Mathematical Social Sciences 35 (1998) 89–103

mathematical
social
sciences

Strategy-proof probabilistic rules for expected utility maximizers

Salvador Barberà[a,*], Anna Bogomolnaia[a], Hans van der Stel[b]

[a]*Universitat Autònoma de Barcelona, Bellaterra, 08193 Barcelona, Spain*
[b]*University of Maastricht, Maastricht, Netherlands*

Received 3 July 1996; received in revised form 15 September 1997; accepted 15 September 1997

Abstract

We consider social choice rules which select a lottery over outcomes for each profile of individual preferences. Agents are assumed to have preferences over lotteries satisfying the axioms of expected utility. We exhibit a large class of rules satisfying strategy-proofness. All these rules are obtained by combining one of the following principles: (1) start from a fixed subset of lotteries, and for each profile let one fixed agent choose her preferred lottery from that subset (we call them unilateral rules); or, (2) start from two fixed lotteries and a rule assigning weights to each of them depending on the coalition of agents which prefer one of the two lotteries to the other; let the outcome at each profile be the convex combination of these two given lotteries according to the weights which correspond to them at that profile (these rules are called duples). All probabilistic mixtures (convex combinations or integrals) of unilateral and duple rules satisfying some additional and natural requirements are strategy-proof. Because we are facing a wide class of procedures, we investigate the possibility of designing some which are not only strategy-proof but also continuous or even smooth in their responses to changes in preferences. Smoothness requirements are not only attractive per se, but they can also be expected to help in telling apart different types of rules. Notice that unilateral rules can be very smooth, while no duple can even be continuous. Yet, continuity can be regained by combining a continuum of duples: we provide an example of a continuous strategy-proof probabilistic rule which is an integral of duples. However, there is a limit as to how smooth a rule can be without resorting to unilateral schemes. We prove that any strategy-proof probabilistic function of class C^2 must indeed be also a convex combination of unilateral schemes. © 1998 Elsevier Science B.V.

Keywords: Social choice rules; Lotteries; Strategy-proofness; Smoothness

---

[*] Corresponding author. Tel.: 34 3 581 1812; fax 34 3 581 2461; e-mail: sbarbera@cc.uab.es

PII S0165-4896(97)00043-7

1. Introduction

Chance is a possible source of equity (see [17]). When a group of agents must decide who will get an indivisible object or who is to perform a dangerous task, establishing a lottery to ultimately determine an outcome may be perceived as more equitable than any set of deterministic rules leading to a unique choice. Yet, what lottery should be used?

When agents have no special entitlements and their preferences are clearly polarized, an even chance lottery becomes a very natural proposal toward an equitable settlement. But there are other circumstances where agents may have different claims, or may hold preferences which do not enter into open conflict: in these cases we may still want to use chance, while not treating all alternatives evenly. We study situations where agents face a finite number of alternatives and are seeking to establish a lottery on these alternatives. We are interested in rules which allow the agents to express their preferences on the set of potential lotteries and determine one lottery for each possible preference profile.

While equity considerations provide an important justification for the introduction of chance as an aid to collective decision-making, probabilistic rules are also attractive for other reasons. Although some of the basic conflicts revealed by Arrow's theorem still survive within probabilistic frameworks [1,5,6,16], aggregating preferences into lotteries allows for much greater flexibility [8,13], even permitting us to encompass the principles of majority and positional voting, which are incompatible in their deterministic versions [3]. Moreover, the incentive properties of collective choice rules are also improved if chance is allowed a role. Strategy-proofness is still in conflict with full efficiency in probabilistic contexts [10–12]. But there are rich classes of strategy-proof probabilistic rules even in a purely ordinal context, and we shall see that these classes are substantially enlarged if we allow agents to express preferences over lotteries.

In this paper we investigate the possibility of designing strategy-proof probabilistic social choice functions. The sharpest results on the subject are still those obtained in [10] for the particular case where agents can only contribute their ordinal preferences on sure outcomes as inputs toward determining the lottery that will be used to determine a final social choice (see [18] for an early contribution). Under this strong limitation on the relevant information to be elicited and the additional restriction that agents cannot be indifferent among sure outcomes, Gibbard obtains a sharp characterization: all strategy-proof probabilistic social choice functions can be expressed as convex combinations of functions belonging to two elementary subclasses, to be called unilateral and duple. To understand the nature of the result, notice that it makes sense to define the convex combination of probabilistic choice functions, since the outcomes of such functions are lotteries, and convex combinations of the lotteries assigned to one profile by different rules will also be lotteries. Moreover, the strategy-proofness requirement can be expressed as a linear inequality, which is preserved under convex combination. Therefore, convex combinations of strategy-proof functions will be strategy-proof: the set to be characterized is a convex set, and Gibbard's result consists in identifying a ‘‘generating’’ set of points for this set. Gibbard's characterization becomes more complicated when sure outcomes may be indifferent. Then, hierarchical rules (for example, serial dictators) may arise. A companion paper [11] considers the possibility of eliciting cardinal preferences, but limits attention to rules with finite ranges.

We eliminate the ‘‘a priori’’ restrictions imposed by Gibbard. We consider all functions defined on preferences over lotteries. We also consider ranges containing an infinity of possible lotteries over outcomes.[1] Our only assumption is that preferences are representable by a von Neumann–Morgenstern utility function. It turns out that the class of strategy-proof probabilistic social choice functions becomes much richer. We cannot achieve a full characterization, but we provide three elements toward the understanding of this rich class.

First, we extend the notion of unilateral and duple probabilistic social choice functions. We show that functions based on the same basic principles are still strategy-proof but now form a much larger class. Several complications must be faced in the way to a complete characterization of all strategy-proof rules, which for the moment seems to be out of reach. We mention two of them. One is that indifference between different alternatives cannot be avoided, due to the continuity of von Neumann–Morgenstern preferences. Thus, a full characterization must allow for complicated hierarchical structures extending the serial dictatorships in Section 3, [10]. The other is that one should not expect functions in our class to be decomposable into a convex combination of a finite number of unilaterals and duples: integrals of these basic objects may also be present in the decomposition of a function, as shown by our example in Section 3.

Second, we investigate the subclass of strategy-proof probabilistic social choice functions which are continuous. Continuity is an attractive property for social choice functions. In our context, it is very easy to obtain continuous functions which are unilateral, or convex combinations of unilaterals. Yet, every duple probabilistic social choice function is discontinuous, and convex combinations of any finite collection of duples will also be. Therefore, one might conjecture that only combinations of unilateral functions can be strategy-proof and continuous. This is not so, because there exist functions which can be expressed as integrals of duple functions, which cannot be decomposed into combination of unilaterals, and yet satisfy both requirements. We provide an example of such a function, which again shows the richness of the classes we describe.

Third, we investigate the subclass of strategy-proof probabilistic social choice functions which are twice continuously differentiable. Now we can prove that, indeed, the only functions satisfying this smoothness condition which are strategy-proof are those which can be expressed as convex combinations of unilateral schemes. This result is interesting in itself, if we are interested in smooth enough rules. Moreover, it can be taken as a criticism to previous work [9], which attempted to investigate the same class of rules we consider by using the ‘‘differentiable approach’’ to mechanism design, as proposed in [14]. This approach requires the assumption that social choice functions are sufficiently differentiable, and then allows us to write incentive constraints in the form of differential equations to be satisfied by the rules. Our results prove that many interesting

---

[1] If only a finite set of lotteries is admitted in the range, then [11] applies.

rules are eliminated by the differentiability assumptions, while those that are not eliminated can be characterized in a much simpler way.

2. Probabilistic social choice functions. Some previous results and a conjecture

Let $I=\{1, 2, \ldots, n\}$ be a finite set of *agents*, or voters.

Let $A=\{a_1, a_2, \ldots, a_m\}$ be a finite set of outcomes.

Let L be the set of *lotteries* over A. A lottery specifies a probability of choice for each outcome a_j. Lotteries will be denoted by $l, l', \ldots$ Given a lottery l, its j-th component l_j stands for the probability of choosing a_j under l.

Agents are endowed with preferences $\succeq$ on L. These preferences are assumed to be representable by a von Neumann–Morgenstern utility function. That is, each preference for agent i will be associated with an m-dimensional vector u^i whose j-th component $u_j^{\,i}$ stands for the utility of the sure outcome a_j, and with the property that, for any two lotteries l, l' on A,

$$l \succeq l' \Leftrightarrow u^i \cdot l \geq u^i \cdot l'.$$

In all that follows, we exclude the possibility of any individual being totally indifferent among all outcomes.

A *preference profile* is any n-tuple of utility functions on L. Under our assumptions, preference profiles are identified with n-tuples of vectors in $\mathbb{R}^m \backslash D$ (where D is the set of vectors in the main diagonal; these elements would represent the case of complete indifference, which is not considered here). Preference profiles are denoted by $\mathbf{u}$, $\hat{\mathbf{u}}$ etc., where $\mathbf{u}=(u^1, \ldots, u^n)$. The standard notation $\mathbf{u}|\hat{u}^i$ denotes the profile obtained from $\mathbf{u}$ by substituting $\hat{u}^i$ for u^i.

A *probabilistic social choice function* is a function f: $(\mathbb{R}^m \setminus D)^n \to L$, assigning a lottery on A to each n-tuple of admissible von Neumann–Morgenstern preferences on L.

A probabilistic social choice function f is *strategy-proof*[2] if and only if for all m, all n-tuples of preferences $\mathbf{u}$ and any $\hat{u}_i$,

$$u^i \cdot f(\mathbf{u}) \geq u^i \cdot f(\mathbf{u}|\hat{u}^i).$$

Our purpose is to investigate the set of strategy-proof probabilistic social choice rules.

---

[2] Notice that the definition of strategy-proofness implies

$$u^i \cdot f(\mathbf{u}) = u^i \cdot f(\mathbf{u}|\hat{u}^i) \text{ and } \hat{u}^i \cdot f(\mathbf{u}) = \hat{u}^i \cdot f(\mathbf{u}|\hat{u}^i),$$

whenever u_i and $\hat{u}_i$ are affine transforms of each other. That is: the images associated to profiles where agents use different utility representations of the same preferences must be indifferent for these agents (although not necessarily identical). This is a natural invariance requirement, since u^i and $\hat{u}^i$ would stand for the same preferences $\succeq$ on lotteries. If we had defined probabilistic social choice functions on profiles of preferences, it would have been natural to require an even stronger form of invariance: that the choice of utility representation should not change the outcome at all. None of our results are affected in any essential way by this choice in the definition of a probabilistic social choice function. Our choice is slightly more economical, because our form of invariance is a direct consequence of strategy-proofness.

We shall describe a very wide class of such rules, based on some elementary constructs. Since the initial ideas for such constructs were first proposed by A. Gibbard in a narrower context, we first describe his initial work and later extend his proposal to our general context. This will also give the reader a feeling for the considerable extension we propose, and provide motivation for the questions we address in later sections.

A probabilistic social choice function f is *ordinal in sure outcomes* if and only if it is invariant to any set of strictly increasing transformations of the agent's utilities. Namely $f(u^1,\dots,u^n)=f(\varphi_1(u^1),\dots,\varphi_n(u^n))$ for all profiles $\mathbf{u}$ and all strictly increasing $\varphi_1,\dots,\varphi_n$.

In contrast, general functions within our class can be sensitive to changes in the ranking of nondegenerate lotteries which preserve that of sure outcomes.

Example 1. Two strategy-proof probabilistic social choice functions which are ordinal in sure outcomes.

1.1. For three outcomes and n agents, let T be the set of lotteries

$$\left\{\left(\tfrac{5}{12},\tfrac{4}{12},\tfrac{3}{12}\right),\left(\tfrac{5}{12},\tfrac{3}{12},\tfrac{4}{12}\right),\left(\tfrac{4}{12},\tfrac{5}{12},\tfrac{3}{12}\right),\left(\tfrac{4}{12},\tfrac{3}{12},\tfrac{5}{12}\right),\left(\tfrac{3}{12},\tfrac{5}{12},\tfrac{4}{12}\right),\left(\tfrac{3}{12},\tfrac{4}{12},\tfrac{5}{12}\right)\right\}.$$

Let $f(u^1,\dots,u^n)\in\arg\max_{l\in T} u^1\cdot l$. If there are several maximizers for agent 1, choose the one assigning the highest probability to alternative 1, or to alternative 2, in that order.

1.2. For m outcomes and n agents, let $S=\{i\in I|u^i_1\geq u^i_2\}$, $p(\{1,\dots,n\})=1$, $p(\emptyset)=0$ and $p(T)>p(T')$ whenever $T'\subset T$. Then, define

$$f_1(u^1,\dots,u^n)=p(S)$$

$$f_2(u^1,\dots,u^n)=1-p(S)$$

$$f_j(u^1,\dots,u^n)=0 \text{ for any } j\neq 1,2.$$

Example 2. Two strategy-proof probabilistic social choice functions which are not ordinal in sure outcomes.

2.1. For three outcomes and n agents, let K be the set of lotteries such that

$$\left(l_1-\frac{1}{3}\right)^2+\left(l_2-\frac{1}{3}\right)^2+\left(l_3-\frac{1}{3}\right)^2=\frac{1}{6}.$$

Let $f(u^1,u^2,\dots,u^n)=\arg\max_{l\in K} u^1\cdot l$.

2.2. For m outcomes and n agents, and two lotteries $l_1=(1,0,0,\dots,0)$ and $l_2=(0,1/(m-1),1/(m-1),\dots,1/(m-1))$, let $S=\{i\in I|u^il_1\geq u^il_2\}$, $p(\{1,\dots,n\})=1$, $p(\emptyset)=0$ and $p(T)>p(T')$ whenever $T'\subset T$. Then, define

$$f(u^1,\dots,u^n)=p(S)l_1+(1-p(S))l_2.$$

Example 1.1 is an instance of what Gibbard called a unilateral scheme, one in which the preferences of only one agent influence the final outcome. Notice that unilateral schemes need not be fully dictatorial because this salient agent does not necessarily get his best alternatives with certainty. However, unilateral schemes are still the result of a maximization by this agent, on an a priori restricted set of lotteries. Example 2.1. is based on the same principle, and we propose to call it unilateral as well. The major difference among 1.1. and 2.1. is that our choice of lotteries is much wider. Specifically, the range of the probabilistic social choice function is the set of possible maximal elements for the salient agent, given constraints. Ordinality in outcomes then forces this range to be finite (to contain at most as many points as the number of orderings of the outcomes). When we drop this requirement, the range can contain an infinity of possible lotteries.

Example 1.2. is an instance of what Gibbard called a duple scheme, one under which only two outcomes ever get a nonzero probability of being chosen. The outcomes of a duple, in Gibbard's definition, are convex combinations of two degenerate lotteries. Similarly, we can extend the notion of a duple to any rule whose image can always be expressed as a convex combination of the same two lotteries: this is the nature of Example 2.2. Notice, again, that duples in Gibbard's sense can only be based on a finite number of lotteries, while our proposed extensions allow for a continuum of pairs of lotteries as a possible basis for the rule.

Notice that some schemes can be expressed both as unilaterals and also as duples. The reader may check that our duple schemes in Examples 1.2. and 2.2. can also be expressed as combinations of unilaterals for some choices of the function $p(S)$. This is the case, for example, if $p(S)=(\#(S))/n$.[3]

We have elaborated on unilateral and duple schemes because they are the building blocks for Gibbard's characterization of strategy-proof probabilistic social choice functions which are ordinal on sure outcomes, and they will also be central for our propositions. Before stating any results, let us give the definitions of (hierarchically) unilateral and duple social choice functions.[4]

The image of a (hierarchical) unilateral function is obtained by letting an agent i_1 choose from a fixed set T^0, letting a second agent i_2 break any possible ties in $1'$s choice, then letting a third agent i_3 choose among alternatives over which both i_1 and i_2 are indifferent, and so on. Any remaining indifferences after the agent i_k are resolved by a fixed tie-breaking order P. We keep the term unilateral by analogy with Gibbard's work, although several agents may eventually play a role in determining the final outcome. Formally,

Definition 1. A probabilistic social choice function is (hierarchically) unilateral if and only if there exist a set $T^0 \subset L$, a nonempty ordered set of agents $i_1, i_2, \ldots, i_k$, and a strict ranking P of T^0, such that, for all $\mathbf{u} \in (\mathbb{R}^m \backslash D)^n$,

$$f(\mathbf{u}) = C(P, T^k),$$

---

[3] Here $\#(S)$ is the cardinality of the set S.

[4] Definitions 1 and 2 imply two properties that Gibbard expresses separately, and calls nonperversity and locality. We shall slightly abuse the language and consider in our statements of Gibbard's results that unilaterals and duples are also required to satisfy them. Hence, in our statement of Gibbard's result we do not mention these requirements explicitly.

where $T^h = \arg\max_{k\in T^{k-1}} u^{i_h} \cdot l$, for any $h=1,\ldots,k$ and $C(P, T^k)$ is the P-maximal element of T^k.

Notice that the final tie-breaking rule could be more complex than just involving a single order, and still preserve strategy-proofness. For details on tie-breaking rules, see [4,7] or [15]. We do not provide the largest possible class, since we are not aiming at a full characterization.

We now turn to duples. The images of duple functions are always convex combinations of the same two basic lotteries. The weights for these two lotteries are determined by the coalitions which support one over the other or are indifferent among the two in such a way that larger support implies a non decreasing weight.

Definition 2. A probabilistic social choice function is duple if and only if there exist two lotteries $l, l' \in L$ and a function

$$w: (2^I)^2 \to [0, 1], \text{ such that}$$

$$f(\mathbf{u}) = w[P(l, l'), (l, l')] \cdot l + (1 - w[P(l, l'), I(l, l')]) \cdot l', \text{ for all } \mathbf{u} \in (\mathbb{R}^m \setminus D)^n$$

where $P(l, l') = \{i \in I | u^i \cdot l > u^i l'\}$, $I(l, l') = \{i \in I | u^i \cdot l = u^i \cdot l'\}$, and w satisfies

1. $S \subset S' \to w(S, T) \leq w(S', T)$
2. $T \subset T' \to w(S, T) \leq w(S, T')$
3. $S \subset S'$ & $S \cup T = S' \cup T' \to w(S, T) \leq w(S', T')$.

A problem similar to the tie-breaking problem addressed after the definition of unilateral functions also arises for duple functions. Again, since we are not seeking a full characterization, the present definition of duples serves our purposes. A more complicated definition involving hierarchies of duples (or even hierarchies of duples and unilaterals, which would also be strategy-proof) would be required in order to capture all strategy-proof rules based on that principle.

Given a family of probabilistic social choice functions $\mathcal{F}$, integrals of elements in this family will also be probabilistic social choice functions. Specifically if elements f_ω of $\mathcal{F}$ are parametrized by elements ω of a probability space (Ω, σ, ν), then

$$f(u^1, \ldots, u^n) = \int_{\omega \in \Omega} f_\omega(u^1, \ldots, u^n) \, \mathrm{d}\nu(\omega)$$

defines a new probabilistic social choice function. We will refer to it as a *probabilistic mixture*.

Notice that convex combinations are particular cases of probabilistic mixtures.

We can now state the following:

Theorem 1. (*Gibbard*) *A probabilistic social choice function is strategy-proof and ordinal in pure outcomes if and only if it is the convex combination of functions, each of which is ordinal in pure outcomes, and either unilateral or duple.*

Proof. (See [10]).

Proposition 1. *All probabilistic mixtures of* (*hierarchically*) *unilateral and duple probabilistic social choice functions are strategy-proof.*

We do not prove this proposition formally. It is a simple consequence of the two following facts:

1. As defined, unilateral and duple schemes are strategy-proof.
2. Probabilistic mixtures of strategy-proof functions are strategy-proof.

In the following sections we describe some subclasses and examples of strategy-proof probabilistic social choice functions. In particular, we discuss the implications of continuity and smoothness conditions on the structure of such functions.

As a final remark, notice that unilateral and duple schemes are also group strategy-proof, but their probabilistic mixtures need not be. We have not carefully explored this issue here. A characterization of group strategy-proof rules for Gibbard's ordinal setup can be found in [2].

3. A continuous probabilistic mixture of duples

In this section we consider a case with three sure outcomes and two voters. Given $\alpha \in [0, 1]$, we define a duple scheme based on the two lotteries

$$\begin{pmatrix} \alpha \\ 1-\alpha \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

as follows:

$$g(u^1, u^2, \alpha) = \begin{cases} \begin{pmatrix} \alpha \\ 1-\alpha \\ 0 \end{pmatrix} & \text{if } u^i \begin{pmatrix} \alpha \\ 1-\alpha \\ 0 \end{pmatrix} \geq u^i \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ for } i = 1, 2 \\ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} & \text{otherwise.} \end{cases}$$

Now define

$$f(u^1, u^2) = \int_{\alpha=0}^{1} g(u^1, u^2, \alpha)\, \mathrm{d}\alpha.$$

Remark 1. *Notice that because f is an integral of duples, it is a strategy-proof probabilistic social choice function.*

Remark 2. *The probabilistic social choice function f is continuous.*

To see this, let

$$S(u^i)=\left\{\alpha\in[0,1] \middle| u^i\cdot\begin{pmatrix}\alpha\\ 1-\alpha\\ 0\end{pmatrix}\geq u^i\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}.$$

Notice that we can rewrite f as follows:

$$f(u^1,u^2)=\int\limits_{S(u^1)\cap S(u^2)}\left(\begin{pmatrix}\alpha\\ 1-\alpha\\ 0\end{pmatrix}-\begin{pmatrix}0\\0\\1\end{pmatrix}\right)\mathrm{d}\alpha+\begin{pmatrix}0\\0\\1\end{pmatrix}. \tag{1}$$

In order to show that f is continuous, we study the intervals $S(u^i)$, which determine the value of $f(u^1, u^2)$. We distinguish several cases.

Case I. $u_1^i, u_2^i \geq u_3^i$. Then

$$u^i\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}\geq u^i\cdot\begin{pmatrix}0\\0\\1\end{pmatrix} \text{ and } u^i\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}\geq u^i\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Hence $S(u^i)=[0,1]$.

Case II. $u_1^i<u_3^i\leq u_2^i$. Then we have:

$$\alpha\in S(u^i)\Rightarrow u^i\cdot\begin{pmatrix}\alpha\\ 1-\alpha\\ 0\end{pmatrix}\geq u^i\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\Rightarrow u_1^i\alpha+u_2^i(1-\alpha)\geq u_3^i\Rightarrow u_2^i-u_3^i$$

$$\geq(u_2^i-u_1^i)\alpha\Rightarrow\alpha\leq\frac{u_2^i-u_3^i}{u_1^i-u_2^i}.$$

So

$$S(u^i)=\left[0,\frac{u_2^i-u_3^i}{u_2^i-u_1^i}\right],$$

Case III. $u_1^i, u_2^i<u_3^i$. Then $S(u^i)=\emptyset$.

Case IV. $u_1^i\geq u_3^i>u_2^i$. Then,

$$\alpha\in S(u^i)\Rightarrow u_1^i\alpha+u_2^i(1-\alpha)\geq u_3^i\Rightarrow(u_1^i-u_2^i)\alpha\geq u_3^i-u_2^i\Rightarrow\alpha\geq\frac{u_3^i-u_2^i}{u_1^i-u_2^i}.$$

So

$$S(u^i)=\left[\frac{u_3^i-u_2^i}{u_1^i-u_2^i},1\right].$$

We now observe that in the cases I, II and IV the upper and lower bounds of the interval $S(u^i)$ vary continuously with u^i. Coming closer to case III, the interval $S(u^i)$ collapses to a set containing only one point (namely $\{0\}$ or $\{1\}$), and diminishes to an empty set if case III is reached. It now follows from Eq. (1) that f depends continuously on u^1 and u^2.

Remark 3. *f is not differentiable everywhere. Specifically, f is not differentiable in many cases where extreme points of the $S(u^i)$ coincide. We illustrate this with the following example.*

Let

$$u^2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, u_2^1 = 1, u_3^1 = 0 \text{ and } u_1^1 < 0.$$

Then

$$S(u^2) = \left[\frac{1}{2}, 1\right] \text{ and } S(u^1) = \left[0, \frac{1}{1-u_1^1}\right].$$

Hence,

$$S(u^1) \cap S(u^2) \begin{cases} 0 & \text{if } u_1^1 < -1, \\ \left[\frac{1}{2}, \frac{1}{1-u_1^1}\right] & \text{otherwise.} \end{cases}$$

Using the Eq. (1), we obtain for the first component of f that:

$$f_1(u^1, u^2) = \begin{cases} 0 & \text{if } u_1^1 < -1, \\ \int_{1/2}^{(1/1-u_1^1)} \alpha \, d\alpha = \frac{1}{2}\left(\frac{1}{1-u_1^1}\right)^2 - \frac{1}{8} & \text{if } -1 \leq u_1^1 < 0. \end{cases}$$

Hence,

$$\frac{\delta}{\delta u_1^1} f_1(u^1, u^2) = \begin{cases} 0 & \text{if } u_1^1 < -1 \\ \frac{1}{(1-u_1^1)^3} & \text{if } -1 < u_1^1 < 0. \end{cases}$$

Therefore,

$$\lim_{u_1^1 \uparrow -1} \frac{\delta}{\delta u_1^1} f_1(u^1, u^2) = 0 \text{ and } \lim_{u_1^1 \downarrow -1} \frac{\delta}{\delta u_1^1} f_1(u^1, u^2) = \frac{1}{8}.$$

Consequently, f is not differentiable at

$$\left(\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\right).$$

Remark 4. *Our example proves that there exist continuous strategy-proof probabilistic social choice rules which have no unilateral components. Notice that this is the case for f, since*

$$f\left(\begin{pmatrix}-1\\-1\\0\end{pmatrix}, u^2\right) = f\left(u^1, \begin{pmatrix}-1\\-1\\0\end{pmatrix}\right) = \begin{pmatrix}0\\0\\1\end{pmatrix} \text{ for all } u^1, u^2.$$

This fact is in sharp contrast with the next section, which applies to twice continuously differentiable functions. It is important to notice that the rule in our example cannot be expressed in an alternative form, involving unilateral rules. Indeed, other rules admit several representations. We elaborate on this point in Section 5.

4. Smooth strategy-proof probabilistic social choice functions

In the preceding section we showed that continuity does not eliminate duple-based rules, and even admits functions with no unilateral components. The purpose of this section is to show that, if we require the probabilistic social choice functions to be smooth enough, then the only methods that preserve strategy-proofness are convex combinations of unilaterals.

Proposition 2. *If a strategy-proof probabilistic social choice function is twice continuously differentiable then it is a convex combination of unilaterals.*[5]

Proof. We give the complete proof for the particular case of 3 alternatives and 2 agents. The extension of the proof to the general case is briefly discussed at the end.

We regard lotteries over k sure outcomes as the elements of the $(k-1)$-dimensional simplex. Indifference curves are straight lines on that simplex. The preferences of agents are characterized by their preferred directions and we can simply identify their preferences with the points on the unit sphere in $\mathbb{R}^{k-1}$.

In the case of three alternatives each agent is characterized by a point on the unit circle $\mathbb{T}$, i.e. by an angle from $[0, 2\pi)$. From now on, we shall let x, x_0, stand for the angle which identifies the preferences of the first agent, and y, y_0 be the angle corresponding to the preferences of the second one. A probabilistic social choice function will now be given by a function $F: \mathbb{T}^2 \to \mathbb{R}^2$, where $F(x, y) = (f(x, y), g(x, y))$. Here $f(x, y)$ will be the probability of the first sure outcome, $g(x, y)$ the probability of the second and $1-(f+g)(x, y)$ the probability of the third one.

Suppose now that F is strategy-proof and that $F \in C^2(\mathbb{T}^2)$. Then, $f, g \in C^2(\mathbb{T}^2)$ as well.

(1) We prove that F can be represented by the form $F(x, y) = (f_1(x), g_1(x)) + (f_2(y), g_2(y))$.

---

[5] It is natural to inquire whether C^2 could be substituted by C^1 in this proposition. It could, and one of the authors (Bogomolnaia) has a proof. However, we felt it was better to keep the present paper technically simple, and leave the C^1 case, which requires more sophisticated arguments, for a separate piece.

Fix some x_0 as the preferred direction of agent 1. Then the options of agent 2, given the choice x_0 of agent 1, are the points $(f(x_0,\cdot), g(x_0,\cdot))$ which form a closed C^2 curve in $\mathbb{R}^2$.

Notice that the strategy-proofness of F means that, when agent 2 chooses y_0 then $F(x_0, y_0)$ must be one of the best points among his options. Thus, the option set of agent 2 lies on one side of any of its tangents, which implies the convexity of $(f(x_0,\cdot), g(x_0,\cdot))$. Next, the tangent to his option curve at the point $F(x_0, y_0)$ has to be perpendicular to his y_0 direction. Since f, g are continuous and the slope of tangent is changing with y_0 along $\mathbb{T}$, we obtain that the curve $(f(x_0,\cdot), g(x_0,\cdot))$ is strictly convex.

Let y_0 be a regular point of the curve (i.e. $(f'_y(x_0, y_0), g'_y(x_0, y_0)) \neq (0, 0)$), if it exists, and let ctg y_0 exist. We have that $\operatorname{tg}(y_0 + \pi/2) = -\operatorname{ctg} y_0$ is the slope of the tangent line to this curve at y_0, which is equal to

$$\frac{-g'_y(x_0, y)}{f'_y(x_0, y)}$$

whenever $f'_y \neq 0$. Similarly,

$$\operatorname{tg} y_0 = \frac{f'_y(x_0, y)}{g'_y(x_0, y)},$$

whenever $g'_y \neq 0$. Since for regular points $f'_y = 0$ implies that $g'_y \neq 0$ and hence $\operatorname{tg} y_0 = 0$, we obtain that $f'_y(x_0, y) \neq 0$, whenever y is a regular point of the curve above defined, and $y \neq 0, \pi$.

By fixing y_0 and proceeding analogously we obtain a similar statement. At the end we have:

$$g'_y(x,y) = f'_y(x, y) \operatorname{ctg} y, \quad g'_x(x, y) = f'_x(x, y) \operatorname{ctg} x \quad (*),$$

whenever y is a regular point of the curve $(f(x,\cdot), g(x,\cdot))$ and $f'_y(x, y)$, $f'_x(x, y) \neq 0$. Since it is obviously true for nonregular points, we have that (*) holds on $\mathbb{T}' = \{(x, y) \in \mathbb{T}^2: x \neq 0, \pi; y \neq 0, \pi\}$. Next, f'_x, $f'_y \in C^1 \mathbb{T}^2)$ and ctg x, ctg y exist, $f'_y(x, y)$, $f'_x(x, y) \neq 0$ almost everywhere (namely on the open set $\mathbb{T}'$), and so the formulae (*) hold and can be differentiable on $\mathbb{T}'$.

When we differentiate the first formula with respect to x and the second one with respect to y we obtain $g''_{xy}(x, y) = f''_{xy}(x, y) \operatorname{ctg} y$, $g''_{xy}(x, y) = f''_{xy}(x, y) \operatorname{ctg} x$. Therefore $f''_{xy}(x, y) \operatorname{ctg} y = f''_{xy}(x, y) \operatorname{ctg} x$ on $\mathbb{T}'$, i.e. almost everywhere on $\mathbb{T}^2$. Since f''_{xy} is continuous, this is possible only if $f''_{xy}(x, y) \equiv 0$ and hence $g''_{xy}(x, y) \equiv 0$ on $\mathbb{T}^2$.

But then we have $f(x, y) = f_1(x) + f_2(y)$, $g(x, y) = g_1(x) + g_2(y)$, i.e. our function F is indeed of the form

$$F(x, y) = (f_1(x), g_1(x)) + (f_2(y), g_2(y)).$$

The last formula means that our social choice function F is the sum of two functions, each depending on the preferences of one agent only.

(2) We prove that F is a probability mixture of two unilateral functions.

Notice that $f_1(x)$, $g_1(x)$ are defined up to an arbitrary constant. So we can choose them in such a way that $\min_{x\in\mathbb{T}} f_1(x) = \min_{x\in\mathbb{T}} g_1(x) = 0$. Let these functions attain their maxima at x_1 and x_2, respectively.

Then, $(f_2(y), g_1(x_1)+g_2(y)) = F(x_1, y)$ is a point in the simplex. So $f_2(y)\geq 0$ and likewise $g_2(y)\geq 0$ for all y. Since $f_1(x)+g_1(x)\leq (f_1(x)+f_2(y))+(g_1(x)+g_2(y))\leq 1$, $(f_1(\cdot), g_1(\cdot))$ is a subset of our simplex. The same holds for $(f_2(\cdot), g_2(\cdot))$.

Since $F(x_0,\cdot) = (f(x_0,\cdot), g(x_0,\cdot)) = (f_1(x_0), g_1(x_0)) + (f_2(\cdot), g_2(\cdot))$ for any x_0, the curve $(f_2(\cdot), g_2(\cdot))$ must be strictly convex and for each y_0 the point $(f_2(y_0), g_2(y_0))$ has to be the best one for agent 2 on this curve. We can say the same about agent 1 and the curve $(f_1(\cdot), g_1(\cdot))$. Thus, in fact we let each agent maximize on some set and then take the vector-sum of their maximal points.

Now, let $p = \max_{x\in\mathbb{T}}(f_1(x)+g_1(x))$, $x^* = \arg\max_{x\in\mathbb{T}}(f_1(x)+g_1(x))$. Note that $0\leq p\leq 1$. Define (f^1_1, g^1_1) and (f^1_2, g^1_2) by the formulae $(f_1(x), g_1(x)) = p(f^1_1(x), g^1_1(x))$, $(f_2(y), g_2(y)) = (1-p)(f^1_2(y), g^1_2(y))$. Since $F(x^*, y)$ is a point in the simplex, we obtain that $(f_1(x^*)+g_1(x^*))+(f_2(y)+g_2(y))\leq 1$. Hence, $f_2(y)+g_2(y)\leq 1-p$. So, $f^1_1(x)+g^1_1(x)$, $f^1_2(y)+g^1_2(y)\leq 1$ and thus these curves lie in our simplex, i.e., they can be regarded as choice sets for individuals 1 and 2, respectively.

We have that $F(x, y) = p(f^1_1(x), g^1_1(x)) + (1-p)(f^1_2(y), g^1_2(y))$, i.e. that $F(x, y)$ is the probability mixture of the two unilateral functions.

(3) Sketch of the proof for an arbitrary number of individuals and alternatives.

The case of 2 agents and $k\geq 3$ alternatives is analogous. We look at the function F: $\mathbb{S}^2\rightarrow\mathbb{R}^{k-1}$, where $\mathbb{S}$ is the $(k-1)$-dimensional unit sphere, points on $\mathbb{S}$ represent preferred directions of agents. Using a similar reasoning, we can prove that

$$\frac{\partial^2 F(x_1,\ldots,x_{k-1}, y_1,\ldots,y_{k-1})}{\partial x_i\,\partial y_j}\equiv 0,$$

for all i, j, and the proposition follows.

The proof for the general case is based on an induction argument, which in the case $n=3$ takes the following form. Our social choice function is $F = F(x_1, x_2, x_3) = (f_1(x_1, x_2, x_3),\ldots,f_k(x_1, x_2, x_3))$. In order to show that $\forall i\ f_i(x_1, x_2, x_3) = f^1_i(x_1)+f^2_i(x_2)+f^3_i(x_3)$ it is sufficient to check that the choice set of agent 3 given that all others have made their choices is always the same up to a shift: $O(x^0_1, x^0_2) = O(x^1_1, x^1_2)$. But when we fix x^0_1, $O(x^0_1, x^0_2) = O(x^0_1, x^1_2)$, whereas when we fix x^1_2, $O(x^0_1, x^1_2) = O(x^1_1, x^1_2)$, both by the induction hypothesis. ■

5. An example of a rule admitting several representations[6]

This section contains an example which should be helpful to better understand our rules. We describe an infinitely smooth rule which is an integral of duples. But we check that this does not contradict Proposition 2, since this rule can also be expressed as a probabilistic mixture of unilaterals.

---

[6] The necessity of this example was pointed out by a referee. We are also grateful to Gilbert Laffond for his comments on this subject.

Suppose we have two agents and 3 outcomes. We identify utility functions with angles and use the same notation as in proposition 2.

Let G be an inner point in the simplex S and r be small enough such that the entire circle $K=\{k: \|k-G\|=r\}$ is included in S.

1. For every $k\in K$, take the following duple:

$$F_k(x, y)=\begin{cases} k, & \text{if both agents prefer } k \text{ to } G \text{ or are indifferent between them} \\ G, & \text{if at least one agent prefers } G \text{ to } k \end{cases}$$

Define

$$F(x, y)=\frac{1}{\mu(K)}\int_K F_k(x, y)\,\mathrm{d}\mu(K),$$

where $\mu(K)$ is a uniform measure over K.

2. Let $f(x)$ and $g(y)$ be respectively the best points of agents 1 and 2 over the same option set $K'=\{k: \|k-G\|=r/\pi\}$. Define $F'(x, y)=\frac{1}{2}f(x)+\frac{1}{2}g(y)$.

It is easy to see that $F(x, y)=F'(x, y)$ is a point in S on a line at angle $(x+y/2)$ and at a distance $(r/\pi)\cos((x-y/2)$ of G. Note that this rule is of class C^{∞}.

Upon reflection, it is not surprising that the same rule may be expressed both as a probabilistic mixture of unilaterals or of duples. In fact, it is simple to construct many such rules, taking the probability mixtures of basic rules, each of which is unilateral on a two-lottery range. The reader may check that the above example is in this class. However, this example also shows that it is not always easy to determine whether or not a rule admits both representations.

6. Final comments

We have shown that the class of strategy-proof probabilistic rules is a very large one and we have exhibited the basic principles that allow the construction of functions in this class. We have remarked that some of our functions admit multiple representations. We have also noticed that continuity is not an exclusive feature of unilateral schemes, since integrals of duples can also be continuous. We have proven that only unilateral schemes and their convex combinations can be twice continuously differentiable.

These results should be helpful in advancing toward a full characterization of strategy-proof probabilistic social choice functions. They also cast doubt on the usefulness of the differential approach as a tool toward this characterization.

Acknowledgements

The work of Barberà is partially supported by DGICYT (Spanish Ministry of Education) and the CIRIT (Comissionat per Universitats i Recerca de la Generalitat de

Catalunya) through grants PB92-590 and GRQ93-2044. Bogomolnaia is supported by an ACE-Tacis fellowship of the European Community. This work was written while van der Stel was visiting UAB supported by a grant of the Human Capital and Mobility program of the European Community. We are thankful to the referees for their probing comments.

References

[1] T. Bandyopadhyay, R. Deb, P.K. Pattanaik, The structure of coalitional power under probabilistic group decision rules, Journal of Economic Theory 27 (1982) 366–375.

[2] S. Barberà, A note on group strategy-proof decision schemes, Econometrica 47 (1979) 637–640.

[3] S. Barberà, Majority and positional voting in a probabilistic framework, Review of Economic Studies 143 (1979) 379–389.

[4] S. Barberà, M. Jackson, A characterization of strategy-proof social choice functions for economies with pure public goods, Social Choice and Welfare 11 (1994) 241–252.

[5] S. Barberà, H. Sonnenschein, Preference aggregation with randomized social orderings, Journal of Economic Theory 18 (1978) 244–254.

[6] S. Barberà, F. Valenciano, Collective probabilistic judgements, Econometrica 51 (1983) 1033–1046.

[7] D. Berga, Strategy-proofness and single-plateaued preferences, Mathematical Social Sciences 35 (1998) 105–120.

[8] P.C. Fishburn, A probabilistic model of social choice comment, Review of Economic Studies 42 (1975) 297–301.

[9] X. Freixas, A cardinal approach to straightforward probabilistic mechanisms, Journal of Economic Theory 34 (1984) 227–251.

[10] A. Gibbard, Manipulation of schemes that mix voting and chance, Econometrica 45 (1977) 665–681.

[11] A. Gibbard, Straightforwardness of game forms with lotteries as outcomes, Econometrica 46 (1978) 595–614.

[12] A. Hylland, Strategy proofness of voting procedures with lotteries as outcomes and infinite sets of strategies, Discussion Paper, University of Oslo, 1980.

[13] M.D. Intriligator, A probabilistic model of social choice, Review of Economic Studies 40 (1973) 553–560.

[14] J.J. Laffont, E. Maskin, A differential approach to dominant strategies mechanisms, Econometrica 48 (1980) 1507–1520.

[15] H. Moulin, Generalized Condorcet-winners for single peaked and single plateau preferences, Social Choice and Welfare 1 (1984) 127–147.

[16] P.K. Pattanaik, B. Peleg, Distribution of power under stochastic social choice rules, Econometrica 54(4) (1986) 909–921.

[17] H.P. Young, Equity, Princeton University Press, 1994.

[18] R. Zeckhauser, Voting systems, honest preferences and Pareto optimality, American Political Science Review 67 (1973) 934–946.