The Uniform Allocation Rule and the Nucleolus

M. Angeles de Frutos*
Boston University

Jordi Massó*
Universitat Autònoma de Barcelona

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Abstract

We show the relationship between the Uniform Allocation Rule for solving the division problem with single-peaked preferences studied by Sprumont (1991) and the Nucleolus of its associated cooperative game à la Aumann and Maschler.

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1 Introduction

The main purpose of this paper is to study the relationship between the Uniform Allocation Rule and the Nucleolus. The Uniform Allocation Rule is considered the best solution to solve the problem of allocating an infinitely divisible good among a group of agents whose preferences are single-peaked. This rule gives to everyone his preferred share within the limits of an upper and a lower bound determined by the feasibility condition that the shares add up to the amount to be divided. The Uniform Allocation Rule has been characterized from several points of view (see, for example, Sprumont[4], Thomson[5], De Frutos and Massó[2], Otten, Peters and Volij [3]).

Following Aumann and Maschler[1] we define, for every division problem, a division TU cooperative game where the set of players will be the agents involved in the distribution problem and the value of a coalition will be the smallest amount that its members will have to contribute in any efficient allocation. This amount will depend on whether the vector of preferred contributions is larger or smaller than the total required. If it is larger, the value of a coalition will be the amount remaining after fulfilling the preferred contributions of the members of the complementary coalition; if it is smaller, the value will just be the sum of the preferred contributions of its members, since that will be the smallest amount they will jointly get.

The cooperative-game set up allows us to find a precise relationship between the Uniform Allocation Rule and the Nucleolus. Thomson[6] had already suggested this by connecting the Uniform Allocation Rule with the rule proposed in the Talmud for the allocation of conflicting claims studied by Aumann and Maschler [1].

The paper is organized as follows. Section 2 contains the main definitions for the division problem and Section 3 provides the main results.

2 Preliminaries

The model, and much of the notation, follow Sprumont[4]. There is one (normalize unit) of some perfectly divisible good that has to be allocated among a set $N = \{1, ..., n\}$ of agents. The preference of every agent $i \in N$ is represented by a complete preordering on $[0, 1]$, denoted R_i. These preference relations are assumed to be single-peaked: for each i, there is $x_i^* \in [0, 1]$ such that for all z, $y \in [0, 1]$ if $y < z \leq x_i^*$ or $x_i^* \geq z > y$ then zP_iy (P_i denotes the strict preference relation associated with R_i). We call x_i^* the peak of R_i and often, to emphasize the dependence upon the preference preordering, we write

$x^*(R_i)$. The symbol $R = (R_i)_{i \in N}$ denotes the vector of announced preferences, and $x^*(R)$ stands for the vector of peaks associated with R. The set of all single-peaked preferences on $[0, 1]$ will be denoted by $\mathcal{R}$ and its elements will be called preference profiles.

A *division problem* will be a pair $(R, 1)$ where $R = (R_1, ..., R_n)$ is the vector of announced single-peaked preferences, and 1 is the amount to be shared.

A *feasible allocation* is a vector $x = (x_i)_{i \in N} \in \Re^n_+$ such that $\sum_{i \in N} x_i = 1$. The set of feasible allocations is denoted by $\triangle^n$.

A *solution* is a mapping Φ which assigns to every admissible preference profile R a unique feasible allocation, $\Phi(R)$, that is, $\Phi : \mathcal{R}^n \to \triangle^n$.

A solution is called *"tops only"* if it is constant on profiles having the same peak; that is, if R and R' are such that $x^*(R_i) = x^*(R'_i)$ for every $i \in N$, then $\Phi(R) = \Phi(R')$. Therefore if Φ is tops-only we can associate to it a function $\Lambda : [0, 1]^n \to \triangle^n$ having the property that $\Lambda(x^*(R)) = \Phi(R)$ for every R.

The main result of Sprumont is that the properties of strategy-proofness, efficiency, and anonymity together characterize a unique allocation rule. In the present context, efficiency simply requires that if the preferred shares add up to more (less) than the amount required, then no agent should get more (less) than his preferred share. He also shows that, alternatively, the anonymity axiom may be replaced by envy-freeness. This unique rule is the Uniform Allocation Rule. The formal definitions are taken from Sprumont[4] and they are as follows.

Efficiency: A solution Φ is efficient if for all $R \in \mathcal{R}^n$,

$$[\textstyle\sum_{i \in N} x^*(R_i) \le 1] \Longrightarrow [\Phi_i(R) \ge x^*(R_i) \text{ for all } i \in N], \text{ and}$$

$$[\textstyle\sum_{i \in N} x^*(R_i) \ge 1] \Longrightarrow [\Phi_i(R) \le x^*(R_i) \text{ for all } i \in N].$$ [1]

Anonymity: A solution Φ is anonymous if for all permutations Π of N, all $R \in \mathcal{R}^n$, $\Phi_i(R^\Pi) = \Phi_{\Pi(i)}(R)$, where $R^\Pi = (R_{\Pi(i)})_{i \in N}$.

Strategy-proofness: A solution Φ is strategy-proof if for all $R \in \mathcal{R}^n$, all $i \in N$, and all $R'_i \in \mathcal{R}$, $\Phi_i(R_i, R_{-i}) R_i \Phi_i(R'_i, R_{-i})$.

Definition 2.1 *(Sprumont[4]) The Uniform Allocation Rule Φ^* is defined as follows:*

$$\Phi^*_i(R) = \begin{cases} \min\{x^*(R_i), \lambda(R)\} & \textit{if } \sum_{i \in N} x^*(R_i) \ge 1 \\ \max\{x^*(R_i), \mu(R)\} & \textit{if } \sum_{i \in N} x^*(R_i) \le 1 \end{cases},$$

for all $i \in N$, where $\lambda(R)$ solves the equation $\sum_{i \in N} \min\{x^*(R_i), \lambda(R)\} = 1$ and $\mu(R)$ solves the equation $\sum_{i \in N} \max\{x^*(R_i), \mu(R)\} = 1$.

[1] It is immediate to see that this definition coincides with Pareto Efficiency under the assumption of single-peakedness.

From the definition it is clear that the Uniform Allocation Rule gives full satisfaction to some of the agents (those with low preferred contributions if $\sum_{i\in N} x^*(R_i) \geq 1$ and those with high preferred contributions if $\sum_{i\in N} x^*(R_i) \leq 1$) at the expense of the others. It is clear, too, that it is a tops-only rule.

Following Aumann and Maschler[1], we will associate to every division problem a characteristic function in the following way.

Definition 2.2 *Let $x^* =(x_1^*, ...x_n^*)$ be a vector of optimal contributions. The Division Cooperative Game is defined as a pair (N, v_{x^*}) where $N = \{1, ..., n\}$ is the set of individuals involved in the division problem and v_{x^*} is the characteristic function defined as follows:*

(a) If $\sum_{i=1}^n x_i^* \geq 1$ then $v_{x^*}(S) = \max\{0, 1 - \sum_{i\in N/S} x_i^*\}$ for every $S \subseteq N$.

(b) If $\sum_{i=1}^n x_i^* \leq 1$ then $v_{x^*}(S) = \begin{cases} \sum_{i\in S} x_i^* & \text{if } S \subset N \\ 1 & \text{if } S = N \end{cases}$.

De Frutos and Massó[2] contains a justification and interpretation of this construction.

Next, for completeness, we provide the formal definition of the Nucleolus.

Definition 2.3 *Let (N, v) be a cooperative T.U. game and let $x \in \Re^n$. For $S \subseteq N$, define $e(x; S) = v(S) - \sum_{i\in S} x_i$ as the excess of coalition S at x. Denote $\theta(x) = (e(x; S_1), e(x; S_2), ..., e(x; S_{2^n}))$, where $e(x; S_k) \geq e(x; S_{k+1})$, $k = 1, ..., 2^n - 1$. Let $X = \{x \in \Re^n | \sum_{i\in N} x_i = v(N)\}$.*

The Nucleolus is defined by $Nu(N, v) = \{x \in X \mid \theta(x) >_l \theta(y)$ for all $y \in X \setminus x\}$ where $>_l$ is the lexicographic order.

3 The Uniform Allocation Rule and the Nucleolus

The Nucleolus has been proposed as a solution concept to solve division problems, for example, in bankruptcy settlements. The egalitarian philosophy underlies this solution concept. De Frutos and Massó[2] have shown that the Uniform Allocation Rule also uses an egalitarian philosophy. In this Section we show the relationship between these two solution concepts.

Consider a division problem where the total amount to be shared is 1. Given a vector of optimal contributions $x^*(R)$ we will denote by $Nu(x^*, 1)$ the Nucleolus of the division game constructed from x^* and 1. Next proposition calculates the Nucleolus shares of a cooperative division game. Its simple proof is omitted.

Proposition 3.1 *The Nucleolus of the division problem proposes the following shares:*

(a) If $\sum_{i \in N} x_i^* \geq 1$*, it distinguishes between two cases:*

(a.1) If $\sum_{i \in N} x_i^* \geq 2$ *then* $Nu_i(x^*, 1) = \min\{x_i^*/2, \lambda\}$ *for all* $i \in N$*, where* λ *solves the equation* $\sum_{i \in N} \min\{x_i^*/2, \lambda\} = 1$.

(a.2) If $\sum_{i \in N} x_i^* \leq 2$ *then* $Nu_i(x^*, 1) = x_i^* - \min\{x_i^*/2, \mu\}$ *for every* $i \in N$*, where* μ *solves the equation* $\sum_{i \in N} \min\{x_i^*/2, \mu\} = \sum_{i \in N} x_i^* - 1$.

(b) If $\sum_{i \in N} x_i^* \leq 1$*, then* $Nu_i(x^*, 1) = x_i^* + \dfrac{1 - \sum_{i \in N} x_i^*}{n}$ *for all* $i \in N$.

Our next proposition shows the relationship between the Nucleolus and the Uniform Allocation Rule.

Proposition 3.2 *Let* $(x^*, 1)$ *be a division problem. We distinguish between two cases:*

(a) If $\sum_{i \in N} x_i^* \geq 1$ *then* $\Phi^*(x^*) = 2Nu(x^*; \frac{1}{2})$.

(b) If $\sum_{i \in N} x_i^* \leq 1$ *then* $\Phi^*(x^*) = 1 - 2Nu(1 - x^*; \frac{n-1}{2})$.

Proof: (a) Let $\sum_{i \in N} x_i^* \geq 1$.

We know that the Nucleolus shares will be $Nu_i(x^*; 0.5) = \min\{x_i^*/2, \lambda\}$.

For this case $2Nu(x^*; 0.5) = \min\{x_i^*, \theta\}$, with $\theta = 2\lambda$. Since θ solves the equation $\sum_{i \in N} \min\{x_i^*, \theta\} = 1$, we get $2Nu(x^*; 0.5) = \Phi^*(x^*)$.

(b) Let $\sum_{i \in N} x_i^* \leq 1$.

$Nu(1 - x^*; \frac{n-1}{2}) = \min\{(1 - x_i^*)/2, \lambda\}$ since $\sum_{i \in N}(1 - x_i^*) = n - \sum_{i \in N} x_i^* > n - 1$. Therefore $2Nu(1 - x^*, (n-1)/2) = \min\{1 - x_i^*, \phi\}$ with $\phi = 2\lambda$.

We have to show that $1 - \min\{1 - x_i^*, \phi\} = \max\{x_i^*, \mu\}$ for every $i \in N$, for any μ that solves the equation $\sum_{i \in N} \max\{x_i^*, \mu\} = 1$, and for any ϕ that solves the equation $\sum_{i \in N} \min\{1 - x_i^*, \phi\} = n - 1$. To prove it, take $\mu = 1 - \phi$.

- If $\max\{x_i^*, \mu\} = x_i^*$, then $1 - \min\{1 - x_i^*, \phi\} = 1 - \min\{1 - x_i^*, 1 - \mu\} = 1 - (1 - x_i^*) = x_i^*$ which is $\Phi^*(x^*)$.

- If $\max\{x_i^*, \mu\} = \mu$, then $1 - \min\{1 - x_i^*, \phi\} = 1 - \min\{1 - x_i^*, 1 - \mu\} = \mu$ which is also $\Phi^*(x^*)$. ■

References

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