Strategy-Proof Allotment Rules

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We consider the problem of allotting shares of a task or good among agents with single peaked preferences over their own shares. Previous characterizations have examined rules, such as the uniform rule, which satisfy various symmetry requirements. We consider the case where agents might begin with natural claims to minimal or maximal allotments, or might be treated with different priorities. We provide characterizations of the rules which are strategy-proof and efficient, but which may treat individuals asymmetrically. *Journal of Economic Literature* Classification Numbers: D71, D78, D63.   © 1997 Academic Press

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1. INTRODUCTION

There are many situations where a group of agents must share a task or a good, including dividing up assets among creditors in a bankruptcy (e.g., Aumann and Maschler, 1985), sharing the cost of a public project or the surplus of a joint venture (e.g., Moulin, 1985a, 1985b, 1987, and Young, 1987), and rationing goods traded at fixed prices (e.g., Benassy, 1982; Barberà and Jackson, 1995). In allotting the task or good there are several issues which are of interest, such as efficiency, equity, and incentive compatibility.[1]

We examine sharing problems when individuals have single-peaked preferences over their desired share. Consider, for instance, a partnership investing in a project, where the benefits from the project are paid out in proportion to the amount each partner invests. Suppose that each partner has an ideal amount that they want to invest and has preferences that are single peaked about that amount. The total of these ideal investment amounts may add up to more or less than the total amount required to undertake the project, and so partners may have to invest more or less than their ideal amounts. There are many different ways in which one can decide how much each partner will be allowed (or required) to invest. One could simply require that each partner make an equal investment. This solution is not very satisfactory, since it is Pareto inefficient in cases where some partners wish to invest more than the equal amount and other partners would like to invest less than that amount. A more sophisticated method for deciding on the investment levels is by the uniform rule. The uniform rule may be thought of as starting from the equal amounts solution, but then correcting it to be efficient. To illustrate the uniform rule, suppose that the total of the partners' ideal investment levels exceeds the total required to undertake the project.[2] If there are any partners who wish to invest less than an equal share, let them invest exactly their ideal amounts. This frees up additional investment opportunities for the remaining partners. Now divide this remaining total equally among the remaining partners. If any of them desire less than this equal amount, then let them have their ideal investment levels. Continue iterating this procedure until all of the remaining partners' ideal investment levels are at least as high as the equal shares of the remaining total. These partners each invest this

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[1] See Moulin and Thomson (1995) for a recent survey of axiomatic approaches to resource allocation.

[2] The case where the total is less than or equal to the required amount is analogous. A complete definition is provided in Section 2.

equal share. The uniform rule has many nice characteristics: it is efficient, strategy-proof, anonymous, and envy-free,[3] and is consistent, individually rational from equal division, and satisfies (one-sided) replacement-domination, and population and resource monotonicity conditions.[4] All of these properties seem to make it an overwhelming candidate to be the "ideal" allotment rule.

However, the anonymity and envy-free properties of the uniform rule make it an inappropriate rule when there are asymmetries among the agents that one wishes to respect.[5] Often partnerships are repeated relationships, where based on historical contributions or tenure some partners may be considered more senior than others. In other applications, other considerations may lead some agents to have natural claims to minimal or maximal allotments. For instance, when sharing a task, some participants may be younger and one might wish to limit their inputs. There are allotment rules which respect such seniority or asymmetry, while still respecting efficiency and strategy proofness. In the case of the partnership, a simple such rule is a queuing rule where partners are lined up according to seniority (see Benassy, 1982). The most senior partners are allowed to choose their investment levels first. The second most senior partners choose next, and so forth. If at some level there is a need to ration, it can be done uniformly. An alternative method which is based on seniority or other asymmetries, and still respects efficiency and strategy-proofness, is to entitle partners with different guaranteed levels of investment. (The uniform rule entitles partners to equal shares: what they obtain is always at least as good as an equal share.) These shares can have guaranteed minimum and maximum levels. A partner would end up with a share outside of this limit only if the total of the ideal amounts permits it and he or she desires it. Feasibility imposes restrictions on how one can choose these minimal and maximal levels across agents, as described in our theorems. Note that such rules, when coupled with existing asymmetries, can result in more equitable allotments than a rule which treats all agents symmetrically in the strong senses implied by anonymity or envy-freeness.

[3] See the characterizations in Sprumont (1991). Ching (1992, 1994) extends Sprumont's characterizations to a domain of single plateaued preferences and weakens the envy-free and anonymity conditions to an equal treatment of equals property.

[4] Consistency requires that a rule's recommendations are unaltered if it is reapplied to the remaining pie after some agents leave with their allotments. The replacement-domination condition requires that agents' welfares all be effected in the same direction as some characteristic of the economy changes, while the monotonicity conditions make similar requirements on the allotments. Characterizations involving these conditions are given by Thomson (1992, 1994a, 1994b, 1995).

[5] Thomson (1994a) provides some other criticisms of the uniform rule.

In this paper we characterize a class of allotment rules that are strategy-proof, efficient, and satisfy replacement monotonicity. Strategy-proofness is quite appealing from the point of view of decentralization; but it is known to be a strong requirement on any collective decision making rule (Gibbard 1973; Satterthwaite, 1975). Yet, under appropriate domain restrictions, nontrivial strategy-proof mechanisms are known to exist. An interesting example for the case of one-dimensional decisions is given by the generalized majority rules (see Moulin, 1980, and Barberà and Jackson, 1994). For the case of two agents, the generalized majority rules and strategy-proof allotment rules are closely related, as we shall describe. This close relationship is due to the fact that with two agents, an agent's preferences can be expressed as preferences over the other agent's allotment since this uniquely determines his or her own allotment. With more than two agents, however, this relationship is no longer true and so the allotment setting is complicated by the fact that the alternatives are multidimensional.

The replacement monotonicity condition applies to situations where one individual's preferences change, which may lead to a change in his or her allotment. In such a situation, there will be a compensating change in the remaining agents' allotments. The replacement monotonicity requires that no two of the remaining agents' allotments move in opposite directions. One might classify this as a very basic symmetry condition. The idea is that an increase in one individual's allotment decreases the amount left to be alloted among the remaining agents. The restriction is that none of the remaining agents' allotments can increase as a result. This does not mean that they have to be affected in a similar manner—some of the remaining allotments might not change at all. It is simply that none of the remaining agents allotments should increase as a result of this change.[6] This condition does not make any requirements on the levels or rates of change of allotments. This is an important distinction from conditions such as equal treatment of equals, anonymity, and envy-freeness. This allows us to capture rules of the type informally described above (thus avoiding possible conflicts with equity), while still imposing enough order (or fairness) to clearly define a class of rules.

The paper proceeds as follows. After this introduction, Section 2 provides notation, definitions, and Sprumont's (1991) result in a form which will help us to describe our procedures. In Section 3 we provide a full

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[6] Much of the normative justification for this condition relies on single-peaked preferences and efficiency. This means that the condition is equivalent to a statement in terms of the welfare of the remaining agents. It should not be that some of them are made better off, while others are made worse off. This condition (one-sided replacement-domination) is defined and discussed by Thomson (1992).

characterization of strategy-proof and efficient allotment rules and examples which illustrate some unappealing aspects of the rules which are admitted if no further conditions are imposed. In Section 4 we introduce and characterize a new subclass of strategy-proof and efficient allotment rules, which can be thought of as natural extensions of the procedure which underlies the uniform rule. Section 5 concludes.

2. NOTATION, DEFINITIONS, AND THE UNIFORM RULE

$N = \{1, \ldots, n\}$ is a finite set of *agents*.

Allotments are n-tuples a in the set $A = \{a \in [0, 1]^n \mid \sum_{i \in N} a_i = 1\}$.

An agent's preferences over allotments are selfish and are thus identified with a complete pre-ordering on $[0, 1]$. Given two allotments a and a', agent i prefers a to a' if and only if he or she prefers a_i to a'_i. Preferences are also assumed to be continuous, and will therefore always be representable by continuous utility functions. These are denoted u_i, u'_i, u_j, etc.

Finally, preferences are assumed to be *single peaked with a unique ideal point*. That is, if u_i represents the preferences of $i \in N$, then there exists $x^* \in [0, 1]$ such that for any $y, z \in [0, 1]$

$$x^* < y < z \Rightarrow u_i(x^*) > u_i(y) > u_i(z),$$

and

$$z < y < x^* \Rightarrow u_i(x^*) > u_i(y) > u_i(z).$$

We call x^* the peak of u_i and denote it by $x^*(u_i)$. Let S denote the set of all continuous utility functions representing single-peaked preferences on $[0, 1]$.

Preference profiles are given by n-tuples of utility functions. They are denoted u, u', etc., with u_{-i} representing the $(n - 1)$-tuple obtained from u by deleting u_i and (u_{-i}, v_i) representing the n-tuple obtained from u by substituting v_i for u_i.

An *allotment rule* associates a vector of shares with each preference profile. It is thus a function $f: S^n \to [0, 1]^n$ satisfying *feasibility*:

$$\sum_{i \in N} f_i(u) = 1 \qquad \text{for all } u \in S^n.$$

We consider only single valued rules, as opposed to correspondences.

A standard requirement on allotment rules is *efficiency*: the selected allotment should be Pareto efficient at each preference profile. It is easy to check that when coupled with the assumption that preferences are single-peaked, the efficiency requirement is equivalent to requiring that at each

$u \in S^n$:

$$\left[\sum_{i \in N} x^*(u_i) \leq 1 \right] \Rightarrow \left[x^*(u_i) \leq f_i(u) \; \forall i \in N \right]$$

and

$$\left[\sum_{i \in N} x^*(u_i) \geq 1 \right] \Rightarrow \left[x^*(u_i) \geq f_i(u) \; \forall i \in N \right].$$

Another basic requirement of allotment rules is *strategy-proofness*:

$$u_i\big(f_i(u)\big) \geq u_i\big(f_i(u_{-i}, v_i)\big) \qquad \forall i \in N, \, u \in S^n, \text{ and } v_i \in S.$$

The virtue of strategy-proofness is that it guarantees that agents will have no incentive to manipulate the allotments. It implies that agents will voluntarily declare their true preferences when asked, which establishes the desired connection between individual preferences and allotments, as expressed by the rule f.

A third property of allotment rules is *anonymity*: for all permutations π of N (π is a function from N onto N) and $u \in S^n$, $f_{\pi(i)}(u^\pi) = f_i(u)$ where $u^\pi = (u_{\pi^{-1}(1)}, \ldots, u_{\pi^{-1}(n)})$. Anonymity implies an equal treatment of equals property which is sometimes desirable: agents with identical preferences are treated identically.[7] As we argued in the Introduction, it need not be desirable in cases where different agents have different entitlements, rights, or endowments. Sprumont (1991) has provided a full characterization of efficient, strategy-proof, and anonymous allotment rules. It is our purpose to extend his results to the nonanonymous case. Before we do, it is helpful to present Sprumont's main result.

The *uniform allotment rule f^**, is defined by

$$f_i^*(u) = \begin{cases} \min\big[x^*(u_i), \lambda(u) \big] & \text{if } \sum_{i \in N} x^*(u_i) \geq 1, \\ \max\big[x^*(u_i), \mu(u) \big] & \text{if } \sum_{i \in N} x^*(u_i) \leq 1, \end{cases}$$

where $\lambda(u)$ solves $\sum_{i \in N} \min[x^*(u_i), \lambda(u)] = 1$, and $\mu(u)$ solves $\sum_{i \in N} \max[x^*(u_i), \mu(u)] = 1$.

THEOREM (Sprumont). *An allotment rule is efficient, anonymous, and strategy-proof, if and only if it is the uniform rule.*

In order to begin to extend the characterization to the nonanonymous case, let us look at the case of two individuals, which does not capture all of the features of the problem but gives us some interesting hints. With

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[7] See Ching (1994) for more on the equal treatment of equals property and its role in characterizing the uniform rule.

two agents, the allotment is fully described by a_1, since $a_2 = 1 - a_1$. Hence, the preferences of agent 2 can be expressed as preferences on a_1 as well, by letting $\tilde{u}_2(a_1) = u_2(1 - a_1)$. Clearly, $\tilde{u}_2$ is continuous and single peaked whenever u_2 is. The allotment problem is now reduced to choosing a single point in $[0, 1]$ when both agents have preferences which are single peaked over the same variable. It is then known (Moulin, 1980; Barberà and Jackson, 1994) that the only strategy-proof efficient and anonymous rule is the one which chooses the median among the ideal points of the agents and a third "phantom" voter at $1/2$. (We leave it to the reader to check that this is indeed equivalent to the uniform rule, given $n = 2$). It is interesting to note that any other median voter rule, with the phantom at any point in $p \in [0, 1]$, is also strategy-proof and efficient. The point p plays a role similar to that of a guaranteed level, since it suffices for either of the two agents to propose p for this to become agent 1's share. Thus, agent 1 can guarantee himself p and agent 2 can guarantee herself $1 - p$.

There are, in fact, more complex two-person strategy-proof and efficient rules, and they correspond to those described by Moulin's theorem. As shown in Barberà and Jackson (1994), such rules depend only on the ideal points of the agents. Let x_2^* represent the ideal point of $\tilde{u}_2$ (agent 2's preference expressed in terms of a_1). Fix $0 \leq p_1 \leq p_2 \leq 1$. Two-person strategy-proof and efficient rules are of the following form:

$$f(x_1^*, x_2^*) = \begin{cases} x_1^* & \text{if } p_1 \leq x_1^* \leq p_2 \\ \min(\max[x_1^*, x_2^*], p_1) & \text{if } x_1^* < p_1 \\ \max(\min[x_1^*, x_2^*], p_2) & \text{if } x_1^* > p_2 \end{cases}$$

The case where $p_1 = p_2$ corresponds to the simple median voter rule, while the dictatorial rule corresponds to the case where $p_1 = 0$ and $p_2 = 1$. A symmetric formula can be written by interchanging the roles of agents 1 and 2.

The above characterization for two agents gives us a hint toward the extension to $n \geq 3$. An agent is offered an interval of values. If one of the values is accepted, the agent is allotted that value. Otherwise we proceed to a next step, in which a median allotment is chosen, with the guarantee that the agent is not penalized for having let the previous opportunity go by.

Let us take a last look at the uniform rule through an example with $n = 5$ agents with ideal points $x_1^* = \frac{3}{20}$, $x_2^* = \frac{5}{20}$, $x_3^* = \frac{2}{20}$, $x_4^* = \frac{6}{20}$, and $x_5^* = \frac{14}{20}$.

As described loosely in the Introduction, the uniform rule can be reached through the following algorithm (see Sönmez, 1994):

Step 1. *Determine whether* $\sum_{i \in N} x^*(u_i)$ *equals, exceeds, or falls short of* 1. *If* $\sum_{i \in N} x^*(u_i) = 1$, *then allot shares equal to the ideal points. If* $\sum_{i \in N} x^*(u_i) > 1$, *allot their ideal points to those agents who demand no more than* $1/n$. *If* $\sum_{i \in N} x^*(u_i) < 1$, *allot their ideal points to those agents who demand at least* $1/n$.

In our case $\sum_{i \in N} x^*(u_i) > 1$, and agents 1 and 3's ideal points are less than $\frac{1}{5}$. Thus $a_1 = \frac{3}{20}$ and $a_3 = \frac{2}{20}$.

Step 2. *Determine the remaining number of agents to be allotted and the remaining share to be allotted. Say there are k agents and an amount s to be shared. Perform the same procedure as in Step* 1, *letting s replace* 1 *and considering only the k agents. Iterate on this step until all of the k′ remaining agents have ideal points exceeding* (*respectively, falling short of*) s'/k'.

In our case, $k = 3$ and $s = \frac{15}{20}$. Agent 2 is allotted $a_2 = \frac{5}{20}$. There are now $k' = 2$ agents remaining with $s' = \frac{10}{20}$. Each has an ideal point which exceeds $s'/k' = \frac{5}{20}$.

Step 3. *Allot the remaining k′ agents s′/k′ each.*

In our case, $a_4 = a_5 = \frac{5}{20}$.

We conclude that agents are allotted the shares $(\frac{3}{20}, \frac{5}{20}, \frac{2}{20}, \frac{5}{20}, \frac{5}{20})$, which corresponds to the outcome of the uniform rule with $\lambda(u) = \frac{5}{20} = \frac{1}{4}$.

The above descriptions suggests a number of possible ways to extend an allotment rule to drop anonymity:

(1) Rather than have $1/n$ as a starting reference point, choose any collection of shares such that $\sum_{i \in N} q_i = 1$.

(2) Rather than having the same reference point for the cases of $\sum_{i \in N} x^*(u_i) < 1$ and $\sum_{i \in N} x^*(u_i) > 1$, choose different reference points q_i^L and q_i^H.

(3) Let the reference levels depend on the share remaining in each iteration of Step 2 (with some qualifications on that dependence, in order to preserve strategy-proofness).

The above remarks give us ideas concerning some of the ways in which we can alter the uniform rule and still retain strategy-proofness and efficiency. The characterization in the next section shows us all of the ways.

3. STRATEGY-PROOF AND EFFICIENT ALLOTMENT RULES

Lemma 2 in Sprumont (1991) provides the basis for a characterization of all strategy-proof and efficient allotment rules. We provide this characteri-

zation below. As we shall see, however, this class is very large and contains some unattractive rules. The class has some of the features discussed at the end of the last section, but also includes rules where the reference points (or guaranteed levels) can depend on u in almost arbitrary ways. This makes it impossible to describe a simple algorithm which results in these rules. We narrow the class to be more manageable in Section 4.

An allotment rule f is strategy-proof and efficient if and only if for each i there exists $a_i: S^{n-1} \to [0, 1]$ and $b_i: S^{n-1} \to [0, 1]$, such that $a_i(u_{-i}) \leq b_i(u_{-i})$ and

$$\sum_{i \in N} \min[x^*(u_i), b_i(u_{-i})] = 1 \qquad \text{for all } u \text{ such that } \sum_{i \in N} x^*(u_i) > 1,$$

$$\sum_{i \in N} \max[x^*(u_i), a_i(u_{-i})] = 1 \qquad \text{for all } u \text{ such that } \sum_{i \in N} x^*(u_i) \leq 1,$$

and

$$f_i(u) = \begin{cases} \min[x^*(u_i), b_i(u_{-i})] & \text{if } \sum_{i \in N} x^*(u_i) > 1, \\ \max[x^*(u_i), a_i(u_{-i})] & \text{if } \sum_{i \in N} x^*(u_i) \leq 1. \end{cases}$$

(The proof of this characterization follows easily from Lemmas 1 and 2 in Sprumont (1991).)

The above characterization, although it provides a functional form for all strategy-proof and efficient allotment rules, does not provide us with much insight into such rules. That is, it does not provide an understanding of an algorithm or procedure which underlies such an allotment rule. The difficulty stems from the fact that there are an enormous number of quite varied strategy-proof and efficient allotment rules. In particular, the choices of the functions $a_i(u_{-i})$ and $b_i(u_{-i})$ can be made in many different ways, some of which are normatively unappealing. Although we desire allotment rules which allow for asymmetric treatment of the agents, the class of all strategy-proof and efficient rules admits some questionable rules. The following examples illustrate such rules and provide suggestions for a normatively appealing subclass.

EXAMPLE 1. Consider an allotment problem with three agents and the allotment rule which is defined as follows.

Agent 1 receives her ideal point $x^*(u_1)$.

If $x^*(u_1)$ is a rational number, then agent 2 receives his most preferred point from $[0, 1 - x^*(u_1)]$ and 3 gets the remainder; or

If $x^*(u_1)$ is an irrational number, then agent 3 receives her most preferred point from $[0, 1 - x^*(u_1)]$ and 2 gets the remainder.

This rule is strategy-proof and efficient. This rule also makes it clear that it will be difficult to find an easy step-by-step procedure which captures all strategy-proof and efficient rules. Note that in addition to strategy-proofness and efficiency, the above described rule is coalitionally strategy-proof, depends only on agents' ideal points, is nonbossy, and is such that f_i is continuous in u_i for each i. One unappealing nature of the example is that a slight change in agent 1's allotment can change which of agents 2 and 3 gets to choose next, and thus can dramatically increase or decrease their respective allotments. Thus, the rule is not fully continuous in u.[8]

As we shall see in the following example, one can find rules which are similar in flavor to the one in Example 1 but are fully continuous.

EXAMPLE 2. Consider an allotment problem with three agents and the allotment rule which is defined as follows.

Agent 1 receives her ideal point $x^*(u_1)$.

If $x^*(u_2) \leq x^*(u_1)[1 - x^*(u_1)]$ then agent 2 receives his ideal point $x^*(u_2)$ with agent 3 getting the remainder.

If $x^*(u_2) \geq x^*(u_1)[1 - x^*(u_1)]$ then agent 2 receives median $[x^*(u_2), x^*(u_1)[1 - x^*(u_1)], 1 - x^*(u_1) - x^*(u_3)]$, with agent 3 getting the remainder.

In addition to being coalitionally strategy-proof, nonbossy, and efficient, the rule described in Example 2 is fully continuous. The part of the rule which is arbitrary is the manner in which the remainder $(1 - x^*(u_1))$ is split between agents 2 and 3. This split depends in a nonmonotonic way on the choice of agent 1. The rule is such that agent 2 gets to choose from some part of $1 - x^*(u_1)$ and is rationed if he wants more than that part. However, the *relative* size of that part depends on $x^*(u_1)$. Here, sometimes when we increase the desired allotment of agent 1, both of the remaining allotments decrease, while at other profiles when we increase the desired allotment of agent 1, agent 2 gets a larger allotment and agent 3 gets a smaller allotment.[9]

In a rule such as the one described in Example 2, we can think of agent 1 taking an allotment and then leaving the rest for the remaining two agents. Thus the remaining allotment problem is simply a two person allotment problem, but with a varying sized pie. It seems normatively appealing to suggest that as the size of the pie (as left by agent 1) increases

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[8] Sprumont (1991) makes use of continuity in his proofs, but the condition he refers to only requires f_i's continuity in u_i for each i.

[9] For example, fix $x^*(u_2) = 1$ and $x^*(u_3) = 1$. If $x^*(u_1) = 1/4$, then agent 2's allotment is $3/16$. If $x^*(u_1) = 1/2$, then agent 2's allotment is $4/16$. If $x^*(u_1) = 3/4$, then agent 2's allotment is $3/16$.

neither of the remaining two agents' allotments should decrease. This condition is violated in Example 2. This replacement monotonicity condition is defined more carefully in the next section and used in our main characterization.

4. SEQUENTIAL ALLOTMENT RULES

In this section, we characterize a collection of rules which are strategy-proof and efficient, and we allow for asymmetries among the treatment of agents. The characterization involves a third condition, replacement monotonicity. This rules out the troublesome examples described in Section 3 and brings us back to procedures which are more clearly variations on the procedure described in Section 2.

Replacement Monotonicity. An allotment rule is *replacement monotonic* if for all $u \in S^n$, $i \in N$, and $v_i \in S$,

$$\left[f_i(u) \leq f_i(u_{-i}, v_i) \right] \Rightarrow \left[f_j(u) \geq f_j(u_{-i}, v_i) \ \forall j \neq i \right].$$

Replacement monotonicity states that if a change in one individual's preferences results in that individual receiving at least as large (respectively, small) an allotment as he received before, then all other individuals' allotments are no larger (respectively, smaller) than before. Note that replacement monotonicity implies the nonbossy condition of Satterthwaite and Sonnenschein (1981) ($[f_i(u) = f_i(u_{-i}, v_i)] \Rightarrow f(u) = f(u_{-i}, v_i)$).

Replacement monotonicity is closely related to the one-sided replacement-domination condition of Thomson (1992).[10] The difference is that Thomson's condition is stated in terms of the welfare of the agents rather than their allotments; however, given efficiency, these two conditions are equivalent. The idea behind replacement monotonicity is that if we think of individual i as walking away with a smaller part of the pie, then the remaining pie has increased for the remaining agents, and so they should all receive at least as large an allotment as they did before.

Thomson (1992) provides a characterization of the uniform rule using one-sided replacement domination. Our characterization turns out to be

[10] There are a number of conditions that are in the spirit of requiring that the allocations or welfare of some group of agents move in the same direction in response to a change in some characteristic of the economy which affects the availability of resources for that group of agents. Examples of such conditions can be found in Kalai (1977), Roemer (1986), Sprumont (1992), Thomson (1994a, 1994b), and Young (1987), to mention a few. See Thomson (1992) for more discussion.

different from Thomson's (Theorem 1)[11] since he marries one-sided replacement-domination with envy-freeness, efficiency, and replication invariance. Envy-freeness implies a symmetric treatment of agents which results in the uniform rule. Our deliberate move away from this type of symmetry and our concentration on strategy-proofness results in a different (larger) class of allotment rules.

There is some flavor of symmetry to the condition of replacement monotonicity, in that it says agents should be affected in a similar direction in response to a change in some agent's preferences. However, this type of symmetry is substantially different from the type of symmetry required by equal treatment of equals, anonymity, or envy-freeness. In particular, equal treatment, anonymity, and envy-free conditions can preclude an equitable treatment in situations where there are existing differences in agents, such as in age or ability. In the presence of such differences, equity may require an allotment which is to some degree asymmetric (and which may involve envy). Such allotments are not precluded by replacement monotonicity, since it only prescribes a symmetry in the directions that agents' allotments should move in in response to a common change, rather than a symmetry in the relative sizes of agents' allotments.

We now define the class of allotment rules which are strategy-proof, efficient, and satisfy replacement monotonicity, which we call ''sequential'' allotment rules. Such rules follow a procedure that considers initial guaranteed levels for the agents and compares these to their ideal points. Consider the case where the sum of the ideal points is at least one (the other case is analogous). In this case agents whose ideal points are less than their initial guaranteed levels will receive those ideal points. This frees up excess to be redistributed among the remaining agents. We can think of this as an adjustment of the guaranteed levels. The levels are adjusted to the ideal points of the first group of agents (who wanted less than their respective guaranteed levels), and the remaining agents' levels are adjusted to incorporate the excess. This is done so that none of the remaining agents' levels decrease (but it is possible that some increase while others stay constant). Thus, at each step the levels are ''guaranteed'' in that an agent will always receive at least that much from the procedure (and possibly more in future steps), unless they desire less in which case they get exactly what they desire. This procedure is repeated as in each step some of the agents whose levels were adjusted may have an ideal point less than their new guaranteed level. This frees up new excess, etc. The process ends when there are no agents whose ideal points are less than their new guaranteed levels. The ending guaranteed levels form the

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[11] See also Thomson (Theorem 1, 1994b) for a characterization involving one-sided resource-monotonicity, envy-freeness, and efficiency.

allotment. Strategy-proofness will impose some constraints on how this adjustment procedure depends on the ideal points of the agents. For instance, an agent whose new guaranteed level is less than his or her ideal point should get the same new guaranteed level if he or she announced an even higher ideal point. The formal description of this adjustment procedure is now given.

First, given $q^L \in A$, $q^H \in A$, and $u \in S^n$, let $q^u = q^H$ if $\sum_{k \in N} x^*(u_k) \geq 1$, and $q^u = q^L$ if $\sum_{k \in N} x^*(u_k) < 1$. Consider a function $g: A \times S^n \to A \times S^n$. The notation g^t denotes g composed with itself t times, with $g^0(q, u) = (q, u)$.

The function g is a *sequential adjustment function* relative to $q^L \in A$ and $q^H \in A$ if the following are true for any (q^t, u) such that $(q^t, u) = g(q^{t-1}, u) = g^t(q^u, u)$ for some $t \geq 1$:[12]

    1. $q_i^t = x^*(u_i)$ if $(1 - \sum_{j \in N} x^*(u_j))(q_i^{t-1} - x^*(u_i)) \leq 0$.

    2. $(q_i^t - q_i^{t-1})(1 - \sum_{j \in N} x^*(u_j)) \leq 0$ if $(1 - \sum_{j \in N} x^*(u_j))(q_i^{t-1} - x^*(u_i)) > 0$.

    3. If $x^*(v_i) \geq x^*(u_i) > q_i^{t-1}$ and $\sum_{j \in N} x^*(u_j) \geq 1$, or if $x^*(v_i) \leq x^*(u_i) < q_i^{t-1}$ and $\sum_{j \in N} x^*(u_j) < 1$, then $g(q^{t-1}, u) = g(q^{t-1}, u_{-i}, v_i)$.

    4. Consider v_i and $\tilde{q}^n$ such that $(\tilde{q}^n, u_{-i}, v_i) = g^n(q^u, u_{-i}, v_i)$. Let $q^n = g^n(q^u, u)$. Then

        if $x^*(v_i) \geq x^*(u_i)$ and $\sum_{k \in N} x^*(u_k) \geq 1$, then $q_j^n \geq \tilde{q}_j^n$ for $j \neq i$.

        if $x^*(v_i) \leq x^*(u_i)$ and $\sum_{k \in N} x^*(u_k) < 1$, then $q_j^n \leq \tilde{q}_j^n$ for $j \neq i$.

Let us briefly paraphrase the above definition for the case where the total of the ideal points is at least 1 ($\sum_j x^*(u_j) \geq 1$). In the definition of g, part 1 says that if at any stage some agents' ideal points are not higher then their guaranteed amounts, then their guaranteed levels adjust to be their ideal points. Part 2 says that the guaranteed levels of the remaining agents (whose ideal points exceed their current guaranteed levels) cannot adjust downward. Part 3 states that if one changes the preferences of an agent who desires more than their guaranteed level at some stage, so that the agent still desires more than their guaranteed level at that stage, then the adjustment is unaffected. Part 4 states that if an agent's ideal point is increased, then the remaining agents' guaranteed levels cannot decrease.

A *sequential allotment rule f* is an allotment rule such that there exist initial guaranteed q^L and q^H in A and g, a sequential adjustment

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[12] Note that a sequential adjustment function is defined over all possible guaranteed levels and utility function profiles. Our requirements only apply to those combinations which can be reached by some iteration applied to initial guaranteed levels and profiles of utility functions. A sequential function could be arbitrary at points never reached and would still respect strategy-proofness, etc.

function relative to q^L and q^H, such that $f_i(u) = q_i$ where

$$(q_i, u) = \begin{cases} g^n(q^H, u) & \text{if } \sum_{i \in N} x^*(u_i) \geq 1 \\ g^n(q^L, u) & \text{if } \sum_{i \in N} x^*(u_i) < 1. \end{cases}$$

Note that a sequential adjustment function will adjust at most n times.[13]

Before stating our characterization theorem, we illustrate the definition of a sequential allotment rule.

EXAMPLE 3. Seven children attend a party and share a cake. We will simply consider the case where the sum of their ideal shares exceeds 1. (In this example we actually have free disposal: the case where the ideal shares sum to less than 1 can be handled by a hungry parent.[14])

Let us name the children Lisa, Emily, Andrea, Claudia, Pablo, Marc, and Quim. Consider a situation where the children line up (in the order Lisa, Emily, Andrea, Claudia, Pablo, Marc, Quim) and then each takes whatever amount of the remaining cake they please when they reach the front of the line. Here, we start with $q^H = (1, 0, 0, 0, 0, 0, 0)$. This follows since Lisa, at the front of the line, can have as much cake as she wishes, but nobody else is initially guaranteed anything. Next, Emily's guarantee is adjusted to be the amount of the cake left after Lisa has left so that

$$q^1 = (x^*(u_1), 1 - x^*(u_1), 0, 0, 0, 0, 0).$$

This continues in the obvious way so that

$$q^2 = (x^*(u_1), \min[1 - x^*(u_1), x^*(u_2)],$$
$$\max[1 - x^*(u_1) - x^*(u_2), 0], 0, 0, 0, 0),$$

and generally

$$q_i^t = \begin{cases} \min[\max[1 - \sum_{j < i} x^*(u_j), 0], x^*(u_i)] & \text{if } i \leq t, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

EXAMPLE 4. Let us reconsider Example 3, assuming that the parents take a less laissez faire attitude about the division of the cake. For

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[13] By parts 1 and 2 of the definition, an adjustment only takes place if some agents ideal points are less than their current guaranteed amounts (in the case where $\sum_{i \in N} x^*(u_i) \geq 1$, and the opposite for the other case), and then those agents' new guaranteed levels adjust to their ideal points and stay there. Since there are only n agents, at most n adjustments take place.

[14] In situations with free disposal the allotment problem can generally be adapted by simply giving agents their ideal points whenever they sum to no more than 1.

simplicity let us consider a party with four children: Lisa, Emily, Marc, and Quim. Emily and Quim are older than their respective siblings, Lisa and Marc, and it is agreed that Emily and Quim be entitled to twice as large shares of the cake. However, Lisa and Emily are guests, so it is also agreed that they have first rights to any excess, should someone desire less than their share. In the case of any further excess (if at least one of the guests does not want all of the excess offered to them), it is to be split uniformly among those not already full.

Given this description, and ordering the children Lisa, Emily, Marc, Quim, we have an initial guaranteed level of $q^H = (1/6, 1/3, 1/6, 1/3)$.

Let us consider any situation where Lisa wants less than her share ($x^*(u_1) < 1/6$), and the others want more than their respective shares ($x^*(u_i) > q_i^H$, $i \geq 2$). In this case the adjustment is to $(q^1, u) = g(q^H, u)$ where $q^1 = (x^*(u_1), 1/3 + 1/6 - x^*(u_1), 1/6, 1/3)$. Now if $x^*(u_2) \geq 1/3 + 1/6 - x^*(u_1)$, then this is the final allotment (so $q^t = q^1$ for all $t > 1$). Otherwise, we move to $q^2 = (x^*(u_1), x^*(u_2), 1/6 + (1/2 - x^*(u_1) - x^*(u_2))/2, 1/3 + (1/2 - x^*(u_1) - x^*(u_2))/2)$. If both Marc and Quim desire as much as their q^2 allotments, then this is the final allotment. If one wants less, then the other receives the undesired portion of the other's allotment. (We know that at most one will want less since we started with the assumption that the total of the ideal points is at least 1). For instance, if $x^*(u_4) < 1/3 + (1/2 - x^*(u_1) - x^*(u_2))/2$, then $q^3 = (x^*(u_1), x^*(u_2), 1 - x^*(u_1) - x^*(u_2) - x^*(u_4), x^*(u_4))$.

Next, consider a situation where Quim wants less than his share ($x^*(u_4) < 1/3$), and the others want more than their respective shares ($x^*(u_i) > q_i^H$, $i \leq 3$). In this case the adjustment is to $(q^1, u) = g(q^H, u)$ where $q^1 = (1/6 + (1/3 - x^*(u_4))/2, 1/3 + (1/3 - x^*(u_4))/2, 1/6, x^*(u_4))$. Now if Emily and Lisa want at least as much as they are allotted here, then this is the final allotment (so $q^t = q^1$ for all $t > 1$). If both want less, then the final allotment would be $q^2 = (x^*(u_1), x^*(u_2), 1 - x^*(u_1) - x^*(u_2) - x^*(u_4), x^*(u_4))$. If just one wants less, say $x^*(u_2) < 1/3 + (1/3 - x^*(u_4))/2$ then this excess is divided in two and added to the guaranteed allotments of Lisa and Marc,[15] so we move to $q^2 = (7/12 - 3x^*(u_4)/4 - x^*(u_2)/2, x^*(u_2), 5/12 - x^*(u_4)/4 - x^*(u_2)/2, x^*(u_4))$. If either Lisa or Marc wants less than this amount, then they receive their ideal and the excess is given to the other. Otherwise, this is the final allotment.

There are many other cases that could arise, which we will not go through, but which follow similar progressions.

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[15] Once the guests have gotten a first excess allocation, then all of those who are still hungry are treated equally if there is any excess at this stage. Note that the rule would still be strategy-proof if instead of treating all equally, one "favored" Marc at this stage, and then went back to favoring Lisa, etc.

THEOREM. *An allotment rule is strategy-proof and efficient and satisfies replacement monotonicity if and only if it is a sequential allotment rule.*

Remark. The theorem can be strengthened by replacing efficiency with the requirement that f map onto the set of possible allotments.[16]

Remark. The sequential allotment rules depend only on the ideal points of the agents, and not on further information concerning preferences.[17] With this in mind, it is easy to see that any sequential allotment rule is fully implemented in dominant strategies by the (direct) mechanism which asks for reports of ideal points.[18]

Remark. The sequential allotment rules are coalitionally strategy-proof. That is, there is no group of agents who could all benefit by coordinating a misrepresentation of their preferences.[19]

In the Introduction we mentioned rules where agents have initial claims or entitlements. The following corollary characterizes this subclass of sequential allotment rules.

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[16] We can see that if an allotment rule is strategy-proof, satisfies replacement monotonicity, and maps onto A, then it is efficient. Suppose that f is not efficient. Then there exists some $u \in S^n$ and partition of the agents into I_1, I_2, I_3 such that $f_i(u) = x^*(u_i)$ for all $i \in I_1$, $f_i(u) < x^*(u_i)$ for all $i \in I_2$, $f_i(u) > x^*(u_i)$ for all $i \in I_3$, and I_2 and I_3 are nonempty. Find $a \in A$ such that $f_i(u) = a_i = x^*(u_i)$ for all $i \in I_1$, $f_i(u) < a_i < x^*(u_i)$ for all $i \in I_2$, and $f_i(u) > a_i > x^*(u_i)$ for all $i \in I_3$. Choose $v \in S^N$ such that $x^*(v_i) = a_i$ for all i. If we change agents' utility functions one at a time from u_i to v_i, it follows from strategy-proofness and replacement monotonicity that $f(u) = f(v)$. However, since f is onto A, there exists some $\hat{u}$ such that $f(\hat{u}) = a$. By the same argument $f(v) = f(\hat{u})$, which leads to a contradiction. Hence our supposition was wrong.

[17] In other words, sequential allotment rules are "tops only," a consequence of strategy-proofness, efficiency, and the nonbossiness implied by replacement monotonicity. By strategy-proofness and efficiency, an individual cannot affect their own allotment by changing to a new preference with the same ideal point. By nonbossiness ($[f_i(u) = f_i(u_{-i}, v_i)] \Rightarrow f(u) = f(u_{-i}, v_i)$) none of the allotments are affected.

[18] This equivalence to full implementation in dominant strategies is not true more generally of strategy-proof social choice functions. This is discussed in Dasgupta *et al.* (1979) and Jackson (1992).

[19] This can be seen using the strategy proofness, efficiency, nonbossiness, and tops-only properties. For instance, consider the case where $\Sigma_i x^*(u_i) > 1$. By efficiency, any coalition C that could improve for all of its members must have $f_j(u) < x^*(u_j)$ for $j \in C$. Consider a change from u_C to v_C, changing one member of the coalition's utility at a time. Let k be the first member whose new announcement changes the allocation. By strategy-proofness and nonbossiness, it must be that $x^*(v_k) < f_k(u)$. If under u_{-C}, v_C the sum of the ideals is still at least one, then by efficiency k will end up worse off ($f(u_{-C}, v_C) \leq x^*(v_k)$). If under u_{-C}, v_C the sum of the ideals is less than one, by efficiency $-C$ will get more than their ideal points, leaving a smaller total pie for C so someone is worse off.

COROLLARY. *An allotment rule is strategy-proof and efficient, satisfies replacement monotonicity, and is individually rational[20] with respect to an allotment e if and only if it is a sequential allotment rule such that $q^H = q^L = e$.*

Remark. Note that the above characterization does not place restrictions on the evolution of guaranteed levels after the first stage of the sequential allotment (beyond those restrictions imposed in the definition of sequential allotment rule). For instance, if ideal points sum to more than 1 and some agent's ideal point is less than his or her entitlement e_i, then this frees up excess so that some other agent(s) may get more than their entitlement. Individual rationality does not have implications for how this excess be allotted. It could be done in any number of ways consistent with a sequential allotment rule. For instance, it could be done according to a queue, according to some further entitlements, or even uniformly. A recent paper by Klaus *et al.* (1995) characterizes this last set of rules, where rationing away from these entitlements is done according to the uniform rule. This rule, which they call the uniform reallocation rule, is the only one to satisfy strategy-proofness, efficiency, and a property of equal treatment.[21]

Proof of the Theorem. We begin by showing that any allotment rule which is strategy-proof and efficient, and which satisfies replacement monotonicity, is a sequential allotment rule. Then we show the converse.

Consider f, which is strategy-proof, efficient, and replacement monotonic. We will construct an associated sequential adjustment function as follows. Let $\bar{u}$ and $\underline{u}$ be such that $x^*(\bar{u}_1) = 1$ and $x^*(\underline{u}_i) = 0$ for each $i \in N$. Let $q_i^H = f_i(\bar{u})$ and $q_i^L = f_i(\underline{u})$. Define g as follows: Given (q, u) let $(\hat{q}, u) = g(q, u)$ be such that $\hat{q}_i = f_i(w)$ where

$$w_i = \begin{cases} u_i & \text{if } \left(1 - \sum_{j \in N} x^*(u_j)\right)\left(q_i - x^*(u_i)\right) \leq 0 \\ \bar{u}_i & \text{if } x^*(u_i) > q_i \text{ and } \sum_{i \in N} x^*(u_i) \geq 1, \text{ and} \\ \underline{u}_i & \text{if } x^*(u_i) < q_i \text{ and } \sum_{i \in N} x^*(u_i) < 1. \end{cases}$$

We show that $f_i(u) = q_i$ where $q_i = g^n(q^u, u)$ and $q^u = q^H$ if $\sum_{i \in N} x^*(u_i) \geq 1$, and $q^u = q^L$ if $\sum_{i \in N} x^*(u_i) < 1$. Then we show that conditions 1 through 4 in the definition of sequential adjustment function are satisfied with respect to the above defined adjustment function.

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[20] $u_i(f_i(u)) \geq e_i$ for each u and i.

[21] Their setting is slightly different from ours in that the consumption set for each agent is unbounded. They use such a specification to allow for a translation invariance condition which is part of their equal treatment condition. Of course, our approaches are complementary in that they characterize different classes of rules.

Let us assume that u is such that $\sum_{i \in N} x^*(u_i) \geq 1$. The proof for the case where $\sum_{i \in N} x^*(u_i) < 1$ is analogous. The following two facts will be useful in the proof.

By efficiency

$$f_i(u) \leq x^*(u_i). \qquad (1)$$

By strategy proofness, for every $u \in S^N$: $f_j(u) \leq f_j(u_{-j}, \bar{u}_j)$, and by replacement monotonicity it follows that for every $i \neq j$: $f_i(u) \geq f_i(u_{-j}, \bar{u}_j)$. Changing u^j to $\bar{u}^j$, one agent at a time, it follows that for any $C \subset N$ and any $i \in C$:

$$f_i(u) \geq f_i(\bar{u}_{-C}, u_C). \qquad (2)$$

Now, let us show that $f_i(u) = q_i$ where $q = g^n(q^u, u)$. Let $(q^t, u) = g^t(q^H, u)$ and let w^t be such that:[22]

$$w_i^t = \begin{cases} u_i & \text{if } x^*(u_i) \leq q_i^{t-1}, \text{ and} \\ \bar{u}_i, & \text{if } x^*(u_i) > q_i^{t-1}. \end{cases}$$

We proceed by induction on t, showing that $f_i(u) = q_i^t = f_i(w^t)$ for all i such that $x^*(u_i) \leq q_i^{t-1}$, and for all $i \in N$ if $x^*(u_i) \geq q_i^{t-1}$ for all i.

We begin with $t = 1$.

Case A. $x^*(u_i) \geq q_i^H$ for all i. We show that $q_i^H = f_i(u) = f_i(w^1)$ $(= q_i^1)$ for each i. By strategy-proofness at $\bar{u}$, $q_i^H = f_i(\bar{u}) \geq f_i(\bar{u}_{-i}, u_i)$ for each $i \in N$. By strategy-proofness at $\bar{u}_{-i}, u_i$, it follows that $q_i^H = f_i(\bar{u}) = f_i(\bar{u}_{-i}, u_i)$. By (2), $f_i(u) \geq f_i(\bar{u}_{-i}, u_i) = q_i^H$ for each $i \in N$. Thus, since $\sum q_i^H = 1$, it follows that $f_i(u) = q_i^H$. By replacement monotonicity (implying nonbossiness), we also have that $f_i(u) = f_i(w^1)$.

Case B. $x^*(u_i) < q_i^H$ for some i. We want to establish that for such an i $f_i(u) = f_i(w^1)$. By (1), $x^*(u_i) \geq f_i(u)$. By (2), $f_i(u) \geq f_i(\bar{u}_{-i}, u_i)$. By strategy-proofness and the fact that $q_i^H = f_i(\bar{u})$, it follows that $q_i^H \geq f_i(\bar{u}_{-i}, u_i)$. By the continuity of $f_i(\bar{u}_{-i}, u_i)$ in u_i (see Sprumont 1991, Lemma 1) and strategy-proofness, it follows that $f_i(\bar{u}_{-i}, u_i) \geq x^*(u_i)$. Thus, $x^*(u_i) \geq f_i(u) \geq f_i(\bar{u}_{-i}, u_i) \geq x^*(u_i)$, so that $f_i(u) = x^*(u_i)$. Since $f_i(\bar{u}_{-i}, u_i) = x^*(u_i) = f_i(u)$, for any such i, it follows from replacement monotonicity and strategy-proofness (considering each such i in sequence) that $f_i(u) = f_i(w^1)$.

We now proceed by induction on t. Suppose that for all $t < T$ $f_i(u) = q_i^t = f_i(w^t)$ for all i such that $x^*(u_i) \leq q^{t-1}$, and for all $i \in N$ if $x^*(u_i) \geq q^{t-1}$ for all i. We must show that the same is true for $t = T$.

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[22] This definition is assuming that $\sum_{i \in N} x^*(u_i) \geq 1$. If $\sum_{i \in N} x^*(u_i) < 1$, then set $w_i^t = u_i$ if $x^*(u_i) \geq q_i^{t-1}$, and $w_i^t = \underline{u}_i$ if $x^*(u_i) < q_i^{t-1}$.

Case A^T. $x^*(u_i) \geq q_i^{T-1}$ for all i. By strategy-proofness at w^{T-1}, q_i^{T-1} $= f_i(w^{T-1}) \geq f_i(w_{-i}^{T-1}, u_i)$ for each $i \in N$. By strategy-proofness at w_{-i}^{T-1}, u_i, it follows that $q_i^{T-1} = f_i(w^{T-1}) = f_i(w_{-i}^{T-1}, u_i)$. By (2), $f_i(u) \geq f_i(w_{-i}^{t-1}, u_i) = q_i^{t-1}$ for each i. Thus, since $\sum_i q_i^{t-1} = 1 = \sum_i f_i(u)$, it follows that $f_i(u) = q_i^{t-1}$ for each i. By replacement monotonicity (implying nonbossiness), we also have that $f_i(u) = f_i(w^t)$.

Case B^T. $x^*(u_i) < q_i^{T-1}$ for some i. Consider such an i. By (1), $x^*(u_i) \geq f_i(u)$, and by (2), $f_i(u) \geq f_i(\bar{u}_{-i}, u_i)$. By strategy-proofness and the fact that $q_i^{T-1} = f_i(w^{T-1})$, it follows that $q_i^T \geq f_i(w_{-i}^{T-1}, u_i)$. By the continuity of f_i in u_i (see Sprumont 1991, Lemma 1) and strategy-proofness, it follows that $f_i(w_{-i}^{T-1}, u_i) \geq x^*(u_i)$. Thus, we have shown that $x^*(u_i) \geq f_i(u) \geq f_i(w_{-i}^{T-1}, u_i) \geq x^*(u_i)$, so that $f_i(u) = x^*(u_i)$. Finally, since $f_i(w_{-i}^{T-1}, u_i) = x^*(u_i) = f_i(u)$ for any such i, it follows from replacement monotonicity and strategy-proofness that $f_i(u) = f_i(w^T)$.

We have shown that f is represented by the above defined adjustment function. It remains to be checked that conditions 1 through 4 in the definition of sequential adjustment function are satisfied.

To see condition 1, we need to show that if $x^*(u_i) \leq f_i(w^{t-1})$ then $x^*(u_i) = f_i(w^t)$. This is shown in both cases of our inductive proof, above. Condition 2, then, follows from replacement monotonicity applied to our inductive proof above. Condition 3 follows from our definition of g and the fact that in this case w_i^t is the same for both u_i and v_i. Condition 4 follows from replacement monotonicity.

Now let us establish the converse. Let f be a sequential allotment rule. We need to show that f is strategy-proof, efficient, and satisfies replacement monotonicity.

Claim. f is efficient.

Proof. Consider u such that $\sum_{i \in N} x^*(u_i) \geq 1$. The case where $\sum_{i \in N} x^*(u_i) < 1$ is analogous. We must show that $f_i(u) \leq x^*(u_j)$ for each i. Consider i such that $f_i(u) \neq x^*(u_i)$. It follows from 1 (in the definition of sequential adjustment rule) that $q_i^t < x^*(u_i)$ for all $t < n$. Then by 2 (in the definition of sequential adjustment rule) it follows that $f_i(u) = q_i^n < x^*(u_i)$. ∎

Claim. f is strategy-proof.

Proof. Consider u such that $\sum_{i \in N} x^*(u_i) \geq 1$. (Again, the case where $\sum_{i \in N} x^*(u_i) < 1$ is analogous.) If $f_i(u) = x^*(u_i)$, then there can be no improving deviation. So consider the case where $f_i(u) \neq x^*(u_i)$. By efficiency (above), $f_i(u) = q_i^n < x^*(u_i)$. By 1 and 2 (in the definition of sequential adjustment rule) it follows that $q_i^{t-1} \leq q_i^t < x^*(u_i)$ for each $1 \leq t \leq n$. Then by 3 (in the definition of sequential adjustment rule), in

order to effect any change in f via v_i at u it would have to be that $x^*(v_i) \leq q_i^t$ for some $1 \leq t < n$. But then by 1, $f_i(u_{-i}, v_i) = q_i^{t+1} \leq q_i^n = f_i(u)$. Thus, $u_i(f_i(u_{-i}, v_i)) \leq u_i(f_i(u))$. ■

Claim. f is replacement monotonic.

Proof.   Consider u, i, and v_i as in the definition of replacement monotonicity. From strategy-proofness (above) it follows that to have $f_i(u) \leq f_i(u_{-i}, v_i)$ (as in the definition of replacement monotonic) it must be that $x^*(v_i) \geq x^*(u_i)$. Thus, $\sum_{j \in N} x^*(u_j) \leq x^*(v_i) + \sum_{j \neq i} x^*(u_j)$. If $\sum_{j \in N} x^*(u_j) \leq 1 \leq x^*(v_i) + \sum_{j \neq i} x^*(u_j)$, then replacement monotonicity follows from efficiency (above). So consider $1 \leq \sum_{j \in N} x^*(u_j)$. Then replacement monotonicity follows from 4 (in the definition of sequential adjustment rule). The same is true for the case where $x^*(v_i) + \sum_{j \neq i} x^*(u_j) \leq 1$.   ■

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