Group strategy-proof social choice functions with binary ranges and arbitrary domains: characterization results[1]

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Abstract: We define different concepts of group strategy-proofness for social choice functions. We discuss the connections between the defined concepts under different assumptions on their domains of definition. We characterize the social choice functions that satisfy each one of them and whose ranges consist of two alternatives, in terms of two types of basic properties.

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1 Introduction

The Gibbard-Satterthwaite Theorem establishes that, when a social choice function is defined on the universal set of preference profiles over k alternatives ($k > 2$), and its range contains at least three alternatives, it can only be strategy-proof if it is dictatorial.

This result is subject to different qualifications. One is that, when the rule is defined on smaller sets of profiles, there may or may not exist other rules that are strategy-proof, in addition to the dictatorial ones. This is the case under a variety of domains, that include the ones formed by the Cartesian product of single-peaked preferences, or of single-dipped preferences, or of separable preferences, among others. Our statements in this paper will be essentially true for functions defined on any domain, however small, asymmetric or special it may be.[1]

A second qualification concerns the range of the social choice function. In this paper we consider the subclass of functions that fail to meet Gibbard and Satterthwaite's requirement that their range should contain at least three alternatives. Specifically, we concentrate on rules that are not constant and whose range consists of exactly two alternatives, x and y. Because of this, it is known that in that case there are possibilities to design non-dictatorial strategy-proof rules. We want to characterize them all. This leads us to notice that the range of a social choice function may be binary because there are only two alternatives in the relevant world, but it may also be binary in the presence of more than two alternatives, in which case this may be considered as one part of the possible choices open to the mechanism designer. As we shall see, the characterization of binary rules in this context requires a number of precisions and careful treatment that can be avoided in worlds where only two alternatives are present to begin with.

A third qualification refers to the notion of strategy-proofness to be used. When we concentrate on rules with binary ranges, there exist a number of attractive strategy-proof rules, and it becomes then much more interesting to explore the extent to which some of them may also be immune to manipulation by groups. We analyze this question carefully, under a number of different possible notions of group strategy-proofness, and also by keeping in mind that we want our statements to hold for functions defined on any type

[1] By "essentially true" we mean that they are either true without qualification, or true under very minor assumptions, to be discussed case by case.

of domains.

One definition of group strategy-proofness requires that it should not be possible for a group of agents to deviate from declaring their true preferences and get a strict gain for each one of them. Social choice functions avoiding this strong type of manipulation are called Weakly Group Strategy-Proof. A second definition starts from considering that a group can profitably deviate if some of its members derive a strict gain from doing so, while others simply remain indifferent while helping their partners. Rules that avoid this weaker form of manipulation are called Strongly Group Strategy-Proof. In an intermediate version of the property, that we simply call Group Strategy-Proofness, we allow that only some agents may gain from the deviation, but we require that all agents involved in getting the change should actively participate in the manipulation by actually deviating from their truthful preference. We provide characterizations of the classes of social choice functions that satisfy each one of these three properties, and also we elaborate on why we single out these particular definitions.

Our main characterization results identify two types of basic properties that these rules must satisfy. These properties must be qualified in each case. Since we allow individuals to be indifferent between x and y, in some cases we will require that they are satisfied "essentially", and in other cases not. By "essentially" we mean that the properties will hold conditional to the fact that the preferences of individuals that are indifferent between the two alternatives in the range remain constant. Our first condition is that of essential xy-monotonicity: if x obtains at a profile, and then some people change their preferences so that the support for x increases, while the support for y does not, then x must still obtain at the new profile. A more demanding requirement in a similar spirit is that of xy-strong monotonicity. In that case if x obtains at a profile, and preference changes induce larger support for x, then x still be chosen at the new profile even if support for y may have also increased.[2] A second type of requirement refers to the type of information on which our rules may be based. We say that they are xy-based if what they choose at each preference profile only depends on the relative position of x with respect to y for each individual. It is essentially xy-based if the property holds when we only compare profiles where individuals indifferent between

[2] In this second definition we drop the qualification for the property being essential because the statement is no longer conditioned to the preferences of indifferent individuals remaining constant.

x and y keep their preferences unchanged. Notice also that the requirement will not apply in the case where all individuals are indifferent between both alternatives in the range.

We establish three characterization results in terms of the above conditions, one for each of our three types of group strategy-proofness requirements. A social choice function is weakly group strategy-proof if and only if it is essentially xy-based and essentially xy-monotonic. It is strongly group strategy-proof if and only if it is xy-based and xy-strong monotonic. Finally, we show that, when $n \geq 3$ and under a mild condition on the richness of the domain, rules that meet our intermediate notion of group strategy-proofness are also strongly group strategy-proof, and thus satisfy the same properties.

The sophisticated reader will realize that our conditions are part of a large set of different requirements that have been used by different authors under different names for the characterization of strategy-proof rules over universal domains. Names like Maskin monotonicity, strong positive association, and others have been used to denote variations of properties that one expects to be satisfied by rules that are strategy-proof. And, indeed, many combinations of properties end up characterizing the same rules when these are defined on rich enough domains. We feel that our choice of properties is especially fit, because they allow us to characterize rules defined on all kinds of domains, possibly very asymmetric and containing few preferences. The equivalence between ours and other properties is not granted under these circumstances. Also notice that we do not insist on individual strategy-proofness as a special case to characterize. This is because by a recent result of ours, it is an established fact that individual and weak group strategy-proofness are equivalent when the range of the social choice function consists of only two (or three) elements (see Barberà, Berga, and Moreno, 2010).

A different type of characterization results are based on descriptions of how the rules would choose alternatives at each preference profile. There exist two relevant papers that take this point of view. One is by Larsson and Svensson (2006), who provide a characterization of strategy-proof rules: under our assumption that the range is binary, strategy-proofness is equivalent to weak group strategy-proofness, as proven in Barberà, Berga, and Moreno (2010). Hence, their characterization in terms of the functional form provides an alternative to the one we present here. A second result, this one due to Manjunath (2009a), characterizes the functional form of strong group strategy-proof rules when there are only two alternatives. We re-state the result with some additional precisions and in order to cover the case where the range is binary but preferences are defined on a larger set of alternatives, and provide a novel proof for it.

The paper proceeds as follows. In Section 2, we provide the framework, we present different versions of group strategy-proofness and discuss their relationships under different domain assumptions. In Section 3 we provide the characterizations in terms of properties. In Section 4 we provide the announced additional characterization of strongly group strategy-proof rules, the novel proof, that also allows us to complete the proof of one of the theorems in the preceding section. Section 5 concludes.

2 The setup and definitions

Let A be a finite set of *alternatives* $A = \{x, y, z, w...\}$. Let N be a finite set of *agents* $N = \{1, 2, ..., n\}$. Let $\mathcal{U}$ be the set of all preorders on A (complete, reflexive, and transitive binary relations on A). Let $\mathcal{R}_i \subseteq \mathcal{U}$ be *the set of admissible preferences for agent* $i \in N$ and let $\mathcal{R} \equiv \times_{i\in N} \mathcal{R}_i$.

For any preference relation $R_i \in \mathcal{R}_i$, we denote by P_i and I_i the strict and indifference part of R_i, respectively. A *preference profile* is denoted by $R = (R_1, .., R_n) \in \mathcal{R}$ or also by $R = (R_C, R_{-C}) \in \mathcal{R}$ when we want to stress the role of a coalition $C \subseteq N$. Then $R_C \in \mathcal{R}^C \equiv \times_{i\in C} \mathcal{R}_i$ and $R_{-C} \in \mathcal{R}^{N\backslash C}$ denote the preferences of agents in C and in $N\backslash C$, respectively.

A *social choice function* (or *rule*) *on a domain* $\mathcal{R}$ is a function $f : \mathcal{R} \to A$. The range of f is denoted by A_f. In this paper we concentrate on the family of social choice functions with *binary range*, that is, whose range consists of exactly two elements, that we call x and y from now on.

Let $\mathcal{R}_i^x \subseteqq \mathcal{R}_i$ be the subset of preferences such that for any $R_i^x \in \mathcal{R}_i^x$, $xP_i^x y$. Similarly, define $\mathcal{R}_i^y$. Let $\mathcal{R}_i^{xy} \subseteqq \mathcal{R}_i$ be the subset of preferences such that for any $R_i^{xy} \in \mathcal{R}_i^{xy}$, $xI_i^{xy} y$.

We state our results under the following **minimal assumption on the domain** of admissible preferences: each individual has at least one admissible preference where x is preferred to y, one where y is preferred to x, and one where he is indifferent between the two. That is, for any $i \in N$ and any $t \in \{x, y, xy\}$, $\mathcal{R}_i^t \neq \varnothing$.[3]

[3]For several of our results, we could even weaken this minimal condition on the domain and allow for some of the sets $\mathcal{R}_i^t$ to be empty.

The best known nonmanipulability axiom is *strategy-proofness*. It requires the truth to be a dominant strategy and it is a necessary condition for implementation in dominant strategies (Gibbard, 1973 and Satterthwaite, 1975).

Definition 1 *An agent $i \in N$ can manipulate a social choice function f on $\mathcal{R}$ at $R \in \mathcal{R}$ if there exists $R_i' \in \mathcal{R}_i$ such that $R_i \neq R_i'$ and $f(R_i', R_{-i}) P_i f(R)$. A social choice function f is* ***strategy-proof*** *on $\mathcal{R}$ if no agent $i \in N$ can manipulate f on $\mathcal{R}$.*

Another form of manipulation is by means of coalitions. The following definitions refer to cases where agents may gain from joint changes of declared preferences. They differ on two accounts: the required gains from manipulation and the actions expected from coalition members. Regarding gains from manipulation we may require that each member from deviating coalitions obtains a strict gain or else that only some of them do with the rest not losing. Regarding deviations we may ask that all members of a coalition misrepresent their preferences or that just some of them do. The three definitions below will reflect these modelling choices.[4]

Definition 2 *A coalition C can strongly manipulate a social choice function f on $\mathcal{R}$ at $R \in \mathcal{R}$ if there exists $R_C' \in \mathcal{R}^C$ such that for all agent $i \in C$, $R_i \neq R_i'$ and $f(R_C', R_{-C}) P_i f(R)$. A social choice function f is* ***weakly group strategy-proof*** *on $\mathcal{R}$ if no coalition $C \subseteq N$ can strongly manipulate f on $\mathcal{R}$.*

Definition 3 *A coalition C can manipulate a social choice function f on $\mathcal{R}$ at $R \in \mathcal{R}$ if there exists $R_C' \in \mathcal{R}^C$ such that for all agent $i \in C$, $R_i \neq R_i'$ and $f(R_C', R_{-C}) R_i f(R)$, and for some $j \in C$, $f(R_C', R_{-C}) P_j f(R)$. A social choice function f is* ***group strategy-proof*** *on $\mathcal{R}$ if no coalition $C \subseteq N$ can manipulate f on $\mathcal{R}$.*

Definition 4 *A coalition C can weakly manipulate a social choice function f on $\mathcal{R}$ at $R \in \mathcal{R}$ if there exists $R_C' \in \mathcal{R}^C$ such that for some agent $l \in C$,*

[4] We shall omit what could have been a fourth version of group strategy-proofness, one that would require all agents to gain but would allow for some of them not to change their preferences. That would turn out to be equivalent to weak group strategy-proofness (see Definition 2).

*$R_l \neq R'_l$, for all agent $i \in C$, $f(R'_C, R_{-C}) R_i f(R)$, and for some $j \in C$, $f(R'_C, R_{-C}) P_j f(R)$. A social choice function f is **strongly group strategy-proof** on $\mathcal{R}$ if no coalition $C \subseteq N$ can weakly manipulate f on $\mathcal{R}$.*

Remarks (1) Strategy-proofness and weak group strategy-proofness are equivalent for social choice functions with binary range (see Proposition 1 and Theorem 1 in Barberà, Berga, and Moreno, 2010).
(2) When indifferences are not allowed, all three definitions of group strategy-proofness collapse in a single one.
(3) Strong group strategy-proofness implies group strategy-proofness and the latter implies weak group strategy-proofness. The converse implications do not hold in general, as shown by the following examples.

Example 1 *A rule that is group strategy-proof but not strongly. Let $n \geq 2$ and $\#A \geq 2$, $x, y \in A$. Then, for any $R \in \mathcal{U}^N$, define the social choice function f as follows:*

$$f(R) = \begin{cases} x & \text{if } xP_iy \text{ for any } i \in N, \\ y & \text{otherwise.} \end{cases}$$

We show that f is not strongly group strategy-proof. Let R be such that each agent strictly prefers x to y and let R' be such that $n-1$ agents strictly prefer x over y, and the other agent is indifferent between x and y. Observe that $f(R) = x$ and $f(R') = y$. Then, coalition N could weakly manipulate f at R' via R. The reader may check that the rule satisfies the two weaker strategic conditions.

Example 2 *A rule that is weakly group strategy-proof but not group. Let $n \geq 2$, $\#A \geq 2$ and agents' preferences such that for any $i \in N$, $\mathcal{R}_i^t \neq \varnothing$ for any $t \in \{x, y, xy\}$. Let k be a dictator on $\{x, y\}$, that is, $f(R) = x$ when $R_k \in \mathcal{R}_k^x \cup \mathcal{R}_k^{xy}$ and $f(R) = y$ otherwise.*
Note that f is (weakly group) strategy-proof. However, coalition $C = \{k, j\}$ $j \neq k$ could manipulate f at $(R_k^{xy}, R_j^y, R_{-\{j,k\}})$ via $(R_k^y, R'_j, R_{-\{j,k\}})$ for any $R'_j \in \mathcal{R}_i^x \cup \mathcal{R}_i^{xy}$ and any $R_{-\{j,k\}} \in \mathcal{R}^{N \setminus \{j,k\}}$. Thus, f is not group strategy-proof (thus not strongly).

Before characterizing the rules that satisfy our different requirements, let us remark that group strategy-proofness and strong group strategy-proofness become equivalent under the mild **complementary domain condition** required in the following proposition.

Proposition 1 *Let $\#A \geq 3$ and $\mathcal{R}$ be such that each individual has at least two admissible preferences in $\mathcal{R}_i$ where x is preferred to y and two where y is preferred to x. Then, any group strategy-proof social choice function f on $\mathcal{R}$ with a binary range is also strongly group strategy-proof.*

Proof. Let f be a group strategy-proof social choice function. Suppose that f is not strongly group strategy-proof. That is, there exist $R \in \mathcal{R}$, a coalition $C \subseteq N$, and $R'_C \in \mathcal{R}^C$ such that for some agent $l \in C$, $R_l \neq R'_l$, for all agents $i \in C$, $f(R'_C, R_{-C})R_i f(R)$, and for some $j \in C$, $f(R'_C, R_{-C})P_j f(R)$. If for any agent $l \in C$, $R_l \neq R'_l$, then we get a contradiction to group strategy-proofness.
Thus, there exist $l \in C$ such that $R_l = R'_l$. Define $C_P = \{j \in C : R_j = R'_j$ and $f(R'_C, R_{-C})P_j f(R)\}$ and $C_I = \{k \in C : R_k = R'_k$ and $f(R'_C, R_{-C})I_k f(R)\}$.
By the complementary domain condition, for any $j \in C_P$, there exists $R''_j \in \mathcal{R}_j \backslash R_j$ such that $f(R'_C, R_{-C})P''_j f(R)$. If $f(R''_{C_P}, R'_{C\backslash C_P}, R_{-C}) = f(R)$ there exist a coalition C_P, a profile $(R''_{C_P}, R'_{C\backslash C_P}, R_{-C}) \in \mathcal{R}$, and $R'_{C_P} = R_{C_P}$ such that for any agent $j \in C_P$ $R''_j \neq R_j$ and $f(R'_C, R_{-C})P_j f(R''_{C_P}, R'_{C\backslash C_P}, R_{-C}) = f(R)$ which is a contradiction to group strategy-proofness.
Thus, $f(R''_{C_P}, R'_{C\backslash C_P}, R_{-C}) = f(R'_C, R_{-C})$.
If $C_I = \varnothing$ then observe that there exist $R \in \mathcal{R}$, a coalition $C \subseteq N$, and $R''_C \equiv (R''_{C_P}, R'_{C\backslash C_P}) \in \mathcal{R}^C$ such that for any agent $i \in C$, $R_i \neq R''_i$ and $f(R''_{C_P}, R'_{C\backslash C_P}, R_{-C})R_i f(R)$, and for some $j \in C$, $f(R''_{C_P}, R'_{C\backslash C_P}, R_{-C})P_j f(R)$. Then we get a contradiction to group strategy-proofness.
Thus, $C_I \neq \varnothing$. By the complementary domain condition, for any $k \in C_I$, there exists $R''_k \in \mathcal{R}_k \backslash R_k$ such that $f(R''_{C_P}, R'_{C\backslash C_P}, R_{-C})P''_k f(R)$. If $f(R''_{C_P \cup C_I}, R'_{C\backslash(C_P \cup C_I)}, R_{-C}) = f(R)$ coalition C_I could manipulate f via R_{C_I} at $(R''_{C_P \cup C_I}, R'_{C\backslash(C_P \cup C_I)}, R_{-C})$, which contradicts group strategy-proofness. Thus, $f(R''_{C_P \cup C_I}, R'_{C\backslash(C_P \cup C_I)}, R_{-C}) = f(R'_C, R_{-C})$. Then observe that there exist $R \in \mathcal{R}$, a coalition $C \subseteq N$, and $R'''_C \equiv (R''_{C_P \cup C_I}, R'_{C\backslash(C_P \cup C_I)}) \in \mathcal{R}^C$ such that for any agent $i \in C$, $R_i \neq R'''_i$ and $f(R''_{C_P \cup C_I}, R'_{C\backslash(C_P \cup C_I)}, R_{-C})R_i f(R)$, and for some $j \in C$, $f(R''_{C_P \cup C_I}, R'_{C\backslash(C_P \cup C_I)}, R_{-C})P_j f(R)$, and then we get a contradiction to group strategy-proofness. ■

Remark 1 *We have assumed in Proposition 1 that $\#A \geq 3$. This is because the complementary domain condition that we assume in our statement can only be satisfied in this case. When $\#A = 2$, this condition cannot be satisfied* *and in fact the equivalence does not hold (the rule in Example 1 when $\#A = 2$ provides a counterexample).*

3 Characterization results: properties

In this section we provide our first set of characterization results. We prove that our different versions of the condition that a rule should be xy-based and monotonic are necessary and sufficient to guarantee that they satisfy our different versions of group strategy-proofness.[5]

For each preference profile $R \in \mathcal{R}$, define the set $X(R) = \{i \in N : xP_i y\}$, $Y(R) = \{j \in N : yP_j x\}$, and $I(R) = \{k \in N : yI_k x\}$.

We now define the conditions that will characterize weak and strong group strategy-proofness.

Definition 5 *A binary social choice function is **essentially** xy**-monotonic**[6] if and only if for any $R, R' \in \mathcal{R}$ such that $R_h = R'_h$ for all $h \in I(R) \cap I(R')$, the following holds:*

$$[X(R') \supseteq X(R),\ Y(R) \supseteq Y(R') \text{ (at least one strict inclusion), and } f(R) = x] \Rightarrow f(R') = x, \text{ and}$$

$$[Y(R') \supseteq Y(R),\ X(R) \supseteq X(R') \text{ (at least one strict inclusion), and } f(R) = y] \Rightarrow f(R') = y.$$

Definition 6 *A binary social choice function is xy**-strongly monotonic** if and only if for any $R, R' \in \mathcal{R}$ the following holds:*

$$\begin{gathered}[\text{if either} X(R') \supseteq X(R),\ Y(R) \supseteq Y(R') \text{ (at least one strict inclusion), or} \\ X(R') \supseteq X(R),\ \varnothing \neq Y(R) \subsetneqq Y(R')] \text{ and } f(R) = x \Rightarrow f(R') = x, \\ [\text{if either } Y(R') \supseteq Y(R),\ X(R) \supseteq X(R') \text{ (at least one strict inclusion), or} \\ Y(R') \supseteq Y(R),\ \varnothing \neq X(R) \subsetneqq X(R')] \text{ and } f(R) = y \Rightarrow f(R') = y.\end{gathered}$$

[5] Examples showing the relationship between the properties defined in this section are available upon request.

[6] Lemma 7 in Manjunath (2009b) shows that when the set of admissible preferences is the set of all single-dipped preferences and a specific binary range restriction, some version of essentially xy-monotonicity is a consequence of strategy-proofness.

Observe that xy-strong monotonicity implies essential xy-monotonicity but the converse does not hold. Moreover, both concepts coincide if indifferences are not allowed.

Definition 7 *A social choice function is* ***essentially*** xy-***based***[7] *if and only if for all* $R, R' \in \mathcal{R}$ *such that* $R_h = R'_h$ *for* $h \in I(R)$

$$[X(R) = X(R') \text{ and } Y(R) = Y(R')] \Rightarrow f(R) = f(R').$$

Definition 8 *A social choice function is* xy-***based*** *if and only if for all* $R, R' \in \mathcal{R}$ *such that* $X(R) \cup Y(R) \neq \varnothing$,

$$[X(R) = X(R') \text{ and } Y(R) = Y(R')] \Rightarrow f(R) = f(R').$$

Observe that if a rule is xy-based it is also essentially xy-based but that the converse does not hold. Again, both concepts coincide if indifferences are not allowed. Moreover, both concepts trivially hold if for any agent $i \in N$, $\#R_i^x \leqslant 1$, $\#R_i^y \leqslant 1$, and $\#R_i^{xy} \leqslant 1$, in particular when $\#A = 2$ they always coincide.

Next, we state our two characterization results using the above properties.[8]

Theorem 1 *A social choice function* f *on* $\mathcal{R}$ *with binary range is strategy-proof, thus also weakly group strategy-proof, if and only if* f *is essentially* xy*-based and essentially* xy*-monotonic.*

Proof. ($\Leftarrow$) Let f be a social choice function with binary range that is essentially xy-based and essentially xy-monotonic. By contradiction, suppose that agent i can manipulate f at R via R'_i, that is, $f(R'_i, R_{-i}) P_i f(R)$, where without loss of generality $f(R) = x$ and $f(R'_i, R_{-i}) = y$. Thus, $i \in Y(R)$. If $i \in Y(R'_i, R_{-i})$, since f is essentially xy-based we obtain that $f(R) = f(R'_i, R_{-i})$. If $i \in N \backslash Y(R'_i, R_{-i})$ by essential xy-monotonicity of f we get

[7] Lemma 6 in Manjunath (2009b) shows that when the set of admissible preferences is the set of all single-dipped preferences and a specific binary range restriction, essentially xy-basedness is a consequence of strategy-proofness.

[8] Examples showing the independence of the properties required to characterize the two different versions of non-manipulability by groups (essentially xy-based and essentially xy-monotonicity on the one hand and xy-based and xy-strong monotonicity on the other hand) are available upon request.

that $f(R'_i, R_{-i}) = f(R)$. Thus, we obtain the desired contradiction.

($\Rightarrow$) Let f be a strategy-proof social choice function with binary range. First, we prove by contradiction that f is essentially xy-based. Suppose not. Let $R, R' \in \mathcal{R}$ such that $X(R) = X(R')$, $Y(R) = Y(R')$, $R_h = R'_h$ for any $h \in I(R)$, $f(R) = x$, and $f(R') = y$. Let S be the set of agents $i \in N$ changing their preferences when going from R_i to R'_i. Note that $S \subseteq X(R) \cup Y(R)$. Without loss of generality, suppose that S is a singleton k.[9] Thus, $f(R) = x$ and $f(R'_k, R_{-k}) = y$. If $k \in X(R) = X(R')$, agent k could manipulate f at R' via R_k. If $k \in Y(R) = Y(R')$, agent k could manipulate f at R via R'_k. This is the desired contradiction.

Now we prove that f is essentially xy-monotonic. Suppose not, that is, there exist $R, R' \in \mathcal{R}$ such that $R_h = R'_h$ for all $h \in I(R) \cap I(R')$, either $X(R') \supseteq X(R)$, $Y(R) \supseteq Y(R')$ (with one inequality strict), $f(R) = x$, and $f(R') = y$, or else $Y(R') \supseteq Y(R)$, $X(R) \supseteq X(R')$ (with one inequality strict), $f(R) = y$, and $f(R') = x$. We analyze the former case since the latter is symmetric and similar arguments apply.

Consider the set S of agents $i \in N$ who change preferences over x and y when going from R_i to R'_i. Any agent $j \in S$ is such that one of the following cases holds: (1) $j \in Y(R)$ and $j \in I(R')$, (2) $j \in Y(R)$ and $j \in X(R')$, or (3) $j \in X(R)$ and $j \in X(R')$. That is, S can be partitioned into three sets of agents, say S_1, S_2, and S_3, satisfying cases 1, 2, and 3, respectively.

Start from profile R and change preferences of all agents $j \in S$ one by one from R_j to R'_j.

Let $j \in S_1$. Then, $f(R'_j, R_{-j}) = x$ (otherwise, agent j could manipulate f at R via R'_j). Repeating the same argument for any $j \in S_1$ we obtain that $f(R'_{S_1}, R_{-S_1}) = x$.

Let $j \in S_2$. Then, $f(R'_{S_1}, R'_j, R_{N\setminus\{S_1 \cup j\}}) = x$ (otherwise, agent j could ma-

[9] If S is not a singleton, observe first that by definition of S, $f(R_S, R'_{-S}) = f(R_S, R_{-S}) = x$. Now, depart from R and first change one by one preferences of agents in $X(R) \cap S$. Then, either $f(R'_{X(R)\cap S}, R_{S\setminus\{X(R)\cap S\}}, R'_{-S}) = x$ or at some step, after changing the preference of some agent $k \in X(R) \cap S$ we would go from x to y. That is, $f(R'_{\{1,\ldots,(k-1)\}}, R_k, R_{S\setminus\{1,\ldots,k\}}, R'_{-S}) = x$ and $f(R'_{\{1,\ldots,k\}}, R_{S\setminus\{1,\ldots,k\}}, R'_{-S}) = y$. Then, we obtain a contradiction to strategy-proofness. Thus, $f(R'_{X(R)\cap S}, R_{S\setminus\{X(R)\cap S\}}, R'_{-S}) = x$. Second, change one by one preferences of agents in $Y(R) \cap S$. Similarly, either $f(R'_{[X(R)\cup Y(R)]\cap S}, R_{S\setminus\{[X(R)\cup Y(R)]\cap S\}}, R'_{-S}) = x$ or at some step after changing the preference of some agent $l \in Y(R) \cap S$ we would go from x to y, and we would also obtain a contradiction to strategy-proofness. Thus, $f(R'_S, R'_{-S}) = x$ which is the desired contradiction.

nipulate f at (R'_{S_1}, R_{-S_1}) via R'_j). By repeating the same argument for any $j \in S_2$ we obtain that $f(R'_{S_1 \cup S_2}, R_{-N\setminus\{S_1 \cup S_2\}}) = x$.
Let $j \in S_3$. Then, $f(R'_{S_1 \cup S_2}, R'_j, R_{N\setminus\{S_1 \cup S_2 \cup j\}}) = x$ (otherwise, agent j could manipulate f at $(R'_{S_1 \cup S_2}, R'_j, R_{N\setminus\{S_1 \cup S_2 \cup j\}})$ via R_j). By repeating the same argument for any $j \in S_3$ we obtain that $f(R'_S, R_{-N\setminus S}) = x$.
Consider the set $T = N\setminus S$ that do not change preferences over x and y when going from R_i to R'_i, but may change their rankings of other alternatives. Any agent $t \in T$ is such that one of the following cases holds: (1) $t \in X(R) \cap X(R')$, (2) $t \in Y(R) \cap Y(R')$. Thus, T can be partitioned into two sets, say T_1 and T_2.
Start from profile R and change the preferences of all agents $t \in T$ one by one from R_t to R'_t.
Let $t \in T_1$. Then, $f(R'_S, R'_t, R_{T\setminus\{t\}}) = x$ (otherwise, agent j could manipulate f at $(R'_S, R'_t, R_{T\setminus\{t\}})$ via R_t). By repeating the same argument for any $t \in T_1$ we obtain that $f(R'_S, R'_{T_1}, R_{T\setminus T_1}) = x$.
Let $t \in T_2$. Then, $f(R'_{S\cup T_1}, R'_t, R_{T\setminus\{T_1 \cup t\}}) = x$ (otherwise, agent j can manipulate f at $(R'_{S\cup T_1}, R_{T\setminus T_1})$ via R'_t). By repeating the same argument for any $t \in T_2$ we obtain that $f(R') = x$ which is the desired contradiction. ■

Our second characterization result has similar flavour than that of Theorem 1. It establishes the equivalence of strong group strategy-proofness with two conditions, one of monotonicity and the other requiring the rule to be range based.

Theorem 2 *Let f be a social choice function on $\mathcal{R}$ with a binary range. Then f is strongly group strategy-proof if and only if f is xy-based and xy-strong monotonic.*

We postpone the proof of Theorem 2. It will be provided jointly with that of Theorems 3 and 4.

4 Characterization results: functional forms

In this Section we provide a second characterization result for strongly group strategy-proof rules. Manjunath (2009a) has already presented essentially the same result for the two-alternative case. We slightly reformulate it here in order to include some additional precisions for the two-agent case, and to cover the case where preferences are defined on more than two alternatives. We also provide a different proof than Manjunath's, and use it as an important piece to establish Theorem 2 in the preceding Section.

The functional form of all strongly group strategy-proof social choice functions is slightly more flexible when $n = 2$ than when $n \geq 3$. Let us first define two relevant classes of social choice functions: serial dictators and veto rules.

Definition 9 *Let $1 \succ 2 \succ \ldots \succ n$ be an ordering of agents. Then, a serial dictator with order $\succ$, say $f_{\succ}$, is defined as follows: $f_{\succ}(R) = x$ if $R \in \mathcal{R}$ is such that either xP_1y, or xI_1y and xP_2y, or xI_iy, $i = 1, 2$ and xP_3y, or so on and so forth if $X(R) \neq \varnothing$; $f_{\succ}(R) = y$ if $R \in \mathcal{R}$ is such that either yP_1x, or xI_1y and yP_2x, or xI_iy, $i = 1, 2$ and yP_3x, or so on and so forth if $Y(R) \neq \varnothing$. The values of the function for profiles where $I(R) = N$ (that is, where all agents are indifferent between x and y) may vary from profile to profile.*

The following example shows that for $n \geq 3$ the serial dictators are not group strategy-proof, thus neither strongly. However, observe that they are always (weakly group) strategy-proof.

Example 3 *Let $n \geq 3$ and let $f_{\succ}$ be a serial dictator with order $1 \succ 2 \succ \ldots \succ n$. Let $R \in \mathcal{R}$ such that xI_1y, xP_2y, yP_3x, and other agents have preference $R_{-\{1,2,3\}}$, $R_{-\{1,2,3\}} \in \mathcal{R}_{N\setminus\{1,2,3\}}$. Let $R' \in \mathcal{R}$ such that $yP_1'x$, $xP_2'y$, $yP_3'x$, where $R_3' \neq R_3$ and for any $j \notin \{1,3\}$, $R_j' = R_j$. Observe that $f_{\succ}(R) = x$ and $f_{\succ}(R') = y$, and then coalition $\{1,3\}$ could manipulate $f_{\succ}$ at R via $R'_{\{1,3\}}$. Thus, $f_{\succ}$ is not group strategy-proof.*

Definition 10 *A veto rule for y is defined as follows:*

$$f(R) = \begin{cases} x & \text{for any } R \in \mathcal{R} \text{ such that } xP_iy \text{ for some } i \in N \\ y & \text{for any } R \in \mathcal{R} \text{ such that } yR_ix \text{ for any } i \in N \text{ and } yP_jx \text{ for some } j \in N \end{cases}$$

The values of the function for profiles where $I(R) = N$ (that is, where all agents are indifferent) may vary from profile to profile.
A veto rule for x is defined symmetrically exchanging x by y, and viceversa.

The following concept and lemmata will be useful in the proof of our results below.

Lemma 1 *Let $n \geq 2$. Any strongly group strategy-proof social choice function f on $\mathcal{R}$ with binary range is xy-Paretian.*[10]

Proof. Suppose first that f is not xy-Paretian. That is, suppose that there exists $R \in \mathcal{R}$ such that xR_iy for any $i \in N$ and xP_jy for some $j \in N$ and $f(R) = y$ (a similar contradiction would be obtained exchanging the roles of x and y). Then, N could weakly manipulate f at R via R' for any R' such that $f(R') = x$, which exists since x is the range. This is the desired contradiction. ■

Definition 11 *Let f be a social choice function on $\mathcal{R}$ with binary range. We say that an agent $i \in N$ is xy-pivotal for a profile $R \in \mathcal{R}$ if there exists $R'_i \in \mathcal{R}_i$ such that $f(R'_i, R_{-i}) \neq f(R)$.*

Lemma 2 *Let $n \geq 3$ and f be a social choice function on $\mathcal{R}$ with binary range. Let $R \in \mathcal{R}$ be such that $X(R) \neq \varnothing$, $Y(R) \neq \varnothing$, and there is an agent $i \in N$ that is xy-pivotal for R such that xI_iy. Then, f is not strongly group strategy-proof.*

Proof. Without loss of generality, suppose that $f(R) = x$. Note that by assumption there exists an agent i that is xy-pivotal for R and such that xI_iy. Then, let $C = \{i\} \cup Y(R)$. Observe that C could weakly manipulate f at R via $R'_C = (R'_i, R_{C\setminus\{i\}})$. ■

Theorem 3 *For $n \geq 3$, veto rules for x or y are the unique strongly group strategy-proof social choice functions with binary ranges.*
For $n = 2$, veto rules for x or y and serial dictators with any order of agents are the unique strongly group strategy-proof social choice functions with binary ranges.

Remark 2 *The veto rules have already been described by Manjunath (2009a). Our main additions are, on the one hand, that we propose a different proof. And, on the other hand, the fact that we consider its extension to the case where there are more than two alternatives, but the range is binary. Notice that in this larger context the rule could choose different alternatives in different profiles where all agents are indifferent between x and y. Also notice that even dictatorships fail to be strongly group strategy-proof when $n \geq 2$ but when $n = 2$ serial dictatorships can also be strongly group strategy-proof.*

[10]Note that this result can be generalized for rules with any range size.

Our proof of Theorems 2 and 3 will also imply a third result that we now state.

Theorem 4 *For $n \geq 3$, veto rules for x or y are the unique xy-based and xy-strong monotonic social choice functions with binary ranges.*
For $n = 2$, veto rules for x or y and serial dictators with any order of agents are the unique xy-based and xy-strong monotonic social choice functions with binary ranges.

We now proceed to the joint proof of Theorems 2, 3, and 4. Our strategy is to show one of the directions in each of the three. Specifically, we start by proving that veto rules (and eventually serial dictatorships when $n = 2$) are strongly group strategy-proof (Step 1). Then, that strongly group strategy-proof rules must be xy-based and xy-strong monotonic (Step 2), and finally that rules satisfying these properties must have the functional forms we started with (Step 3).

We now present the formal proof.

Proof of Theorems 2, 3, and 4.
Step 1 We first show that, for $n \geq 2$, any veto rule is strongly group strategy-proof and that for $n = 2$, any serial dictatorship is also strongly group strategy-proof.
Proof of Step 1:
By contradiction, let f be a veto rule for x (a similar argument would follow if f was a veto rule for y) that is not strongly group strategy-proof. That is, there exist $R \in \mathcal{R}$, $C \subseteq N$, and $R'_C \in \mathcal{R}^C$ such that for some agent $l \in C$, $R_l \neq R'_l$, for all agent $i \in C$, $f(R'_C, R_{-C}) R_i f(R)$, and for some $j \in C$, $f(R'_C, R_{-C}) P_j f(R)$. Clearly, $f(R'_C, R_{-C}) \neq f(R)$. By definition of f as a veto rule for x, $f(R'_C, R_{-C}) = x$ and $f(R) = y$ (otherwise, if $f(R'_C, R_{-C}) = y$ and $f(R) = x$, since for some agent $j \in C$, $y P_j x$, then by definition of a veto rule for x, $f(R) = y$ which is a contradiction). In order that $f(R) = y$ there must exist $j \in N \backslash C$ such that $y P_j x$. Since $R'_j = R_j$ for any $j \in N \backslash C$, $f(R'_C, R_{-C}) = y$ which is the desired contradiction. Thus, a veto rule is strongly group strategy-proof.

Next, we show that, for $n = 2$, any serial dictatorship is strongly group strategy-proof. By strategy-proofness no individual deviation is beneficial: only two agents' deviations might exist. Notice that the first agent in the order will never participate in a deviating coalition unless he is indifferent between x and y. But then the second agent obtains his best outcome.

This ends the proof of Step 1.

Step 2 Any strongly group strategy-proof social choice function f with binary range is xy-based and xy-strong monotonic.

Proof of Step 2:

By Theorem 1, we know that f is essentially xy-based and essentially xy-monotonic.

We now prove by contradiction that f is xy-based. Suppose not, then there exist $R, R' \in \mathcal{R}$ such that $X(R) \cup Y(R) \neq \varnothing$, $X(R) = X(R')$, $Y(R) = Y(R')$, $f(R) \neq f(R')$. Suppose first that $X(R) = \varnothing$ and thus $Y(R) \neq \varnothing$ (a similar argument applies if $Y(R) = \varnothing$). By Lemma 1, $f(R) = f(R') = y$ which is a contradiction. Thus, $X(R) \neq \varnothing$ and $Y(R) \neq \varnothing$.

By essentially xy-based, $f(R'_{X(R)\cup Y(R)}, R_{I(R)}) = f(R)$.

If $n = 2$, $R' = (R'_{X(R)\cup Y(R)}, R_{I(R)})$ and we get the desired contradiction since $f(R')$ must be different from $f(R)$.

If $n \geq 3$, since $f(R) \neq f(R')$ there must exist an agent $i \in N$ such that xI_iy that is xy-pivotal for $(R'_{X(R)\cup Y(R)}, R_{I(R)})$. By Lemma 2, f is not strongly group strategy-proof which is a contradiction.

We now prove by contradiction that f is xy-strong monotonic. Suppose not, that is there exist $R, R' \in \mathcal{R}$ such that either (1) $X(R') \supseteq X(R)$, $Y(R) \supseteq Y(R')$ (at least one inclusion strict), $f(R) = x$ but $f(R') = y$, or else (2) $X(R') \supseteq X(R)$, $\varnothing \neq Y(R) \subsetneqq Y(R')$, $f(R) = x$ and $f(R') = y$. A similar argument holds for the other possibility where the roles of x and y are exchanged. First observe that by Lemma 1, $X(R) \neq \varnothing$ and $Y(R) \neq \varnothing$ (otherwise, if $Y(R) = \varnothing$, then $Y(R') = \varnothing$ and $X(R') \supsetneqq X(R)$. By Lemma 1, $f(R') = x$ which is the desired contradiction. If $X(R) = \varnothing$ and $Y(R) \neq \varnothing$, then by Lemma 1 $f(R) = y$ which is the desired contradiction).

If case (1) holds, by essential xy-monotonicity and essential xy-basedness, $f(R'_{X(R)\cup Y(R)}, R_{I(R)}) = x$ (define $R'' = (R'_{X(R)\cup Y(R)}, R_{I(R)})$, if either $X(R'') \supsetneqq X(R)$ and $Y(R'') \subsetneqq Y(R)$ or $X(R'') = X(R)$ and $Y(R'') \subsetneqq Y(R)$ we apply essential xy-monotonicity. If $X(R'') = X(R)$ and $Y(R'') = Y(R)$ we apply essential xy-basedness).

If $n = 2$, $R' = (R'_{X(R)\cup Y(R)}, R_{I(R)})$ and we get the desired contradiction since $f(R')$ must be different from $f(R)$.

If $n \geq 3$, since $f(R) \neq f(R')$ there must exist an agent $i \in N$ such that xI_iy that is xy-pivotal for $(R'_{X(R)\cup Y(R)}, R_{I(R)})$. By Lemma 2, f is not strongly

group strategy-proof which is a contradiction.
If case (2) holds, by essential xy-monotonicity and essential xy-basedness, $f(R'_{X(R')}, R_{N\setminus X(R')}) = x$ (if $X(R') \supsetneqq X(R)$, that is, $X(R')$ includes some agents in $I(R)$, we apply essential xy-monotonicity. If $X(R') = X(R)$ we apply essential xy-basedness). Then, $f(R'_{X(R')}, R'_{Y(R)}, R_{N\setminus\{X(R')\cup Y(R)\}}) = x$ by essential xy-basedness.
If $n = 2$, $R' = (R'_{X(R')}, R'_{Y(R)}, R_{N\setminus\{X(R')\cup Y(R)\}})$ and we get the desired contradiction since $f(R')$ must be different from $f(R)$.
If $n \geq 3$, since $f(R) \neq f(R')$ there must exist an agent $i \in N$ that is xy-pivotal agent for $(R'_{X(R')}, R'_{Y(R)}, R_{N\setminus\{X(R')\cup Y(R)\}})$ such that xI_iy. By Lemma 2, f is not strongly group strategy-proof which is a contradiction.

This ends the proof of Step 2.

Step 3 Any xy-based and xy-strong monotonic social choice function f with binary range can be described as a veto rule when $n \geq 3$. When $n = 2$, f is either a veto rule or a serial dictator.

To show Step 3, we use the following claims.

Observe first that since f is xy-based then for any $R_i^t, \overline{R}_i^t \in \mathcal{R}_i^t$, $f(R_i^t, R_{-i}) = f(\overline{R}_i^t, R_{-i})$ for any $R_{-i} \in \mathcal{R}^{N\setminus\{i\}}$ where $t \in \{x, y, xy\}$.
In what follows, when we use R_i^t we refer to any $R_i^t \in \mathcal{R}_i^t$ without loss of generality. This is because all the statements we make in this proof from now on hold whatever the representative of the set $\mathcal{R}_i^t$ is.

Claim 1 Let $n \geq 2$. If f is xy-based and xy-strong monotonic then f is xy-Paretian.
Proof of Claim 1 Let $R^x \in \times_{i\in N}\mathcal{R}_i^x$, that is, $X(R^x) = N$. Suppose to get a contradiction that $f(R^x) = y$. Note that by xy-based and xy-strong monotonicity, for any other profile $R \in \mathcal{R}$, $f(R) = y$ which contradicts that f has a binary range. Thus, $f(R^x) = x$.
Suppose that there is R such that xR_iy for any $i \in N$ and xP_jy for some $j \in N$, $X(R) \neq N$ and $f(R) = y$. Then $X(R) \neq \varnothing$ and $Y(R) = \varnothing$. Note that $X(R) \subsetneqq X(R^x)$ and $Y(R) = Y(R^x)$, for any $R^x \in \times_{i\in N}\mathcal{R}_i^x$. By xy-strong monotonicity, $f(R^x) = y$ which contradicts what we have just proved. This ends the proof of Claim 1.

Note that the counterpart results to Claims 2 and 3 below exchanging the roles of x and y do also hold.

Claim 2 Let $n \geq 2$. If for some $i \in N$ and some $R_i^y \in \mathcal{R}_i^y$, $f(R_i^y, R_{-i}^x) = y$,

then $f(R_i^y, R'_{-i}) = y$ for any $R'_{-i} \in \mathcal{R}^{N\backslash\{i\}}$.
Proof of Claim 2 Let $R'_{-i} \in \mathcal{R}^{N\backslash\{i\}}$. Observe that $f(R_i^y, R'_{-i}) = y$ either by xy-based if $X(R_i^y, R'_{-i}) = N\backslash\{i\} = X(R_i^y, R_{-i}^x)$, or else by xy-strong monotonicity if $X(R_i^y, R'_{-i}) \subsetneqq N\backslash\{i\} = X(R_i^y, R_{-i}^x)$. This ends the proof of Claim 2.

Claim 3 Let $n \geq 3$. If for some $i \in N$ and some $R_i^y \in \mathcal{R}_i^y$, $f(R_i^y, R_{-i}^x) = y$, then for any $j \in N$ we have that $f(R_j^y, R_{-j}^x) = y$.
Proof of Claim 3 By contradiction, suppose that $f(R_i^y, R_{-i}^x) = y$ and $f(R_j^y, R_{-j}^x) = x$. If $f(R_i^{xy}, R_j^y, R_{-\{i,j\}}^x) = y$ then $f(R_j^y, R_{-j}^x) = y$ by xy-strong monotonicity since $\varnothing \neq X(R_i^{xy}, R_j^y, R_{-\{i,j\}}^x) \subsetneqq X(R_j^y, R_{-j}^x)$ and $Y(R_i^{xy}, R_j^y, R_{-\{i,j\}}^x) = Y(R_j^y, R_{-j}^x)$. Thus, $f(R_i^{xy}, R_j^y, R_{-\{i,j\}}^x) = x$. By xy-strong monotonicity, $f(R_i^y, R_j^y, R_{-\{i,j\}}^x) = x$, since $X(R_i^{xy}, R_j^y, R_{-\{i,j\}}^x) = X(R_i^y, R_j^y, R_{-\{i,j\}}^x)$ and $\varnothing \neq Y(R_i^{xy}, R_j^y, R_{-\{i,j\}}^x) \subsetneqq Y(R_i^y, R_j^y, R_{-\{i,j\}}^x)$. By Claim 2, since $f(R_i^y, R_{-i}^x) = y$ then $f(R_i^y, R_j^y, R_{-\{i,j\}}^x) = y$ which contradicts what we obtained above. This ends the proof of Claim 3.

Claim 4 Let $n \geq 3$. If for some $i \in N$ and some $R_i^y \in \mathcal{R}_i^y$, $f(R_i^y, R_{-i}^x) = x$ then $f(R_C^y, R_{-C}^x) = x$ for any C, $\varnothing \subsetneqq C \subsetneqq N$.
Proof of Claim 4 Suppose, to get a contradiction, that for some C, $\varnothing \subsetneqq C \subsetneqq N$, $f(R_C^y, R_{-C}^x) = y$. Clearly, $C \neq \{i\}$. Note also that C can not be a singleton (otherwise, if $C = \{j\}$, $j \neq i$, we would get a contradiction by Claim 3). Thus, $\#C > 1$. Note also that $f(R_k^y, R_{-k}^x) = x$ for any $k \in N$ (otherwise, $f(R_k^y, R_{-k}^x) = y$, by Claim 4, $f(R_i^y, R_{-i}^x) = y$ which is not the case). Therefore, without loss of generality, we can suppose that $i \in C$. We distinguish two subcases:
Subcase 1 Let $f(R_i^y, R_{C\backslash\{i\}}^{xy}, R_{-C}^x) = x$. Note that $X(R_i^y, R_{C\backslash\{i\}}^{xy}, R_{-C}^x) = X(R_C^y, R_{-C}^x)$ and $\varnothing \neq Y(R_i^y, R_{C\backslash\{i\}}^{xy}, R_{-C}^x) \subsetneqq Y(R_C^y, R_{-C}^x)$. Therefore, by xy-strong monotonicity $f(R_C^y, R_{-C}^x) = x$, which is a contradiction.
Subcase 2 Let $f(R_i^y, R_{C\backslash\{i\}}^{xy}, R_{-C}^x) = y$. Note that $Y(R_i^y, R_{C\backslash\{i\}}^{xy}, R_{-C}^x) = Y(R_i^y, R_{-i}^x)$ and $\varnothing \neq X(R_i^y, R_{C\backslash\{i\}}^{xy}, R_{-C}^x) \subsetneqq X(R_i^y, R_{-i}^x)$. Therefore, by xy-strong monotonicity, $f(R_i^y, R_{-i}^x) = y$, which is a contradiction. This ends the proof of Claim 4.

<u>*Proof of Step 3:*</u>
First, by Claim 1, $f(R) = x$ for any R such that xR_iy for any $i \in N$ and xP_jy for some $j \in N$ and $f(R) = y$ for any R such that yR_ix for any $i \in N$ and yP_jx for some $j \in N$. Second, $f(R)$ can be any outcome for any R where

all agents are indifferent. Third, the argument differs depending on n being two or higher.

If $n = 2$, suppose first that f is such that for some profile (R_1^y, R_2^x), where $R_1^y \in \mathcal{R}_1^y$ and $R_2^x \in \mathcal{R}_2^x$, $f(R_1^y, R_2^x) = y$ and for some profile (R_2^y, R_1^x), where $R_2^y \in \mathcal{R}_2^y$ and $R_1^x \in \mathcal{R}_1^x$, $f(R_2^y, R_1^x) = y$. By Claim 2, for any $R_1^y \in \mathcal{R}_1^y$ and $R_2^x \in \mathcal{R}_2^x$, $f(R_1^y, R_2) = y$ and $f(R_2^y, R_1) = y$ for any $R_2 \in \mathcal{R}_2$ and $R_1 \in \mathcal{R}$. Thus, f can be rewritten as a veto rule for x.
Second, suppose that for some profile (R_2^y, R_1^x), where $R_2^y \in \mathcal{R}_2^y$ and $R_1^x \in \mathcal{R}_1^x$, $f(R_2^y, R_1^x) = y$ and for any profile (R_1^y, R_2^x), where $R_1^y \in \mathcal{R}_1^y$ and $R_2^x \in \mathcal{R}_2^x$, $f(R_1^y, R_2^x) = x$. By Claim 2, for any $R_2^y \in \mathcal{R}_2^y$, $f(R_2^y, R_1) = y$ and for any $R_1 \in \mathcal{R}_1$. Note that this rule f can be rewritten as a serial dictator with order $2 \succ 1$.
Third, suppose that for some profile (R_1^y, R_2^x), where $R_1^y \in \mathcal{R}_1^y$ and $R_2^x \in \mathcal{R}_2^x$, $f(R_1^y, R_2^x) = y$ and for any profile (R_2^y, R_1^x), where $R_2^y \in \mathcal{R}_2^y$ and $R_1^x \in \mathcal{R}_1^x$, $f(R_2^y, R_1^x) = x$. By Claim 2, for any $R_1^y \in \mathcal{R}_1^y$, $f(R_1^y, R_2) = y$ and for any $R_2 \in \mathcal{R}_2$. Note that this rule f can be rewritten as a serial dictator with order $1 \succ 2$.
Finally, suppose that for any profile (R_2^y, R_1^x), where $R_2^y \in \mathcal{R}_2^y$ and $R_1^x \in \mathcal{R}_1^x$, $f(R_2^y, R_1^x) = x$ and for any profile (R_1^y, R_2^x), where $R_1^y \in \mathcal{R}_1^y$ and $R_2^x \in \mathcal{R}_2^x$, $f(R_1^y, R_2^x) = x$. By the counterpart of Claim 2, for any $R_2^y \in \mathcal{R}_2^y$, $f(R_2^y, R_1) = x$ for any $R_1 \in \mathcal{R}_1$, and for any $R_1^y \in \mathcal{R}_1^y$, $f(R_1^y, R_2) = x$ for any $R_2 \in \mathcal{R}_2$. Thus, f can be rewritten as a veto rule for y.

If $n \geq 3$, suppose first that f is such that for some profile (R_i^y, R_{-i}^x), where $R_i^y \in \mathcal{R}_i^y$ and $R_j^x \in \mathcal{R}_j^x$ for any $j \in N\backslash\{i\}$, $f(R_i^y, R_{-i}^x) = y$. Then, by Claims 2 and 3, $f(R_k^y, R_{-k}) = y$ for any k, any $R_k \in \mathcal{R}_k^y$, and any $R_j \in \mathcal{R}_j$, $j \in N\backslash\{k\}$. That is, the outcome will be y for any profile where there is one agent that strictly supports y over x. Thus, f is a veto rule for x.
Let now suppose that f is such that for all profiles (R_i^y, R_{-i}^x), where $R_i^y \in \mathcal{R}_i^y$ and $R_j^x \in \mathcal{R}_j^x$ for any $j \in N\backslash\{i\}$, $f(R_i^y, R_{-i}^x) = x$. Then, by Claim 4 and the counterparts of Claims 2 and 3, the outcome will be x for any profile where there is one agent that strictly supports x over y. Thus, f is a veto rule for y.

This ends proof of Step 3, and hence the proof of Theorems 2, 3, and 4. ■

5 Final Remarks

In this paper we have provided different definitions of strategy-proofness in front of possible manipulations by groups, and several characterizations of rules satisfying these properties when their range is restricted to cover two alternatives.

We feel that, when attainable, non-manipulability by groups (in its different forms) is an attractive property, since in many contexts different agents can be expected to explore the possibility of benefiting from joint actions, in addition to individual ones. Early authors on the issue of strategy-proofness did indeed refer to the interest of avoiding such joint strategic behavior (Pattanaik, 1978, Dasgupta, Hammond, and Maskin, 1979, Peleg, 1984 and 2002). True, in many domains, and for functions with non-binary ranges, it may be excessive to ask for these properties. But not always! For interesting cases when they may be fulfilled because of domain restrictions, see Moulin (1999), Pápai (2000), Barberà and Jackson (1995). In fact, our paper contemplates another case where joint manipulations can be avoided, this time because the ranges of our functions are restricted.

We have allowed for agents to have preferences over other alternatives that are not in the range, and been careful in following up the implications of that extension in the domains of the rules. This is in contrast with the work of authors who assume that only two alternatives are available when the range consists of two of them. We insist in the difference, because we want to emphasize that the choice to restrict the range is indeed a possible tool for the mechanism designer, even when more than two choices are in principle socially available.

We have also looked for characterizations that are essentially independent of the characteristics of the domains of definition of the rules. This is because the sets of rules satisfying our different versions of non-manipulability by groups could in principle be varying as the domains of definition change from one application to another. By selecting properties that are necessary and sufficient for our conditions to be satisfied, we go to the essentials of the question. And, when needed, our qualifications on the minimal requirements on domains for our results to hold are made explicit at each point.

We have also insisted in examining the role of individuals who are indifferent between the alternatives in the range (but not identical in other respects). The presence of indifferences is always a source of problems in social choice, and it also complicates and enriches our analysis here.

We leave it to the interested reader to examine how our analysis would be simplified (and sometimes reduced to previously existing results) when only two alternatives are present at all, and/or when indifferences among alternatives are ruled out.

Let us also mention that we have concentrated on the notions of weak and strong group strategy-proofness, The intermediate notion of group strategy-proofness has been proven to be equivalent to the strong version under mild domain assumptions, but not for the particular case of two alternatives only. Characterizations of rules satisfying the intermediate property in this particular case are left as an open problem.

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