

TOP MONOTONICITY: A COMMON ROOT FOR SINGLE PEAKEDNESS, SINGLE CROSSING AND THE MEDIAN VOTER RESULT*

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Abstract

When the members of a voting body exhibit single peaked preferences, majority winners exist. Moreover, the median(s) of the preferred alternatives of voters is (are) indeed the majority (Condorcet) winner(s). This important result of Duncan Black (1958) has been crucial in the development of public economics and political economy, even if it only provides a sufficient condition. Yet, there are many examples in the literature of environments where voting equilibria exist and alternative versions of the median voter results are satisfied while single peakedness does not hold. Some of them correspond to instances

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where other relevant conditions, apparently not connected with single peakedness, are satisfied. For example preferences may satisfy the single-crossing property (Mirrlees, 1971, Gans and Smart, 1996, and Milgrom and Shannon, 1994), intermediateness (Grandmont, 1978) or order restriction (Rothstein, 1990). Still other interesting cases of existence of voting equilibria do not fall in any of these categories.

We present a new and weak domain restriction which encompasses all the above mentioned ones, allows for new cases, still guarantees the existence of Condorcet winners and preserves a version of the median voter result. We illustrate how this new condition, that we call top monotonicity, arises naturally in different economic contexts.

Keywords: Single peaked, single crossing and intermediate preferences, majority (Condorcet) winners. JEL Classification: D720, D710

1 Introduction

The existence of voting equilibria is crucial in all models of the economy where some of the variables are determined by the political process, rather than set by the market. The tax rates, the level of provision of public goods or the location of public facilities are examples of such variables. Some simple models just try to describe the partial process leading to determine one of these variables. Others incorporate their choice into a larger picture, where markets coexist with the political process and these variables are jointly determined with many others. Public economics, political economy, public choice and other strands of economic analysis base their predictions on the study of equilibria in models of this kind, and yet the existence problems pop out even in the simplest and most stylized versions of reality. In particular, voting equilibria fail to exist when voting is by majority and cycles arise. The possibility of simple majority cycles is just one instance of the pervasive difficulties that lure behind the use of any type of voting rule, with unrestricted domain. Arrow's impossibility theorem is a warning that some restrictions are needed to get any form of existence results.

Luckily, there are many models of economic interest where individual preferences can be expected to meet the type of requirements that would avoid social preference cycles. A major instance is the case where it can be proven or assumed that preferences of voters are single peaked. Then, majority voting leads to well defined equilibrium outcomes. These are called the Condorcet winners, and they are those alternatives that do not lose by

majority to any other. Moreover, Condorcet winners under single peakedness are the medians of the distribution of preferred alternatives for the different voters, and they are unique under well defined circumstances.

Single peakedness is the oldest and probably the best known restriction on agents' preferences guaranteeing the existence of voting equilibria (Black, 1958). It is sometimes predicated by assumption, but most often it is derived as the natural consequence on a reduced model, where there is only one variable to choose, of assuming convex preferences on a larger space, from which the reduced model is derived.

Much in the same way as convexity induces single peakedness in some reduced models, other general assumptions regarding preferences also induce alternative domain restrictions when applied to simple enough frameworks. This is the case, for example, when preferences satisfy the single crossing property (Mirrlees, 1971, Gans and Smart, 1996, and Milgrom and Shannon, 1994), or the condition of intermediateness (Grandmont, 1978, Rothstein, 1990). Thanks to the implications of these assumptions, it is possible to prove that Condorcet winners exist in many models of political economy, and to identify these winners with the best alternative for some median voter. Also, the single-crossing and the single-peaked properties are meaningful domain restrictions where majority voting works with incentive properties (see Moulin, 1980 and Saporiti, 2009 among many others).

Notice that single peakedness, single crossing and intermediateness appeared independently of each other in the economics literature, that they do not imply one another, and that each one results from its own underlying logic.

In this paper we propose a new condition on preference profiles over one-dimensional alternatives, which we call top monotonicity. We prove that top monotonicity can be viewed as the common root of all these classical restrictions, which had been perceived till now as rather different and unrelated to each other. Specifically, we'll show that single peakedness, as well as the one-dimensional versions of intermediateness and single crossing all imply top monotonicity. In addition, we'll prove that top monotonicity is sufficient to guarantee the existence of Condorcet winners, and that these will be closely connected to an extended notion of the median voter.

Therefore, we claim to have achieved a double goal. One is to clarify the connections among different restrictions that guarantee the existence of Condorcet winners, by finding their common root. The other is to extend the median-based existence result to preference profiles which allow for much

richer combinations of individual preferences than those previously considered. In fact, classical conditions are encompassed by our restriction, but many other profiles will also pass our test (and thus guarantee existence), while not meeting any of the traditional requirements. A third and nontrivial contribution of the paper is of a more technical nature. Our restriction allows for agents to exhibit indifferences to an extent that classical domain restrictions do not. To the extent that indifferences on subsets of alternatives may arise in many natural settings, this ability to deal with them may well be considered substantial, in addition to being an obvious technical improvement.

We are aware that a new domain condition, especially when presented in its most general form, raises some immediate questions. Why do we need it? Is it easy to check? When shall it be useful to know about it? Who can find it of interest?

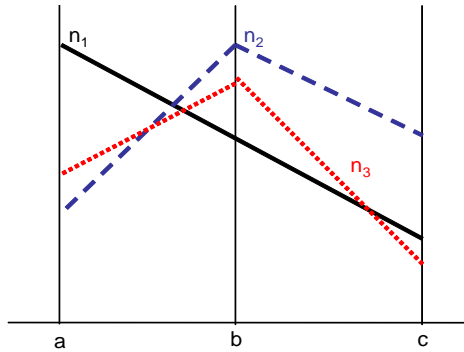


Figure 1. n_1 , n_2 , and n_3 stand for the number of agents having each one of the above preferences.

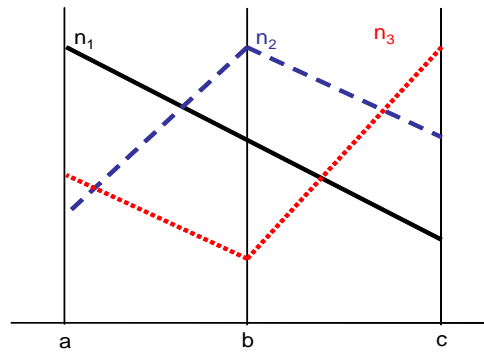


Figure 2. n_1 , n_2 , and n_3 stand for the number of agents having each one of the above preferences.

Let us try to make our case. We must acknowledge from the start that knowing of a sufficient but not necessary condition is never a determining factor: the results we get when it holds may also apply even when the condition does not. Therefore, the defense of a sufficient condition must be in terms of the work it saves us when it happens to be satisfied. For example, knowing that the general shape of a preference profile is as in Figure 1 immediately tells us that we can count on the existence of a Condorcet winner, regardless of the number of voters involved (i.e., for whatever n_1, n_2, n_3). This is because we have been trained to recognize profiles of single peaked preferences and

we know that they deliver the nice existence result. Notice that, if presented with profiles where preferences have the alternative shapes in Figure 2, then existence is neither guaranteed nor precluded, and depends on the relative sizes of n_1 , n_2 and n_3 .

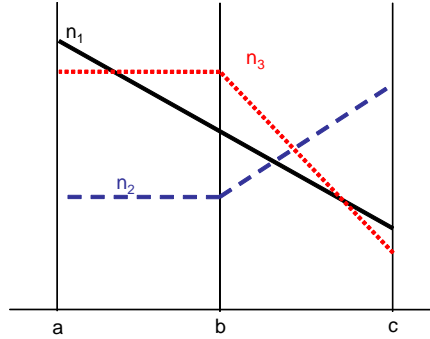


Figure 3. n_1 , n_2 , and n_3 stand for the number of agents having each one of the above preferences.

Similarly, if we are given preferences as in Figure 3, the trained public economist will recognize that the shape of this profile, though not satisfying single peakedness, meets the requirements of single crossing. Again, this will guarantee existence of Condorcet winners at the median, and avoid the need for tedious and specific calculations. In this second case checking for the condition in more complex situations may require more work than in the case of single peakedness. This does not, in our view, make the condition any less interesting, as it results naturally from a different type of models, ones where the shapes of indifference curves for different individuals happen to be adequately connected.

To finish with our argument, consider the two shapes below for possible preference profiles. Could one tell "a priori" whether any of the profiles will lead to the existence of Condorcet winners at the median of the most preferred point for the voters? Not immediately from the shape, although of course one could always proceed to a computation, one for each n_1 , n_2 , and n_3 . Except if you know our condition. Then, it becomes an easy exercise to check that the answer is positive for Example 4, because our new condition holds and is easy to check!

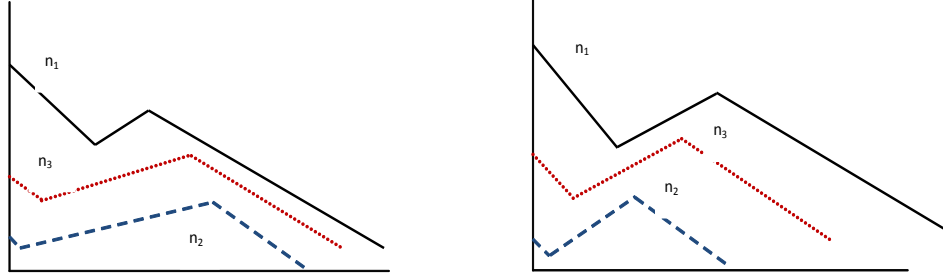


Figure 4. n_1 , n_2 , and n_3 stand for the number of agents having each one of the above preferences.

Figure 5. n_1 , n_2 , and n_3 stand for the number of agents having each one of the above preferences.

Moreover, checking that our condition cannot hold in Example 5 is a simple matter. It is enough to remark that the distribution of the peaks is not itself single peaked. Therefore, we can identify the latter as a case where we must revert to a case by case analysis, one of the many instances where existence needs to be proved by specific arguments.

Thus, we argue that

1. Our condition may be easy to check for some cases.
2. That these cases may arise from models of interest: in particular, the shapes we describe in Figures 4 and 5 are those that many authors (Epple and Romano, 1996 and Stiglitz, 1974 among others, see Appendix) do get when considering decisions that involve two modes of supplying and paying for a public good (private or public).
3. That we know of simple criteria to identify those cases where our condition does not hold, in addition to understanding that checking for it may sometimes be hard, but sometimes not.

We elaborate on these practical reasons in Section 4. Of course, we agree that they are relevant to evaluate the potential practical impact of our findings for the many analysts that usually make standard assumptions as a matter of fact. We certainly hope to be of use to them, in suggesting that some additional generality may be lying ahead at little cost.

But we also insist in our other justifications, in particular the original one regarding the intellectual challenge of unifying the different setups where the

median voter result does hold. We hope that the combination of theoretical interest, potential practical uses and technical advances regarding the treatment of indifferences are sufficient to justify our effort.

We proceed as follows. After this introduction, in Section 2 we present the basic framework to be discussed, introduce the classical restrictions for the purpose of reference, present our new condition and prove that it encompasses all the previous ones. In Section 3 we show that the median voter result extends to our new framework, with appropriate qualifications. In Section 4, we discuss why we think that knowing about this new restriction is useful, beyond the obvious fact that it involves an extension and unification of known results. Specifically, we argue that some necessary conditions for our restriction to be satisfied are easy to check, and we also present some stylized economic models where top monotonic profiles arise naturally, while the previously known domain restrictions would not hold.

2 The Model

Let A be a set of alternatives and N be a set of agents.

Agents' preferences on the alternatives are complete, reflexive and transitive binary relations on A . We denote the preference of i by \succsim_i . Its strict part \succ_i is defined so that, for any $x, y \in A$, $x \succ_i y \iff (x \succsim_i y \text{ and not } y \succsim_i x)$. Its indifference part \sim_i is defined so that, for any $x, y \in A$, $x \sim_i y \iff (x \succsim_i y \text{ and } y \succsim_i x)$. The set of all preferences on A is denoted by \mathfrak{R} .

Preference profiles are elements of \mathfrak{R}^n , and they are denoted by $\succsim = (\succsim_1, \dots, \succsim_i, \dots, \succsim_n)$, $\succsim' = (\succsim'_1, \dots, \succsim'_i, \dots, \succsim'_n)$, etc.

For all $i \in N$, for any $S \subset A$, we denote by $t_i(S)$ the set of maximal elements of \succsim_i on S . That is, $t_i(S) = \{x \in S \text{ such that } x \succsim_i y \text{ for all } y \in S\}$, the set of maximal elements on S for each \succsim_i . We call $t_i(S)$ the top of i in S . When $t_i(S)$ is a singleton, $t_i(S)$ will be called i 's peak on S .

For each preference profile \succsim , let $A(\succsim)$ be the family of sets containing A itself, and also all triples of alternatives which are top on A for some agent $k \in N$ according to \succsim .

Before introducing our new condition, let us recall some classical ones that it will encompass as particular cases. We begin by single peakedness.

Definition 1 *A preference profile \succsim is single peaked iff there exists a linear order $>$ of the set of alternatives such that*

- (1) Each of the voters' preferences has a unique maximal element $p_i(A)$, called the peak of i , and
(2) For all $i \in N$, for all $p_i(A)$, and for all $y, z \in A$

$$[z < y < p_i(A) \text{ or } z > y > p_i(A)] \rightarrow y \succ_i z.$$

When convenient, we'll say that a preference profile is single peaked relative to $>$.

Single peakedness requires each agent to have a unique maximal element. Moreover, it must be true for any agent that any alternative z to the right (left) of its peak is preferred to any other that is further to the right (left) of it. In particular, this implies that no agent is indifferent between two alternatives on the same side of its peak. Moreover, indifference classes may consist of at most two alternatives (one to the right and one to the left of the agent's peak).

There are situations where it would be natural to allow for larger indifference classes. Yet, weakening the notion of single peakedness to allow for indifferences is a delicate matter, because it may destroy all regularities of the majority rule. This is a well known fact (see Exercise 21.D.14 in Mas-Colell, Whinston and Green, 1995), but the distinction between indifferences that do not create cycles and others that do has not been studied systematically. Still, we know that one very important source of breakdown arises when indifferences result from the existence of outside options (see Cantala, 2004). Barberà (2007) describes the complex role of indifferences in domain restrictions.

Among other things, our definition of top monotonicity will stretch the extent to which one may accommodate indifferences and still obtain positive results regarding Condorcet winners. The careful reader will be able to notice at different points that we actually are able to include in our analysis many combinations of preference profiles where indifferences would preclude the satisfaction of conditions stronger than ours. In particular, we never need to exclude individuals whose preferences are flat over large sets of alternatives. At this point, though, we simply remind the reader of a non-controversial extension that allows for indifferences among top alternatives: it is the idea of single plateaued preferences.

Definition 2 *A preference profile \succsim is single plateaued iff there exists a linear order $>$ of the set of alternatives such that*

- (1) The set of alternatives in the top of each of the voters is an interval $t_i(A) = [p_i^-(A), p_i^+(A)]$ relative to $>$, called the plateau of i , and
(2) For all $i \in N$, for all $t_i(A)$, and for all $y, z \in A$

$$[z < y \leq p_i^-(A) \text{ or } z > y \geq p_i^+(A)] \rightarrow y \succ_i z.$$

When convenient, we'll say that a preference profile is single plateaued relative to $>$.

An important result of Black is that Condorcet winners exist under single peaked preferences, and that they coincide with the median(s) of the distribution of the voters' peaks. An elegant extension of the result to the case of single plateaued preferences is due to Fishburn (1973, Theorem 9.3).

Let us now turn to other types of domain restrictions that have already been proven to be related among them, but are usually considered to be quite separate from the logic of single peakedness. We refer specifically to the one dimensional versions of single crossing and of intermediate preferences. The latter appears in the social choice literature under the name of order restriction.

Definition 3 A preference profile \succ satisfies the single crossing condition iff there exist a linear order $>$ of the set of alternatives and a linear order $>'$ of the set of agents such that for all $i, j \in N$ such that $j >' i$, and for all $x, y \in A$ such that $y > x$

$$\begin{aligned} y \succ_i x &\rightarrow y \succ_j x, \text{ and} \\ y \succ_i x &\rightarrow y \succ_j x. \end{aligned}$$

When convenient, we'll say that a preference profile is single crossing relative to $>$ and $>'$.

Definition 4 If B and C are sets of integers, let $B \gg C$ mean that every element of B is greater than every element of C . A preference profile \succ is order restricted on A iff there is a permutation $\pi : N \rightarrow N$ such that for all distinct $x, y \in A$,

$$\{\pi(i) : x \succ_i y\} \gg \{\pi(i) : x \sim_i y\} \gg \{\pi(i) : y \succ_i x\},$$

or

$$\{\pi(i) : x \succ_i y\} \ll \{\pi(i) : x \sim_i y\} \ll \{\pi(i) : y \succ_i x\}.$$

Remark 1 *Single crossing and order restriction have been proven to be equivalent (Gans and Smart, 1996). We shall use one or the other in our reasonings and comparisons with other conditions, depending on which version is more amenable to treatment in each case.*

Both requirements have been frequently used in the political economy literature to prove the existence of Condorcet winners. Indeed, a median result also holds under both preference conditions, since they coincide with the top alternative(s) of the median agent(s) in the order of voters implied by these conditions.

It is now time to present our top monotonicity condition.

Definition 5 *A preference profile \succsim is top monotonic iff there exists a linear order $>$ of the set of the alternatives, such that $t_i(A) \neq \emptyset$ for all $i \in N$, and for all $S \in A(\succsim)$, for all $i, j \in N$, all $x \in t_i(S)$, all $y \in t_j(S)$, and any $z \in S$*

$$[z < y < x \text{ or } z > y > x] \rightarrow \begin{array}{l} y \succsim_i z \quad \text{if } z \in t_i(S) \cup t_j(S) \\ \text{and} \\ y \succsim_i z \quad \text{if } z \notin t_i(S) \cup t_j(S). \end{array}$$

When convenient, we'll say that a preference profile is top monotonic relative to $>$.

Remark 2 *When, in addition of satisfying the requirements of Definition 5, the profile \succsim is such that $t_i(A)$ is a singleton for all i , we will say that it is peak monotonic relative to $>$.*

We can begin by comparing top monotonicity with single peakedness and single plateauedness to see that it represents a significant weakening of these conditions. Single peakedness requires each agent to have a unique maximal element. Moreover, it must be true for any agent that any alternative y to the right (left) of its peak is strictly preferred to any other that is further to the right (left) of it. In particular, this implies that no agent is indifferent between two alternative on the same side of its peak. Hence, indifference classes may consist of at most two alternatives (one to the right and one to the left of the agent's peak).

In contrast, our definition of top monotonicity allows for individual agents to have nontrivial indifference classes, even among alternatives out of the top. In that respect, it allows for many more indifferences than single plateaued

preferences do. Most importantly, top monotonicity relaxes the requirement imposed on the ranking of two alternatives lying on the same side of the agent's top. Under our preference condition, this requirement is only effective for triples where the alternative that is closest to the top of the agent is itself a top element for some other agent. Moreover, the implication is only in weak terms when the alternative involved in the comparison is top for one or for both agents.

A similar, although less direct comparison can be made between top monotonicity and intermediateness or order restriction. The original conditions involve comparisons between pairs of alternatives, regardless of their positions in the ranking of agents. Top monotonicity is also a strict weakening of these requirements, involving the comparison of only a limited number of pairs.

Finally, let us remark that our new definition is predicated on the set of all alternatives, and also on $A(\succ)$, i.e., on triples of alternatives which are top for some agent. As we shall see in Section 3, this additional requirement is needed because the property of top monotonicity on a set is not necessarily inherited on its subsets.

We can now state the following result, proving that top monotonicity is a common root for all the above conditions above as it is implied by any of them.

Theorem 1 *If a preference profile is single peaked, single plateaued, single crossing or order restricted, then it also satisfies top monotonicity.*

P roof. It is obvious from the definition that single peaked and single plateaued preferences satisfy top monotonicity. Gans and Smart (1996) prove that single crossing and order restriction are equivalent. Therefore, it will be sufficient to show that single crossing preferences satisfy top monotonicity¹. This implies showing that top monotonicity holds for the set of all alternatives, and also for each triple of alternatives which are top. Notice that single crossing on all alternatives implies single crossing on triples. Therefore, our argument below, which does not appeal to the size of the set of alternatives covers all cases simultaneously.

Let \succ be a single crossing preference profile relative to a linear order $>$ of the set of alternatives and to a linear order $>'$ of the set of agents. We now

¹A direct proof that order restriction implies top monotonicity is available from the authors.

show that \succsim is top monotonic relative to the linear order $>$ of the set of alternatives. Suppose not. Then, there exist $i, j \in N$, $x \in t_i(S)$, $y \in t_j(S)$, and $z \in S$ such that $y > x$ and $z > y$ (the case in which $y < x$ and $z < y$ is equivalent) but (a) it is not the case that $y \succsim_i z$ if $z \in t_i(S) \cup t_j(S)$ and/or (b) $y \succ_i z$ if $z \notin t_i(S) \cup t_j(S)$. We first consider part (a). Suppose that $z \in t_i(S) \cup t_j(S)$ but $z \succ_i y$. If $j >' i$ we have that $z > y$, and $z \succ_i y$, a contradiction since $y \in t_j(S)$ implies $y \succ_j z$. Second, if $i >' j$ we have that $y > x$ and $y \succ_j x$, contradicting the assumption that $x \succ_i y$. We now consider part (b). Suppose that $z \notin t_i(S) \cup t_j(S)$ but $z \succ_i y$. If $j >' i$ we have that $z > y$, and $z \succ_i y$, a contradiction since $y \in t_j(S)$ and $z \notin t_j(S)$ implies $y \succ_j z$. If $i >' j$ we have that $y > x$ and $y \succ_j x$ but since $z \notin t_i(S)$ we have that $x \succ_i y$, our last contradiction. ■

Before we finish the section, let us present some examples and provide some further precisions regarding the requirement of top monotonicity. Examples 1 and 2 show that neither single peakedness implies single crossing nor the converse.

Example 1 *Single peakedness without single crossing.* Suppose $A = \{x, y, z, w\}$ and $N = \{1, 2, 3\}$. Assume that preferences are as in Table I and Figure 6. It is easy to see that the profile is single peaked relative to $z < y < x < w$. However, the profile violates single crossing relative to $z < y < x < w$, for any order $>'$ of the agents (and therefore it is not order restricted either). If $2 <' 3$, $w \succ_2 z$ and $z \succ_3 w$ constitute a violation of single crossing. If $3 <' 2$, $x \succ_3 y$ and $y \succ_2 x$ constitute a violation of single crossing. Similar arguments apply for any other order of alternatives.

Table I

\succsim_1	\succsim_2	\succsim_3
z	y	x
y	x	y
x	w	z
w	z	w

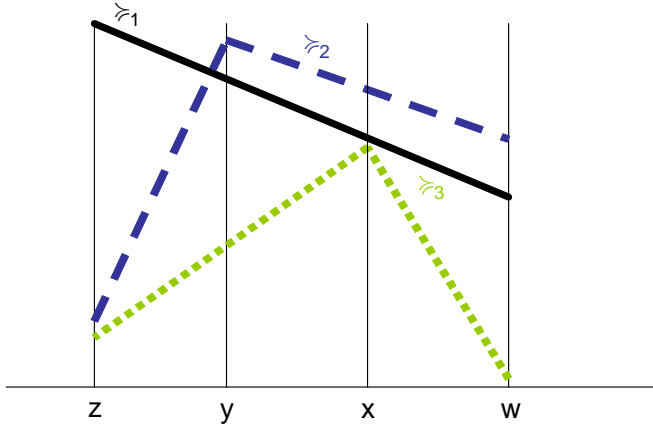


Figure 6. Preference profile of Example 1.

Example 2 *Single crossing without single peakedness.* Suppose $A = \{x, y, z\}$ and $N = \{1, 2, 3\}$. Assume that preferences are as in Table II and Figure 7. It is easy to see that this preference profile satisfies single crossing on A , relative to $x < y < z$ and $1 <' 2 <' 3$. However, the reader can check that this preference profile is neither single peaked, nor single plateaued.

Table II

\succ_1	\succ_2	\succ_3
x	xy	z
y	z	xy
z		

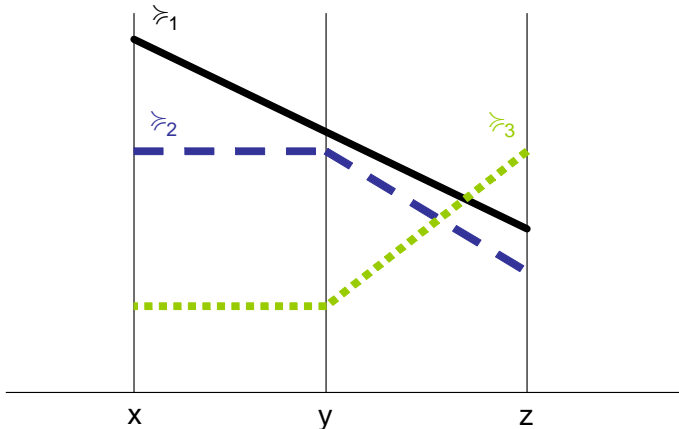


Figure 7. Preference profile of Example 2.

In Examples 1 and 2, references to single crossing could be changed to order restriction, because the equivalence between both properties. The reader can also check by inspection that, as expected from Theorem 1, top monotonicity is satisfied in both examples.

Example 3 shows that top monotonicity can be satisfied even none if the previously considered conditions hold.

Example 3 *Top monotonicity without single peakedness or single crossing.* Suppose $A = \{x, y, z, w\}$ and $N = \{1, 2, 3\}$. Assume that preferences are as in Table III and Figure 8. It is easy to see that the profile is top monotonic relative to $x < y < z < w$. The preference profile is not single peaked on A because there are triples of alternatives such that any of the three are last for the preferences of some agent.² Finally, we show that the preference profile is not order restricted for any order of the set of agents, and therefore it is not single crossing. To prove it, consider the three distinct permutations for 1, 2 and 3. If $1 < 2 < 3$, $x \succ_1 y$ and $y \succ_2 x$ but $x \succ_3 y$; if $1 < 3 < 2$, $z \succ_1 w$ and $w \succ_3 z$ but $z \succ_2 w$; finally, if $2 < 1 < 3$, $w \succ_2 y$ and $y \succ_1 w$ but $w \succ_3 y$.

²This violates a necessary condition for single peakedness when preferences are strict. The condition says that, for any triple of alternatives, one of them cannot be last for any voter. The violation, in our case, appears for all triples.

Table III

\succsim_1	\succsim_2	\succsim_3
x	z	w
y	w	z
z	y	x
w	x	y

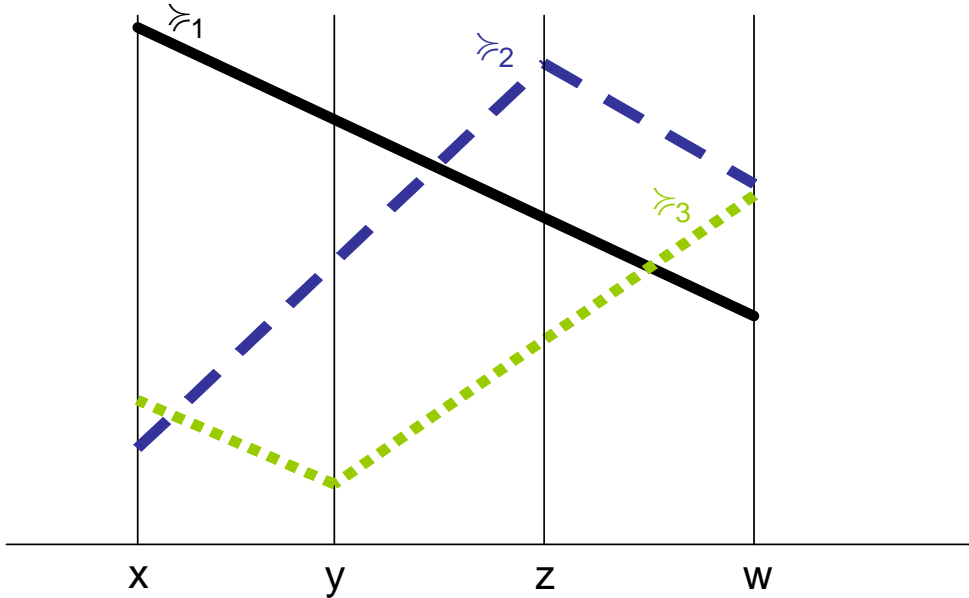


Figure 8. Preference profile of Example 3.

The standard conditions that we have proven to be special cases of top monotonicity share a common feature: when they hold for the universal set of alternatives, they also apply when we restrict attention to any of its subsets, and to triples in particular.³ In contrast, such inheritance properties from the large to the small do not hold in our case. Example 4 below shows that

³Defining domain restrictions of triples has a long tradition (see, for example, Sen and Pattanaik (1969)). The part of single peakedness that is actually used in proving the quasitransitivity of the majority relation, and thus the existence of Condorcet winners is precisely the fact that it holds for triples, once it is assumed to hold globally. Notice that the converse statement, ensuring that single peakedness on triples implies single peakedness for the whole preference profile is only true under an additional assumption, involving pairs of agents and 4-tuples of alternatives (see Ballester and Haeringer (2007)).

top monotonicity can be satisfied in a four-alternative profile, and yet not hold when we look at the profile’s restriction to a triple. This, of course, extends to larger sets, where we may have top monotonicity at large, and yet not for some subset of alternatives.

Example 4 *Let $A = \{x, y, z, w\}$, and $N = \{1, 2, 3\}$. Assume that preferences are as in Table IV. It is easy to see that the profile is peak monotonic (and therefore top monotonic) relative to $x < y < z < w$. However, it violates peak monotonicity on $\{x, y, z\}$ not only relative to $x < y < z$, but also for any other order of the alternatives.*

Table IV

\succ_1	\succ_2	\succ_3
x	w	w
y	z	y
w	x	z
z	y	x

What is going on in this Example is that there is a cycle between x , y and z , and yet top monotonicity is satisfied in the presence of the fourth alternative w , which is actually the Condorcet winner. Indeed, our domain restriction does not preclude cyclical patterns, but just guarantees that these do not occur at the top of the majority relation. Another way to understand why top monotonicity on a set of alternatives is not inherited on its subsets is by realizing that, as we change the size of relevant sets, we also change the collection of top alternatives, and therefore of the pairs to be compared. This is why, in order to control some limited relationships among some triples of alternatives, top monotonicity is required to hold not only on the universal set A , but also on triples of alternatives which are top for some agents.

One last interesting point regarding our new definition is that, when we restrict attention to preference profiles defined on triples, and where no agent is fully indifferent, then top monotonicity is equivalent to single crossing.⁴ This shows that it is important to preserve the level of generality that we have achieved in our definition, if we want to have a domain restriction which really supersedes all those that we have considered as predecessors. In particular, the encounter of single peakedness and single crossing could not have been reached if we had insisted on looking at triples as the starting point of our analysis.

⁴A formal proposition and a proof of that fact is available from the authors.

3 Weak medians of the tops and the existence of Condorcet winners

In this Section, we show that top monotonicity guarantees the existence of Condorcet winners and preserves a version of the median voter result. Before stating this second result of the paper, we introduce some notation.

Let $>$ be a linear order of the set of alternatives and \succsim be a preference profile. For any $z \in A$, we define the following three sets

$$N_{\{z\}} = \{j \in N : z \in t_j(A)\},$$

$$N_{\{z\}^-} = \{k \in N : z > x \text{ for all } x \in t_k(A)\},$$

and

$$N_{\{z\}^+} = \{h \in N : z < x \text{ for all } x \in t_h(A)\}.$$

We remark that when \succsim is top monotonic relative to $>$, and z is in the top of some agent i , then $N_{\{z\}} \neq \emptyset$ and the three sets $(N_{\{z\}^-}, N_{\{z\}}, N_{\{z\}^+})$ constitute a partition of the of voters N . Indeed, $N_{\{z\}}$ contains all voters, including i , for whom z is in the top. $N_{\{z\}^-}$ (resp. $N_{\{z\}^+}$) contains all voters for which all top elements are to the left (resp. to the right) of z . Clearly, then, these three sets are disjoint. To prove that their union contains all elements of N , suppose not. Then, for some agent l , z should not be in l 's top, while some alternatives x and y , one to the right and one to the left of z , should belong to the top of l . But then, by top monotonicity we would have $z \succsim_l x$ and also $z \succsim_l y$. Since x and y are both top for l , so is z , a contradiction⁵.

Let n , $n_{\{z\}}$, $n_{\{z\}^-}$, and $n_{\{z\}^+}$ be the cardinalities of N , $N_{\{z\}^-}$, $N_{\{z\}}$ and $N_{\{z\}^+}$, respectively. From the remark above, we know that if z is in the top of some agent, then $n_{\{z\}} + n_{\{z\}^-} + n_{\{z\}^+} = n$. The following definition will allow us to establish an analogue of the classical median voter result for the case of top monotonic profiles.

⁵Notice that our definition of top monotonicity does not preclude the possibility that an agent's top might be non-connected relative to the order of $>$. Informally, what it demands is that, if an agent has two peaks with a valley in between, then no other agent's peak lies in that valley. In that sense also, our condition is less demanding than that of single plateaued, where we assumed that the tops are connected.

Definition 6 *An alternative z is a weak median top alternative in a top monotonic profile \succsim relative to an order $>$ of the set of alternatives iff*

- (1) *z is a top alternative in \succsim for some agent, and*
- (2) *$n_{\{z\}^-} + n_{\{z\}} \geq n_{\{z\}^+}$ and $n_{\{z\}} + n_{\{z\}^+} \geq n_{\{z\}^-}$.*

We will denote by $MT(\succsim)$ the set of weak median top alternatives at that profile. We define m^- and m^+ as the lowest and the highest elements in this set according to the order $>$ at that profile.

Definition 7 *An alternative z is an extended weak median in a top monotonic profile \succsim relative to an order $>$ of the set of alternatives iff $m^- \leq z \leq m^+$.*

We will denote by $M(\succsim)$ the set of extended weak median alternatives at that profile.

We can now state and prove the following result.

Theorem 2 (1) *Whenever a profile of preferences is top monotonic relative to some order $>$, the majority relation has Condorcet winners, which belong to the set of extended weak alternatives at that profile.*

(2) *If the profile of preferences is peak monotonic, the median(s) of the distribution of agents' peaks is (are) Condorcet winners.*

P roof. Statement (2) is an immediate corollary of (1) when all agents' tops are singletons.

We now prove statement (1). Let the preference profile $\succsim \in \mathfrak{R}^n$ be top monotonic on A relative to $>$. The strategy of proof involves showing that

- (a) There are alternatives in $MT(\succsim)$ which do not loose against any element of $MT(\succsim)$. To establish that, we show that the majority relation on $MT(\succsim)$ is quasitransitive. As we shall see, the argument uses the part in the definition of top monotonicity which requires the property to hold on triples of alternatives that are top for some agent.

- (b) Alternatives in $MT(\succsim)$ do not loose against alternatives in $M(\succsim) \setminus MT(\succsim)$,
- (c) All alternatives outside $M(\succsim)$ are defeated by some element in $MT(\succsim)$,

and

- (d) All the alternatives in $MT(\succsim)$ which do not loose against any element of $MT(\succsim)$ do not loose against any alternative outside $M(\succsim)$ either.

Steps (a), (b) and (d) imply that the undefeated elements in $MT(\succsim)$ that we identify in (a) are indeed Condorcet winners and (c) proves that no alternative outside $M(\succsim)$ can be. Notice that conclusion (b) does not preclude

the possibility of additional elements in $M(\succsim)$ but not in $MT(\succsim)$ also being Condorcet winners.

(a) To prove that there are alternatives in $MT(\succsim)$ which do not loose against any other element of $MT(\succsim)$, it is enough to show that the majority relation is quasitransitive on that set.

Let, w.l.o.g., $x, y, z \in MT(\succsim)$ be such that $x < y < z$. Top monotonicity, by definition, holds on each such triple, since all elements in $MT(\succsim)$ are tops for some agent. First notice that if one of the admissible preference relations in the profile has y as its unique top alternative, then top monotonicity requires that all preferences in this profile should be single plateaued. In that case, it is well known that the majority preference relation is quasitransitive.

Also notice that, since \succsim is top monotonic relative to $x < y < z$, preferences where $x \succ_i z \succ_i y$, $z \succ_i x \succ_i y$ and $x \sim_i z \succ_i y$ cannot be part of the profile. In view of the preceding remarks, we are left with the cases where our preference profile is a combination of the preferences that appear below:

\succsim_1	\succsim_2	\succsim_3	\succsim_4	\succsim_5	\succsim_6
x	xy	x	z	zy	z
y	z	yz	y	x	xy
z			x		

To finish our argument, let n_i be the number of agents of type \succsim_i . We write $a \succ b$ iff $\#\{i \in N : a \succ_i b\} > \#\{i \in N : b \succ_i a\}$ for all $a \neq b$, $a, b \in \{x, y, z\}$. We must prove that: $x \succ y$ and $y \succ z$ implies $x \succ z$, $x \succ z$ and $z \succ y$ implies $x \succ y$, $y \succ z$ and $z \succ x$ implies $y \succ x$, $y \succ x$ and $x \succ z$ implies $y \succ z$, $z \succ x$ and $x \succ y$ implies $z \succ y$ and $z \succ y$ and $y \succ x$ implies $z \succ x$.

We provide the argument for the case $x \succ y$ and $y \succ z$, proving that this implies $x \succ z$. Other proofs are left to the reader. Since $x \succ y$,

$$n_1 + n_3 > n_4 + n_5 \tag{1}$$

and since $y \succ z$,

$$n_1 + n_2 > n_4 + n_6. \tag{2}$$

We must show that $n_1 + n_2 + n_3 > n_4 + n_5 + n_6$.

In fact, not all combinations of these preferences are compatible, given that our profile satisfies top monotonicity. Specifically, \succsim_2 and \succsim_3 or \succsim_5 and \succsim_6 are not mutually compatible: that is, either n_2 or n_3 must equal 0, and either n_5 or n_6 must equal 0. Therefore $n_1 + n_2 + n_3 = \max\{n_1 + n_2, n_1 + n_3\}$ and $n_4 + n_5 + n_6 = \max\{n_4 + n_5, n_4 + n_6\}$. From (1) and (2), it follows that

$$\max\{n_1 + n_2, n_1 + n_3\} > \max\{n_4 + n_5, n_4 + n_6\}.$$

(b) We now show that no $y \in MT(\succsim)$ loses against alternatives in $M(\succsim) \setminus MT(\succsim)$. Suppose that $x > y$ (the case $y > x$ is identical). Because x is not a top alternative, by top monotonicity $y \succ_i x$ for all $i \in N_{\{y\}^-} \cup N_{\{y\}}$. Since y is a weak median top alternative, $n_{\{y\}^-} + n_{\{y\}} \geq n_{\{y\}^+}$. Therefore y is not defeated by x in pairwise comparisons.

(c) We'll show that m^- defeats by majority any alternative to its left, and that m^+ defeats any alternative to its right.

We provide the argument for m^- and $x < m^-$. Notice that, since m^- is the lowest weak median top alternative

$$n_{\{m^-\}^-} < n_{\{m^-\}} + n_{\{m^-\}^+}.$$

By top monotonicity, notice that,

$$\{i \in N : x \succ_i m^-\} \subseteq N_{\{m^-\}^-},$$

that

$$\begin{aligned} & \{i \in N : x \sim_i m^- \text{ and } m^- \in t_i(S)\} \cup \\ & \{i \in N : m^- \succ_i x \text{ and } m^- \in t_i(S)\} = N_{\{m^-\}}, \end{aligned}$$

and that

$$\{i \in N : m^- \succ_i x \text{ and } m^- < t_i(S)\} = N_{\{m^-\}^+}.$$

Hence, the number of votes for x against m^- (including indifferent agents, which we take to vote on both directions) is at most $n_{\{m^-\}^-}$. The number of votes for m^- against x (again counting indifferences) is $n_{\{m^-\}} + n_{\{m^-\}^+}$. Therefore, m^- defeats any $x < m^-$ by majority.

(d) Finally, it is not hard to prove that if an alternative w in $MT(\succsim)$ is never defeated by others in $MT(\succsim)$, it will not be defeated by m^- and m^+ , and it will not be defeated by any alternative not in $M(\succsim)$. ■

4 How useful is our new restriction?

In Section 2 we have proven that top monotonicity is a weakening of classical domain restrictions. This gain in generality is clarifying, since it exhibits the common root of conditions that have been till now perceived as quite

unrelated and that are indeed independent from each other, as shown by Examples 1 and 2. In Section 3 we have shown that this gain in scope still allows for an existence result for Condorcet winners where medians play a central role. In the present section, we want to address the following two questions: (1) When confronted with a given preference profile, can we easily recognize whether it satisfies top monotonicity? (2) Are there interesting economic models where top monotonicity holds, while previously known conditions do not?

4.1 Necessary conditions for top monotonicity

To answer the first question, we present a condition that is easy to check and that is necessary for a profile to be top monotonic. A simple version of this result applies for peak monotonic profiles. In this case, where each agent has a singleton top, the condition requires that the profile of preferences over the peaks of individuals to be single peaked. This is easy to check. Establishing peak monotonicity may be harder, but discarding it is a simple matter. In the general case of top monotonicity, we can establish a similar necessary condition, which is close to requiring that preferences of tops should be single plateaued. Proposition 2, whose proof we leave to the reader, will make this intuition more precise.

Given a preference profile let $F(\succsim)$ be the set of alternatives that belong to the top set of some agent.

Definition 8 *A preference profile \succsim is weakly single plateaued on $S \subset A$ relative to a linear order $>$ of the set of alternatives, iff*

- (1) *Each of the voters' preferences has a unique maximal interval $t_i(S) = [p_i^-(S), p_i^+(S)]$, called the plateau of i , and*
- (2) *For all $i \in N$, for all $t_i(S)$, and for all $y, z \in S$*

$$[z < y \leq p_i^-(S) \text{ or } z > y \geq p_i^+(S)] \rightarrow y \succsim_i z.$$

Proposition 1 *A preference profile \succsim is top monotonic on A relative to a linear order $>$ of the set of alternatives only if it is weakly single plateaued on $F(\succsim)$ relative to the same linear order when restricted to the set of alternatives in $F(\succsim)$.*

In the case of peak monotonic profiles, weak single plateauedness on $F(\succsim)$ collapses to the standard condition of single peakedness on $F(\succsim)$.

4.2 Economic models giving rise to top monotonicity

Top monotonicity leaves room for new types of preferences that arise from the analysis of economic models.

Take, as in Figure 9, for example an agent who can guarantee herself the maximum of two utilities on an interval $[0, T]$.

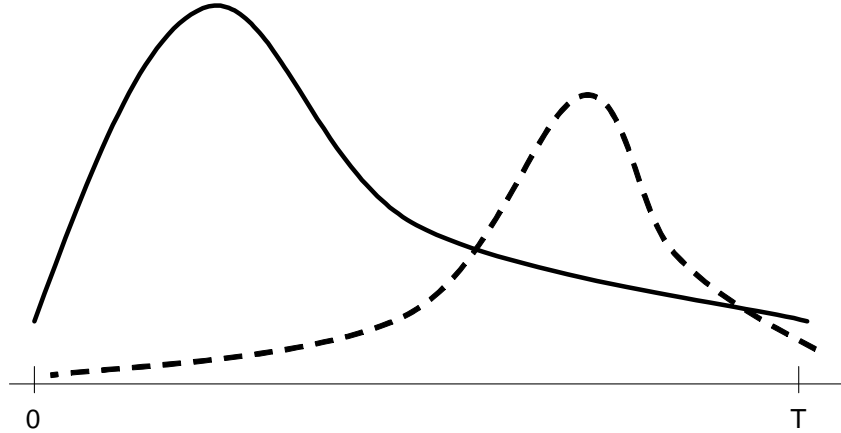


Figure 9. Two utilities whose envelope represents the preference of an agent.

Then, the attainable utilities by her choices on $[0, T]$ are represented by the upper envelope of two curves. This agent will have two local peaks, one which (at least) will be global.

Assume that, in addition to this general structure, the specific shapes of the preferences of different agents are such that:

(a) There exist two points B and B' , $B < B'$, such that the left global peaks of the agents will be attained below B , and the right global peaks will be attained above B' , and

(b) If the global peak of an agent i is below B this agent prefers B' to any alternative above B' and if the global peak of an agent i is above B' this agent prefers B to any alternative below B .

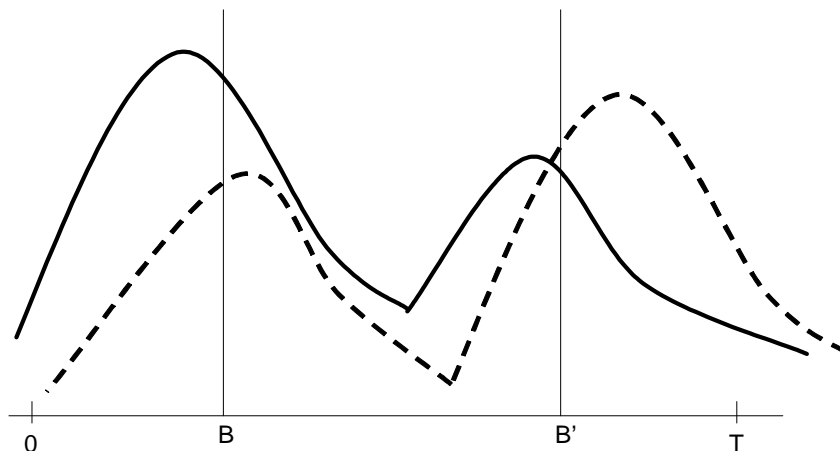


Figure 10. General structure of fundamental top monotonicity.

Then, the reader can check that profiles of these double peaked preferences, as in Figure 10, arising from such a construction do satisfy top monotonicity. We propose this particular structure because it captures the main features of profiles that arise when solving for the preferences of individuals in models of public economics where two modes of provision of a service are possible. Example of these include the choice of a tax rate to finance a public good (Stiglitz, 1974) or the choice of a tax rate to finance public schooling in the presence of an option to buy private schooling (Epple and Romano, 1996)⁶.

In both cases, one of the maxima is attained at 0 (which plays the role of our B point), and the other at some point beyond that which would make the individual indifferent between the public and the private option (this is our point B' above). The additional connections between the preferences of different agents establishing whether or not they satisfy top monotonicity as a whole profile depend on well defined economic variables.

Admittedly, our condition is not always equally useful. In the models where it is, there will exist regions where no global peak is to be found. This is the role of B and B' . We are aware that, in some applications, it is useful to assume that all alternatives are the unique peak for some agent. When this is the right modelling decision, we have little new to offer, since then top monotonicity collapses to single peakedness (and so do all other classical domain conditions). Similarly, if any subset of the set of alternatives

⁶A detailed analysis of the models in these papers and the assumptions under which top monotonicity holds is available from the authors.

is a plateau for some agent, top monotonicity collapses to single plateaued preferences.

Even in these cases, where preferences domains are assumed to be so rich, we have something to contribute. Our previous analysis shows that, if one is ready to work under the assumption that any subset is a top for someone, then all other classical conditions collapse to that of single peakedness. This gives special value to that classical condition.

5 Appendix. Two models of choosing a tax giving rise to top monotonic preferences

I. Voting for a level of expenditure

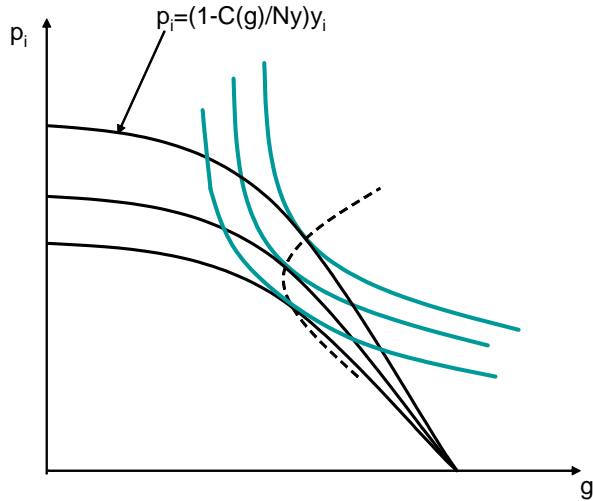
This is based on Stiglitz (1974). There are two goods, a public good, g and a private good, p . The set of voters and of taxpayers are identical. Wealth (given exogenously) of the i th individual is denoted by y_i . Thus total national wealth is just $\sum y_i = \bar{y}N$, where N is the number of voters in the economy, and \bar{y} is the mean wealth. Let the level of expenditure on the public good, $C(g)$, be such that $C'(g) > 0$ and $C(0) = 0$. If public expenditures are financed by proportional income taxed, and t is the tax rate, then $C(g) = t\bar{y}N$.

The i th individual votes for the level of public expenditure which maximizes his utility. We represent utility as a function of the expenditure on the public good and on the private good, p_i . p_i is just his after tax wealth, $p_i = (1 - t)y_i$. Thus she maximizes $U^i(g, (1 - \frac{C(g)}{\bar{y}N})y_i)$ with respect to g , and her utility is maximized when

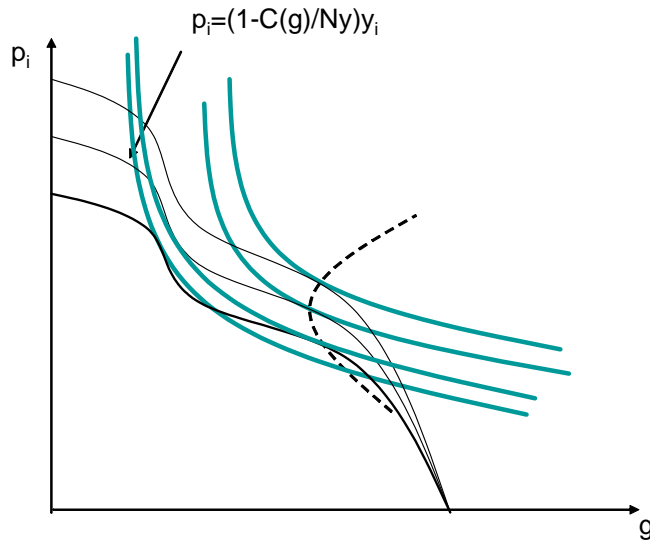
$$\frac{U_g^i}{U_{p_i}^i} = C'(g) \frac{y_i}{\bar{y}N}. \quad (3)$$

Suppose that individuals differ only with respect to their endowment, we can trace out the demand for public good as a function of y_i . In particular, if U is quasi-concave on g , then preferences are single-peaked, and the majority voting equilibrium will be the level of demand of the individual with median wealth, provided that the demand for g is monotonic in y_i .

However, the level of g voted for may not be monotonic in y . Since preferences are single-peaked, there is still a majority voting equilibrium. But the "median voter" is not the individual with median income.



Alternatively, if U is not quasi-concave on g and the level of g voted for is not monotonic in y , the preferences are not single-peaked, and the "median voter" is not the individual with median income. However, it is quite easy to find situations in which the preference profiles are top monotonic. In particular, under non-convexities in the technology of production of the public good, as we see in the following picture.



II. Choosing between public and private provision

This is based on Epple and Romano (1996). There are two goods, educational services, x , and the numeraire commodity, b . All individuals have

the same strictly increasing, strictly quasi-concave, and twice continuously differentiable utility function $U(x, b)$. It is also assumed that:

Assumption A1. Educational services are a normal (or superior) good.

Assumption A2. For $x > 0$, $b > 0$, \bar{x} and $\bar{b} \geq 0$, $U(x, b) > U(0, \bar{b})$ and $U(x, b) > U(\bar{x}, 0)$.

Households differ in endowed income (i.e. numeraire commodity), y . The p.d.f. and c.d.f. of household income are denoted $f(y)$ and $F(y)$, respectively, with support $[y, \bar{y}] \in [0, \infty)$. We assume that $f(y)$ is continuous and positive over its support. We normalize the number of households to one and denote aggregate income by $Y = \int_{\underline{y}}^{\bar{y}} yf(y)dy$, which is also then equal to mean income.

One unit of publicly provided educational services is produced with one unit of numeraire and all consumers of public school services obtain the same level of education services. Public school inputs are financed by a proportional tax, t , on income:

$$tY = NE,$$

where N is the number of households using public schools, and E is per household public school services. The level of public school expenditure is determined by majority voting of all households, whether or not they utilize public schools.

Private school services are provided by price-taking suppliers. The cost per unit of educational services provided by private schools is one unit of the numeraire. A household consuming private school services can choose as many units as it desires at price one per unit. A household can consume either public or private school services but not both.

A household that consumes private school services chooses x to maximize $U(x, b)$ subject to the budget constraint $y(1 - t) = x + b$. Let $v(y(1 - t))$ be the indirect utility function of a household with income y that chooses private schooling. The preferred tax rate for an agent choosing private services is $t = 0$.

A household with income y choosing public schooling obtains utility:

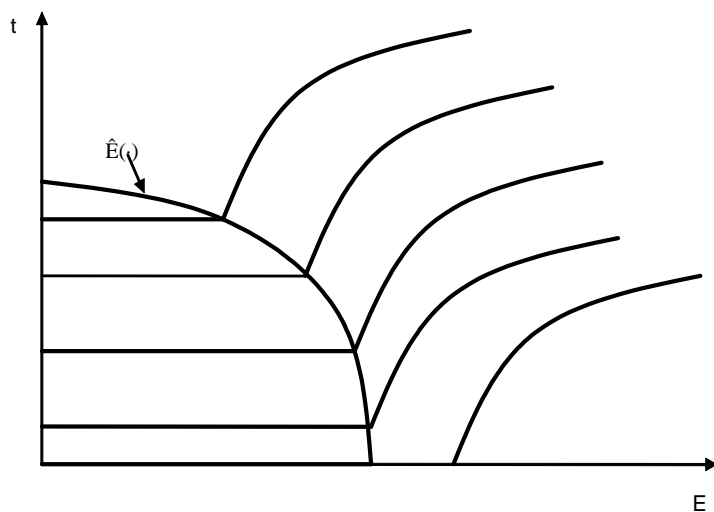
$$U(E, y(1 - t)).$$

Let $E(t^*(y))$ be educational expenditure per household at the preferred tax rate for an agent choosing public services.

Hence, the induced utility function of a household with income y that can choose between public and private alternatives is

$$V(E, y(1-t)) = \max\{v(y(1-t)), U(E, y(1-t))\}. \quad (4)$$

Let $\hat{E}(y(1-t))$ be the locus of (E, t) pairs along which household y is indifferent between public and private school. A typical indifference map in the (E, t) plane is illustrated in the following picture.



A typical indifference map in the (E, t) plane.

Let $E^*(t)$ be educational expenditure per household for those attending public school when all households make utility-maximizing choices. Note that $E^*(t)$ need not be everywhere increasing.

We now denote the slope of an indifference curve of $U(E, y(1-t))$ in the (E, t) plane be denoted by $M(E, y, t)$. Hence,

$$M(E, y, t) = \frac{U_1(E, y(1-t))}{yU_2(E, y(1-t))}.$$

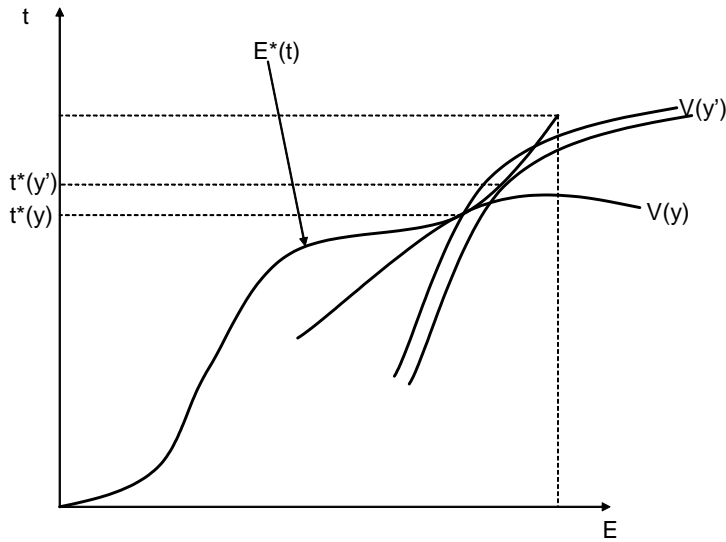
Epple and Romano (1996) assume that for all y the slope of the $U(E, y(1-t))$ function in the (E, t) plane is monotonic in y . In particular, they assume that one of the following alternatives holds:

Assumption A3 or Slope Declining in Income (SDI). $\frac{\partial M(E, y, t)}{\partial y} \leq 0$ for all y .

Assumption A4 or Slope Rising in Income (SRI). $\frac{\partial M(E,y,t)}{\partial y} \geq 0$ for all y .

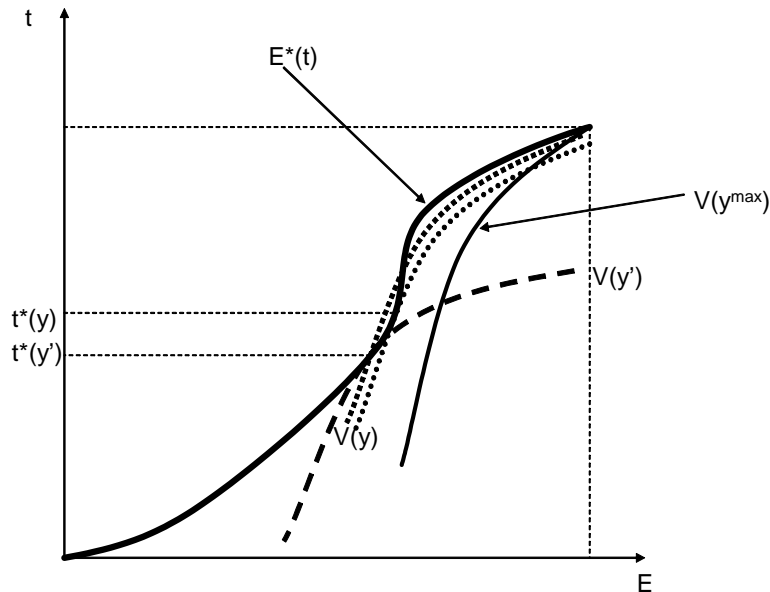
SRI results if the income elasticity of the implied demand for public education exceeds the (absolute value of the) price elasticity of the same, and SDI results is the reverse holds. Epple and Romano show that if the utility function satisfies the single crossing condition SDI, then the indifference curves of the utility function defined in [3] also satisfy the same single crossing condition. Therefore, when SDI holds, a majority voting equilibrium exist, and the median voter is decisive. From Theorem 1, we know that any preference profile satisfying the single crossing condition is top monotonic (and since for this particular example essentially all agents have one maximal element, also peak monotonic).

We now illustrate how to check that if the utility function satisfies the single crossing condition SDI, then the preference profile is peak monotonic. Let $y > y'$ be two households.



Suppose that either both households, y and y' , choose public provision or household y chooses private provision and household y' chooses public provision. Then household y is such that either $t^*(y) < t^*(y')$ or $0 < t^*(y')$, and it is clear from the picture that household y prefers $t^*(y')$ to any $t^* > t^*(y')$. Household y' is such that either $t^*(y) < t^*(y')$, and it is clear from the picture that household y' prefers $t^*(y)$ to any $t^* < t^*(y)$ or $0 < t^*(y')$, and condition (2) in the definition of peak monotonicity is vacuously satisfied. Finally, since $\widehat{E}(y) > \widehat{E}(y')$ it can not be the case that household y chooses public provision and household y' chooses private provision.

We finally present a graphical example in which the utility function satisfies the SRI condition, the preference profile is not single crossing but it is still peak monotonic. We also assume that the highest income individual prefers $t = 1$ to private provision. It is clear that all individuals prefer public to private provision. Let $y > y'$ be two households.



Household y is such that $t^*(y) > t^*(y')$, and it is clear from the picture that household y prefers $t^*(y')$ to any $t^* < t^*(y')$. Household y' is such that $t^*(y) > t^*(y')$, and it is clear from the picture that household y' prefers $t^*(y)$ to any $t^* > t^*(y)$.

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