\mathbf{R})$ solves $\sum \varphi_j(\mathbf{R}) = M$.*

Ching [8] characterized the uniform allocation rule on $\mathcal{R}_s(M)$ as the unique one satisfying strategy-proofness, efficiency, and symmetry.[4]

The second definition of preferences is a bit weaker since it allows for indifferences on the top.

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[3]Ching [8] and Ching and Serizawa [9] name this property *equal treatment of equals* and *strong symmetry*, respectively. Ching and Serizawa [9] use the name of symmetry when the condition $\Phi_i(\mathbf{R}) = \Phi_j(\mathbf{R})$ is replaced by $\Phi_i(\mathbf{R}) I_i \Phi_j(\mathbf{R})$. However, they show that, under efficiency, both properties are equivalent.

[4]See Ching [7], Schummer and Thomson [11], Sprumont [13], Thomson [15], [16], and [17] for alternative characterizations of the uniform allocation rule. In a recent paper, Weymark [18] shows that Sprumont's characterization using efficiency, strategy-proofness, and anonimity still holds even if the continuity of the preferences is not required.

**Definition 6** *A preference $R_i \in \mathcal{R}(M)$ is single-plateaued if $p(R_i) = \left[\underline{p}(R_i), \overline{p}(R_i)\right]$ and for all $x, y \in [0, M]$ we have $xP_iy$ whenever $y < x < \underline{p}(R_i)$ or $\overline{p}(R_i) < x < y$.*[5]

Let $\mathcal{R}_{sp}(M)$ be the set of single-plateaued preferences. The following rule on $\mathcal{R}_{sp}(M)$ constitutes a natural extension of the uniform allocation rule to the domain of single-plateaued preferences.

**Definition 7** *The uniform allocation rule on $\mathcal{R}_{sp}(M)$, $\psi : \mathcal{R}_{sp}(M)^n \to Z(M)$, is defined as follows: for all $\mathbf{R} \in \mathcal{R}_{sp}(M)^n$ and $i \in N$,*

$$\psi_i(\mathbf{R}) = \begin{cases} \min\left\{\underline{p}(R_i), \lambda(\mathbf{R})\right\} & \text{if } M \leq \sum_j \underline{p}(R_j), \\ \min\left\{\overline{p}(R_i), \underline{p}(R_i) + \lambda(\mathbf{R})\right\} & \text{if } \sum_j \underline{p}(R_j) \leq M \leq \sum_j \overline{p}(R_j), \\ \max\left\{\overline{p}(R_i), \lambda(\mathbf{R})\right\} & \text{if } \sum_j \overline{p}(R_j) \leq M, \end{cases}$$

*where $\lambda(\mathbf{R})$ solves $\sum \psi_j(\mathbf{R}) = M$.*

Finally, our third definition of preferences, the weakest one, refers to the following interval $\Theta(R_i)$, which will play a fundamental role in the sequel:

$$\Theta(R_i) = \left[\min\left\{\frac{M}{n}, \underline{p}(R_i)\right\}, \max\left\{\frac{M}{n}, \overline{p}(R_i)\right\}\right].$$

**Definition 8** *A preference $R_i \in \mathcal{R}(M)$ is restricted-monotonic on $\Theta(R_i)$ if for all $x, y \in [0, M]$:*
*(a) If $[x < y$ and $M/n \leq y \leq \underline{p}(R_i)]$ then $[yR_ix$ and if $yI_ix$ then there exists $[x_0, y_0] \supseteq [x, y]$ such that $x_0 + y_0 > M$ and $x'I_iy_0$ for all $x' \in [x_0, y_0]]$.*
*(b) If $[x < y$ and $\overline{p}(R_i) \leq x \leq M/n]$ then $[xR_iy$ and if $xI_iy$ then there exists $[x_0, y_0] \supseteq [x, y]$ such that $x_0 + y_0 < M$ and $x'I_iy_0$ for all $x' \in [x_0, y_0]]$.*
*(c) If $x \in \left[\underline{p}(R_i), \overline{p}(R_i)\right]$ then $xI_i\overline{p}(R_i)$.*

Notice that the number of agents $n$ also plays a role in conditions (a) and (b) of the above definition. Therefore, given $M$ and $n$, we denote by $\mathcal{R}_{rm}^{\Theta}(M, n)$ the set of preferences satisfying Definition 8 and we name it *the set of restricted-monotonic preferences on $\Theta$*; that is, $R_i \in \mathcal{R}_{rm}^{\Theta}(M, n)$ if and only if $R_i$ is restricted-monotonic on $\Theta(R_i)$. We will show in Theorem 1 that the set of restricted-monotonic preferences on $\Theta$ is the maximal domain of preferences admitting strategy-proof, efficient, and symmetric rules. Figure 1 illustrates three possible types of restricted-monotonic preferences on $\Theta$ depending on whether $M/n \leq \underline{p}(R_i)$, $\overline{p}(R_i) \leq M/n$, or $\underline{p}(R_i) \leq M/n \leq \overline{p}(R_i)$.

Insert Figure 1 here

Following Ching and Serizawa [9] we can define, given $(M, n)$ and a list of properties that rules may satisfy, the concept of maximal domain of preferences.

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[5]See Moulin [10] and Berga [4] for characterizations of strategy-proof rules under this domain restriction in a public good context.

**Definition 9** *A set $\mathcal{R}_m(M,n)$ of preferences is a maximal domain for a list of properties if: (1) $\mathcal{R}_m(M,n) \subseteq \mathcal{R}(M)$; (2) there exists a rule on $\mathcal{R}_m(M,n)$ satisfying the properties; and (3) there is no rule on $\mathcal{R}'$ satisfying the same properties such that $\mathcal{R}_m(M,n) \subsetneq \mathcal{R}' \subseteq \mathcal{R}(M)$.*

**Theorem 1** *The set of restricted-monotonic preferences on $\Theta$, $\mathcal{R}^{\Theta}_{rm}(M,n)$, is the unique maximal domain including $\mathcal{R}_s(M)$ for the properties of strategy-proofness, efficiency, and symmetry.*

Before proving Theorem 1 we illustrate, in Example 1 below, the reason why the properties of efficiency and symmetry together force the domain to contain only preferences with intervals of indifference of a very special type outside the top.

**Example 1.** Consider the case where $M = 8$ and the set of agents is $N = \{1,2\}$. Let $\Phi$ be any efficient and symmetric rule. Consider the preference $\bar{R}$ on $[0,8]$ defined by:

$$y\bar{P}x \text{ for all } 0 \leq x < y \leq 2 \text{ and all } 5 \leq x < y \leq 8,$$
$$y\bar{I}x \text{ for all } x,y \in [2,5].$$

Notice that $\bar{R} \notin \mathcal{R}^{\Theta}_{rm}(8,2)$ since condition (a) of Definition 8 is not satisfied because $2\bar{I}5$ and we can not find an interval of indifference $[x_0,y_0] \supseteq [2,5]$ such that $x_0+y_0 > 8$. A maximal domain of preferences can not contain $\bar{R}$ because by symmetry $\Phi\left(\bar{R},\bar{R}\right) = (4,4)$ but the allocation $(2,6)$ contradicts the efficiency of $\Phi$ because $2\bar{I}4$ and $6\bar{P}4$. Consider now the preference $\hat{R}$ on $[0,8]$ defined by:

$$y\hat{P}x \text{ for all } 0 \leq x < y \leq 3 \text{ and all } 6 \leq x < y \leq 8,$$
$$y\hat{I}x \text{ for all } x,y \in [3,6].$$

Notice now that $\hat{R} \in \mathcal{R}^{\Theta}_{rm}(8,2)$ since condition (a) of Definition 8 is satisfied because the sum of the extremes of the indifference interval $[3,6]$ is larger than 8. In contrast, the symmetric allocation $\Phi\left(\hat{R},\hat{R}\right) = (4,4)$ is now efficient.

To illustrate the role of condition (b) in Definition 8 consider the preference $\tilde{R}$ on $[0,8]$ defined by:

$$x\tilde{P}y \text{ for all } 0 \leq x < y \leq 3 \text{ and all } 6 \leq x < y \leq 8,$$
$$x\tilde{I}y \text{ for all } x,y \in [3,6].$$

In this case $\tilde{R} \notin \mathcal{R}^{\Theta}_{rm}(8,2)$ because now the sum of the extremes of the indifference interval is larger than 8. By symmetry $\Phi\left(\tilde{R},\tilde{R}\right) = (4,4)$ but $2\tilde{P}4$ and $6\tilde{I}4$ which indicates that $\Phi$ is not efficient. Finally, consider the preference $R'$ on $[0,8]$ defined by:

$$xP'y \text{ for all } 0 \leq x < y \leq 2 \text{ and all } 5 \leq x < y \leq 8,$$
$$xI'y \text{ for all } x,y \in [2,5].$$

Now $R' \in \mathcal{R}^{\Theta}_{rm}(8,2)$ since condition (b) of Definition 8 is satisfied because the sum of the extremes of the indifference interval $[2,5]$ is smaller than 8. In this case the symmetric allocation $\Phi(R',R') = (4,4)$ is also efficient.

## 3 The Proof of Theorem 1

Before proving Theorem 1 we state, in the following Remark, a consequence of Ching's characterization (Ching [8]) that we will repeatedly use in this section.

**Remark 1** *Let $\Phi : \mathcal{U}^n \to Z(M)$ be any rule on $\mathcal{U}$ ($\supseteq \mathcal{R}_s(M)$) satisfying strategy-proofness, efficiency, and symmetry. If $\mathbf{R} \in \mathcal{R}_s(M)^n$ then $\Phi(\mathbf{R}) = \varphi(\mathbf{R})$; that is, $\Phi$ coincides with the uniform allocation rule on the subset of single-peaked preferences.*

Now, let $(M, n)$ be given and let $\mathcal{R}_m(M, n)$ be a subset of preferences satisfying the following condition: $\mathcal{R}_s(M) \subsetneq \mathcal{R}_m(M, n) \subseteq \mathcal{R}(M)$. Suppose that there exists a rule on $\mathcal{R}_m(M, n)$, $\Phi : \mathcal{R}_m(M, n)^n \longrightarrow Z(M)$, satisfying strategy-proofness, efficiency, and symmetry. Moreover, let $R^0, R^M \in \mathcal{R}_s(M)$ be the two single-peaked preferences such that $p(R^0) = 0$ and $p\left(R^M\right) = M$.

First, to show that $\mathcal{R}_m(M, n) = \mathcal{R}^{\ominus}_{rm}(M, n)$ we will use the following Lemmata.

**Lemma 1** *Let $R_0 \in \mathcal{R}_m(M, n)$ and $x, y \in [0, M]$ be arbitrary.*
*(a) If $M/n \le x < y \le \overline{p}(R_0)$ then $yR_0x$.*
*(b) If $\underline{p}(R_0) \le y < x \le M/n$ then $yR_0x$.*

**Proof of Lemma 1.** Case (a): Suppose otherwise; that is, there exist $R_0 \in \mathcal{R}_m(M, n)$ and $\overline{x}, \overline{y} \in [0, M]$ such that $M/n \le \overline{x} < \overline{y} \le \overline{p}(R_0)$ and $\overline{x}P_0\overline{y}$. We can also find (see Figure 2) $x_0, y_0 \in [0, M]$ such that:

(a.1) $M/n \le x_0 < y_0 \le \overline{p}(R_0)$,
(a.2) $x_0 I_0 y_0$,
(a.3) $x_0 R_0 x$ for all $x \in [M/n, x_0]$, and
(a.4) $x_0 P_0 x$ for all $x \in (x_0, y_0)$.

Insert Figure 2 here

Notice that $x_0$ is the smallest value below $\overline{p}(R_0)$ and above $M/n$ at which $R_0$ starts decreasing to its right.[6] Since $R_0$ is continuous and $\overline{p}(R_0) R_0 \overline{x}$, the existence of such $y_0$ follows. Obviously, $x_0$ could be equal to $M/n$, $y_0$ equal to $\overline{p}(R_0)$, or both.

Note that for all $z_0 \in (x_0, y_0)$ the following inequalities hold:

$$\frac{M - y_0}{n-1} < \frac{M - z_0}{n-1} < \frac{M - x_0}{n-1} \le \frac{M}{n}. \tag{1}$$

Now, fix $z_0 \in (x_0, y_0)$ and let $\bar{R} \in \mathcal{R}_s(M)$ be such that $p\left(\bar{R}\right) = \frac{M-z_0}{n-1}$ and $\left(\frac{M-y_0}{n-1}\right) \bar{P} \left(\frac{M-x_0}{n-1}\right)$. The existence of such a preference $\bar{R}$ follows from condition (1).

Let $\hat{R} \in \mathcal{R}_s(M)$ be any preference such that $p\left(\hat{R}\right) = x_0$. By Remark 1, $\Phi\left(\hat{R}, \bar{R}, ..., \bar{R}\right)$ coincides with $\varphi\left(\hat{R}, \bar{R}, ..., \bar{R}\right)$, the uniform allocation rule, and since

$$x_0 + (n-1) \cdot \frac{M - z_0}{n-1} < M,$$

[6] We often abuse language by using the utility representation terminology to refer to properties of preference relations.

we have that $\Phi_1\left(\hat{R}, \bar{R}, ..., \bar{R}\right) = x_0$. By the strategy-proofness of $\Phi$ we have that

$$\Phi_1\left(R_0, \bar{R}, ..., \overline{R}\right) R_0 x_0. \tag{2}$$

Again, by Remark 1, we have that $\Phi_1\left(R^M, \bar{R}, .., \bar{R}\right) = z_0$ and by strategy-proofness of $\Phi$ we also have that $z_0 R^M \Phi_1\left(R_0, \bar{R}, ..., \bar{R}\right)$ implying that

$$\Phi_1\left(R_0, \bar{R}, ..., \bar{R}\right) \leq z_0. \tag{3}$$

Finally, by Remark 1, we have that $\Phi_1\left(R^0, \bar{R}, ..., \bar{R}\right) = M/n$ and by strategy-proofness of $\Phi$ we must have that $M/nR^0\Phi_1\left(R_0, \bar{R}, ..., \bar{R}\right)$ implying that

$$\Phi_1\left(R_0, \bar{R}, ..., \bar{R}\right) \geq \frac{M}{n}. \tag{4}$$

Then, by (2), (3), and (4) we have that

$$\Phi\left(R_0, \bar{R}, ..., \bar{R}\right) = \left(x_1, \frac{M - x_1}{n-1}, ..., \frac{M - x_1}{n-1}\right)$$

with $M/n \leq x_1 \leq x_0$ and $x_1 I_0 x_0$. But the allocation $\left(y_0, \frac{M-y_0}{n-1}, ..., \frac{M-y_0}{n-1}\right)$ contradicts efficiency of $\Phi$.

Case (b): Its proof is omitted since follows an argument which is symmetric to the one used to prove Case (a). ■

**Lemma 2** *Let $R_0 \in \mathcal{R}_m(M,n)$ and $x \in [0, M]$ be arbitrary.*
*(a) If $x < M/n \leq \overline{p}(R_0)$ then $M/nR_0x$.*
*(b) If $\underline{p}(R_0) \leq M/n < x$ then $M/nR_0x$.*

**Proof of Lemma 2.** Case (a): Suppose otherwise; that is, there exist $R_0 \in \mathcal{R}_m(M,n)$ and $x_0 < M/n \leq \overline{p}(R_0)$ such that $x_0 P_0 M/n$.

First, assume that $M/n$ is a minimal element on $[x_0, M/n]$ relative to $R_0$; that is

$$yR_0\frac{M}{n} \text{ for all } y \in \left[x_0, \frac{M}{n}\right]. \tag{5}$$

Since $\Phi$ is symmetric, we have that $\Phi(R_0, ..., R_0) = (M/n, ..., M/n)$. By condition (5) and Lemma 1 we have that for all $\varepsilon \in \left(0, \min\left\{\frac{M}{n} - x_0, \overline{p}(R_0) - \frac{M}{n}\right\}\right]$

$$\left(\frac{M}{n} - \varepsilon\right) R_0 \frac{M}{n} \text{ and } \left(\frac{M}{n} + \varepsilon\right) R_0 \frac{M}{n}.$$

Let $\bar{\varepsilon} = \min\left\{\frac{M}{n} - x_0, \overline{p}(R_0) - \frac{M}{n}\right\}$. Then, either

$$\left(\frac{M}{n} - \bar{\varepsilon}\right) P_0 \frac{M}{n} \text{ or } \left(\frac{M}{n} + \bar{\varepsilon}\right) P_0 \frac{M}{n},$$

depending on whether $\bar{\varepsilon}$ is either equal to $\frac{M}{n} - x_0$ or to $\overline{p}(R_0) - \frac{M}{n}$, respectively. Then the allocation $\left(\left(\frac{M}{n} + \bar{\varepsilon}\right), \left(\frac{M}{n} - \bar{\varepsilon}\right), M/n, ..., M/n\right)$ contradicts the efficiency of $\Phi$.

Second, assume that there exists $y_0 \in (x_0, M/n)$ such that $M/nP_0y_0$. Then, there exist $x_1$, $y_1$ and $z_1$ such that:

(a.1') $0 \leq x_1 < z_1 < y_1 \leq M/n$,
(a.2') $x_1 I_0 y_1 I_0 M/n$,
(a.3') $x_1 P_0 x$ for all $x \in (x_1, y_1)$, and
(a.4') $y_1 I_0 x$ for all $x \in [y_1, M/n]$.

Note that

$$\frac{M}{n} \leq \frac{M-y_1}{n-1} < \frac{M-z_1}{n-1} < \frac{M-x_1}{n-1}.$$

Now, let $\bar{R} \in \mathcal{R}_s(M)$ be any single-peaked preference such that $p(\bar{R}) = \frac{M-z_1}{n-1}$ and $\frac{M-x_1}{n-1} \bar{P} \frac{M-y_1}{n-1}$. By Remark 1, $\Phi_1(R^M, \bar{R}, ..., \bar{R})$ coincides with $\varphi_1(R^M, \bar{R}, ..., \bar{R}) = M/n$, the uniform allocation rule. By the strategy-proofness of $\Phi$, we have that

$$\Phi_1(R_0, \bar{R}, ..., \bar{R}) R_0 \frac{M}{n}. \tag{6}$$

Again, by Remark 1, we have that $\Phi_1(R^0, \bar{R}, ..., \bar{R}) = z_1$ and by the strategy-proofness of $\Phi$ we also have that $z_1 R^0 \Phi_1(R_0, \bar{R}, ..., \bar{R})$ implying that

$$\Phi_1(R_0, \bar{R}, ..., \bar{R}) \geq z_1. \tag{7}$$

Then, by (6) and (7) we have that

$$\Phi_1(R_0, \bar{R}, ..., \bar{R}) \geq y_1. \tag{8}$$

Finally, by Remark 1, we have that $\Phi_1(R^M, \bar{R}, ..., \bar{R}) = M/n$ and by the strategy-proofness of $\Phi$ we must have that $M/n R^M \Phi_1(R_0, \bar{R}, ..., \bar{R})$ implying that

$$\Phi_1(R_0, \bar{R}, ..., \bar{R}) \leq \frac{M}{n}. \tag{9}$$

Then by (8) and (9) we have that

$$\Phi(R_0, \bar{R}, ..., \bar{R}) = \left(x_2, \frac{M-x_2}{n-1}, ..., \frac{M-x_2}{n-1}\right)$$

with $y_1 \leq x_2 \leq M/n$ and $x_2 I_0 M/n$ (by construction). But then, the allocation $\left(x_1, \frac{M-x_1}{n-1}, ..., \frac{M-x_1}{n-1}\right)$ contradicts the efficiency of $\Phi$, since $\frac{M-x_2}{n-1} \leq \frac{M-y_1}{n-1}$, $\bar{R} \in \mathcal{R}_s(M)$, and all preference orderings are transitive.

Case (b): Its proof is omitted since follows an argument which is symmetric to the one used to prove Case (a). ■

**Lemma 3** *Let $R_0 \in \mathcal{R}_m(M, n)$ and $x \in [0, M]$ be arbitrary.*
*(a) If $x < M/n \leq \overline{p}(R_0)$ is such that $x I_0 M/n$ then $M/n I_0 x'$ for all $x' \in [x, M/n]$.*
*(b) If $\underline{p}(R_0) \leq M/n < x$ is such that $x I_0 M/n$ then $M/n I_0 x'$ for all $x' \in [M/n, x]$.*

**Proof of Lemma 3.** Case (a): Suppose otherwise; that is, there exist $R_0 \in \mathcal{R}_m(M,n)$ and $x_1 < M/n \leq \overline{p}(R_0)$ such that $x_1 I_0 M/n$ and $M/n P_0 z_1$ for at least one $z_1 \in (x_1, M/n)$. Notice that by Lemma 2 we already know that $M/n R_0 z_1$. Without loss of generality we can assume that there exists $y_1 \in [x_1, M/n]$ such that $M/n I_0 y$ for all $y \in [y_1, M/n]$, $M/n P_0 y$ for all $y \in (x_1, y_1)$, and $z_1 \in (x_1, y_1)$. Note that

$$\frac{M}{n} \leq \frac{M-y_1}{n-1} < \frac{M-z_1}{n-1} < \frac{M-x_1}{n-1}.$$

Now, let $\bar{R} \in \mathcal{R}_s(M)$ be any single-peaked preference such that $p\left(\bar{R}\right) = \frac{M-z_1}{n-1}$ and $\frac{M-x_1}{n-1} \bar{P} \frac{M-y_1}{n-1}$.

By Remark 1, $\Phi\left(R^M, \bar{R}, ..., \bar{R}\right)$ coincides with $\varphi\left(R^M, \bar{R}, ..., \bar{R}\right)$, the uniform allocation rule; therefore, we have that $\Phi_1\left(R^M, \bar{R}, ..., \bar{R}\right) = M/n$. By the strategy-proofness of $\Phi$ we have that

$$\Phi_1\left(R_0, \bar{R}, ..., \bar{R}\right) R_0 \frac{M}{n}. \tag{10}$$

Again, by Remark 1, we have that $\Phi_1\left(R^0, \bar{R}, ..., \bar{R}\right) = z_1$ and by the strategy-proofness of $\Phi$ we also have that $z_1 R^0 \Phi_1\left(R_0, \bar{R}, ..., \bar{R}\right)$ implying that

$$\Phi_1\left(R_0, \bar{R}, ..., \bar{R}\right) \geq z_1. \tag{11}$$

Then, by (10) and (11) we have that

$$\Phi_1\left(R_0, \bar{R}, ..., \bar{R}\right) \geq y_1. \tag{12}$$

Finally, since $\Phi_1\left(R^M, \bar{R}, ..., \bar{R}\right) = M/n$, by the strategy-proofness of $\Phi$ we must have that $M/n R^M \Phi_1\left(R_0, \bar{R}, ..., \bar{R}\right)$ implying that

$$\Phi_1\left(R_0, \bar{R}, ..., \bar{R}\right) \leq \frac{M}{n}. \tag{13}$$

Then by (12) and (13) we have that

$$\Phi\left(R_0, \bar{R}, ..., \bar{R}\right) = \left(x_2, \frac{M-x_2}{n-1}, ..., \frac{M-x_2}{n-1}\right)$$

with $y_1 \leq x_2 \leq M/n$ and $\frac{M-x_2}{n-1} \leq \frac{M-y_1}{n-1}$. Because $\frac{M-x_1}{n-1} \bar{P} \frac{M-y_1}{n-1} \bar{P} \frac{M-x_2}{n-1}$ and $x_1 I_0 x_2$ we have that the allocation $\left(x_1, \frac{M-x_1}{n-1}, ..., \frac{M-x_1}{n-1}\right)$ contradicts efficiency of $\Phi$.

Case (b): Its proof is omitted since follows an argument which is symmetric to the one used to prove Case (a). ■

**Lemma 4** *Let $R_0 \in \mathcal{R}_m(M,n)$ and $x, y \in [0, M]$ be arbitrary.*
*(a) If $M/n \leq x < y \leq \underline{p}(R_0)$ is such that $x I_0 y$ then there exists an interval $[x_0, y_0] \supseteq [x, y]$ such that $x_0 + y_0 > M$ and $x' I_0 y_0$ for all $x' \in [x_0, y_0]$.*
*(b) If $\overline{p}(R_0) \leq x < y \leq M/n$ is such that $x I_0 y$ then there exists an interval $[x_0, y_0] \supseteq [x, y]$ such that $x_0 + y_0 < M$ and $x' I_0 y_0$ for all $x' \in [x_0, y_0]$.*

To prove Lemma 4 we need the following definition.

**Definition 10** *Given a preference $R_0 \in \mathcal{R}(M)$ we say that the interval $[x_0, y_0]$ is a maximal interval of indifference for $R_0$ if $x' I_0 x_0$ for all $x' \in [x_0, y_0]$ and if $[x_1, y_1] \supseteq [x_0, y_0]$ is such that $x I_0 x_0$ for all $x \in [x_1, y_1]$, then $[x_0, y_0] = [x_1, y_1]$.*

**Proof of Lemma 4.** Case (a): Let $R_0 \in \mathcal{R}_m(M, n)$ and suppose that $x$ and $y$ are such that $M/n \leq x < y \leq \underline{p}(R_0)$ and $x I_0 y$. By Lemmas 1, 2 and 3 there exists a maximal interval of indifference for $R_0$, $[x_0, y_0]$, containing $[x, y]$. Notice that $x' I_0 y_0$ for all $x' \in [x_0, y_0]$ and $M/n < y_0$.

In order to obtain a contradiction, assume that $x_0 + y_0 \leq M$. Let $z_0 \in (x_0, y_0)$ be any share such that $M/n \leq z_0$ and

$$(z_0 - x_0) > (y_0 - z_0).$$

Case (a.1): Assume that $z_0$ is such that there exists an integer $n'$ with the properties that $n \geq n' \geq 3$ and

$$(n' - 1)\, z_0 \leq M \leq n' z_0.$$

Notice that this is possible as long as $x_0 + y_0 \leq M$.

Let $\bar{R} \in \mathcal{R}_s(M)$ be such that

$$p(\bar{R}) = M - (n' - 1) z_0 = z_1 \text{ and } \frac{M}{n'} \bar{P} y_1 = M - (n' - 1)\, y_0.$$

Notice that $M/n' \leq z_0$ implies $z_1 = M - (n' - 1) z_0 \leq M/n'$. Therefore, we have

$$y_1 = M - (n' - 1)\, y_0 < M - (n' - 1) z_0 = z_1 \leq \frac{M}{n'}.$$

Define $\mathbf{R}_0 = (\underbrace{R_0, ..., R_0}_{(n'-1)-times}, R^0, ..., R^0, \bar{R}) \in \mathcal{R}_m(M, n)^n$. To show that $\Phi(\mathbf{R}_0) = (\underbrace{z_0, ..., z_0}_{(n'-1)-times}, 0, ..., 0, z_1)$ suppose first that $\Phi(\mathbf{R}_0) = (t_1, ..., t_1, t_2, ..., t_2, t_3)$ with $t_2 > 0$. Since $\Phi(R_0, ..., R_0, R^0, ..., R^0, R_0) = (M/n', ..., M/n', 0, ..., 0, M/n')$ we have that $\Phi_n(\mathbf{R}_0) = t_3 \bar{R} M/n'$, which implies

$$M - (n' - 1)\, y_0 \leq t_3 \leq \frac{M}{n'}. \tag{14}$$

But the allocation $\left(t'_1, .., t'_1, 0, .., 0, t_3\right)$ contradicts efficiency of $\Phi$. To see this, first notice that $0 P^0 t_2$. Moreover, condition (14) implies $M/n' \leq t'_1 \leq y_0$. Therefore, we have that $t'_1 I_0 y_0 R_0 t_1$ since $t_1 \leq y_0$. Hence, $t'_1 R_0 t_1$.

Now assume that $\Phi(\mathbf{R}_0) = \left(\hat{t}_1, .., \hat{t}_1, 0, .., 0, \hat{t}_3\right)$ and $\hat{t}_3 \neq z_1 = M - (n' - 1)\, z_0$. Since $\Phi(R_0, ..., R_0, R^0, ..., R^0, R_0) = (M/n', ..., M/n', 0, ..., 0, M/n')$ we have that $\Phi_n(\mathbf{R}_0) = \hat{t}_3 \bar{R} M/n'$, which implies

$$M - (n' - 1)\, y_0 \leq \hat{t}_3 \leq \frac{M}{n'}. \tag{15}$$

But the allocation $(z_0,..,z_0,0,..,0,z_1)$ contradicts efficiency of $\Phi$. To see this, first notice that $z_1\bar{P}\hat{t}_3$ since $p\left(\bar{R}\right) = z_1$ and $z_1 \neq \hat{t}_3$. Moreover, condition (15) implies $M/n' \leq \hat{t}_1 \leq y_0$. Therefore, we have that $z_0 I_0 y_0 R_0 \hat{t}_1$ since $\hat{t}_1 \leq y_0$. Hence, $z_0 R_0 \hat{t}_1$. Therefore, $\Phi(\mathbf{R_0}) = (\underbrace{z_0,...,z_0}_{(n'-1)-times},0,...,0,z_1)$.

To finish with Case (a.1), suppose first that $y_0 < M$. Let $\varepsilon > 0$ be such that $(z_0 - x_0) > \varepsilon > (y_0 - z_0)$. Because $(z_0 - \varepsilon) \in [x_0, z_0]$, $(z_0 + \varepsilon) > y_0$, and by Lemma 1, we have that $(z_0 - \varepsilon) I_0 z_0$ and $(z_0 + \varepsilon) P_0 z_0$, since $[x_0, y_0]$ is a maximal interval of indifference for $R_0$. Therefore, the allocation

$$((z_0 - \varepsilon), (z_0 + \varepsilon), \underbrace{z_0,...,z_0}_{(n'-3)-times},0,...,0,z_1)$$

contradicts the efficiency of $\Phi$. Now, assume that the extreme case $y_0 = M$ holds. Then, $x_0 = 0$ because our contradiction hypothesis says that $x_0 + y_0 \leq M$. In this case $R_0$ is such that $xI_0y$ for all $x, y \in [0, M]$. But then, the statement of Lemma 4 follows, since for any $x_0' \in (0, M/n)$ we have that $[x_0', y_0] \supseteq [x, y]$, $x_0' + y_0 > M$, and $x' I_0 y_0$ for all $x' \in [x_0', y_0]$.

Case (a.2): Assume that $n > 2$ and that $z_0$ satisfies the following inequalities: $z_0 < M < 2z_0$. Using arguments similar to the ones already used in Case (a.1) it is possible to show that $\Phi(R_0, R_0, R^0, ..., R^0) = (M/2, M/2, 0, ..., 0)$. Since $x_0 + y_0 \leq M$ we have that $y_0 - \frac{M}{2} \leq \frac{M}{2} - x_0$, which implies that we can find an $\varepsilon > 0$ such that $\frac{M}{2} + \varepsilon > y_0$ and $\frac{M}{2} - \varepsilon > x_0$. As before, we can assume that $\frac{M}{2} + \varepsilon \leq M$ because if $y_0 = M$ the statement follows trivially as in Case (a.1). By Lemma 1, $\left(\frac{M}{2} - \varepsilon\right) I_0 M/2$ and $\left(\frac{M}{2} + \varepsilon\right) P_0 M/2$ hold since $[x_0, y_0]$ is a maximal interval of indifference for $R_0$. Therefore, the allocation $\left(\frac{M}{2} - \varepsilon, \frac{M}{2} + \varepsilon, 0, ..., 0\right)$ contradicts the efficiency of $\Phi$.

Case (a.3): Assume that $n = 2$ and remember that we can suppose that $M/n < y_0 < M$. By symmetry, $\Phi(R_0, R_0) = (M/2, M/2)$. We can also find $\varepsilon > 0$ such that $y_0 < z_0 + \varepsilon$, $x_0 < z_0 - \varepsilon$, $(z_0 + \varepsilon) P_0 M/2$, and $(z_0 - \varepsilon) I_0 M/2$. Therefore, the allocation $(z_0 - \varepsilon, z_0 + \varepsilon)$ contradicts the efficiency of $\Phi$.

Case (b): Its proof is omitted since follows an argument which is symmetric to the one used to prove Case (a). ■

**Proof of Theorem 1:** Let $R_0 \in \mathcal{R}_m(M, n)$ be arbitrary. We have to show that $R_0$ is restricted-monotonic on $\Theta(R_0)$. Consider the following cases:

Case (1): Assume that $M/n \leq \underline{p}(R_0)$. Then, $\Theta(R_0) = [M/n, \overline{p}(R_0)]$. To show that property (a) of Definition 8 holds, suppose first that $M/n \leq x < y \leq \underline{p}(R_0)$. Then, by Lemma 1 (part (a)), we have $yR_0x$. If $yI_0x$ then, by Lemma 4 (part (a)), there exists an interval $[x_0, y_0] \supseteq [x, y]$ such that $x_0 + y_0 > M$ and $x' I_0 y_0$ for all $x' \in [x_0, y_0]$. Assume now that $x < M/n \leq y \leq \underline{p}(R_0)$. Then, by Lemma 2 (part (a)), we have that $M/nR_0x$. Moreover, $yR_0M/n$, by Lemma 1 (part (a)). Therefore, since $R_0$ is transitive we must have that $yR_0x$. If $yI_0x$ then, by Lemma 4 (part (a)), there exists an interval $[x_0, y_0] \supseteq [x, y]$ such that $x_0 + y_0 > M$ and $x' I_0 y_0$ for all $x' \in [x_0, y_0]$. To show that property (c) of Definition 8 holds, suppose that $x \in \left(\underline{p}(R_0), \overline{p}(R_0)\right)$.

Then, $M/n \leq \underline{p}(R_0) < x < \overline{p}(R_0)$ which implies, by Lemma 1 (part (a)), that $xR_0\underline{p}(R_0)$, and hence $xI_0\bar{P}(R_0)$.

Case (2): Assume that $\underline{p}(R_0) \leq M/n \leq \overline{p}(R_0)$. Then, $\Theta(R_0) = \left[\underline{p}(R_0), \overline{p}(R_0)\right]$. To show that property (c) of Definition 8 holds, assume first that $\underline{p}(R_0) = M/n$ and let $x$ be any share such that $\underline{p}(R_0) < x \leq \overline{p}(R_0)$. By Lemma 1 (part (a)) $xR_0\underline{p}(R_0)$ which implies that $xI_0\overline{p}(R_0)$. Assume now that $\underline{p}(R_0) < M/n \leq \overline{p}(R_0)$. By Lemma 2 (part (a)) we have that

$$\frac{M}{n}R_0\underline{p}(R_0). \tag{16}$$

First, let $x$ be any share such that $\underline{p}(R_0) < x < M/n \leq \overline{p}(R_0)$. By Lemma 1 (part (b)) $xR_0M/n$ and by condition (16) we have that $xI_0\overline{p}(R_0)$. Second, let $x$ be any share such that $\underline{p}(R_0) < M/n < x \leq \overline{p}(R_0)$. By Lemma 1 (part (a)) $xR_0M/n$ and by condition (16) we have that $xI_0\overline{p}(R_0)$.

Case (3): Assume that $\overline{p}(R_0) \leq M/n$. Then, $\Theta(R_0) = \left[\underline{p}(R_0), M/n\right]$. The proof that properties (b) and (c) of Definition 8 hold is symmetrical to that of Case (1), using the respective parts (b)'s of Lemmata 1, 2, and 4.

The proof of Theorem 1 is completed by exhibiting a rule on the set of restricted-monotonic preferences on $\Theta$, $\mathcal{R}^{\Theta}_{rm}(M,n)$, that satisfies the properties of strategy-proofness, efficiency, and symmetry. We obtain such a rule by extending the uniform allocation rule $\psi$ on the domain of single-plateaued preferences, $\mathcal{R}_{sp}(M)$, to this larger domain.

The *extended uniform rule on* $\mathcal{R}^{\Theta}_{rm}(M,n)$, $\Psi : \mathcal{R}^{\Theta}_{rm}(M,n)^n \to Z(M)$, is defined by the following algorithm: let $\mathbf{R} = (R_1, ..., R_n) \in \mathcal{R}^{\Theta}_{rm}(M,n)^n$ be any profile of restricted-monotonic preferences on $\Theta$.

Stage 0: Let $\bar{\mathbf{R}} = \left(\bar{R}_1, ..., \bar{R}_n\right) \in \mathcal{R}_{sp}(M)^n$ be any profile of single-plateaued preferences such that $\left[\underline{p}(R_i), \overline{p}(R_i)\right] = \left[\underline{p}\left(\bar{R}_i\right), \overline{p}\left(\bar{R}_i\right)\right]$ for all $i \in N$. Compute $\psi\left(\bar{\mathbf{R}}\right)$ and let $S^0$ be the set of agents receiving an amount on the interior of a maximal interval of indifference for $R_i$ (the original preference), denoted by $[x^0_i, y^0_i]$, such that $[x^0_i, y^0_i] \neq \left[\underline{p}(R_i), \overline{p}(R_i)\right]$; that is,

$$S^0 = \left\{ i \in N \mid \begin{array}{l} \psi_i\left(\bar{\mathbf{R}}\right) \in (x^0_i, y^0_i) \text{ where } [x^0_i, y^0_i] \text{ is a maximal interval} \\ \text{of indifference for } R_i \text{ and } \underline{p}(R_i)\, P_i x \text{ for all } x \in [x^0_i, y^0_i] \end{array} \right\}.$$

If $S^0 = \emptyset$ then define $\Psi(\mathbf{R}) = \psi\left(\bar{\mathbf{R}}\right)$ and stop. If $S^0 \neq \emptyset$ then select any profile $\mathbf{R}^1 = (R^1_1, ..., R^1_n) \in \mathcal{R}^{\Theta}_{rm}(M,n)^n$ such that $R^1_i = R_i$ for all $i \notin S^0$ and for all $i \in S^0$

$$R^1_i = \begin{cases} R_i \text{ on } [0, y^0_i] \text{ and } y^0_i P^1_i x \text{ for all } x > y^0_i & \text{if } M \leq \sum \underline{p}(R_j) \\ R_i \text{ on } [x^0_i, M] \text{ and } x^0_i P^1_i x \text{ for all } x < x^0_i & \text{if } \sum \underline{p}(R_j) \leq M \end{cases}.[7]$$

Go to stage 1.

[7] Notice that the efficiency of $\psi$ implies that if $M \leq \sum \underline{p}(R_j)$ then $\psi_i(\mathbf{R}) \leq \underline{p}(R_i)$ and therefore $y^0_i < \underline{p}(R_i)$. Symmetrically, if $\sum \underline{p}(R_j) \leq M$ then $\underline{p}(R_i) \leq \psi_i(\mathbf{R})$ and therefore $\overline{p}(R_i)$. The same argument will apply also in all stages.

Now, for k$\geq$ 1, and given that the algorithm has not stopped yet at stage k-1, stage k is as follows.

Stage k: Given the preference profile $\mathbf{R}^k = \left(R_1^k, ..., R_n^k\right) \in \mathcal{R}_{rm}^{\Theta}(M,n)^n$, the outcome of stage k-1, let $\bar{\mathbf{R}}^k = \left(\bar{R}_1^k, ..., \bar{R}_n^k\right) \in \mathcal{R}_{sp}(M)^n$ be any profile of single-plateaued preferences such that $\left[\underline{p}\left(R_i^k\right), \overline{p}\left(R_i^k\right)\right] = \left[\underline{p}\left(\bar{R}_i^k\right), \overline{p}\left(\bar{R}_i^k\right)\right]$ for all $i \in N$. Compute $\psi\left(\bar{\mathbf{R}}^k\right)$. If $\psi\left(\bar{\mathbf{R}}^k\right) = \psi\left(\bar{\mathbf{R}}^{k-1}\right)$, define $\Psi(\mathbf{R}) = \psi\left(\bar{\mathbf{R}}^k\right)$ and stop. Otherwise, let $S^k$ be the set of agents receiving an amount on the interior of a maximal interval of indifference for $R_i^k$, denoted by $\left[x_i^k, y_i^k\right]$, such that $\left[x_i^k, y_i^k\right] \neq \left[\underline{p}\left(R_i^k\right), \overline{p}\left(R_i^k\right)\right]$; that is,

$$S^k = \left\{ i \in N \mid \begin{array}{l} \psi_i\left(\bar{\mathbf{R}}^k\right) \in \left(x_i^k, y_i^k\right) \text{ where } \left[x_i^k, y_i^k\right] \text{ is a maximal interval} \\ \text{of indifference for } R_i^k \text{ and } \underline{p}\left(R_i^k\right) P_i^k x \text{ for all } x \in \left[x_i^k, y_i^k\right] \end{array} \right\}.$$

If $S^k = \emptyset$ then define $\Psi(\mathbf{R}) = \psi\left(\bar{\mathbf{R}}^k\right)$ and stop. If $S^k \neq \emptyset$ then select any profile $\mathbf{R}^{k+1} = \left(R_1^{k+1}, ..., R_n^{k+1}\right) \in \mathcal{R}_{rm}^{\Theta}(M,n)^n$ such that $R_i^{k+1} = R_i^k$ for all $i \notin S^k$ and for all $i \in S^k$

$$R_i^{k+1} = \begin{cases} R_i^k \text{ on } \left[0, y_i^k\right] \text{ and } y_i^k P_i^{k+1} x \text{ for all } x > y_i^k & \text{if } M \leq \sum \underline{p}\left(R_j^k\right) \\ R_i^k \text{ on } \left[x_i^k, M\right] \text{ and } x_i^k P_i^{k+1} x \text{ for all } x < x_i^k & \text{if } \sum \underline{p}\left(R_j^k\right) \leq M \end{cases}.$$

Go to stage k+1.

The algorithm stops after at most $n$ stages. This is because the sets $S^k$ only contain players whose stage $k$ proposed shares are not maximal. Hence, for all $K \geq 2$

$$S^K \cap \left(\bigcup_{k=0}^{K-1} S^k\right) = \emptyset.$$

Note that the rule $\Psi$ satisfies strategy-proofness and symmetry. To show that it satisfies efficiency, let $\mathbf{R} = (R_1, ..., R_n) \in \mathcal{R}_{rm}^{\Theta}(M,n)^n$ be arbitrary and consider the following cases:

Case (1): Assume that $\sum \underline{p}(R_j) \leq M \leq \sum \overline{p}(R_j)$. Then, efficiency is clearly satisfied because $\psi_i\left(\bar{\mathbf{R}}^0\right) \in \left[\underline{p}(R_i), \overline{p}(R_i)\right]$ for all $i \in N$ implies that $S^0 = \emptyset$ and the process stops at stage 0 after setting $\Psi(\mathbf{R}) = \psi\left(\bar{\mathbf{R}}^0\right)$. Therefore, $\overline{p}(R_i) I_i \Psi_i(\mathbf{R})$ for all $i \in N$, which means that $\Psi$ is efficient.

Case (2): Assume that $M \leq \sum \underline{p}(R_j)$. Then, it is easy to show that $\Psi_i(\mathbf{R}) \leq \underline{p}(R_i)$ for all $i \in N$. Let $S$ be the subset of agents who are rationed; that is,

$$S = \left\{ i \in N \mid \Psi_i(\mathbf{R}) < \underline{p}(R_i) \right\}.$$

If $S = \emptyset$, then $\sum \underline{p}(R_j) = M$ and $\Psi_i(\mathbf{R}) = \underline{p}(R_i)$ for all $i \in N$, in which case, the efficiency of $\Psi$ follows. Therefore, suppose $S \neq \emptyset$ and assume that $\Psi$ violates efficiency at profile $\mathbf{R}$; that is, there exist a feasible allocation $r = (r_1, .., r_n) \in Z(M)$ and $j \in N$ such that:

$$r_i R_i \Psi_i(\mathbf{R}) \text{ for all } i \in N \text{ and}$$

$$r_j P_j \Psi_j(\mathbf{R}). \tag{17}$$

However, (17) and the definition of $\Psi$ imply that $\Psi_j(\mathbf{R}) < r_j$. Therefore, there exists $k \in N$ such that $\Psi_k(\mathbf{R}) > r_k$, since $\Psi(\mathbf{R}) \in Z(M)$. Then $\Psi_k(\mathbf{R}) I_k r_k$. By definition of $\Psi$ we have that $\underline{p}(R_i) = \Psi_k(\mathbf{R})$ for all $i \in N$ such that $\underline{p}(R_i) \leq M/n$ and

$$\Psi_j(\mathbf{R}) = \Psi_k(\mathbf{R}), \tag{18}$$

since $\Psi_j(\mathbf{R}) \neq \underline{p}(R_j)$ and $\Psi_k(\mathbf{R}) \neq \underline{p}(R_k)$. If $\Psi_k(\mathbf{R}) + r_k \geq M$, then $\Psi_k(\mathbf{R}) > M/2$ and by (18) we have that $\Phi_j(\mathbf{R}) > M/2$ which implies that $\Psi(\mathbf{R}) \notin Z(M)$, a contradiction. Assume that $\Psi_k(\mathbf{R}) + r_k < M$. First, $r_k R_k \Psi_k(\mathbf{R})$ and $r_k < \Psi_k(\mathbf{R})$ imply that $r_k I_k \Psi_k(\mathbf{R})$. Therefore, by Definition 8, there exists a maximal interval of indifference for $R_k$, call it $[x_k, y_k]$, such that $\Psi_k(\mathbf{R}) \in [x_k, y_k]$. But then, the definition of $\Psi$ implies that $r_k \in [x_k, y_k]$, a contradiction.

Case (3): Assume that $\sum \overline{p}(R_j) \leq M$. Then, an argument symmetric to the one used in Case (2) proves that $\Psi$ is efficient. ■

# 4 Concluding Remarks

We finish this paper with two remarks. First, we show how to obtain Ching and Serizawa [9] result as an implication of our Theorem. While we have considered $M$ as an exogenous data, they formulate the division problem for all possible values of $M$ by letting rules depend not only on preferences profiles but also on all possible amounts of the good to be allocated. This distinction has important consequences for the maximality problem since their approach implies that preferences have to be defined over all positive shares, and consequently the same domain of preferences has to be maximal for *all* values of $M$, while our approach allows to find the maximal domain of preferences (on $[0, M]$) for *each* value $M$. Therefore, to formulate the division problem in their setting, assume now that every agent $i \in N$ has a continuous preference ordering over the interval $[0, \infty)$ and denote by $\mathcal{R}(\infty)$ the set of all these preference orderings.

A *rule on* $\mathcal{U} \subseteq \mathcal{R}(\infty)$ and $(0, \infty)$ is a function $\Phi^{\infty} : \mathcal{U}^n \times (0, \infty) \longrightarrow \Re^n$ such that $\sum \Phi_i^{\infty}(\mathbf{R}, M) = M$ for all $(\mathbf{R}, M) \in \mathcal{U}^n \times (0, \infty)$.

Consider the natural extensions of strategy-proofness, efficiency, and symmetry to this new setting where rules are defined on $\mathcal{U}$ and $(0, \infty)$.[8] Denote them by $sp(\infty)$, $eff(\infty)$, and $sy(\infty)$.

Definition below adapts our concept of maximal domain of preferences to their setting.

**Definition 11** *A set $\mathcal{R}_m$ of preferences is a maximal (infinite) domain for a list of properties if: (1) $\mathcal{R}_m \subseteq \mathcal{R}(\infty)$; (2) there exists a rule on $\mathcal{R}_m$ and $[0, \infty)$ satisfying the properties; and (3) there is no rule on $\mathcal{R}'$ and $[0, \infty)$ satisfying the same properties such that $\mathcal{R}_m \subsetneq \mathcal{R}' \subseteq \mathcal{R}(\infty)$.*

[8]This means that we have to replace, in Definitions 1, 2, and 3, the expression "for all $\mathbf{R} \in \mathcal{U}^n$" by the expression "for all $(\mathbf{R}, M) \in \mathcal{U}^n \times (0, \infty)$".

Now, Ching and Serizawa [9] result can be stated (and proved as a Corollary of our Theorem) as follows.

**Theorem 2** *(Ching-Serizawa) The set of single-plateaued preferences, $\mathcal{R}_{sp}(\infty)$, is the unique maximal (infinite) domain including single-peaked preferences for $sp(\infty)$, $eff(\infty)$, and $sy(\infty)$.*

**Proof.** Let $\mathcal{R}_a$ be a domain such that there is a rule $\Phi^\infty$ on $\mathcal{R}_a$ and $(0,\infty)$ satisfying $sp(\infty)$, $eff(\infty)$, and $sy(\infty)$. Assume also that $\mathcal{R}_s(\infty) \subseteq \mathcal{R}_a$. Then, for each $M \in (0,\infty)$, we have that $\Phi_M : \mathcal{R}_a(M)^n \longrightarrow Z(M)$ satisfies strategy-proofness, symmetry and efficiency, where $\mathcal{R}_a(M)$ is the restriction to $[0, M]$ of preferences in $\mathcal{R}_a$ and after setting $\Phi_M(\mathbf{R}) = \Phi^\infty(\mathbf{R}, M)$. Then, by Theorem 1, $\mathcal{R}_a(M) = \mathcal{R}^{\Theta}_{rm}(M,n)$ for every $M \in (0,\infty)$. Since this is true for every $M$ we have that $\mathcal{R}_a = \bigcap_{M>0} \mathcal{R}^{\Theta}_{rm}(M,n)$. Finally, one sees immediately that $\mathcal{R}_{sp}(\infty) = \bigcap_{M>0} \mathcal{R}^{\Theta}_{rm}(M,n)$. Hence $\mathcal{R}_a = \mathcal{R}_{sp}(\infty)$. ■

Second, the interval $\Theta(R_i)$ is intimately related with "option" sets, where given a rule $\Phi$ on $\mathcal{U}$ and a preference $R_i \in \mathcal{U}$ we define the *option set left by $R_i$ at $\Phi$* as

$$\sigma^{\Phi}(R_i) = \left\{x \in [0, M] \mid \exists R_{-i} \in \mathcal{U}^{n-1} \text{ such that } \Phi_i(R_i, R_{-i}) = x\right\}.$$

This is not surprising, since option sets also play a fundamental role to describe maximal domains in voting environments. The main two ideas are the following. Given a preference $R_i$, alternatives at the left (right) of the top and outside the option set have to be worse than the smallest (largest) alternative in the option set. Moreover, the preference $R_i$ has to be single-peaked on the option set.

It is easy to show here that, given a preference $R_i \in \mathcal{R}^{\Theta}_{rm}(M,n)$ and a strategy-proof, efficient and symmetric rule on $\mathcal{R}^{\Theta}_{rm}(M,n)$, the relationship between $\Theta(R_i)$ and $\sigma^{\Phi}(R_i)$ is as follows. Suppose that $R_i$ is such that $M/n$ does not belong to an indifference interval, then $\Theta(R_i) = \sigma^{\Phi}(R_i)$. However, if $M/n$ belongs to an indifference interval, then $\sigma^{\Phi}(R_i) = \Theta(R_i) \cup [x_0, y_0]$, where $[x_0, y_0]$ is the maximal interval of indifference for $R_i$ that contains $M/n$.

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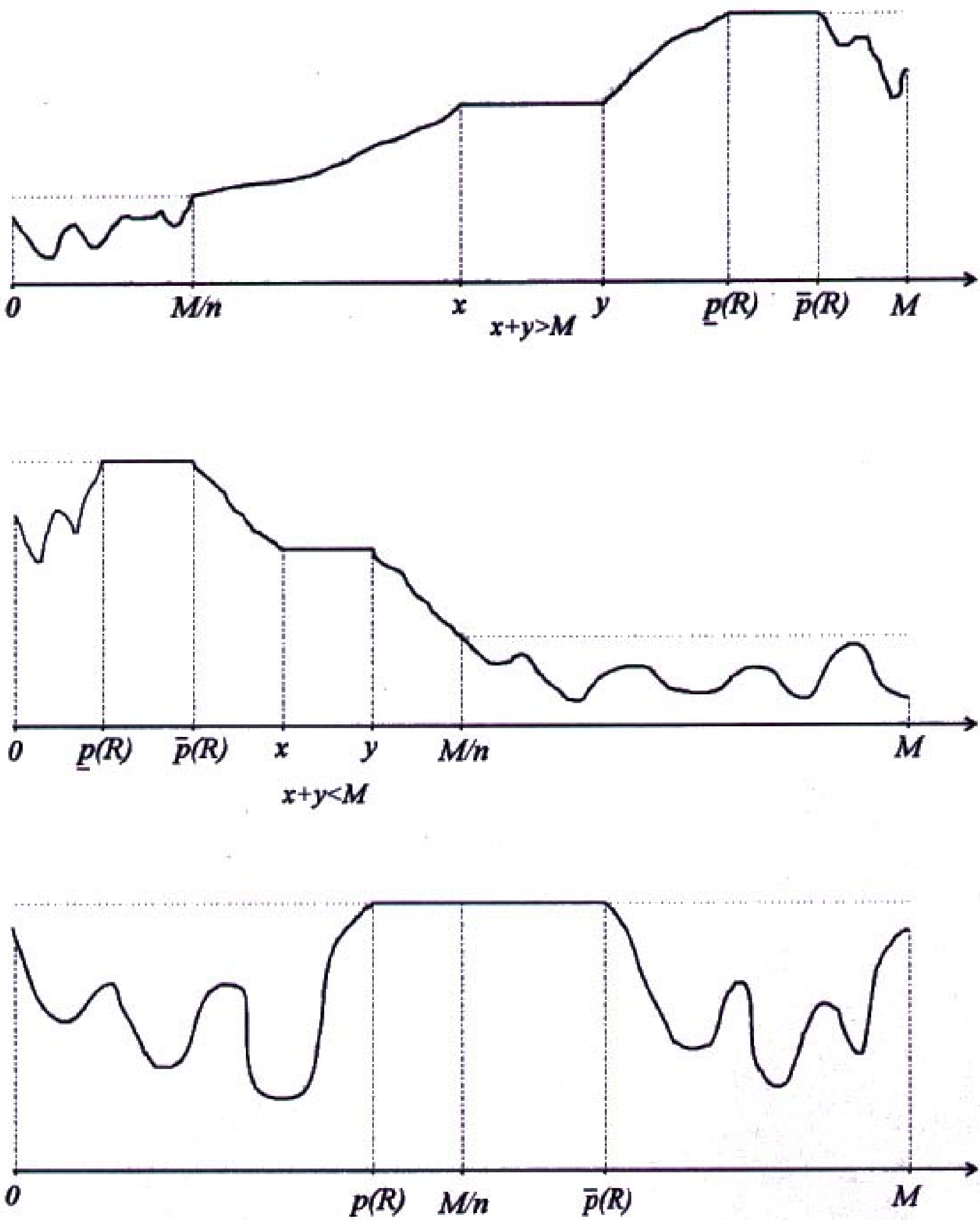


**Figure 1**

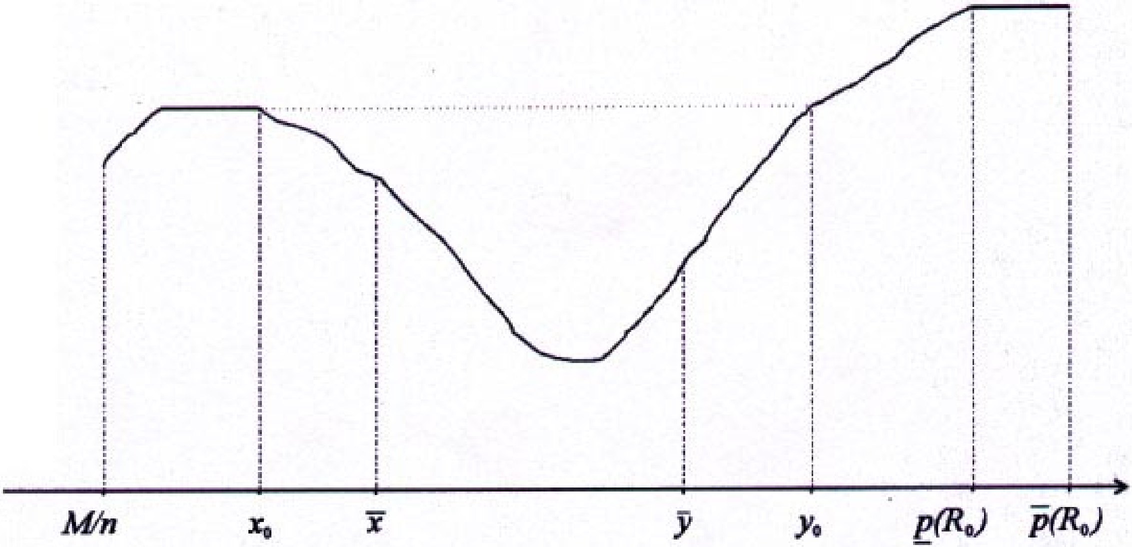


Figure 2