

Multi-dimensional screening in insurance—with an application to gender discrimination.*

Pau Olivella[†] and Fred Schroyen[‡]

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Abstract: In this paper, we consider a population of insurees who differ in two dimensions: their risk type (expected loss) and their risk aversion. We solve for the profit maximising menu of contracts that a monopolistic insurer wants to put out on the market. One result is that it is in general optimal to separate insurees with different risk aversion, despite the fact that they have identical risk types and therefore equally costly to a risk neutral insurer. Secondly, we apply the framework to analyse the welfare consequences of allowing the insurer to practise gender discrimination. Here, we find that if even when the average man and woman differ only in risk aversion, gender discrimination may lead to a Pareto improvement in the insurance market.

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[†]Departament d'Economia i d'Historia Econòmica and CODE, Universitat Autònoma de Barcelona, Edifici B, E-08193 Bellaterra (Spain). E-mail: pau.olivella@uab.es.

[‡]Department of Economics, Norwegian School of Economics & Business Administration, Helleveien 30, N-5045 Bergen (Norway). E-mail: fred.schroyen@nhh.no.

1 Introduction

People that seek insurance differ from each other in many dimensions. For insurance companies, and for the outcome on the insurance market, at least two of these dimensions are of central importance: the distribution of losses that insurees face, and the willingness to bear the risk of those losses.¹ Insurance market theory has focussed on the first type of heterogeneity and, in particular, has traced out the consequences of private information about the loss distribution for the equilibrium set of contracts. Rothschild and Stiglitz (1976) did so for a competitive insurance market, while Stiglitz (1977, sections 3 and 4) characterised the menu of contracts which a monopolist insurer puts on offer.²

Of course, one may ask whether private information is really limited to risks and whether it should not include risk aversion. Even though the risk aversion of an insuree does not in itself determine the profitability of a particular insurance contract, it will determine which choice is made from those contracts that are put out on the market. The same car insurance contract that is marketed for a careful driver with high risk aversion, may be found attractive by a careless driver with low risk aversion, thereby affecting its profitability.

Adding risk aversion heterogeneity to the analysis of insurance markets calls for a multi-dimensional adverse-selection model. Such analysis is technically not straightforward, because the existence of private information in two or more dimensions breaks the natural ordering of agents and monotonicity of the contract attributes is no longer a necessary condition for implementability.³ Recently, however, Smart (2000), Wambach (2000) and Villeneuve (2003) have challenged this hurdle and looked at the consequences of such double heterogeneity for the equi-

¹A third dimension, that will not be discussed here, is the moral stance of insurees, determining the amount of false claims which insurers each year have to deal with.

²Villeneuve (2004) considers the competitive insurance market when the private information about risk resides with the insurer. Stiglitz (1977, section 5) analysis the monopolist problem when insurees only differ in risk aversion.

³Though Armstrong & Rochet's (1999) "user's guide" provides some guidance, their set-up (in particular, the additive separability of agents' preferences) does not lend itself easily for an insurance context.

librium outcome on a competitive insurance market. Their main finding is that in equilibrium, if it exists, firms may earn positive variable profits by attracting some low risk type.⁴

One question that is interesting in its own right is how these results change when the insurance market is not perfectly competitive, and particularly, how a monopolistic insurance company would design a contract menu to maximise profits. Answering this question is the first aim of this paper.

Its second purpose is to analyse the welfare consequences of a recent proposal by the European Commission to “implement the principle of equal treatment between women and men in the access to and supply of goods and services” (Commission of the European Communities, 2003). Indeed, insurance is one of the sectors which would be affected by this proposal because of the common practice to differentiate premia (and other components of the insurance contract) according to gender when underwriting life, health and car accident risks.

Regarding life insurance, for example, it is argued that if one controls for lifestyle, environmental factors, and social class, “the difference in average life expectancy between men and women lies between zero and two years” and therefore that “the practice of insurers to use sex as a determining factor in the evaluation of risk is based on ease of use rather than real value as a guide to life expectancy.” (Commission of the European Communities, 2003: 6) Not surprisingly, European insurer carriers have reacted fiercely to the proposed ban, arguing that removing gender would weaken their ability to assess risk and that gender-neutral calculation would increase the premia for many of their products, especially for women (Financial Times, November 3, 2003, p 2). Insurance companies thus regard gender as a *signal* that is informative about the insurees risk type, while the Commission claims that, once one controls for other factors, gender is not even an informative signal about risk type.

⁴To sustain this as an equilibrium, these authors assume some degree of barriers to entry. First, each firm can only offer a single contract, and second, either the number of firms is fixed or entry is at some fixed cost. Villeneuve (2003) points out that the introduction of menus of contracts would greatly compromise equilibrium existence. This is not an issue in our analysis of monopolistic screening.

Insurance theory has mainly studied the effects of categorisation for a competitive setting. The basic result is that when market outcome is governed by a Nash (separating) equilibrium (if it exists), firms will ignore even informative signals, because what matters for the profitability of the contracts is the risk type of the insuree. If, as is common to assume, insurees know their type, and if insurers anticipate that their potential customers will make a choice according to their type, signals will in equilibrium be ignored (see, e.g. Glazer & McGuire, 2000: 1061).⁵ From this perspective, the worries of the Commission about the use of gender as a signal (and the reactions of the industry for that matter) can only be grounded if insurance companies possess some market power. We are then led to question how the presence of market power will give insurers the incentive to use signals when underwriting risks. Our model of a monopolistic insurer is a first attempt to address this question. We will discuss in this framework the effects of gender discrimination under the hypothesis that gender is only an informative (but not necessarily a perfect) signal about a person's *risk aversion*. This hypothesis not only fits with the Commission's observation that gender is hardly informative about risk type but is at the same time in line with recent econometric and experimental evidence that women tend to be more risk averse than men.⁶ We show that allowing the monopolist to condition the terms of the insurance contract on gender may be Pareto improving, and provide necessary and sufficient conditions for such an improvement.

From a methodological point of view, we have taken a new approach to the analysis of screening insurees that simplifies the problem and is at the same time, we believe, quite appealing. Rather than following the standard set up where the

⁵ On the other hand, if the market allocation is determined by a Wilson reactive equilibrium, Hoy (1982) showed that risk categorisation may lead to a (weak) Pareto improvement in the market outcome. This happens when the no-categorisation equilibrium is separating, while after categorisation the low risk group has so few high risk people that this part of the market moves to a (Pareto superior) pooling equilibrium, and the high risk group stays at the separating equilibrium. Changing the equilibrium concept thus leads to different conclusions about the use of signals in a competitive market. Hellwig (1987) has pointed out the *ad hoc* nature of such dynamic extensions of the equilibrium concept.

⁶Eckel & Grossman (2003), Hartog, Ferrer-i-Carbonell & Jonker (2002).

individual faces two states of the world—loss and no loss—we assume that the loss is normally distributed and that agents differ in the *expected* loss, which can be high or low. If the insurance indemnity is linear in the loss, as is the case under a reimbursement insurance scheme with a constant coinsurance rate, final income will be normally distributed as well. Endowing agents with a utility function that displays constant absolute risk aversion (r), which also can be high or low, their preferences over uncertain income prospects can be represented as

$$E(\textit{final income}) - \frac{r}{2}\text{var}(\textit{final income}).$$

An important consequence of this approach is that preferences over insurance contracts become quasi-linear in the insurance premium and therefore in the information rent. Readers familiar with contract theory will acknowledge the usefulness of linearity in the information rent in writing down the incentive compatibility constraints. An additional advantage of mean-variance preferences is that it allows for an explicit characterisation of the optimal menu of contracts.

The limitations of our approach follow immediately from these assumptions. We do not consider insurance contracts with either a deductible or a cap since that would destroy the normality of net income. Secondly, the normality assumption implies a positive likelihood of negative losses, although this problem may be of second order by considering sufficiently high means and/or low variances for the loss. Perhaps the most important objection is that we have no skewness in the loss distribution, and in particular no strictly positive probability mass for a zero loss. Still we feel these are minor blemishes when compared to the considerable advantages the approach offers for characterising the solution to a two-dimensional screening problem.

The remainder of the paper is organised as follows. In section 2, we model the preferences of insureds and specify reimbursement contracts. In section 3 we set up the problem faced by a monopolistic insurer, and discuss a condition (A) that under which the population of high risk people (whatever their risk aversion) may be regarded as one. Section 4 characterises the optimal menu of contracts

under this condition. In section 5 we trace out the consequences of allowing the monopolistic insurer to gender discriminate. Section 6 briefly discusses in which direction the analysis should proceed if a central condition is no longer satisfied. Section 7 concludes.

2 Consumers/insurees and reimbursement contracts

Insurance takers

We assume that individuals are endowed with initial wealth e and a negative exponential vNM utility function defined on final wealth y : $u(y) = -\exp(-ry)$, where $r > 0$ is the (constant) degree of absolute risk aversion. Initial wealth is subject to a random loss ℓ that follows a normal distribution with mean μ and variance σ^2 .

Agents have access to reimbursement insurance. A typical reimbursement contract pays out a compensation of β per Euro loss, in return for a premium P . *Ex post*, final wealth is then given by

$$y = e - P - (1 - \beta)\ell, \quad (1)$$

which *ex ante* is also normally distributed. For convenience, we denote by c the coinsurance rate, that is, we let $c \stackrel{\text{def}}{=} 1 - \beta$, and express contracts as pairs of coinsurance rate and premium. We denote contracts by $\mathbf{C} = (c, P)$.

It is well known that under the assumptions made, the expected utility of the agent is representable by the mean-variance function $U = \mathbb{E}(y) - \frac{r}{2}\text{var}(y)$. By replacing the mean and variance of final wealth, expected utility is given by

$$U = e - P - c\mu - \frac{r}{2}c^2\sigma^2. \quad (2)$$

From now on, we write $\nu \stackrel{\text{def}}{=} r\sigma^2$, which can be thought of as the consumer's perceived loss variance. We assume that this perceived loss variance can be either high or low, and likewise for the expected loss:

$$\mu \in \{\mu_L, \mu_H\} \text{ and } \nu \in \{\nu_L, \nu_H\},$$

where $\mu_L < \mu_H$ and $\nu_L < \nu_H$. The model can thus be interpreted in two ways: either people are equally risk averse but their losses have different variances, or the loss variance is identical but people have different degrees of risk aversion. Throughout we will stick to the second interpretation.

A person with characteristics (μ_i, ν_j) is said to be of type ij . The share of ij people in the population is given by α_{ij} ($i, j = H, L, \sum_{i,j} \alpha_{ij} = 1$). We denote by α_k the fraction of people with expected loss μ_k ($\alpha_k = \alpha_{kL} + \alpha_{kH}$); likewise $\alpha_{\cdot k}$ is the fraction of people with perceived variance ν_k ($\alpha_{\cdot k} = \alpha_{Lk} + \alpha_{Hk}$).

Incentive compatible contracts

When a person of type ij ($i, j \in \{H, L\}$) signs the contract $\mathbf{C} = (c, P)$, her expected utility, is

$$U^{ij}(c, P) \stackrel{\text{def}}{=} e - P - c\mu_i - \frac{1}{2}c^2\nu_j. \quad (3)$$

If instead she decides to remain uninsured, her expected utility becomes $e - \mu_i - \frac{1}{2}\nu_j$, which is of course equivalent to accepting the contract $(c, P) = (1, 0)$, where the agent bears the full loss but pays no premium.

The utility rent that the agent enjoys from contract (c, P) is

$$R^{ij}(c, P) \stackrel{\text{def}}{=} U^{ij}(c, P) - U^{ij}(0, 0) = -P + (1 - c)\mu_i + \frac{1}{2}[1 - c^2]\nu_j. \quad (4)$$

Hence, rent decreases with the coinsurance rate both via the expected loss and via the perceived variance (if $c > 0$).

The marginal willingness to pay for a slightly lower coinsurance rate c is thus

$$MWP^{ij}(c) \stackrel{\text{def}}{=} -\frac{dP}{dc} \Big|_{dR^{ij}=0} = \mu_i + c\nu_j, \quad (5)$$

and increases in c :

$$\frac{dMWP^{ij}(c)}{dc} \Big|_{dR^{ij}=0} = \nu_j > 0. \quad (6)$$

Indifference curves in the contract space (c, P) are thus concave in c , and downward-sloping for non-negative coinsurance rates. Also, individuals with a

higher expected loss and/or a higher variance have a higher marginal willingness to pay. Figure 1 illustrates the indifference curve that passes through the no-insurance point $N = (1, 0)$. Since the slope of the indifference curve when it passes the P -axis is μ , it is easy to decompose the total willingness to pay for full insurance into the expected loss and the risk premium $\nu/2$.

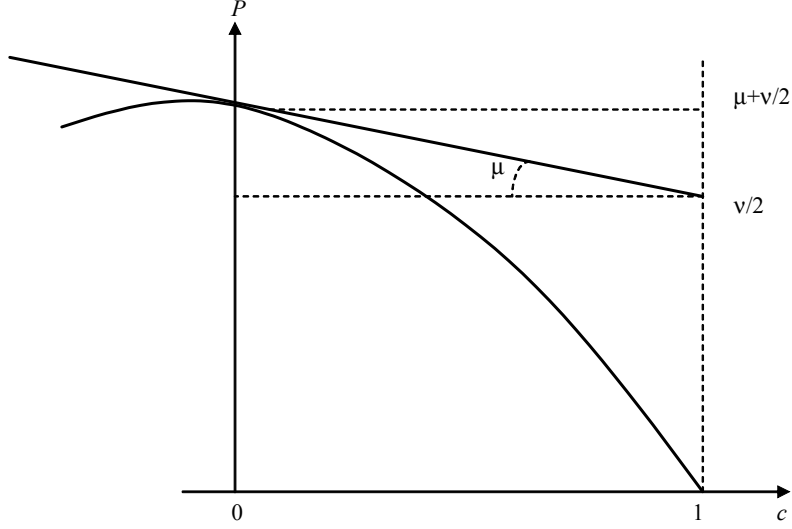


Figure 1: An indifference curve and an iso-profit line in the contract space.

When agent ij signs a contract intended for agent kl , the rent that the former receives is given by

$$R^{ij}(c_{kl}, P_{kl}) = -P_{kl} + (1 - c_{kl})\mu_i + \frac{1}{2}(1 - c_{kl}^2)\nu_j. \quad (7)$$

It is useful to define the following function

$$\delta(c_{kl}, \mu_i - \mu_k, \nu_j - \nu_l) \stackrel{\text{def}}{=} (1 - c_{kl})(\mu_i - \mu_k) + \frac{1}{2}(1 - c_{kl}^2)(\nu_j - \nu_l). \quad (8)$$

Suppose now that type kl is truthful and gets rent $R^{kl}(c_{kl}, P_{kl})$. Which rent does ij obtain when choosing the contract for kl ? Using (4) and (8), the answer is given by

$$R^{ij}(c_{kl}, P_{kl}) \stackrel{\text{def}}{=} R^{kl}(c_{kl}, P_{kl}) + \delta(c_{kl}, \mu_i - \mu_k, \nu_j - \nu_l), \quad (9)$$

i.e. δ is the extra rent that ij earns from kl 's contract. From now on we simply write R^{ij} for $R^{ij}(c_{ij}, P_{ij})$ ($i, j = L, H$).

Self-selection between contracts (c_{ij}, P_{ij}) and (c_{kl}, P_{kl}) requires that

$$\begin{aligned} R^{ij} &\geq R^{kl} + \delta(c_{kl}, \mu_i - \mu_k, \nu_j - \nu_l), \\ R^{kl} &\geq R^{ij} + \delta(c_{ij}, \mu_k - \mu_i, \nu_l - \nu_j), \end{aligned}$$

which taken together imply

$$0 \geq \delta(c_{kl}, \mu_i - \mu_k, \nu_j - \nu_l) + \delta(c_{ij}, \mu_k - \mu_i, \nu_l - \nu_j),$$

or, using (8),

$$\int_{c_{kl}}^{c_{ij}} [(\mu_i - \mu_k) + c(\nu_j - \nu_l)] dc \leq 0.$$

A necessary condition for incentive compatibility between contracts Hj and Lj ($j = H, L$) is that

$$\int_{c_{Lj}}^{c_{Hj}} \Delta\mu dc \leq 0 \iff c_{Hj} \leq c_{Lj}, \quad (10)$$

with $\Delta\mu \stackrel{\text{def}}{=} \mu_H - \mu_L > 0$. Similarly, incentive compatibility between contracts iH and iL ($i = H, L$) is that

$$\int_{c_{iL}}^{c_{iH}} c\Delta\nu dc \leq 0 \iff c_{iH} \leq c_{iL}, \quad (11)$$

with $\Delta\nu \stackrel{\text{def}}{=} \nu_H - \nu_L$ and where it is assumed that $c \geq 0$ (on which more below).

Finally, we mention that the double dimensionality leads in general to double crossing of the indifference curves of HL and LH. Solving $MWP^{HL}(c) = MWP^{LH}(c)$ for c yields $c = \frac{\Delta\mu}{\Delta\nu}$. I.e. in the (c, P) -space, the locus of tangency points between HL's and LH's indifference curves is a vertical line at $\frac{\Delta\mu}{\Delta\nu}$. For lower coinsurance rates, HL's indifference curve crosses that of LH downwards from above, while for higher rates, this happens from below—see figure 2.

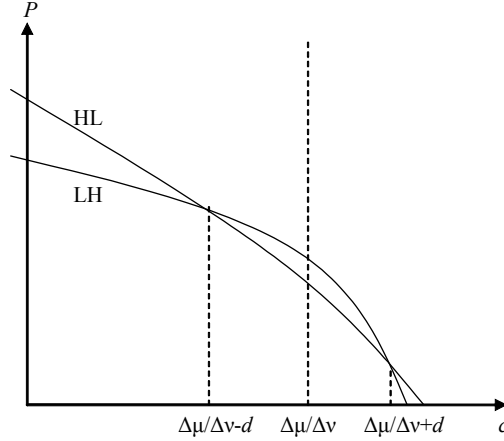


Figure 2: Double crossing of indifference curves in the contract space.

3 The insurance company

We consider a single, risk-neutral insurer with monopoly power on the market for reimbursement contracts.

The iso-profit line for the insurer is given by

$$\pi = P - \beta\mu = P - (1 - c)\mu.$$

and has slope $-\mu$ in the contract space (c, P) .

With *full* information, the monopolist will provide ij with full insurance ($c_{ij} = 0$) at a premium that sets her rent equal to zero. Hence using (4), $P_{ij} = \mu_i + \frac{1}{2}\nu_j$. This yields payoffs per capita equal to $\pi = \frac{1}{2}\nu_j$. The tangency line in Figure 1 thus corresponds to the maximal iso-profit line, and the profit which the insurer makes can be read of from the dashed vertical axis on the right hand side. Under full information, the insurer can extract the entire risk premium $\nu/2$. In general, when offering a contract with a coinsurance rate c to agent ij , the maximal premium $P^{ij}(c)$ that the insurer can charge this agent is (again using (4))

$$P^{ij}(c) = (1 - c)\mu_i + \frac{1}{2}(1 - c^2)\nu_j. \quad (12)$$

In contrast, when the type of an agent is her *private* information, the monopolist will in general have to reduce the maximal premium $\mu_i + \frac{1}{2}\nu_j$ by a rent R^{ij}

necessary to make this agent self-select.

In the remainder, we will characterize the optimal coinsurance rates and the optimal rents. The corresponding premia can then be found with the help of (4).

The insurer's total profit is equal to $\sum_{i,j} \alpha_{ij} [P_{ij} - (1 - c_{ij})\mu_i]$. Using (12) and (4), this can be rewritten as

$$\sum_{i,j} \alpha_{ij} [P^{ij}(c_{ij}) - R^{ij} - (1 - c_{ij})\mu_i]. \quad (13)$$

This objective function is to be maximized with respect to (c_{ij}, R_{ij}) ($ij = H, L$), subject to the usual voluntary participation and incentive compatibility constraints.

To these constraints we add an additional set of constraints that are needed to solve an *ex ante* moral hazard problem. If a coinsurance rate is negative, the insurer refunds losses for more than 100%, and the insured will obviously have a very strong incentive to manipulate the size of their losses (driving less carefully). To avoid such behavior it is enough to restrict insurance rates not to exceed 100%, i.e. to restrict coinsurance rates to be non-negative.⁷

The monopolist thus solves the following problem:

$$\begin{aligned} & \max_{\{c_{ij}, R_{ij}\}} \sum_{i,j} \alpha_{ij} [P^{ij}(c_{ij}) - R^{ij} - (1 - c_{ij})\mu_i] \\ & \text{s.t.} \\ & R^{ij} \geq 0 \quad (i, j, = H, L) \\ & R^{ij} \geq R^{kl} + \delta(c_{kl}, \mu_i - \mu_k, \nu_j - \nu_l) \quad (i, j, k, l, = H, L) \\ & c_{ij} \geq 0 \quad (i, j, = H, L) \end{aligned}$$

The first set of constraints ensures voluntary participation, while the second ensures that all types self-select. The third set consists of the (reduced form) *ex-ante* moral hazard constraints.

⁷Insurance markets may also be subject to an *ex post* moral hazard, where insureds falsify losses, i.e. hands in false claims. Such behaviour is ruled out in our analysis (as in most of the insurance literature). Clearly, negative coinsurance rates would exacerbate the incentive to present false claims. But it is also true that constraining coinsurance rates to be non-negative does not get rid of this moral hazard problem.

A crucial variable in characterizing the profit maximizing set of contracts, is the ratio of ν_L to ν_H , which we define as $x \stackrel{\text{def}}{=} \frac{\nu_L}{\nu_H} \in (0, 1]$.

In the standard monopoly problem with just two types (insurees with a low or a high expected loss), the insurer wants to distort the contract for the low risk type, in order to extract as much rent as possible from the high risk type, who is offered full insurance. The low risk type is thus offered a contract with a positive coinsurance rate.

In our model, people not only differ in their risk (expected loss), but also in their perceived loss variance. Since the insurer is risk neutral, all she cares about is the expected profitability of a contract. This suggests that she should offer type HL the same contract as type HH. After all, if x is close to one, both types can be considered as almost identical, and there should be no reason to discriminate between them. The following lemma (proven in appendix) specifies when this conjecture is correct.

Lemma 1 *Suppose we have an incentive compatible menu of contracts $\{(c_{ij}, R_{ij})\}$ where (i) $c_{HH} = c_{HL}$; (ii) HL is indifferent between her own contract and that of LH, i.e. $R^{HL} = R^{LH} + \delta(c_{LH}, \Delta\mu, -\Delta\nu)$; and (iii) $c_{HH} < \frac{\Delta\mu}{\Delta\nu}$. Then total profits derived from HH and HL cannot be increased by offering HL a contract with a higher coinsurance rate (than the contract for HH) without upsetting LH's incentive compatibility constraint $R^{LH} \geq R^{HH} + \delta(c_{HH}, -\Delta\mu, 0)$ if and only if*

$$(x - \frac{\alpha_{HH}}{\alpha_H})(c_{LH} - c_{HH}) \geq 0. \quad (\text{A})$$

According to condition (ii) in the lemma, HL's rent is fixed by her possibility to mimic LH. Offering HL a higher coinsurance rate has two effects on profits. First, it will expose her to more risk and therefore has to be compensated by a lower premium in order for her not to reject the contract \mathbf{C}_{HL} , a negative effect. Second, because of HH's rent is given by $R^{HH} = R^{HL} + \delta(c_{HL}, 0, \Delta\nu)$, an increase in c_{HL} will allow for more rent extraction from type HH (through a higher premium), a positive effect. The latter effect is the stronger, the more the two types differ in perceived variance (i.e. the lower x is). These two effects

are multiplied by the respective population shares, α_{HL} and α_{HH} . A simple cost-benefit computation then shows that increasing c_{HL} above c_{HH} is not profitable iff $x > \frac{\alpha_{HH}}{\alpha_H}$. However, if $c_{HH} = c_{LH}$, increasing c_{HL} above c_{HH} will compromise the incentive compatibility of contract \mathbf{C}_{LH} , since LH will regard HL's contract superior to her own when $c_{HL} < \frac{\Delta\mu}{\Delta\nu}$. Taking this qualification into account leads to condition (A).

When characterizing the optimal menu of contracts, we thus have to distinguish between the case where the population of HH types and HL types are *alike*, in the sense of condition (A), or not. The main focus in the remainder of the paper will be on the former case. We briefly come back to the other case in section 6.

4 Optimal contracts when condition (A) holds

In this section we provide a full characterisation of the profit maximising menu of contracts when the populations of HH and HL types are alike. As argued earlier, monotonicity requires that of all contracts put on offer, the one with the highest coinsurance rate is selected by type LL . If this coinsurance rate is 100%, this practically means that type LL is excluded from the market. Whether this happens or not impinges on the structure of the remaining contracts. In Proposition 1 we give the optimal coinsurance rates for each type when type LL is *not* excluded ($c_{LL} < 1$). The case where LL is excluded is dealt with in Proposition 2. Each proposition begins by stating an assumption on the parameters to meet condition (A). Propositions 3 and 4 then specify the corresponding rent that each of the four types receive from the optimal contracts. By means of (4) the different premia can be deduced.

While the proofs of the propositions are given in appendix, we comment them by in the text by a graphical argument.

Proposition 1 *Let $\frac{1+\alpha_{LH}-\alpha_{LL}}{1+\alpha_{LH}+\alpha_{LL}} > \frac{\alpha_{HH}}{\alpha_H}$. At a profit maximizing set of contracts, HH and HL always receive a full insurance contract; $c_{HH} = c_{HL} = 0$. The*

contracts for the low loss types, depend on x in the following way:

Regime 1. $x \in (1 - \alpha_{LL}, 1]$: LH and LL are pooled and have a coinsurance rate

$$c_L = \frac{\Delta\mu}{\nu_L} \frac{\alpha_H}{1 - \alpha_H};$$

Regime 2. $x \in (\frac{1+\alpha_{LH}-\alpha_{LL}}{1+\alpha_{LH}+\alpha_{LL}}, 1 - \alpha_{LL}]$: LH and LL are separated with the following coinsurance rates

$$\begin{aligned} c_{LH} &= \frac{\Delta\mu}{\nu_L} \frac{2(x - \alpha_{LH}) - (1 + x)\alpha_H}{(1 - x)(1 - \alpha_H)}, \\ c_{LL} &= \frac{\Delta\mu}{\nu_L} \left[\frac{2\alpha_{LH}}{(1 - x)(1 - \alpha_H)} + \frac{\alpha_H}{1 - \alpha_H} \right]; \end{aligned}$$

Regime 3. $x \in (\frac{1-\alpha_{LL}}{1+\alpha_{LL}}, \frac{1+\alpha_{LH}-\alpha_{LL}}{1+\alpha_{LH}+\alpha_{LL}}]$: LH is separated from LL , but pooled with HH and HL :

$$\begin{aligned} c_{LH} &= 0, \\ c_{LL} &= \frac{\Delta\mu}{\nu_L} \frac{2x}{1 - x}; \end{aligned}$$

Regime 4. $x \in (0, \frac{1-\alpha_{LL}}{1+\alpha_{LL}}]$: LH is separated from LL , but pooled with HH and HL :

$$\begin{aligned} c_{LH} &= 0, \\ c_{LL} &= \frac{\Delta\mu}{\nu_L} \frac{(1 - \alpha_{LL})}{\alpha_{LL}}. \end{aligned}$$

Remarks on proposition 1.

1. The condition for exclusion of LL can be obtained by letting c_{LL} approach 1.
2. For regimes 1 and 2, condition (A) is satisfied because $x > \frac{1+\alpha_{LH}-\alpha_{LL}}{1+\alpha_{LH}+\alpha_{LL}}$, which by assumption is larger than $\frac{\alpha_{HH}}{\alpha_H}$; for regimes 3 and 4, (A) is also met since the optimal coinsurance rates for c_{LH} and c_{HH} coincide (at zero).

3. Figure 3 summarises the optimal coinsurance rates for the low loss agents as a function of x . The discontinuity at $\alpha_{LH} + \alpha_H$ for LL's coinsurance rate points at a non-convexity of the problem, which will be explained below.

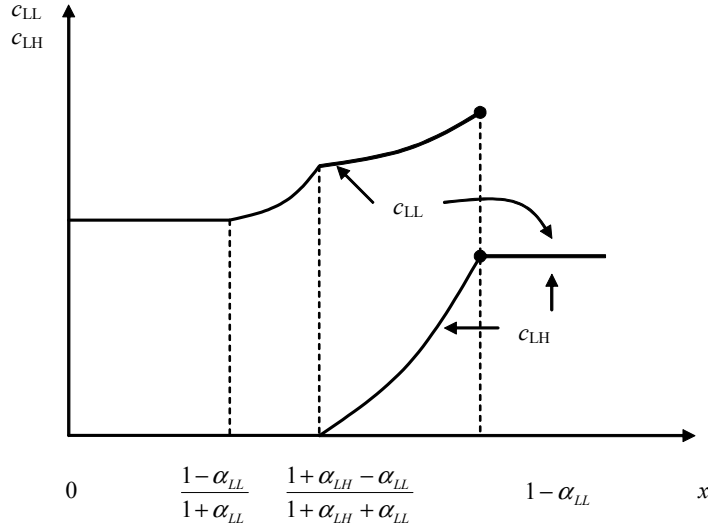


Figure 3: Optimal coinsurance rates for low risk people as a function of x .

For very high values of x (regime 1), the differences in types is mainly due to differences in expected loss. In the limit, as $x \rightarrow 1$, agents differ only in expected loss and we are back in the Stiglitz (1977) model. Figure 4 displays the optimal set of contracts. Full (dashed) lines refer to high (low) risk averse types, while bold (thin) lines to high (low) loss types; N indicates the no-insurance alternative.

The optimal menu of contracts consists of \mathbf{C}_H , which is meant for the high loss types, and \mathbf{C}_L , which is meant for the low loss types. Since x is by assumption larger than $1 - \alpha_{LL}$, $c_L < \frac{\Delta\mu}{\Delta\nu}$.

The pooling contracts \mathbf{C}_H and \mathbf{C}_L are optimal for $x > 1 - \alpha_{LL}$, i.e. for a sufficiently high share of LL people. Suppose now that this share falls just below $1 - x$. The insurance company can now exploit the existence of a second crossing to the right of the $\frac{\Delta\mu}{\Delta\nu}$ -line. We now refer to figure 5 which shows three additional (lower-valued) indifference curves than figure 4 does. By increasing the premium

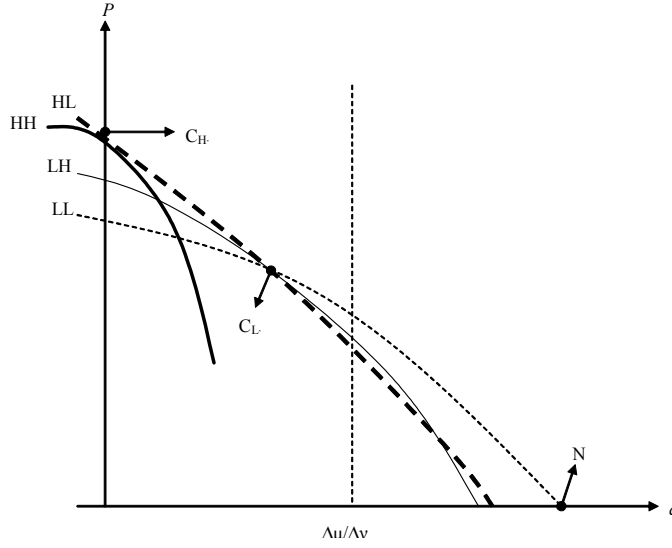


Figure 4: Illustration of the solution in regime 1.

on high loss types with ΔP , say (the vertical distance between C_H and C'_H in the figure), but not touching their (zero) coinsurance rate, and doing the same for LH types (offering them C'_{LH} in stead of C_L .) incentive compatibility between these types is preserved. When simultaneously offering contract C'_{LL} to LL , HL will still not have an incentive to mimic LL . But, while previously LH was indifferent between C_H and C_L , she now strictly prefers C'_{LH} to C'_{LL} . Thus one incentive constraint is relaxed by the operation. What are the costs and benefits? The cost is a much lower profit from LL . The benefit is an increase ΔP in the profit from the three other types. But since by assumption there are now less LL types than before, the net benefit is positive. This explains the discontinuity in the use of the coinsurance rate on LL types. Further optimization leads to a contract menu as depicted in figure 6.

When the differences in risk aversion becomes sufficiently large, i.e. a sufficiently low x , we get a situation like in figure 7, where the left hand crossing of the indifference curve of LH and HL takes place to the *left* of the vertical axis. Ignoring the moral hazard constraint, it would be optimal to provide LH with contract E while providing HL and HH with the over-insurance contract

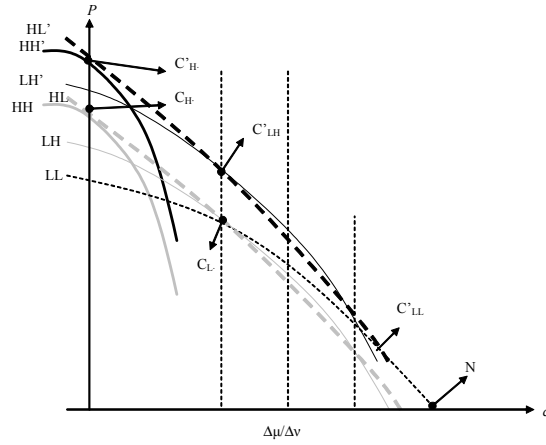


Figure 5: Transition from the solution in regime 1 to a feasible menu for regime 2.

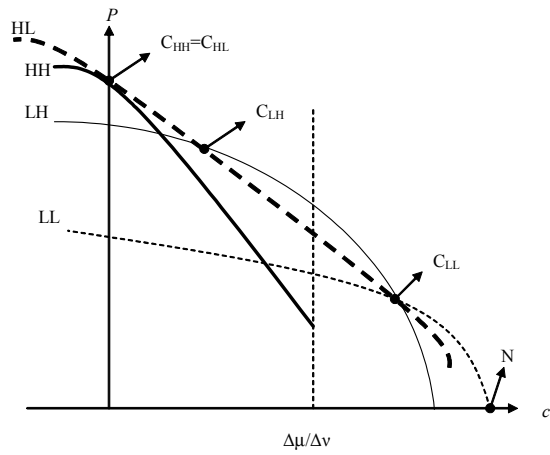


Figure 6: Illustration of the solution in regime 2.

D (providing them the full insurance contract F would clearly upset incentive compatibility for LH). Imposing the moral hazard constraint leads to a pooling solution between contracts E and F , as in figure 8.

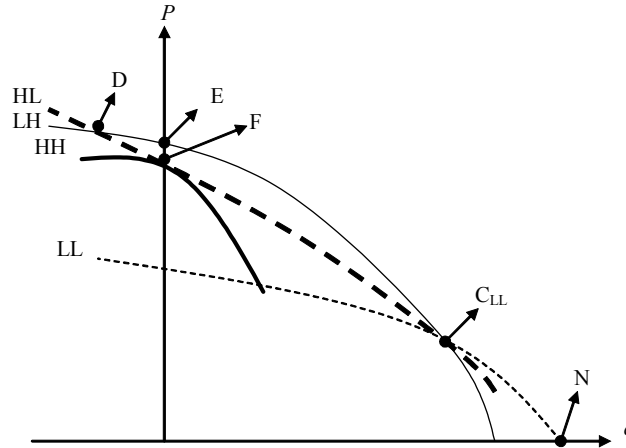


Figure 7: Over-insurance of high risk types may be optimal when the moral hazard constraint is ignored.

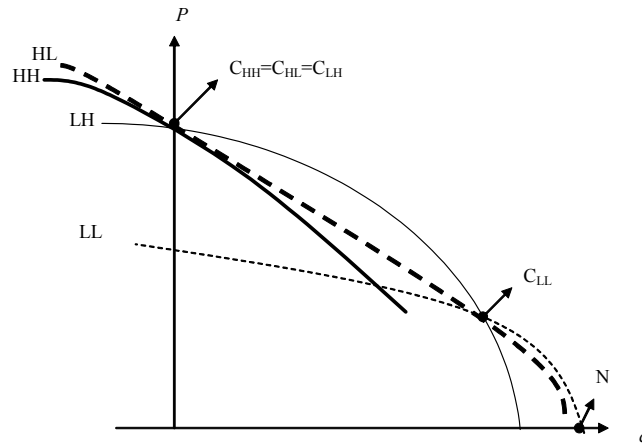


Figure 8: Illustration of the solution in regime 3.

Finally, when x is very low, the curvature in the indifference curve of LH is so strong that this agent finds contract C_{LL} strictly inferior to her own contract (see figure 9). This would suggest that the company could extract a higher premium

from this type, but such reasoning is wrong. Increasing the premium would also mean increasing the premium on HL and, for incentive compatibility reasons, this would require an even larger distortion of the contract for LL . If there are a sufficient amount of LL people around (α_{LL} sufficiently large), this does not pay.⁸

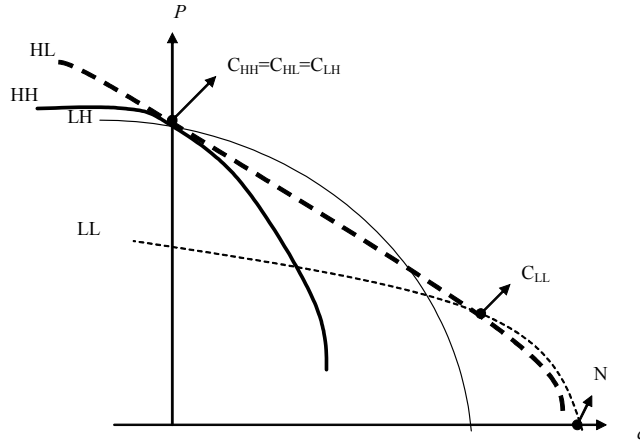


Figure 9: Illustration of the solution in regime 4.

Of course, it need not to be optimal to offer insurance to all four types. In particular, it may pay to exclude a “lower” type in order to extract extra rent from the “higher” types. Exclusion is the largest distortion that can be imposed on an insuree; it corresponds to a coinsurance rate of 100%. The conditions for exclusion of LL to take place where implicitly given by proposition 1 (cf remark 1). In the following proposition, we specify how this exclusion affects the coinsurance rates for the remaining types.

Proposition 2 *Let $\frac{\alpha_{HH}}{\alpha_H} \geq (1 + 2\frac{\Delta\mu}{\nu_L})^{-1}$. At a profit maximizing set of contracts, HH and HL always receive a full insurance contract; $c_{HH} = c_{HL} = 0$. Depending*

⁸Here again, if negative co-payment rates were feasible, it would be profitable to increase the premium on LH till strict indifference between C_{LH} and C_{LL} is obtained, and move the contracts C_{HH} and C_{HL} in north-west direction (i.e. provides overinsurance) to re-establish incentive compatibility w.r.t. the new C_{LH} .

on x , the coinsurance rate LH is as follows:

1. $x \in (\frac{1}{\frac{\Delta\mu}{\nu_L}+1} \frac{1-\alpha_{LL}}{\alpha_H}, 1]$: LH is excluded: $c_{LH} = 1$;

2. $x \in (\hat{x}, \frac{1}{\frac{\Delta\mu}{\nu_L}+1} \frac{1-\alpha_{LL}}{\alpha_H}]$: the coinsurance rate for LH is

$$c_{LH} = \frac{\Delta\mu}{\nu_L} \frac{x\alpha_H}{\alpha_{LH} + (1-x)\alpha_H};$$

3. $x \in (\frac{1}{1+2\frac{\Delta\mu}{\nu_L}}, \hat{x}]$: the coinsurance rate for LH is

$$c_{LH} = 2\frac{\Delta\mu}{\nu_L} \frac{x}{1-x} - 1;$$

4. $x \in (0, \frac{1}{1+2\frac{\Delta\mu}{\nu_L}}]$: LH is pooled with HH and HL , i.e. $c_{LH} = 0$.

Here \hat{x} is the lowest root of $-a(1+b)x^2 + (2a+ab+2b+1)x - (1+a) = 0$ with

$$a \stackrel{\text{def}}{=} \frac{\alpha_H}{\alpha_{LH}} \text{ and } b \stackrel{\text{def}}{=} \frac{\Delta\mu}{\nu_L}.$$

Figure 10 sketches the optimal coinsurance rate for LH as a function of x when type LL is excluded.

We have now characterised the optimal coinsurance rates for the different regimes. Using expression (4), and keeping in mind who mimics whom, it is fairly straightforward to calculate the corresponding rents that are left in equilibrium.

Proposition 3 *Let $\frac{1+\alpha_{LH}-\alpha_{LL}}{1+\alpha_{LH}+\alpha_{LL}} > \frac{\alpha_{HH}}{\alpha_H}$. Without exclusion of LL , the rents for the different agents are as follows:*

$$\begin{aligned} R^{LL} &= 0, \\ R^{LH} &= \max\left\{\frac{1}{2}(1-c_{LL}^2)\Delta\nu, -c_{LL}\Delta\mu + \frac{1}{2}\Delta\nu\right\} \\ R^{HL} &= (1-c_{LL})\Delta\mu, \\ R^{HH} &= (1-c_{LL})\Delta\mu + \frac{1}{2}\Delta\nu. \end{aligned}$$

R_{LH}, R_{LH} and R_{HH} are thus falling in the coinsurance rate for LL .

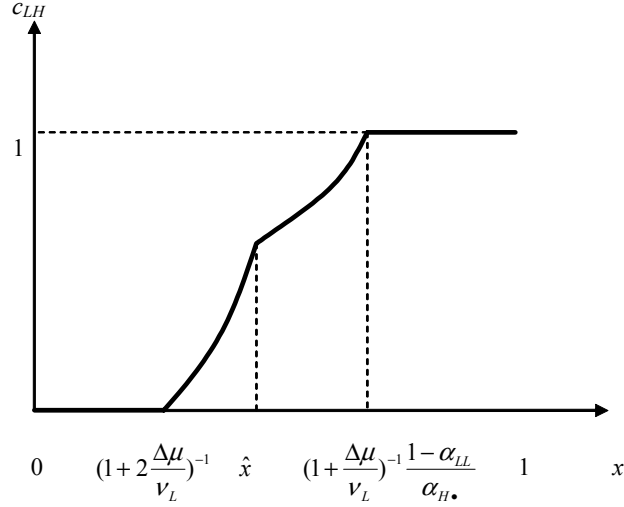


Figure 10: The optimal coinsurance rate for LH as a function of x (under exclusion of LL).

Proposition 4 Let $\frac{\alpha_{HH}}{\alpha_H} \geq (1 + 2\frac{\Delta\mu}{\nu_L})^{-1}$. When LL is excluded, the rents are as follows:

$$\begin{aligned}
 R^{LL} &= 0, \\
 R^{LH} &= \max\{0, -\Delta\mu + \frac{1}{2}\Delta\nu\} \\
 R^{HL} &= \max\{0, (1 - c_{LH})\Delta\mu - \frac{1}{2}(1 - c_{LH}^2)\Delta\nu\}, \\
 R^{HH} &= R^{HL} + \frac{1}{2}\Delta\nu.
 \end{aligned}$$

The rents for the high loss people are non-increasing in the coinsurance rate for LH .

5 Gender discrimination

In this section, we want to explore the effects of gender discrimination, also called gender categorisation, on people's welfare. Let us write $p(\mu, \nu, g)$ as the likelihood function that an arbitrary insuree has expected loss μ , a risk aversion ν and gender $g \in \{m, w\}$.

A monopolist who is allowed to condition on gender will for each of the two genders g design an optimal contract menu based on the risk aversion ratio x and the probability matrix⁹

$$\begin{pmatrix} p(\mu_L, \nu_L|g) & p(\mu_L, \nu_H|g) \\ p(\mu_H, \nu_L|g) & p(\mu_H, \nu_H|g) \end{pmatrix}.$$

We now make two assumptions:

$$p(g|\nu, \mu) = p(g|\nu). \quad (\text{S})$$

$$p(\mu|\nu) = p(\mu) \quad (\text{I})$$

Assumption (S) means that risk aversion is a sufficient statistic for gender w.r.t. expected loss; if expected loss of a person could be observed, this carries no extra information about the gender of that person on top of the observation of the person's risk aversion. This alone, however, is not sufficient to break the link between gender and expected loss. If female drivers are highly risk averse, and if this attitude leads them to careful driving, then there will still be a connection between gender and expected loss. This last connection is broken by assumption (I) which says that expected loss is independently distributed of risk aversion—that risk aversion has no impact on driving. The following result then immediately follows:

Lemma 2 *If the likelihood function $p(\cdot)$ satisfies (S) and (I) then expected loss and gender will be independently distributed: $p(\mu|g) = p(\mu)$.*

Thus, under assumptions (S) and (I), one should be led to the same conclusion as the European Commission—that a person's gender is insignificant in explaining her risk type.

Since a gender discriminating firm will use the probability functions $p(\mu, \nu|g)$ ($g = m, w$), rather than the single function $p(\mu, \nu)$ to design menus, and since

⁹Here, we assume that the support of the distribution of types does *not* vary with the signal. Alternatively, the support could be made dependent on the signal. This, in effect, amounts to assuming that the support consists of more than four (μ, ν) -pairs, some of which have zero probability mass, depending on the observation of the signal.

profits and consumer rents depend on the coinsurance rate c_{LL} , it is important to trace out changes in $p(\cdot)$ on c_{LL} . From proposition 1, it follows that without discrimination, the boundaries of the 4 regimes are determined by the parameters α_{LL} and α_{LH} . Fixing x , α_L , (and thus $\alpha_H = 1 - \alpha_L$), and $\frac{\Delta\mu}{\nu_L}$, it is possible to trace out the optimal value of c_{LL} as a function of α_{LL} . Doing so yields figure 11, where it is assumed that $\frac{1-x}{1+x} < \frac{1}{2}(1 + \alpha_L)(1 - x)$. This means that curve for LL's optimal coinsurance rate is flat for some range of α_{LL} values; it is equivalent to assuming that

$$\frac{1 - \alpha_L}{1 + \alpha_L} < x. \quad (\text{F})$$

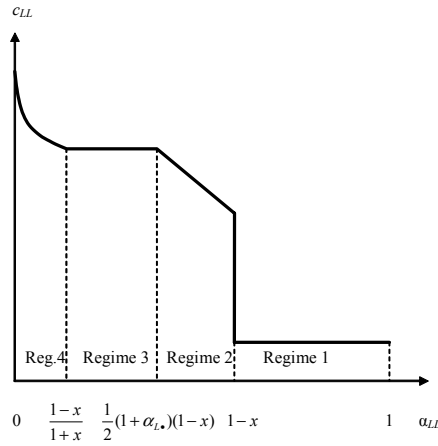


Figure 11: The optimal coinsurance rate for LL as a function of α_{LL} .

Let us define ω_L (ω_H) as the likelihood that an arbitrary person with low (high) risk aversion is a female, i.e. $\omega_L \stackrel{\text{def}}{=} p(w|\nu_L)$ and $\omega_H \stackrel{\text{def}}{=} p(w|\nu_H)$. With half of the population consisting of women, we have that $p(w) = \omega_H\alpha_H + \omega_L(1 - \alpha_H) = \frac{1}{2}$.

Hartog, Ferrer-i-Carbonell & Jonker (2002) and Eckel & Grossman (2003) report evidence that women are more risk averse than men. For our model, this means that $\omega_L < \frac{1}{2} < \omega_H$. When the insurance company is allowed to gender discriminate, it will upon observation of a customer's gender g update the

probability α_{LL} in the following way:

$$\begin{aligned}\alpha_{LL|w} &\stackrel{\text{def}}{=} p(\mu_L, \nu_L|w) = 2\omega_L\alpha_{LL} (< \alpha_{LL}), \text{ and} \\ \alpha_{LL|m} &\stackrel{\text{def}}{=} p(\mu_L, \nu_L|m) = 2(1 - \omega_L)\alpha_{LL} (> \alpha_{LL}).\end{aligned}$$

By lemma 2, we also have that $p(\mu_i|g) = p(\mu_i)$ ($i = L, H$), meaning that α_L and α_H do not change when gender is observed.

Suppose now that (F) holds, that the fractions of LL people for the entire population, the population of men and that of women, are α_{LL} , $\alpha_{LL|m}$, and $\alpha_{LL|w}$, respectively, and that these fractions are as in figure 12.

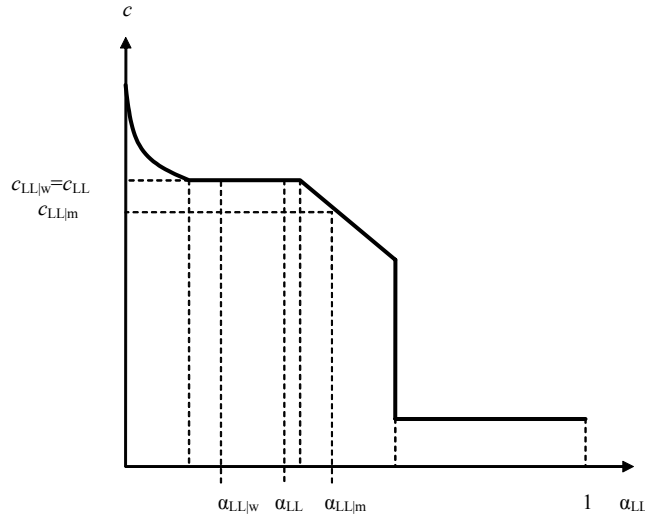


Figure 12: A configuration of *a priori* and updated probabilities that an insuree is of type LL .

Then we can conclude that since

$$\frac{1-x}{1+x} < \alpha_{LL|w} < \alpha_{LL} < \frac{1}{2}(1 + \alpha_L)(1-x), \quad (14)$$

the coinsurance rate for LL women will remain at its no-discrimination value, the rents of LH , HL and HH women will not change due to discrimination (and LL women continue to receive zero rent). On the other hand, because

$$\frac{1-x}{1+x} < \alpha_{LL} < \frac{1}{2}(1 + \alpha_L)(1-x) < \alpha_{LL|m}, \quad (15)$$

the optimal coinsurance rate for LL men will drop below its no-discrimination value, and therefore all men will get more rent when offered the optimal contract menu for men (except LL men, who continue to receive zero rent). Finally, the insurance company will increase total profit since it found it optimal to choose a new menu for its male clientele—it could have stuck to same menu as in the no-discrimination case. Thus, a (weak) Pareto improvement is possible by allowing for gender discrimination. As can be seen from the figure, conditions (14) and (15) are not only sufficient for a (weak) Pareto improvement, but also necessary. We summarise this result as

Proposition 5 *Suppose that (F) holds. For given values of x, α_L , and $\frac{\Delta\mu}{\nu_L}$, allowing for gender discrimination will lead to a (weak) Pareto improvement in the insurance market if and only if conditions (14) and (15) hold.*

Condition (F) will be satisfied when the fraction of low risk people, α_L , is not too small in relation to x .

6 Optimal contracts when condition (A) is not satisfied

When condition (A) is not satisfied, the population of HL people is not similar enough to that of HH people for pooling of the two to be optimal. Formally, the marginal effect on profits of increasing the coinsurance rate for HL above that of HH is

$$(1 - \beta_{HL}) \left(\frac{\alpha_{HH}}{\alpha_H} - x \right) \nu_H \alpha_H > 0$$

and therefore the second order effect is also positive, meaning that the insurance company would like to increase the insurance rate on HL until HH ceases to envy the contract for HL.

When (A) is not satisfied, regimes 3 and 4 of proposition 1 can occur, but several new configurations of contracts may show up. We mention here

- full insurance for HH, intermediate insurance for LH, pooling of HH and HL at low insurance;
- full insurance for HH, LH and HL pooled at intermediate insurance, low insurance for LL;
- full insurance for HH, pooling of LH, HL and LL at low insurance.

To decide which contract is optimal under which parameter constellation is not trivial because of discontinuities in the transition from one regime to another. This requires a pair-wise comparison of the optimal profits for each regime. This will be taken up in a follow up to this paper.

7 Concluding remarks

In this paper we have defined a new setting for analysing screening problems in a monopolistic insurance market when crucial information is of a private nature. The novel elements are (i) a mean-variance approach to the insuree's preferences and the risks she faces, and (ii) a bi-dimensional heterogeneity of insurees in terms of expected loss and risk aversion.

For a wide range of parameters, we have given a complete characterisation of the optimal menu of contracts which a monopolistic insurer wishes to market. Making use of these results, we have then analysed a recent policy proposal of the European Commission to prohibit gender discrimination when underwriting risks. The arguments leading to this proposal emphasise the weak linkages between gender and risk type (i.e. expected loss) when one controls for other socio-economic factors. We have shown that even when gender and risk type are independently distributed in the population, the fact that women tend to exhibit a higher risk aversion than men may be sufficient for gender discrimination to lead to a Pareto improvement in a monopolistic insurance market.

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8 Appendix

Proof of Lemma 1

Using (13) and (12), the profits per agent which the insurance company derives from types HH and HL are given by, respectively,

$$\begin{aligned}\pi^{HH} &= \frac{1}{2} (1 - c_{HH}^2) \nu_H - R^{HH}, \\ \pi^{HL} &= \frac{1}{2} (1 - c_{HL}^2) \nu_L - R^{HL}.\end{aligned}$$

Substitute R^{HH} by $R^{HL} + \delta(c_{HL}, 0, \Delta\nu) = R^{HL} + \frac{1}{2}(1 - c_{HL}^2)\Delta\nu$. Differentiate the resulting expression for $\alpha_{HH}\pi^{HH} + \alpha_{HL}\pi^{HL}$ with respect to c_{HL} . This yields the effect on the profit coming from these two types when the coinsurance rate c_{HL} is marginally increased while R^{HL} is kept fixed, or

$$c_{HL} (\alpha_{HH}\Delta\nu - \alpha_{HL}\nu_L).$$

This is non-positive iff $x \geq \frac{\alpha_{HH}}{\alpha_H}$. However, if $c_{LH} = c_{HH}$ then increasing c_{HL} above c_{HH} will make LH start to envy the contract for HL . This is because by assumption $c_{LH} < \frac{\Delta\mu}{\Delta\nu}$. Hence either $x \geq \frac{\alpha_{HH}}{\alpha_H}$ or $c_{LH} = c_{HH}$ are sufficient to rule out a profit increase by giving HL a higher coinsurance rate than HH .

■

Proof of proposition 1.

We assume in this proof that LL is not excluded, in other words that the constraint that the optimal co-payment rate is below 1 is not binding. The proof uses the methodology proposed by Armstrong & Rochet (1999). It goes in several steps.

1. Suppose that $x \geq \frac{1+\alpha_{LH}-\alpha_{LL}}{1+\alpha_{LH}+\alpha_{LL}}$. We then claim that the active constraints are the voluntary participation constraint for LL ($R^{LL} \geq 0$), and the incentive compatibility constraints that are represented by arrows in figure 13, that is, HH should not have an incentive to mimic HL , HL should not have an incentive to mimic both LH and LL , and LH should not have an incentive to mimic LL .

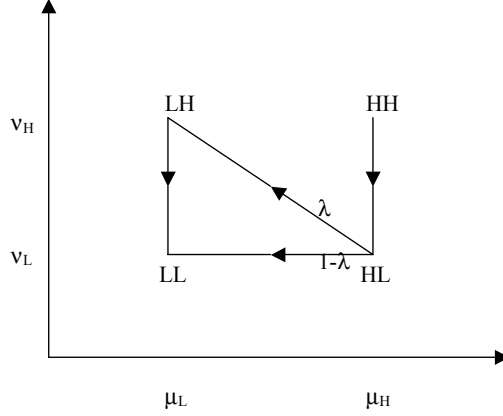


Figure 13: The active incentive constraints for regimes 1 and 2 (no exclusion).

The list of active constraints is thus:

$$\begin{aligned}
R^{LL} &\geq 0 \\
R^{HH} &\geq R^{HL} + \delta(c_{HL}, 0, \Delta\nu) \\
R^{LH} &\geq R^{LL} + \delta(c_{LL}, 0, \Delta\nu) \\
R^{HL} &\geq R^{LL} + \delta(c_{LL}, \Delta\mu, 0) \\
&\geq R^{LH} + \delta(c_{LH}, \Delta\mu, -\Delta\nu).
\end{aligned}$$

We are going to deal with the last two constraints by summarizing them as

$$R^{HL} = (1 - \lambda) [R^{LL} + \delta(c_{LL}, \Delta\mu, 0)] + \lambda [R^{LH} + \delta(c_{LH}, \Delta\mu, -\Delta\nu)]. \quad (16)$$

Notice that, by imposing (16), (a) if neither of the two constraints are binding then $\lambda \notin [0, 1]$; (b) if *exactly* one of them is binding then λ must be either 0 or 1.¹⁰ Therefore, if $\lambda \in (0, 1)$ then both must simultaneously bind.

It is easy to see that the first three constraints together with (16) must be binding and that at least one of the incentive constraints for HL must be also binding. Therefore, we know that $\lambda \in [0, 1]$ and we have the following system of

¹⁰Use $R^{HL} > R^{LL} + \delta(c_{LL}, \Delta\mu, 0)$ and $R^{HL} = R^{LH} + \delta(c_{LH}, \Delta\mu, -\Delta\nu)$ in the linear combination to imply $\lambda = 1$. The other case follows an analogous argument.

equations in rents:

$$\begin{aligned}
R^{LL} &= 0 \\
R^{HH} &= R^{HL} + \delta(c_{HL}, 0, \Delta\nu) \\
R^{LH} &= R^{LL} + \delta(c_{LL}, 0, \Delta\nu) \\
R^{HL} &= (1 - \lambda) [R^{LL} + \delta(c_{LL}, \Delta\mu, 0)] + \lambda [R^{LH} + \delta(c_{LH}, \Delta\mu, -\Delta\nu)].
\end{aligned}$$

Substituting out the rents in the objective function (13) then gives

$$\begin{aligned}
\mathcal{L} &= \alpha_{HH} \{ P^{HH}(c_{HH}) - (1 - c_{HH})\mu_H \\
&\quad - [\delta(c_{HL}, 0, \Delta\nu) + (1 - \lambda)\delta(c_{LL}, \Delta\mu, 0) + \lambda(\delta(c_{LL}, 0, \Delta\nu) + \delta(c_{LH}, \Delta\mu, -\Delta\nu))] \} \\
&\quad + \alpha_{HL} \{ P^{HL}(c_{HL}) - (1 - c_{HL})\mu_H \\
&\quad - [(1 - \lambda)\delta(c_{LL}, \Delta\mu, 0) + \lambda(\delta(c_{LL}, 0, \Delta\nu) + \delta(c_{LH}, \Delta\mu, -\Delta\nu))] \} \\
&\quad + \alpha_{LH} \{ P^{LH}(c_{LH}) - (1 - c_{LH})\mu_L - \delta(c_{HL}, 0, \Delta\nu) \} \\
&\quad + \alpha_{LL} \{ P^{LL}(c_{LL}) - (1 - c_{LL})\mu_L \}.
\end{aligned}$$

The first derivatives w.r.t. the four co-payment rates are:

$$\frac{\partial \mathcal{L}}{\partial c_{HH}} = -\alpha_{HH}[\mu_H + c_{HH}\nu_H - \mu_H], \quad (17)$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_{HL}} &= \alpha_{HH}c_{HL}\Delta\nu - \alpha_{HL}(\mu_H + c_{HL}\nu_L - \mu_H) \\
&= -c_{HL}[\alpha_{HL}\nu_L - \alpha_{HH}\Delta\nu], \quad (18)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_{LH}} &= \alpha_{HH}\lambda(\Delta\mu - c_{LH}\Delta\nu) - \alpha_{HL}\lambda(\Delta\mu - c_{LH}\Delta\nu) \\
&\quad - \alpha_{LH}[\mu_L + c_{LH}\nu_H - \mu_L] = 0, \quad (19)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_{LL}} &= \alpha_{HH}[(1 - \lambda)\Delta\mu + \lambda c_{LL}\Delta\nu] + \alpha_{HL}[(1 - \lambda)\Delta\mu + \lambda c_{LL}\Delta\nu] \\
&\quad + \alpha_{LH}c_{LL}\Delta\nu + \alpha_{LL}[\mu_L + c_{LL}\nu_L - \mu_L]. \quad (20)
\end{aligned}$$

Setting (17) to zero immediately gives $c_{HH} = 0$. Because $\frac{1+\alpha_{LH}-\alpha_{LL}}{1+\alpha_{LH}+\alpha_{LL}} \geq \frac{\alpha_{HH}}{\alpha_H}$, condition (A) will be satisfied and the square bracket term (18) is negative, so that $c_{HL} = 0$ is a local optimum. Equating (18) and (20) to zero and making use of the definition $x = \frac{\nu_L}{\nu_H}$ gives

$$c_{LH} = \frac{\Delta\mu}{\nu_L} \frac{\lambda\alpha_H \cdot x}{\alpha_{LH} + \lambda\alpha_H(1-x)} \quad (21)$$

and

$$c_{LL} = \frac{\Delta\mu}{\nu_L} \frac{(1-\lambda)\alpha_H \cdot x}{\alpha_{LL}x - (\alpha_{LH} + \lambda\alpha_H)(1-x)}. \quad (22)$$

2. Suppose first that $\lambda \in (0, 1)$, i.e., that the rents when HL mimics either LH or LL are identical:

$$\begin{aligned} \delta(c_{LL}, \Delta\mu, 0) &= \delta(c_{LL}, 0, \Delta\nu) + \delta(c_{LH}, \Delta\mu, -\Delta\nu) \\ &\Updownarrow \\ (c_{LH} - c_{LL})\Delta\mu &= \frac{1}{2} (c_{LH}^2 - c_{LL}^2) \Delta\nu. \end{aligned}$$

2a. Assume that $c_{LH} \neq c_{LL}$. Then the previous equality reduces to

$$2 \frac{\Delta\mu}{\nu_L} \frac{x}{1-x} = c_{LH} + c_{LL}.$$

Making use of (21) and (22), this equality becomes

$$\begin{aligned} 2\alpha_{LH}\alpha_{LL}x - 2\alpha_{LH}^2(1-x) - \alpha_{LH}\alpha_H\lambda(1-x) + \alpha_H\alpha_{LL}\lambda x(1-x) \\ - \alpha_{LH}\alpha_H\lambda\nu_H(1-x)^2 = \alpha_{LH}\alpha_H(1-x) + \lambda\alpha_H^2(1-x)^2. \end{aligned}$$

Solving for λ , and making use of the fact that $\alpha_{LL} = 1 - \alpha_{LH} - \alpha_{HL} - \alpha_{HH}$, we get

$$\lambda = \frac{\alpha_{LH}}{\alpha_H} \frac{1}{(1-x)} \frac{(2 - \alpha_H)x - (\alpha_H + 2\alpha_{LH})}{(\alpha_H + 2\alpha_{LH}) - x} \quad (23)$$

To obtain the optimal co-payment rates for LL and LH , we substitute (23) into (21) and (22):

$$c_{LH} = \frac{\Delta\mu}{\nu_L} \frac{2(x - \alpha_{LH}) - (1+x)\alpha_H}{(1-x)(1-\alpha_H)}, \quad (24)$$

$$c_{LL} = \frac{\Delta\mu}{\nu_L} \frac{2\alpha_{LH} + (1-x)\alpha_H}{(1-x)(1-\alpha_H)}. \quad (25)$$

Comparing these two co-payment rates, we may conclude that $c_{LH} < c_{LL}$ because the upper bound for x is $1 - \alpha_{LL} = \alpha_{LH} + \alpha_{H\cdot}$. The assumption that $c_{LH} \neq c_{LL}$ is thus confirmed. The above solution is valid as long as the *rhs* of (25) is below one, i.e. when LL is not excluded.

We still have to check when $\lambda \in (0, 1)$. Equating (23) to 1 gives a quadratic equation in x ,

$$\alpha_{H\cdot} x^2 - (\alpha_{H\cdot}^2 + 3\alpha_{LH}\alpha_{H\cdot} + \alpha_{H\cdot} - 2\alpha_{LH})x + (\alpha_{H\cdot}^2 + 3\alpha_{LH}\alpha_{H\cdot} + 2\alpha_{LH}^2) = 0,$$

which has two roots, only one of which lies below 1: $1 - \alpha_{LL}$. This is the upper bound for x .

Alternatively, equating (23) to zero, we obtain the lower bound for x :

$$\frac{1 + \alpha_{LH} - \alpha_{LL}}{1 + \alpha_{LH} + \alpha_{LL}}.$$

The expressions (25) and (24) thus provide the optimal coinsurance rates for LL and LH for $x \in (\frac{1 + \alpha_{LH} - \alpha_{LL}}{1 + \alpha_{LH} + \alpha_{LL}}, 1 - \alpha_{LL})$, as stated under 'Regime 2' in the proposition.

2b. Suppose that we have a pooling solution, $c_{LH} = c_{LL}$. Equating (21) and (22), yields

$$\lambda = \frac{\alpha_{LH}}{x - \alpha_{H\cdot}}. \quad (26)$$

Imposing that λ lies in the unit interval gives the following condition on x :

$$x \geq 1 - \alpha_{LL}.$$

Substituting the solution for λ back into (21) and (22) gives the optimal (common) co-payment rates for regime 1 in the proposition:

$$c_{LH} = c_{LL} = \frac{\Delta\mu}{\nu_L} \frac{\alpha_{H\cdot}}{1 - \alpha_{H\cdot}}, \quad (27)$$

which is valid for $\frac{\Delta\mu}{\nu_L} \frac{\alpha_{H\cdot}}{(1 - \alpha_{H\cdot})} \leq 1$.

Are the neglected incentive constraints satisfied by the solution? The easiest way to check this is diagrammatically, and figure 4 and 6 in the text provide all the information to do so.

This proves parts 1 and 2 in the proposition.

3. As argued under 2a, if $x \rightarrow \frac{1+\alpha_{LH}-\alpha_{LL}}{1+\alpha_{LH}+\alpha_{LL}}$, then $\lambda \rightarrow 0$ and $c_{LH} \rightarrow 0$ (see (21)) and LH 's contract will coincide with that of HH and HL . Then LH has become indifferent between C_{LH} on the one hand and C_{HH} and C_{HL} on the other. We then claim that the active incentive constraints are

$$R^{HH} = R^{HL} + \delta(c_{HL}, 0, \Delta\nu) \quad (28)$$

$$R^{LH} \geq R^{HH} + \delta(c_{HH}, -\Delta\mu, 0) \quad (29)$$

$$= R^{HL} + \delta(c_{HL}, -\Delta\mu, \Delta\nu) \quad (30)$$

$$\geq R^{LL} + \delta(c_{LL}, 0, \Delta\nu). \quad (31)$$

which can be sketched as in figure 14.

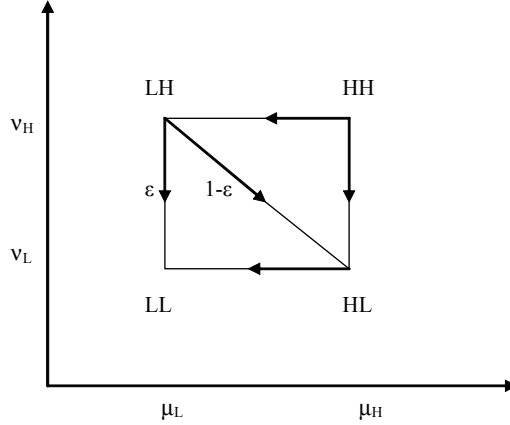


Figure 14: The active incentive constraints for regimes 3 and 4.

We now make use of the following lemma:

Lemma 3 *If HH is indifferent between C_{HH} and C_{HL} ($R^{HH} = R^{HL} + \delta(c_{HL}, 0, \Delta\nu)$), then at an incentive compatible solution the rent that LH can obtain when mimicking HL cannot be lower than the rent that she can obtain when mimicking*

HH:

$$R^{HL} + \delta(c_{HL}, -\Delta\mu, \Delta\nu) \geq R^{HH} + \delta(c_{HH}, -\Delta\mu, 0).$$

Proof of lemma 3.

Suppose that it is not true, i.e.

$$\begin{aligned} R^{HH} &= R^{HL} + \delta(c_{HL}, 0, \Delta\nu), \text{ and} \\ R^{HL} + \delta(c_{HL}, -\Delta\mu, \Delta\nu) &< R^{HH} + \delta(c_{HH}, -\Delta\mu, 0). \end{aligned}$$

Then it follows that

$$\begin{aligned} R^{HL} + \delta(c_{HL}, -\Delta\mu, \Delta\nu) &< R^{HL} + \delta(c_{HL}, 0, \Delta\nu) + \delta(c_{HH}, -\Delta\mu, 0) \\ &\Downarrow \\ \delta(c_{HL}, -\Delta\mu, 0) &< \delta(c_{HH}, -\Delta\mu, 0) \\ &\Downarrow \\ c_{HL} &< c_{HH}, \end{aligned}$$

which contradicts monotonicity. \blacksquare

Lemma 3 thus allows us to ignore (29) if we take the monotonicity constraint $c_{HH} \leq c_{HL}$ into account.

Combining (30) and (31) into

$$R^{LH} = \varepsilon\delta(c_{LL}, 0, \Delta\nu) + (1 - \varepsilon)[\delta(c_{LL}, \Delta\mu, 0) + \delta(c_{HL}, -\Delta\mu, \Delta\nu)] \text{ with } \varepsilon \in [0, 1], \quad (32)$$

the Lagrange function of the insurance company can be written as

$$\begin{aligned} \mathcal{L} &= \alpha_{HH}\{P^{HH}(c_{HH}) - (1 - c_{HH})\mu_H - \delta(c_{HL}, 0, \Delta\nu) - \delta(c_{LL}, \Delta\mu, 0)\} \\ &\quad + \alpha_{HL}\{P^{HL}(c_{HL}) - (1 - c_{HL})\mu_H - \delta(c_{LL}, \Delta\mu, 0)\} \\ &\quad + \alpha_{LH}\{P^{LH}(c_{LH}) - (1 - c_{LH})\mu_L - \varepsilon\delta(c_{LL}, 0, \Delta\nu) \\ &\quad \quad - (1 - \varepsilon)[\delta(c_{LL}, \Delta\mu, 0) + \delta(c_{HL}, -\Delta\mu, \Delta\nu)]\} \\ &\quad + \alpha_{LL}\{P^{LL}(c_{LL}) - (1 - c_{LL})\mu_L\} \\ &\quad + \eta_{HL}\{c_{HL} - c_{HH}\} + \eta_{HH}\{c_{HH}\}, \end{aligned}$$

where the last but one term is the monotonicity constraint and the final term is the moral hazard constraints $c_{HH} \geq 0$, with Lagrange multipliers η_{Hj} ($j = L, H$).

The derivatives of the Lagrange function are as follows.

$$\frac{\partial \mathcal{L}}{\partial c_{HH}} = -\alpha_{HH}c_{HH}\nu_H - \eta_{HL} + \eta_{HH}, \quad (33)$$

$$\frac{\partial \mathcal{L}}{\partial c_{HL}} = \alpha_{HH}c_{HL}\Delta\nu - \alpha_{HL}c_{HL}\nu_L - \alpha_{LH}(1 - \varepsilon)[\Delta\mu - c_{HL}\Delta\nu] + \eta_{HL}, \quad (34)$$

$$\frac{\partial \mathcal{L}}{\partial c_{LH}} = -\alpha_{LH}c_{LH}\nu_H, \quad (35)$$

$$\frac{\partial \mathcal{L}}{\partial c_{LL}} = \alpha_H\Delta\mu + \alpha_{LH}[\varepsilon c_{LL}\Delta\nu + (1 - \varepsilon)\Delta\mu] - \alpha_{LL}c_{LL}\nu_L. \quad (36)$$

From (35) we immediately get that an optimal menu has

$$c_{LH} = 0. \quad (37)$$

Together with the monotonicity requirement $c_{HH} \geq c_{LH}$, this implies that also $c_{HH} = 0$. Setting (33) to zero then gives

$$\eta_{HL} = \eta_{HH}.$$

Equating (34) to zero yields

$$c_{HL}[\alpha_{HH}\Delta\nu - \alpha_{HL}\nu_L + \alpha_{LH}(1 - \varepsilon)\Delta\nu] = \alpha_{LH}(1 - \varepsilon)\Delta\mu - \eta_{HL}. \quad (38)$$

If the Lagrange function is concave in c_{HL} such that the square bracket term is negative, then $c_{HL} > 0$ would mean that the monotonicity condition $c_{HH} \leq c_{HL}$ is slack and that $\eta_{HL} = 0$. But then (38) prescribes a negative value for c_{HL} , a contradiction. On the other hand, if the square bracket term is positive, then total profit is convex in c_{HL} and the optimal coinsurance rate for HL is either zero or 1. If $c_{HL} = 1$, then by monotonicity also $c_{LL} = 1$. This configuration is however, clearly suboptimal: giving HL the same contract as HH and LH will

increase profits from HL and leave all other profits unaffected. We are therefore left with the only possibility that also $c_{HL} = 0$.

If the rent which LH obtains when mimicking HL is the same as when mimicking LL , then

$$\begin{aligned}\delta(c_{LL}, 0, \Delta\nu) &= \delta(0, -\Delta\mu, \Delta\nu) + \delta(c_{LL}, \Delta\mu, 0), \text{ or} \\ \delta(c_{LL}, -\Delta\mu, \Delta\nu) &= \delta(0, -\Delta\mu, \Delta\nu).\end{aligned}$$

This equality has two solutions. The trivial solution that $c_{LL} = 0$ would mean that all four types are pooled at a full insurance contract. But by an argument similar to that of Stiglitz (1977, proposition 3), it can be shown that the best pooling contract cannot provide complete insurance. Hence we are left with the other solution

$$c_{LL} = 2\frac{\Delta\mu}{\Delta\nu} = 2\frac{\Delta\mu}{\nu_L} \frac{x}{1-x}. \quad (39)$$

But as rearranging (40) gives

$$c_{LL} = \frac{[\alpha_H + \alpha_{LH}(1-\varepsilon)]\Delta\mu}{\alpha_{LL}\nu_L - \alpha_{LH}\varepsilon\Delta\nu}. \quad (40)$$

we can solve for the equilibrium value of ε :

$$\varepsilon = \frac{2x - (1 - \alpha_{LL})(1 + x)}{\alpha_{LH}(1 - x)} \quad (41)$$

It can easily be checked that $\varepsilon \rightarrow 1$ as $x \rightarrow \frac{1+\alpha_{LH}-\alpha_{LL}}{1+\alpha_{LH}+\alpha_{LL}}$, the previous lower bound for x . On the other hand $\varepsilon \rightarrow 0$ as $x \rightarrow \frac{1-\alpha_{LL}}{1+\alpha_{LL}}$, which is the new lower bound for x in this new regime. Checking whether the ignored incentive constraints are verified can be done with the help of figure 8 in the text. This proves part 3 proposition 1.

4. Once x falls below $\frac{1-\alpha_{LL}}{1+\alpha_{LL}}$, we enter a fourth regime where LH is no longer interested in mimicking LL , and the optimal coinsurance rate for LL is (40) evaluated at $\varepsilon = 0$:

$$c_{LL} = \frac{(\alpha_H + \alpha_{LH})\Delta\mu}{\alpha_{LL}\nu_L} = \frac{\Delta\mu}{\nu_L} \frac{1 - \alpha_{LL}}{\alpha_{LL}} \quad (42)$$

This proves the 4th part of proposition 1. ■

Proof of proposition 2.

This proof proceeds in a similar way to that of proposition 1.

1.. Starting with the situation where the binding incentive constraints are given as in figure 13, a similar reasoning then leads to the conclusion that $c_{HH} = c_{HL} = 0$.

2. Suppose that LL is excluded so that $c_{LL} = 1$. If HL is indifferent between the contracts C_{LL} and C_{LH} then

$$\begin{aligned} R^{LL} + \delta(1, \Delta\mu, 0) &= R^{LH} + \delta(c_{LH}, -\Delta\mu, \Delta\nu) \\ &= R^{LL} + \delta(1, 0, \Delta\nu) + \delta(c_{LH}, -\Delta\mu, \Delta\nu), \end{aligned}$$

or

$$[c_{LH} - 1]\Delta\mu = \frac{1}{2}[c_{LH}^2 - 1]\Delta\nu.$$

This has two solutions: $c_{LH} = 1$ and

$$c_{LH} = 2\frac{\Delta\mu}{\Delta\nu} - 1 = 2\frac{\Delta\mu}{\nu_L} \frac{x}{1-x} - 1. \quad (43)$$

2a. Suppose that LH is separated from LL and (43) holds. Using (21), and solving for λ then gives

$$\lambda = \frac{\alpha_{LH}}{\alpha_H} \frac{2\frac{\Delta\mu}{\nu_L} - \frac{1-x}{x}}{\frac{1-x}{x} - \frac{\Delta\mu}{\nu_L}} \frac{1}{1-x}. \quad (44)$$

If LH envies both LL and LH then we need $\lambda \in (0, 1)$. Solving for $\lambda \leq 1$ results in a quadratic expression for x :

$$-a(1+b)x^2 + (2a+ab+2b+1)x - (1+a) \leq 0, \quad (45)$$

with

$$a \stackrel{\text{def}}{=} \frac{\alpha_H}{\alpha_{LH}} \text{ and } b \stackrel{\text{def}}{=} \frac{\Delta\mu}{\nu_L}.$$

The *lhs* of (25) is increasing on the $[0,1]$ interval and has two roots, one above and one below unity. The latter is given by

$$\widehat{x} \stackrel{\text{def}}{=} 1 + \frac{1 + 2b - \sqrt{(1 + 2b)^2 + ab[ab + 2(3 + 2b)]}}{2a(1 + b)}.$$

On the other hand, solving $\lambda \geq 0$ for x gives

$$x \geq \left(1 + 2\frac{\Delta\mu}{\nu_L}\right)^{-1}.$$

Thus, for values of x in the interval $\left[\left(1 + 2\frac{\Delta\mu}{\nu_L}\right)^{-1}, \widehat{x}\right]$, *HL* will envy both *LL* and *LH*, and the optimal coinsurance rate for *LH* is (43).

2b. When $x \geq \widehat{x}$, *HL* only wants to mimic *LH*. Evaluating (21) at $\lambda = 1$, gives

$$c_{LH} = \frac{\Delta\mu}{\nu_L} \frac{\alpha_H \cdot x}{\alpha_{LH} + \alpha_H \cdot (1 - x)} \quad (46)$$

This expression is increasing in x . This means that there will be some value for x for which c_{LH} will become unity, meaning that *LH* and *LL* are pooled (at an exclusion contract). Solving for this x gives

$$x = \frac{1 - \alpha_{LL}}{\alpha_H} \frac{1}{1 + \frac{\Delta\mu}{\nu_L}}$$

This is the upper bound for regime 2, and the lower bound for regime 1.

3. For $x \leq \left(1 + 2\frac{\Delta\mu}{\nu_L}\right)^{-1}$, *HL* will no longer want to mimic *LH*. Evaluating (21) at $\lambda = 0$, gives $c_{LH} = 0$. The active incentive constraints are now as in figure 14. But because *LL* is excluded, the rent which *LH* obtains when mimicking *HL* is now $\delta(0, -\Delta\mu, \Delta\nu) + \delta(1, \Delta\mu, 0) = -\Delta\mu + \frac{1}{2}\Delta\nu$ while her rent when mimicking *LL* is $\delta(1, 0, \Delta\nu) = 0$, . Because $x \leq \left(1 + 2\frac{\Delta\mu}{\nu_L}\right)^{-1}$, the former rent is positive and *LH* will thus never mimic *LL* so that $\varepsilon = 0$. Expression (38) then reduces to

$$c_{HL}[\alpha_{HH}\Delta\nu - \alpha_{HL}\nu_L + \alpha_{LH}\Delta\nu] = \alpha_{LH}\Delta\mu - \eta_{HL}. \quad (47)$$

But a similar reasoning as in step 3 in the proof of proposition 2 leads to the conclusion that $c_{HL} = 0$.

This proves the 4th part of proposition 2. ■

Proof of proposition 3

Because the participation constraint for LL is always binding, $R_{LL} = 0$. HL always has an incentive to mimic LL , therefore

$$\begin{aligned} R_{HL} &= R_{LL} + \delta(c_{LL}, \Delta\mu, 0) \\ &= (1 - c_{LL})\Delta\mu. \end{aligned}$$

If LH envies in equilibrium LL , then

$$R_{LH} = R_{LL} + \delta(c_{LL}, 0, \Delta\nu).$$

while if LH envies only HL in equilibrium, then (using the fact that $c_{HL} = 0$)

$$\begin{aligned} R_{LH} &= R_{HL} + \delta(0, -\Delta\mu, \Delta\nu) \\ &= R_{LL} + \delta(c_{LL}, \Delta\mu, 0) + \delta(0, -\Delta\mu, \Delta\nu). \end{aligned}$$

LH 's rent is therefore

$$R_{LH} = \max\left\{\frac{1}{2}(1 - c_{LL}^2)\Delta\nu, -c_{LL}\Delta\mu + \frac{1}{2}\Delta\nu\right\}.$$

As HH always envies HL , her rent is (using the fact that $c_{HL} = 0$)

$$\begin{aligned} R_{HH} &= R_{HL} + \delta(c_{HL}, 0, \Delta\nu) \\ &= R_{LL} + \delta(c_{LL}, \Delta\mu, 0) + \delta(0, 0, \Delta\nu) \\ &= (1 - c_{LL})\Delta\mu + \frac{1}{2}\Delta\nu. \end{aligned}$$

This proves proposition 3. ■

Proof of proposition 4

Because the participation constraint for LL is always binding, $R_{LL} = 0$. If HL has an incentive to mimic LL , then as LL is excluded, $c_{LL} = 1$ and

$$R_{HL} = R_{LL} + \delta(1, \Delta\mu, 0) = 0.$$

But for $x > \hat{x}$, HL only has an incentive to mimic LH . Since for such high x -values LH envies LL , we have

$$\begin{aligned} R_{HL} &= R_{LH} + \delta(c_{LH}, \Delta\mu, -\Delta\nu) \\ &= R_{LL} + \delta(1, 0, \Delta\nu) + \delta(c_{LH}, \Delta\mu, -\Delta\nu) \\ &= (1 - c_{LH})\Delta\mu - \frac{1}{2}(1 - c_{LH}^2)\Delta\nu. \end{aligned}$$

To summarise:

$$R_{HL} = \max\{0, (1 - c_{LH})\Delta\mu - \frac{1}{2}(1 - c_{LH}^2)\Delta\nu\}.$$

If LH envies in equilibrium LL , then

$$R_{LH} = R_{LL} + \delta(1, 0, \Delta\nu) = 0,$$

while if LH envies only HL in equilibrium (which happens when $x < \left(1 + 2\frac{\Delta\mu}{\nu_L}\right)^{-1}$) we have (using that $c_{HL} = 0$)

$$\begin{aligned} R_{LH} &= R_{HL} + \delta(0, -\Delta\mu, \Delta\nu) \\ &= R_{LL} + \delta(1, \Delta\mu, 0) + \delta(0, -\Delta\mu, \Delta\nu) \\ &= -\Delta\mu + \frac{1}{2}\Delta\nu > 0. \end{aligned}$$

LH 's rent is therefore

$$R_{LH} = \max\{0, -\Delta\mu + \frac{1}{2}\Delta\nu\}.$$

As HH always envies HL , her rent is (using the fact that $c_{HL} = 0$)

$$\begin{aligned} R_{HH} &= R_{HL} + \delta(c_{HL}, 0, \Delta\nu) \\ &= R_{LL} + \delta(1, \Delta\mu, 0) + \delta(0, 0, \Delta\nu) \\ &= \frac{1}{2}\Delta\nu. \end{aligned}$$

This proves proposition 4. ■