9. STRUCTURAL VAR: THEORY

1 Structural Vector Autoregressions

Impulse response functions are interpreted under the assumption that *all the other shocks are held constant*. However in the Wold representation the shocks are not orthogonal. So the assumption is not very realistic!.

This is why we need Structural VAR in order to perform policy analysis. Ideally we would like to have

1) orthogonal shock

2) shocks with economic meaning (technology, demand, labor supply, monetary policy etc.)

1.1 Statistical Orthogonalizations

There are two easy way to orthogonalize shocks.

1) Cholesky decomposition

2) Spectral Decomposition

1.2 Cholesky decomposition

Let us consider the matrix Ω. The Cholesky factor, S, of Ω is defined as the unique lower triangular matrix such that $SS' = \Omega$. This implies that we can rewrite the VAR in terms of orthogonal shocks $\eta_t = S^{-1}\epsilon_t$ with identity covariance matrix

$$A(L)Y_t = S\eta_t$$

Impulse response to orthogonalized shocks are found from the MA representation

$$\begin{aligned} Y_t &= C(L)S\eta_t \\ &= \sum_{j=0}^{\infty} C_j S\eta_{t-j} \end{aligned} \tag{1}$$

where C_jS has the interpretation

$$\frac{\partial Y_{t+j}}{\partial \eta_t} = C_jS \tag{2}$$

That is, the row i, column k element of C_jS identifies the consequences of a unit increase in η_{kt} for the value of the ith variable at time $t+j$ holding all other shocks constant.

1.3 Spectral Decomposition

Let V and be a matrix containing the eigenvectors of Ω and Λ a diagonal matrix with the eigenvalues of Ω on the main diagonal. Then we have that $V\Lambda V' = \Omega$. This implies that we can rewrite the VAR in terms of orthogonal shocks $\xi_t = (VD^{1/2})^{-1}\epsilon_t$ with identity covariance matrix

$$A(L)Y_t = VD^{1/2}\xi$$

Impulse response to orthogonalized shocks are found from the MA representation

$$\begin{aligned} Y_t &= C(L)VD^{1/2}\xi_t \\ &= \sum_{j=0}^{\infty} C_j S \eta_{t-j} \end{aligned} \tag{3}$$

where $C_j VD^{1/2}$ has the interpretation

$$\frac{\partial Y_{t+j}}{\partial \xi_t} = C_j VD^{1/2} \tag{4}$$

That is, the row i, column k element of $C_j VD^{1/2}$ identifies the consequences of a unit increase in ξ_{kt} at date t for the value of the ith variable at time $t+j$ holding all other shocks constant.

Problem: what is the economic interpretation of the orthogonal shocks? What is the economic information contained in the impulse response functions to orthogonal shocks?

Except for special cases not clear.

1.4 The Class of Orthonormal Representations

From the class of invertible MA representation of Y_t we can derive the class of orthonormal representation, i.e. the class of representations of Y_t in term of orthonormal shocks. Let H any orthogonal matrix, i.e. $HH' = H'H = I$. Defining $w_t = (SH)^{-1}\epsilon_t$ we can recover the general class of the orthonormal representation of Y_t

$$
\begin{aligned}
Y_t &= C(L)SHw_t \\
&= F(L)w_t
\end{aligned}
$$

where $F(L) = C(L)SH$ and $w_t \sim WN$ with

$$
\begin{aligned}
E(w_t w_t') &= E((SH')^{-1}\epsilon_t\epsilon_t'(SH')^{-1\prime}) \\
&= HS^{-1}E(\epsilon_t\epsilon_t')H'(S')^{-1} \\
&= HS^{-1}\Omega H'(S')^{-1} \\
&= HS^{-1}SS'(S')^{-1}H' \\
&= I
\end{aligned}
$$

Problem: H can be any, so how should we choose one?

2 The Identification Problem

• Identifying the VAR means fixing a particular matrix H, i.e. choosing one particular representation of Y_t in order to recover the structural shocks from the VAR innovations

• Therefore structural economic shocks are linear combinations of the VAR innovations.

• In order to choose a matrix H we have to fix $n(n-1)/2$ parameters since there is a total of n^2 parameters and a total of $n(n+1)/2$ restrictions implied by orthonormality.

• Use economic theory in order to derive some restrictions on the effects of some shock on a particular variables to fix the remaining $n(n-1)/2$.

2.1 Zero restrictions: contemporaneous restrictions

• An identification scheme based on zero contemporaneous restrictions is a scheme which imposes restrictions to zero on the matrix F_0, the matrix of the impact effects.

Example. Let us consider a bivariate VAR. We have a total of $n^2 = 4$ parameters to fix. $n(n+1)/2 = 3$ are pinned down by the ortnonormality restrictions so that there are $n(n-1)/2 = 1$ free parameters. Suppose that the theory tells us that shock 2 has no effect on impact (contemporaneously) on Y_1 equal to 0, that is $F_0^{12} = 0$. This is the additional restriction that allows us to identify the shocks. In particular we will have the following restrictions:

$$HH' = I$$

$$S_{11}H_{12} + S_{12}H_{22} = 0$$

Since $S_{12} = 0$ the solution is $H_{11} = H_{22} = 1$ and $H_{12} = H_{21} = 0$.

• A common identification scheme is the Cholesky scheme (like in this case). This implies setting $H = I$. Such an identification scheme creates a recursive contemporaneous ordering ordering among variables since S^{-1} is triangular.

• This means that any variable in the vector Y_t does not depend contemporanously on the variables ordered after.

- Results depend on the particular ordering of the variables.

2.2 Zero restrictions: long run restrictions

• An identification scheme based on zero long run restrictions is a scheme which imposes restrictions on the matrix $F(1) = F_0 + F_1 + F_2 + ...$, the matrix of the long run coefficients.

Example. Again let us consider a bivariate VAR. We have a total of $n^2 = 4$ parameters to fix. $n(n+1)/2 = 3$ are pinned down by the ortnonormality restrictions so that there are $n(n-1)/2 = 1$ free parameters. Suppose that the theory tells us that shock 2 does not affect Y_1 in the long run, i.e. $F_{12}(1) = 0$. This is the additional restriction that allows us to identify the shocks. In particular we will have the following restrictions:

$$
\begin{aligned}
HH' &= I \\
D_{11}(1)H_{12} + D_{12}(1)H_{22} &= 0
\end{aligned}
$$

where $D(1) = C(1)S$ represents the long run effects of the Cholesky shocks.

2.3 Signs restrictions

• The previous two examples yield just identification in the sense that the shockis uniquely identified, there exists a unique matrix H yielding the structural shocks.

• Sign identification is based on qualitative restriction involving the sign of some shocks on some variables. In this case we will have sets of consistent impulse response functions.

Example. Again let us consider a bivariate VAR. We have a total of $n^2 = 4$ parameters to fix. $n(n+1)/2 = 3$ are pinned down by the ortnonormality restrictions so that there are $n(n-1)/2 = 1$ free parameters. Suppose that the theory tells us that shock 2, which is the interesting one, produce a positive effect on Y_1 for k periods after the shock $F_j^{12} > 0$ for $j = 1, ..., k$. We will have the following restrictions:

$$\begin{aligned} HH' = I \\ S_{11}H_{12} + S_{12}H_{22} &> 0 \\ D_{j,12}H_{12} + D_{j,22}H_{22} &> 0 \quad j = 1, ..., k \end{aligned}$$

where $D_j = C_jS$ represents the effects at horizon j.

• In a classical statistics approach this delivers not exact identification since there can be many H consistent with such a restriction. That is for each parameter of the impulse response functions we will have an admissible set of values.

• Increasing the number of restrictions can be helpful in reducing the number of H consistent with such restrictions.

2.4 Parametrizing H

• A useful way to parametrize the matrix H in order to include orthonormality restrictions is using rotation matrices. Let us consider the bivariate case. A rotation matrix in this case is the unity matrix

$$H = \begin{pmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{pmatrix}$$

• Note that such a matrix incorporates the orthonormality conditions. The parameter θ will be found by imposing the additional economic restriction.

• In general the rotation matrix will be found as the product of $n(n-1)/2$ rotation matrices. For the case of three shocks the rotation matrix can be found as the product of the following three matrices

$$\begin{pmatrix} cos(\theta_1) & sin(\theta_1) & 0 \\ -sin(\theta_1) & cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} cos(\theta_2) & 0 & sin(\theta_2) \\ 0 & 1 & 0 \\ -sin(\theta_2) & 0 & cos(\theta_2) \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos(\theta_3) & sin(\theta_3) \\ 0 & -sin(\theta_3) & cos(\theta_3) \end{pmatrix}$$

Example. Suppose that $n = 2$ and the restriction we want to impose is that the effect of the first shock on the second variable has a positive sign, i.e.

$$S_{21}H_{11} + S_{22}H_{21} > 0$$

Using the parametrization seen before the restriction becomes

$$S_{21}\cos(\theta) - S_{22}\sin(\theta) > 0$$

Which implies

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} < \frac{S_{21}}{S_{22}}$$

If $S_{21} = 0.5$ and $S_{22} = 1$ then all the impulse response fcuntions obtained with $\theta < \text{atan}(0.5)$ satisfy the rstriction and should be kept.

2.5 Partial Identification

• In many cases we might be interested in identifying just a single shock and not all the n shocks.

• Since the shock are orthogonal we can also partially identify the model, i.e. fix just one (or a subset of) column of H. In this case what we have to do is to fix $n-1$ elements of H, all but one elements of a column of the identifying matrix. The additional restriction is provided by the norm of the vector equal one.

Example Suppose $n = 3$. We want to identify a single shock using the restriction that such shock has no effects on the first variable on impact a positive effect on the second variable and negative on the third variable.

First of all we notice that the first column of the product of orthogonal matrices seen before is

$$H_1 = \begin{pmatrix} \cos(\theta_1)\cos(\theta_2) \\ -\sin(\theta_1)\cos(\theta_2) \\ -\sin(\theta_2) \end{pmatrix}$$

therefore we have that the impact effects ofthe first shock are given by

$$\begin{pmatrix} S_{11} & 0 & 0 \\ S_{21} & S_{22} & 0 \\ S_{31} & S_{32} & S_{33} \end{pmatrix} \begin{pmatrix} \cos(\theta_1)\cos(\theta_2) \\ -\sin(\theta_1)\cos(\theta_2) \\ -\sin(\theta_2) \end{pmatrix}$$

To implement the first restriction we can set $\theta_1 = \pi/2$, i.e. $\cos(\theta_1) = 0$. This imples that

$$\begin{pmatrix} S_{11} & 0 & 0 \\ S_{21} & S_{22} & 0 \\ S_{31} & S_{32} & S_{33} \end{pmatrix} \begin{pmatrix} 0 \\ -\cos(\theta_2) \\ -\sin(\theta_2) \end{pmatrix}$$

The second restriction implies that

$$-S_{22}\cos(\theta_2) > 0$$

and the third

$$-S_{32}\cos(\theta_2) - S_{33}\sin(\theta_2) < 0$$

All the values of θ_2 satisfying the two restrictions yield impulse response functions consistent with the identification scheme.

2.6 Variance Decomposition

• The second type of analysis which is usually done in SVAR is the variance decomposition analysis.

• The idea is to decompose the total variance of a time series into the percentages attributable to each structural shock.

• Variance decomposition analysis is useful in order to address questions like "What are the sources of the business cycle?" or "Is the shock important for economic fluctuations?".

Let us consider the MA representation of an identified SVAR

$$Y_t = F(L) w_t$$

The variance of Y_{it} is given by

$$\begin{aligned}
\mathrm{var}(Y_{it}) &= \sum_{k=1}^{n} \sum_{j=0}^{\infty} F_{ik}^{j2} \mathrm{var}(w_{kt}) \\
&= \sum_{k=1}^{n} \sum_{j=0}^{\infty} F_{ik}^{j2}
\end{aligned}$$

where $\sum_{j=0}^{\infty} F_{ik}^{j2}$ is the variance of Y_{it} generated by the kth shock. This implies that

$$\frac{\sum_{j=0}^{\infty} F_{ik}^{j2}}{\sum_{k=1}^{n} \sum_{j=0}^{\infty} F_{ik}^{j2}}$$

is the percentage of variance of Y_{it} explained by the kth shock.

It is also possible to study the of the series explained by the shock at different horizons, i.e. short vs. long run. Consider the forecast error in terms of structural shocks. The horizon h forecast error is given by

$$Y_{t+h} - Y_{t+h|t} = F_0 w_{t+1} + F_2 w_{t+2} + ... + F_k w_{t+h}$$

the variance of the forecast error of the ith variable is thus

$$\begin{aligned} \mathrm{var}(Y_{it+h} - Y_{it+h|t}) &= \sum_{k=1}^{n} \sum_{j=0}^{h} F_{ik}^{j2} \mathrm{var}(w_{kt}) \\ &= \sum_{k=1}^{n} \sum_{j=1}^{h} F_{ik}^{j2} \end{aligned}$$

Thus the percentage of variance of Y_{it} explained by the kth shock is

$$\frac{\sum_{j=0}^{h} F_{ik}^{j2}}{\sum_{k=1}^{n} \sum_{j=1}^{h} F_{ik}^{j2}}$$