More on the Uniform Allocation Rule: Equality and Consistency

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Abstract

We study the distributive and consistency properties of the Uniform Allocation Rule for solving the division problem with single-peaked preferences studied by Sprumont (1991). We show that under efficiency, strategy-proofness and anonymity can be replaced by Lorenz dominance. We also characterize the Uniform Allocation Rule as the unique solution satisfying efficiency, consistency and the Upper-Bound Rationing property for two agents.

Resumen

En este trabajo estudiamos las propiedades distributivas y de consistencia de la Regla de Asignación Uniforme caracterizada por Sprumont (1991) como solución a los problemas de división con preferencias unimodales. Demostramos que bajo el supuesto de eficiencia, los axiomas de anonimidad y de no manipulabilidad pueden ser reemplazados por el axioma de dominación en el sentido de Lorenz. También demostramos que dicha regla es la única solución que satisface eficiencia, consistencia y una cierta propiedad de acotación en el racionamiento para dos agentes.

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1. Introduction

We study the problem of allocating an infinitely divisible good among a group of agents whose preferences are single-peaked. Each agent has a preferred consumption level; if he moves away from it he will be worse off, but the sum of the individual preferred consumptions may be greater (smaller) than the amount to be divided. The question is how to achieve a division taking into account the agents' preferences.

This model was initially considered by Sprumont[13]. He offered the two following interpretations. The first one is distribution of goods at disequilibrium prices: consider an economy with two commodities which are allocated via prices. However, prices are not necessarily in equilibrium. If distribution must take place, a rationing rule has to be defined. When preferences are strictly convex, then, they become single-peaked, when restricted to the budget constraints. The second interpretation is that of a group of agents who must supply a quantity of labor in order to complete some task. If workers are paid an hourly wage and their disutility of labor is concave, then their induced preferences on the labor they supply are single-peaked. There are many other situations that fit with this model, for example, the allocation of a commodity when preferences become satiated at some point and free disposal is not allowed. A common characteristic of all these examples is that the preferred shares may not be compatible: they might add up to more or less than the total amount required or needed.

Within this framework Sprumont established the existence of a unique efficient, anonymous and strategy-proof solution which he named the Uniform Allocation Rule. It gives to everyone his preferred share within the limits of an upper and a lower bound determined by the feasibility condition that the shares add up to one. He also proved that the anonymity axiom may be replaced by the no-envy one.

The Uniform Allocation Rule has also been broadly studied by Thomson[14] from the normative or axiomatic viewpoint. His main results are two. He first showed that the Uniform Allocation Rule is the unique solution satisfying non-envy, efficiency, consistency and continuity to changes in the amount to be divided. He also showed that non-envy may be replaced by individual rationality from equal division. He states that the Uniform Allocation Rule can wholeheartedly be advocated as the best solution to the problem of fair allocation in economies with single-peaked preferences.

Here our focus is also normative. We would like to understand, in Sprumont's model, the powerful distributive implications of the strategy-proofness axiom. Indeed, this axiom helps to single out the Uniform Allocation Rule, when combined with anonymity and efficiency, which seem very weak in distributive terms. Several examples have shown that there exists a tension, on taking collective choices, between efficiency and strategy-proofness. Multidimensional voting schemes and exchange economies are good examples of it.[1] We found this division problem interesting because it allows to meet both properties simultaneously. Our first result (Theorem 3.1) is that one can replace strategy-proofness and anonymity, under efficiency, by the axiom of Lorenz dominance which has a clear distributive content. This axiom states that the solution should select the allocation that Lorenz dominates all the other efficient allocations. We want to emphasize that even though Lorenz dominance is a partial order, we obtain a maximal element in this case. This is related with the Egalitarian Solution proposed by Dutta and Ray[6].

We consider a "characteristic-like" function derived from the division problem and using this cooperative framework we show that the Uniform Allocation Rule coincides with the Egalitarian Solution. We say "characteristic-like" function to warn the reader that we do not obtain a cooperative game with transferable utility for two reasons. First, coalitions cannot make treats, they do not have the power to get away and block proposed allocations. Second, the good that agents can transfer among themselves is time units of work of which they may be satiated. In this "characteristic-like" function the value of a coalition, given a preference profile, will indicate the smallest amount of work that a coalition can guarantee in any efficient allocation. The cooperative game defined by this characteristic function is convex and its core coincides with the set of efficient allocations. Surprisingly, in a recent paper, Otten, Peters and Volij [10] prove that the Uniform Allocation Rule coincides with the Lexicographic Egalitarian Solution of an auxiliary bargaining problem.

Our next set of results shows, using also this cooperative framework, that the Uniform Allocation Rule has strong consistency properties. It is consistent under all possible definitions of reduced game. Moreover, it satisfies the converse consistency property. We show that consistency, efficiency and the Upper-Bound Rationing property for two agents single out the Uniform Allocation Rule.

The paper is organized as follows. Section 2 contains the main definitions for the division problem. Section 3 provides the main results related to the distributive aspects of the Uniform Allocation Rule. Finally, Section 4 contains those related to the consistency axiom.

2. The Division Problem and the Uniform Allocation Rule

The model, and much of the notation, follow Sprumont[13]. There are M units of some perfectly divisible good that has to be allocated among a set $N = \{1, ..., n\}$ of agents. The preference of every agent $i \in N$ is represented by a complete preordering on $[0, M]$,

[1] See, for instance, Barberà, Sonnenschein and Zhou[5], Barberà, Gul and Stacchetti[2], Barberà, Massó and Neme[4] and Barberà and Jackson[3].

denoted R_i. These preference relations are assumed to be single-peaked: for each i, there is $x_i^* \in [0, M]$ such that for all x, $x' \in [0, M]$ if $x' < x \leq x_i^*$ or $x_i^* \geq x > x'$ then xP_ix' (P_i denotes the strict preference relation associated with R_i). We call x_i^* the peak of R_i and often, to emphasize the dependence upon the preference preordering, we write $x^*(R_i)$. The symbol $R = (R_i)_{i \in N}$ denotes the vector of announced preferences, and $x^*(R)$ stands for the vector of peaks associated with R. The set of all single-peaked preferences on $[0, M]$ will be denoted by $\mathcal{R}$ and its elements will be called preference profiles.

A *division problem* will be a pair (R, M) where $R = (R_1, ..., R_n)$ is the vector of announced single-peaked preferences, and M is the amount to be shared. From now on, and except in Section 4, we will normalize M to one.

A *feasible allocation* is a vector $x = (x_i)_{i \in N} \in \Re_+^n$ such that $\sum_{i \in N} x_i = 1$. Therefore, the set of feasible allocations is the n-dimensional simplex, denoted by $\triangle^n$.

A *solution* is a mapping Φ which assigns to every admissible preference profile R a unique feasible allocation, $\Phi(R)$, that is, $\Phi : \mathcal{R}^n \rightarrow \triangle^n$.

A solution is called "*tops-only*" if it is constant on profiles having the same peak; that is, if R and R' are such that $x^*(R_i) = x^*(R_i')$ for every $i \in N$, then $\Phi(R) = \Phi(R')$. Therefore if Φ is tops-only we can associate to it a function $\Lambda : [0, 1]^n \rightarrow \triangle^n$ having the property that $\Lambda(x^*(R)) = \Phi(R)$ for every R.

The main result of Sprumont is that the properties of strategy-proofness, efficiency, and anonymity together characterize a unique allocation rule. In the present context, efficiency simply requires that if the preferred shares add up to more (less) than one, then no agent should get more (less) than his preferred share. He also shows that, alternatively, the anonymity axiom may be replaced by non-envy. This unique rule is the Uniform Allocation Rule. The formal definitions are taken from Sprumont[13] and they are as follows.

Efficiency: A solution Φ is efficient if for all $R \in \mathcal{R}^n$,

$$[\textstyle\sum_{i \in N} x^*(R_i) \leq 1] \Longrightarrow [\Phi_i(R) \geq x^*(R_i) \text{ for all } i \in N], \text{ and}$$

$$[\textstyle\sum_{i \in N} x^*(R_i) \geq 1] \Longrightarrow [\Phi_i(R) \leq x^*(R_i) \text{ for all } i \in N].$$ [2]

Anonymity: A solution Φ is anonymous if for all permutations Π of N, all $R \in \mathcal{R}^n$, $\Phi_i(R^\Pi) = \Phi_{\Pi(i)}(R)$, where $R^\Pi = (R_{\Pi(i)})_{i \in N}$.

Strategy-proofness: A solution Φ is strategy-proof if for all $R \in \mathcal{R}^n$, all $i \in N$, and all $R_i' \in \mathcal{R}$, $\Phi_i(R_i, R_{-i}) R_i \Phi_i(R_i', R_{-i})$.

[2] It is immediate to see that this definition coincides with Pareto Efficiency under single-peakedness.

Definition 2.1. *(Sprumont[13]) The Uniform Allocation Rule Φ^* is defined as follows:*

$$\Phi_i^*(R) = \begin{cases} \min\{x^*(R_i), \lambda(R)\} & \text{if } \sum_{i\in N} x^*(R_i) \geq 1 \\ \max\{x^*(R_i), \mu(R)\} & \text{if } \sum_{i\in N} x^*(R_i) \leq 1 \end{cases},$$

for all $i \in N$, where $\lambda(R)$ solves the equation $\sum_{i\in N} \min\{x^*(R_i), \lambda(R)\} = 1$ and $\mu(R)$ solves the equation $\sum_{i\in N} \max\{x^*(R_i), \mu(R)\} = 1$.

From the definition it is clear that the Uniform Allocation Rule gives full satisfaction to some of the agents (those with low preferred contributions if $\sum_{i\in N} x^*(R_i) \geq 1$ and those with high preferred contributions if $\sum_{i\in N} x^*(R_i) \leq 1$) at the expense of the others. It is clear, too, that it is a tops-only rule.

3. A Characterization with a Distributive Axiom

The main distributive concern is equality. Two criteria have been widely accepted as embodying a set of minimal ethical judgments that "should" be made in carrying out inequality comparisons. They are the leximin and the Lorenz criteria. The distributive concern, together with efficiency, implies that the outcome of the rule should be either the Lorenz or the leximin dominant element from within the set of efficient shares. We show in this section that, under efficiency, the Lorenz criterion chooses the allocation selected by the Uniform Allocation Rule.[3]

The Lorenz dominance criterion has been applied by Dutta and Ray[6] and [7] to define the so called Egalitarian Solution. This is a solution concept for transferable utility games which marries commitment for egalitarianism and promotion of individual interests in a consistent manner. They show that, in convex games, this allocation Lorenz dominates every other point in the core.

Before formally stating the axiom, we need the following definitions. Suppose agents have the profile R. The set of efficient allocations only depends on the associated vector of preferred contributions $x^* = x^*(R)$. We denote this set by $Eff(x^*)$, that is,

$$Eff(x^*) = \left\{ y \in \triangle^n \;\middle|\; \begin{array}{ll} \text{If } \sum_{i\in N} x_i^* \leq 1 & \text{then } y_i \geq x_i^* \text{ for all } i \in N \\ \text{If } \sum_{i\in N} x_i^* \geq 1 & \text{then } y_i \leq x_i^* \text{ for all } i \in N \end{array} \right\}.$$

Given a vector $y \in \triangle^n$, denote by $\widehat{y}$ the vector obtained by rearranging the coordinates of y in decreasing order.

Given two vectors $y, z \in \triangle^n$, we say that y *Lorenz dominates* z, $y >_L z$, if $\sum_{j=1}^k \widehat{y}_j \leq$

[3]Recall that Lorenz domination implies leximin domination.

$\sum_{j=1}^{k} \widehat{z}_j$ for every $k = 1, ..., n$, with at least one strict inequality.[4]

Notice that $>_L$ is a partial order. It is clear that $e = (1/n, ..., 1/n)$ Lorenz dominates every vector in $\triangle^n \backslash e$.

Formally the axiom we consider is the following.

Lorenz dominance: A solution Φ is Lorenz dominant if for all $R \in \mathcal{R}^n$, $\Phi(R) >_L y$ for every $y \in Eff(x^*(R)) \setminus \Phi(R)$.

Our main result can be stated as follows:

Theorem 3.1. *A solution Φ is efficient and Lorenz dominant if and only if Φ is the Uniform Allocation Rule.*

Proof: To show that the Uniform Allocation Rule is efficient and Lorenz dominant is immediate. Assume that Φ is an efficient and Lorenz dominant solution. We will show that it has to be the Uniform Allocation Rule.
Let R be given and let x^* be the corresponding tops. Denote by x the allocation proposed by Φ. By efficiency we can divide the population among those who get what they want and those who get more (if $\sum_{i=1}^{n} x_i^* < 1$), or less (if $\sum_{i=1}^{n} x_i^* > 1$) of what they want. If $\sum_{i=1}^{n} x_i^* = 1$, efficiency requires the uniform distribution, which coincides with the Uniform Allocation Rule. Since both cases are symmetric we only provide the details for the case $\sum_{i=1}^{n} x_i^* > 1$.
Anybody who gets rationed (by receiving more or less than its preferred contribution) must get the same. Suppose not, i.e., let us assume that there exists $i, j \in N$ such that $x_i < x_j$ and $x_i < x_i^*$, $x_j < x_j^*$. Then, $(x_1, ..., x_i + \epsilon, ..., x_j - \epsilon, ..., x_n)$ is also efficient for sufficiently small $\epsilon > 0$, and it Lorenz dominates the original $(x_1, ..., x_n)$. Hence, $x = \Phi(R)$ has the property that there exists a q such that either $x_i = x_i^*$ or $x_i = q$. Denote by S the set of rationed agents, i.e., $S = \{i \in N \mid x_i = q\}$. It only remains to be shown that x has the property that $q > x_j$ for all j such that $q \neq x_j$. Assume not, i.e., $q < x_j$ for some $j \notin S$. Then, there exist $q' > q$ and y such that $y_j = q'$, $y_i = q'$ for $i \in S$, and $y_i = x_i$ for $i \notin S \cup j$. Then y is efficient and Lorenz dominates the original allocation. But this is the allocation chosen by the Uniform Allocation Rule. ∎

The remainder of Section 3 is devoted to present an alternative proof of Theorem 3.1. We do that for two reasons. First, we find this alternative proof interesting since it allows to relate the Uniform Allocation Rule with the Ray and Dutta Egalitarian Solution for cooperative games. Second, in this proof we construct a cooperative game that will be used in Section 4 to study the consistency properties of the rule in terms of its reduced game properties.

[4] In this definition we follow Dutta and Ray[6] in arranging the vectors in decreasing order. Some authors define the Lorenz curves by arranging the vectors in increasing order.

Following Aumann and Maschler[1], we will associate to every division problem something that "looks like" a characteristic function. We can apply to the division problem what Aumann and Maschler say for the bankruptcy problem: "As it stands, the bankruptcy problem considered here is not a game; coalitions do not appear explicitly in its formulation." As they did we have found a natural way to associate a game with the division problem. We obtain a "division cooperative game" where the set of players will be the agents involved in the distribution problem and the value of a coalition will be the smallest amount of work that its members will have to contribute to in any efficient allocation. This amount will depend on whether the vector of preferred contributions is bigger or smaller than the total required. If it is bigger, the value of a coalition will be the remainder amount of work after fulfilling the preferred contributions of the members of the complementary coalition; if it is smaller, the value will just be the sum of the preferred contributions of its members, since, at least, that will be the smallest amount they will jointly work in any efficient allocation.

Definition 3.2. *Let $x^* = (x_1^*, ... x_n^*)$ be a vector of preferred contributions. The Division Cooperative Game is defined as a pair (N, v_{x^*}) where $N = \{1, ..., n\}$ is the set of individuals involved in the division problem and v_{x^*} is the characteristic function defined as follows:*

(a) If $\sum_{i=1}^n x_i^* \geq 1$ then $v_{x^*}(S) = \max\{0, 1 - \sum_{i \in N/S} x_i^*\}$ for every $S \subseteq N$.

(b) If $\sum_{i=1}^n x_i^* \leq 1$ then $v_{x^*}(S) = \begin{cases} \sum_{i \in S} x_i^* & \text{if } S \subset N \\ 1 & \text{if } S = N \end{cases}$.

We want to emphasize that $v(S)$ is a number that measures time units of work and that agents, in contrast with the common transferable utility cooperative game interpretation, may become satiated by this good. When it is clear from the context, as in this case, we will omit x^* in v_{x^*}. We will denote by Γ the family of division cooperative games. Let us illustrate this definition with the help of two examples.

Consider case (a) first. Assume there are two agents with preferred contributions 0.4 and 0.7. In this case we will have $v(1) = 0.3$ and $v(2) = 0.6$ with $v(12) = 1$. By efficiency the shares have to be $y_1 \leq 0.4$ and $y_2 \leq 0.7$ while feasibility implies $y_1 + y_2 = 1$. Therefore, it follows that $y_1 \geq 0.3$ and $y_2 \geq 0.6$. The characteristic function assigns to every player (in general coalitions) the smallest contribution of all possible "equilibrium" binding agreements in the sense of Ray and Vohra[12].

Next consider case (b). Using the same v, namely $v(S) = \max\{0, 1 - \sum_{i \in N/S} x_i^*\}$ for every $S \subseteq N$, the interpretation would be different: it would give the highest amount of work that S would do in any efficient allocation, and not the smallest one as in case (a). Consider, for instance, two agents with preferred contributions 0.2 and 0.6. By efficiency the shares have to be $y_1 \geq 0.2$ and $y_2 \geq 0.6$. Therefore, we have $v(1) = 0.2$, $v(2) = 0.6$ and $v(12) = 1$. In this case, contrary to the previous one, the smallest efficient allocation is the one that gives highest satisfaction.

This cooperative game has good properties; for instance, it is convex, and therefore, has a nonempty core. From now on we fix a vector of preferred contributions, x^*, from which we obtain a division cooperative game. We will focus on solution concepts for this game and on the relationship with the solutions of the original division problem. For example, the core of this game coincides with the set of efficient allocations of the division problem.

Definition 3.3. *The core of a division cooperative game* (N, v) *is the following set:*

$$C(N, v) = \{y \in \triangle^n \mid \sum_{i \in S} y_i \geq v(S) \text{ for all } S \subset N\}.$$

Lemma 3.4. $Eff(x^*) = C(N, v_{x^*})$ *for every* x^*.

Proof: Let $y \in Eff(x^*)$ and suppose $y \notin C(N, v_{x^*})$. Without loss of generality we assume that $\sum_{i \in N} x_i^* \geq 1$. Since $y \notin C(N, v_{x^*})$ there must exist $S \subset N$, $S \neq N$, such that $\sum_{i \in S} y_i < v(S) = 1 - \sum_{j \in S^C} x_j^*$. Therefore $\sum_{i \in S} y_i + \sum_{j \in S^C} x_j^* < 1 = \sum_{i \in S} y_i + \sum_{j \in S^C} y_j$. This implies that there exists $j \in S^C$ such that $x_j^* < y_j$, but this contradicts efficiency. Assume now $y \in C(N, v_{x^*})$. If $\sum_i x_i^* \leq 1$ then $y_i \geq v(i) = x_i^*$, and henceforth it is efficient. If $\sum_{i \in N} x_i^* \geq 1$, we have that $\sum_{i \in S} y_i \geq v(S)$ for all $S \subseteq N$ in particular for the coalitions with $n - 1$ members. This implies that for every $j \in N$, $1 - y_j = \sum_{i \neq j} y_i \geq v(N \backslash j) = \max\{0, 1 - x_j^*\} \geq 1 - x_j^*$. Therefore, $x_j^* \geq y_j$ for all $j \in N$. ∎

In what follows we describe an algorithm to construct a feasible imputation $\xi^* \in \triangle^n$ from a division game. This imputation, as we will see, coincides with the selected by the Uniform Allocation Rule. Besides, in Lemma 3.7, we will show that it Lorenz dominates any other imputation in the core. This result implies that it coincides with the Egalitarian Solution of Dutta and Ray[6]. This fact will be the clue to prove Theorem 3.1.

For any characteristic function $v \in \Gamma$, the algorithm works as follows.

Step 1: For any coalition $S \subseteq N$, define $w_1(S, v) = \dfrac{v(S)}{|S|}$, so that $w_1(S, v)$ is the average worth of S under v. Let T_1 be the largest coalition with the highest average worth. Then $\xi_i^* = w_1(T_1, v)$ for all $i \in T_1$. Note that if $T_1 = N$ then $\xi_i^* = \dfrac{1}{n}$ for all i.

Step 2: Consider all the coalitions S such that $S \supset T_1$, and define

$$w_2(S, v) = \frac{v(S) - \sum_{i \in T_1} \xi_i^*}{|S| - |T_1|} = \frac{v(S) - v(T_1)}{|S| - |T_1|}.$$

Let us denote by T_2 the largest coalition with the highest $w_2(S, v)$. Then $\xi_i^* = w_2(T_2, v)$ for all $i \in T_2 \setminus T_1$.

Step k: Consider all the coalitions S such that $S \supset T_{k-1}$, and define

$$w_k(S,v) = \frac{v(S) - \sum_{i \in T_{k-1}} \xi_i^*}{|S| - |T_{k-1}|} = \frac{v(S) - v(T_{k-1})}{|S| - |T_{k-1}|}.$$

We denote by T_k the largest coalition with the highest $w_k(S,v)$, then $\xi_i^* = w_k(T_k, v)$ for all $i \in T_k \setminus T_{k-1}$.

Clearly, at some step K, smaller than n, we will have that the grand coalition is the largest with the highest $w_K(S,v)$. Hence, at this step, $\xi_j^* = w_K(N,v)$ for all $j \in N \setminus T_{K-1}$, and thus the algorithm stops.

Lemma 3.5. $\xi^* = \Phi^*(x^*)$.

Proof: We will distinguish between two cases:
Case (a): $\sum_{i \in N} x_i^* \geq 1$.
Pick an arbitrary $i \in N$. If $i \in T_1$, then $\xi_i^* = \dfrac{v(T_1)}{|T_1|}$. Let us denote by α the average worth of T_1, that is, $\alpha = \dfrac{v(T_1)}{|T_1|} = \dfrac{1 - \sum_{j \in T_1^C} x_j^*}{|T_1|}$. If $i \notin T_1$, then $\xi_i^* = \dfrac{v(T_k) - v(T_{k-1})}{|T_k| - |T_{k-1}|}$ for the first k such that T_k contains i. Since $v(T_l) \geq v(T_{l-s})$ for every $s = 1, ..., l-1$, and $v(T_1) > 0$, we have that

$$\frac{v(T_k) - v(T_{k-1})}{|T_k| - |T_{k-1}|} = \frac{\sum_{j \in T_k / T_{k-1}} x_j^*}{|T_k| - |T_{k-1}|}.$$

From the definition of T_k it follows that for every $j \in T_k \setminus T_{k-1}$

$$\xi_j^* = \frac{v(T_k) - v(T_{k-1})}{|T_k| - |T_{k-1}|} \geq v(T_{k-1} \cup \{j\}) - v(T_{k-1}) = x_j^*.$$

Since $\sum_{k=2}^{K} \sum_{j \in T_k / T_{k-1}} \xi_j^* = 1 - \sum_{j \in T_1} \xi_j^* = \sum_{k=2}^{K} \sum_{j \in T_k / T_{k-1}} x_j^*$, and since $\xi_j^* \geq x_j^*$ for every $j \in T_k \setminus T_{k-1}$ and every $k > 1$, we can conclude that $\xi_i^* = x_i^*$ whenever $i \notin T_1$. Therefore $\xi_i^* = \min\{x_i^*, \alpha\}$, which is $\Phi_i^*(x^*)$ since α is such that $\sum_{i \in N} \min\{x_i^*, \alpha\} = 1$.
Case (b): $\sum_{i \in N} x_i^* \leq 1$.
Pick an arbitrary $i \in N$. If $i \in T_1$ and $|T_1| < n$ then $\xi_i^* = \dfrac{v(T_1)}{|T_1|} = \dfrac{\sum_{j \in T_1} x_j^*}{|T_1|} = x_i^*$. If $i \notin T_1$ then $\xi_i^* = \dfrac{v(T_k) - v(T_{k-1})}{|T_k| - |T_{k-1}|}$ for the first k such that T_k contains i.
From the definition of T_k it follows that for every $j \in T_k \setminus T_{k-1}$

$$\xi_j^* = \frac{v(T_k) - v(T_{k-1})}{|T_k| - |T_{k-1}|} = \lambda \geq v(T_{k-1} \cup \{j\}) - v(T_{k-1}) = x_j^*.$$

Since $\sum_{j \in N} \xi_j^* = 1$, and $\sum_{j \in T_1} \xi_j^* = \sum_{j \in T_1} x_j^*$ then $\xi_j^* \geq x_j^*$ $\forall j \notin T_1$ with at least one strict inequality. Therefore $\xi_i^* = \max\{x_i^*, \lambda\} = \Phi_i^*(x^*)$ for all $i \in N$. ∎

Remark 3.6. *ξ^* is a core imputation by Lemmas 3.4 and 3.5.*

Let us illustrate this algorithm with the help of two examples.

Example 1: Let $n = 3$ and $x^* = (0.16, 0.25, 0.66)$. From these preferred contributions we can derive the following division game: $v(1) = 0.083$, $v(2) = 0.166$, $v(3) = 0.583$, $v(12) = 0.333$, $v(23) = 0.833$, $v(13) = 0.75$ and $v(123) = 1$. In a first step $T_1 = \{3\}$ and $w_1(\{3\}, v) = 0.583$; therefore $\xi_3^* = 0.583$. In a second step $T_2 = \{23\}$ and $w_2(\{23\}, v) = v(23) - 0.583 = 0.25$; therefore $\xi_2^* = 0.25$. Finally, $\xi_1^* = 0.16$. That is, $\xi^* = \Phi^*(x^*) = (0.16, 0.25, 0.583)$.

Example 2: Let $n = 3$ and $x^* = (0.1, 0.3, 0.5)$. From these preferred contributions we can derive the following division game: $v(1) = 0.1$, $v(2) = 0.3$, $v(3) = 0.5$, $v(12) = 0.4$, $v(23) = 0.8$, $v(13) = 0.6$, and $v(123) = 1$. In a first step $T_1 = \{3\}$ and $w_1(\{3\}, v) = 0.5$. Next, $T_2 = \{23\}$ and $w_2(\{23\}, v) = v(23) - 0.5 = 0.3$, therefore $\xi_2^* = 0.3$. Finally, $\xi_1^* = 0.2$. Hence, $\xi^* = \Phi^*(x^*) = (0.2, 0.3, 0.5)$.

Next lemma shows that the imputation constructed by the algorithm is Lorenz dominant among the imputations in the core of the division game.

Lemma 3.7. *$\xi^* >_L y$ for every $y \in C(N, v) \backslash \xi^*$.*

Proof: The proof is by contradiction. Assume, without loss of generality that $\xi^* = \widehat{\xi}^*$. Let $y \in C(N, v) \backslash \xi^*$ be such that ξ^* does not Lorenz dominates it. This implies that there exists k (the smallest one) such that $\sum_{j=1}^{k} \hat{\xi}_j^* > \sum_{j=1}^{k} \widehat{y}_j$. Let T_l be the smallest coalition used in the algorithm to obtain ξ^* such that $|T_l| \geq k$. Let us define $|T_l| = W$. Notice that since ξ^* is a vector with decreasing components it is the case that $\sum_{j=1}^{W} \hat{\xi}_j^* = \sum_{j \in T_l} \xi_j^* = v(T_l)$. Since for every $k < j \leq W$, $\hat{\xi}_j^* = \hat{\xi}_k^*$ and $\widehat{y}_j \leq \widehat{y}_k$, then $\sum_{j=1}^{W} \hat{\xi}_j^* = \sum_{j \in T_l} \xi_j^* > \sum_{j=1}^{W} \widehat{y}_j$. Also, since $\widehat{y}$ is a vector with decreasing components we have that $\sum_{j=1}^{W} \widehat{y}_j \geq \sum_{j \in S} y_j$ for every $S \subset N$ with $|S| = W$. In particular, for $S = T_l$. Thus, we have $\sum_{j \in T_l} \xi_j^* = v(T_l) > \sum_{j=1}^{W} \widehat{y}_j \geq \sum_{j \in T_l} y_j$. This contradicts the assumption $y \in C(N, v) \backslash \xi^*$. ∎

With the help of the above lemmas, we now provide the proof of our main result.

Proof of Theorem 3.1: .We first prove necessity. Efficiency follows from Theorem 1 in Sprumont. The property of Lorenz dominance follows from Lemmas 3.4, 3.5 and 3.7.
To prove sufficiency let Φ be an efficient and Lorenz dominant allocation rule. We want to show that $\Phi(R) = \Phi^*(x^*)$ for every $R \in \mathcal{R}^n$ where $x^* = x^*(R)$. Let R be given. Since Φ is Lorenz dominant we have that $\Phi(R) >_L y$ for every $y \in Eff(x^*) \setminus \Phi(R)$. By Lemma 3.4 if $y \in Eff(x^*)$ then $y \in C(N, v)$. Appealing to Theorems 2 and 3 in Dutta and Ray[6] we know that in convex games there exists a unique imputation that Lorenz dominates every other imputation in the core. Therefore, Lemmas 3.5 and 3.7 imply that $\Phi(R) = \Phi^*(x^*)$. ∎

4. A Characterization with a Consistency Axiom

Consider a group of people playing a cooperative game. Assume that players live in a society that believes in a point-valued solution concept Θ. The consistency principle means the following: If a subset S of players gather together and observe what they have received under Θ, they will decide that they have no motivation to defect, because what they have received is the payoff vector of the solution Θ for "their own game" or "their reduced game". This principle can be thought of as a stability requirement. If Θ is not a consistent solution then there will be games with payoff $x = \Theta(N, v)$ and coalitions S whose members will disagree with x claiming that for their own game the solution Θ yields a payoff vector different from x^S, the restriction of x to S.

We first show, by considering the division game, that the Uniform Allocation Rule is consistent under the main definitions of reduced game properties; this is not the case of the Nucleolus, or the Shapley value, for instance.

Definition 4.1. *A solution Θ is consistent (has the reduced game property, RGP) if for all coalitions S and for every payoff x it satisfies the following*:

If $x = \Theta(N, v)$ then $(x_i)_{i \in S} = \Theta(S, v_{x,S})$.

$(S, v_{x,S})$ is called the "reduced game on S", and consistency is also called "having the reduced game property". The question is how to define the reduced game $(S, v_{x,S})$.

Two definitions have so far proved fruitful.

Definition 4.2. *(Peleg [11]) Let (N, v) be a game, let $S \subset N, S \neq \emptyset$, and let $x = \Theta(N, v)$. The reduced game RG_1 with respect to S and x is the game $(S, v^1_{x,S})$ where*

$$v^1_{x,S}(T) = \begin{cases} x(S) & \text{if } T = S \\ \max_{R \in S^C} \{v(T \cup R) - x(R)\} & \text{if } T \subset S \end{cases}.$$

Definition 4.3. *(Hart and Mas-Colell[9]) Let (N, v) be a game, let $S \subset N, S \neq \emptyset$, and let $x = \Theta(N, v)$. The reduced game RG_2 with respect to S and x is the game $(S, v^2_{x,S})$ where*

$$v^2_{x,S}(T) = \begin{cases} x(S) & \text{if } T = S \\ v(T \cup S^C) - \sum_{i \in S^C} \Theta_i(T \cup S^C, v) & \text{if } T \subset S \end{cases}.$$

The solution Θ has the RGP_k if it is consistent with respect to the RG_k, for $k = 1, 2$.

The intrinsic difference between the definitions above is the way the subcoalitions, the subsets $T \subset S \subset N$, interpret their own game. In 1 they choose the "best partners", while in 2 they go with all the others. In 1 they pay them their original payoff, while in 2 they "figure out" that a new game will be played $(T \cup S^C, v)$ and S^C will take home $\sum_{i \in S^C} \Theta_i(T \cup S^C, v)$.

One of the most fascinating results in cooperative game theory is the characterization of its main solutions by means of the axioms of anonymity and zero-independence combined with some of these consistency properties. For instance, RGP_1, anonymity and zero-independence characterize the Nucleolus; RGP_2, anonymity and zero-independence characterize the Shapley value.

Definition 4.4. *(Peleg[11]) Let Θ be a solution. It has the converse reduced game property (CRGP) if for every $x \in \Re^n$ and for every $S \in \Pi(N)$, $\Pi(N) = \{S \subseteq N \mid |S| = 2\}$, if $x^S = \Theta(S, v_x^1)$ then $x = \Theta(N, v)$.*

Next proposition shows that the Uniform Allocation Rule is consistent under the above definitions of reduced game properties and that it does also satisfy the $CRGP$. Since the definition of this rule is based on the division problem and not on the game, we will make use of the results in Section 3 to prove the Proposition.

Proposition 4.5. *Φ^* satisfies the RGP_k for $k = 1, 2$. Moreover, Φ^* satisfies the $CRGP$.*

Proof: From Dutta [8] it is known that the Egalitarian Solution satisfies RGP_1, RGP_2 and $CRGP$ for convex games. Hence, the Uniform Allocation Rule will also satisfy them, since in Section 3 we showed that division games are convex and that the Uniform Allocation Rule coincides with the Egalitarian Solution for these games. ∎

Remark 4.6. *Since Φ^* is anonymous and satisfies the RGP_1 and the RGP_2, it does not have the zero-independence property.*

We now focus on the characterization of the Uniform Allocation Rule by means of the RGP_1 and two other axioms (Individual Rationality and the Unanimity Sharing Bounds) to hold only for the two-players case.

Definition 4.7. *Let Θ be a solution in the class of games Γ. It is Individually Rational for two players (IR^2) if $\Theta_i(N, v) \geq v(i)$ for all $i \in N$, whenever $|N| = 2$.*

Definition 4.8. *Let Θ be a solution in the class of games Γ. It has the Unanimity Sharing Bound property for two players (USB^2) if it satisfies that $\Theta_i(\{1,2\}, v) \leq \max\left\{\frac{v(12)}{2}, v(i)\right\}$ for $i = 1, 2$.*

Proposition 4.9. *The Uniform Allocation Rule is the unique solution that satisfies RGP_1, $IR^2$and USB^2.*

Proof: One direction is immediate. For the other one, assume θ satisfies the three properties. Then, we have to show that it coincides with the Uniform Allocation Rule, that is, $\theta = \Phi^*$. Let $n = 2$. By USB^2 and IR^2 if $v(1) = v(2)$ then $\theta_1 = \Phi_1^* = \Phi_2^* = \theta_2 = \frac{v(12)}{2} = \frac{1}{2}$. If $v(1) < v(2)$, by USB^2, we have that $\theta_1 \leq \max\left\{\frac{v(12)}{2}, v(1)\right\}$ and $\theta_2 \leq \max\left\{\frac{v(12)}{2}, v(2)\right\}$. We distinguish two cases:

Case (a): $\max\left\{\frac{v(12)}{2}, v(2)\right\} = \frac{v(12)}{2}$. In this case, $\Phi_1^* = \Phi_2^* = \frac{v(12)}{2} = \frac{1}{2}$. Let $\theta_2 = v(2) + \alpha$. By IR^2 we know that $\alpha \geq 0$. Assume that $\theta_2 = v(2) + \alpha < \frac{v(12)}{2}\cdot$ Efficiency implies that $\theta_1 > \frac{v(12)}{2}$, but this contradicts that θ satisfies the USB^2 property since by hypothesis $v(1) < v(2) < \frac{v(12)}{2}$. Therefore $\theta_2 = \theta_1 = \frac{v(12)}{2}\cdot$

Case (b): $\max\left\{\frac{v(12)}{2}, v(2)\right\} = v(2)$. Since in this case $\xi_2^* = v(2)$, it is sufficient to show that $\theta_2 = v(2)$. But this follows immediately since USB^2 and IR^2 together imply that $\theta_2 = v(2)$.

Now let $x = \theta(N, v)$. By the RGP_1 we know that $x^S = \theta(S, v_x)$ for every $S \in \Pi(N)$. Hence $x^S = \Phi^*(S, v_x)$ for every $S \in \Pi(N)$. But since Φ^* has the $CRGP$ then $x = \Phi^*(N, v)$. ∎

Remark 4.10. *The characterization is tight. The Nucleolus is an example of a value that satisfies all the properties of the Theorem except USB^2. The equal split is an example of a value that satisfies USB^2 and RGP_1, but fails to satisfy IR^2. A solution composed by a mixture of the Nucleolus and the Uniform Allocation Rule (it applies the Nucleolus whenever $v(N) \geq \sum_{i\in N}(v(N) - v(N \setminus i))$ and the Uniform Allocation Rule whenever $v(N) \leq \sum_{i\in N}(v(N) - v(N \setminus i)))$ satisfies all the properties, but the RGP_1.*

Before stating the Theorem it will be useful to interpret the axioms in terms of the original division problem. IR^2 states that the solution has to be efficient. USB^2 imposes an upper bound on the "rationing". We will refer to this property as the *Upper-Bound Rationing* property. RGP_1 is, under efficiency, equivalent to the consistency axiom used in Thomson[14] (this claim is proved in Lemma 4.13). Therefore, we can rewrite for the original setting the characterization obtained in the cooperative framework as follows: the Uniform Allocation Rule is the unique solution satisfying efficiency, consistency, and the Upper-Bound Rationing property for two players. We will also see that efficiency and the Upper-Bound Rationing property imply the Median Voter property, which is the

Uniform Allocation Rule for two players; that is, what player 1 gets is the median among his preferred contribution (x_1^*), what player 2 concedes to him ($1-x_2^*$) and $\frac{1}{2}\cdot$ The formal definitions are the following.

Definition 4.11. *Let Φ be a solution of the division problem. We say that Φ satisfies the Upper-Bound Rationing property for two players if it satisfies the following condition:*

-If preferences are such that $x_1^*(R)+x_2^*(R)\geq 1$, then $\Phi_i(R)\leq \max\left\{1-x_j^*,\ \frac{1}{2}\right\}.$

-If preferences are such that $x_1^*(R)+x_2^*(R)\leq 1$, then $\Phi_i(R)\leq \max\left\{x_i^*,\ \frac{1}{2}\right\}\cdot$

Definition 4.12. *Let Φ be a solution of the division problem defined on the pairs (R, M) such that $\sum_{i\in N}\Phi_i(R,M)=M$. Then Φ is consistent if $\Phi_i(R,M)=\Phi_i((R)^S,\sum_{i\in S}\Phi_i(R,M))$ for every $i\in S$ and every $S\subset N$.*

Lemma 4.13. *Let Φ be an efficient solution of the division problem. For every $R\in R^n$, denote by $\theta(N,v_x^*)$ the allocation $\Phi(R)$, where $x^*=x^*(R)$. Then, Φ is consistent if and only if θ satisfies the RGP_1.*

Proof: Let Φ be an efficient solution and let $R\in R^n$ be such that $\sum_{i\in N}x_i^*\geq 1$, where $x^*=x^*(R)$.
For every $S\subseteq N$, $S\neq\emptyset$, define the restricted characteristic function $u_S(T)$ as follows: $u_S(T)=\max\{0,\sum_{j\in S}\Phi_j(R,M)-\sum_{j\in S\setminus T}x_j^*\}$, for all $T\subseteq S$. Since consistency means that $\Phi_i(R,M)=\Phi_i((R)^S,\sum_{i\in S}\Phi_i(R,M))$, and since the RGP_1 of θ implies $\theta_i(S,v_x)=\theta_i(S,v_{x,S}^1)$ for all $i\in S$, then to prove the lemma, i.e., to show that $\theta_i(S,v_{x,S}^1)=\theta_i(S,u_S)$, it is sufficient to show that $v_{x,S}^1(T)=u_S(T)$ for all $T\subseteq S$, where

$$v_{x,S}^1(T)=\begin{cases}\sum_{j\in T}\Phi_j(R,M) & \text{if } T=S,\\ \max_{Q\subseteq S^C}\left\{v(T\cup Q)-\sum_{i\in Q}\Phi_i(R,M)\right\} & \text{if } T\subset S.\end{cases}$$

Let $x=\Phi(R,M)$. Following the proof of Lemma 6.2 in Aumann and Maschler[1] consider the three possible cases:

(i) Let $T=S$; in this case we have $v_{x,S}^1(T)=\sum_{j\in T}x_j=u_S(T)$.

(ii) Let $T=\emptyset$; the efficiency of Φ implies that $u_S(\emptyset)=0$, and we also have that $v_{x,S}^1(T)=\max_{Q\in S^C}\left\{\max\{0,M-\sum_{j\in Q^C}x_j^*\}-\sum_{j\in Q}x_j\right\}$.
Letting $Q=\emptyset$ we get $v_{x,S}^1(T)=\max_{Q\in S^C}\{\max\{-\sum_{j\in Q}x_j,\sum_{j\in Q^C}x_j-\sum_{j\in Q^C}x_j^*\}\}=0$.

(iii) Let $T\subset S$, and let $Q\subset S^C$ be such that $v_{x,S}^1(T)=v(T\cup Q)-\sum_{j\in Q}x_j$.
Then, $v_{x,S}^1(T)=\max\{0,M-\sum_{j\in(T\cup Q)^C}x_j^*\}-\sum_{j\in Q}x_j\leq\max\{0,M-\sum_{j\in Q^C}x_j^*-\sum_{j\in Q}x_j\}$

$= \max\ \{0, \sum_{j\in S} x_j + \sum_{j\in S^C} x_j - [\sum_{j\in S\setminus T} x_j^* - \sum_{j\in Q} x_j^* + \sum_{j\in S^C} x_j^*] - \sum_{j\in Q} x_j\}$
$= \max\ \{0, \sum_{j\in S} x_j - \sum_{j\in S\setminus T} x_j^* + \sum_{j\in N\setminus S\cup Q} x_j - \sum_{j\in N\setminus S\cup Q} x_j^*\}$
$\leq \max\ \{0, \sum_{j\in S} x_j - \sum_{j\in S\setminus T} x_j^*\} = u_S(T)$. Hence, we have $v^1_{x,S}(T) \leq u_S(T)$.
On the other hand, letting $Q = N \setminus S$, we have that
$v^1_{x,S}(T) \geq v(T \cup (N \setminus S)) - \sum_{j\in N\setminus S} x_j =\max\ \{0, M - \sum_{j\in(T\cup(N\setminus S))^C} x_j^*\} - \sum_{j\in N\setminus S} x_j$
$\geq M - \sum_{j\in S\setminus T} x_j^* - (M - \sum_{j\in S} x_j) = \sum_{j\in S} x_j - \sum_{j\in S\setminus T} x_j^*$.
Also, letting $Q = \emptyset$, we have $v^1_{x,S}(T) \geq \max\ \{0, M - \sum_{j\in N\setminus T} x_j^*\} \geq 0$.

Therefore, we get that $v^1_{x,S}(T) \geq \max\ \{0, \sum_{j\in S} x_j - \sum_{j\in S\setminus T} x_j^*\} = u_S(T)$. Hence, $v^1_{x,S}(T) = u_S(T)$ for $T \subseteq S$.

Assume now that $R \in \mathcal{R}^n$ is such that $\sum_{i\in N} x_i^* < M$, where $x^* = x^*(R)$. For every $S \subseteq N$, $S \neq \emptyset$, define the restricted characteristic function $u_S(T)$ as follows:

$$u_S(T) = \begin{cases} \sum_{j\in S} \Phi_j(R) & \text{if } T = S, \\ \sum_{j\in T} x_j^* & \text{if } T \subset S. \end{cases}$$

Let $x = \Phi(R, M)$. To show that $v^1_{x,S}(T) = u_S(T)$ for all $T \subseteq S$, we have to consider the three possible cases:

(i) Let $T = S$; in this case we have $v^1_{x,S}(T) = \sum_{j\in T} x_j = u_S(T)$.

(ii) Let $T = \emptyset$; from the definition of the reduced game and for $Q = \emptyset$ we have that

$$v^1_{x,S}(T) = \max_{Q\subseteq S^C} \left\{ v(Q) - \sum_{j\in Q} x_j \right\} = \max_{Q\subseteq S^C} \{\sum_{j\in Q} x_j^* - \sum_{j\in Q} x_j\} = 0.$$

Therefore, $v^1_{x,S}(T) = u_S(\emptyset) = 0$.

(iii) Let $T \subset S$ and let $Q \subseteq S^C$ be such that

$$v^1_{x,S}(T) = v(T \cup Q) - \sum_{j\in Q} x_j = \sum_{j\in(T\cup Q)} x_j^* - \sum_{j\in Q} x_j \leq \sum_{j\in T} x_j^* = u_S(T).$$

Now, letting $Q = \emptyset$, we get $v^1_{x,S}(T) \geq v(T) = \sum_{j\in T} x_j^* = u_S(T)$. ∎

Theorem 4.14. *The Uniform Allocation Rule is the unique solution that satisfies consistency, efficiency and the upper bound rationing property.*

Proof: The proof follows immediately from the previous Proposition and Lemma. ∎

Before finishing this Section we comment about some related work. We showed in Section 3 that the Uniform Allocation Rule coincides with the Egalitarian Solution of

Dutta and Ray[6]. This solution has been axiomatized over the class of convex games by Dutta[8]. He shows that the Egalitarian Solution, and henceforth the Uniform Allocation Rule, is the only solution satisfying any one of the two reduced game properties, RGP_1 or RGP_2, and agreeing with the Egalitarian Solution for two-person games. Our result here shows the axioms that characterize this solution for two-person games: IR^2 and USB^2. He also shows that there is no solution satisfying symmetry, individual rationality and a monotonicity condition for two-person games that also satisfies the two reduced game properties. In this context, we conjecture that the the Uniform Allocation Rule is the only solution that satisfies RGP_1, RGP_2 and that for two-person games it is a core selection, and has the Equal Treatment Property.

References

[1] Aumann, R. and M. Maschler. *"Game Theoretic Analysis of a Bankruptcy Problem from the Talmud".* Journal of Economic Theory 36 (1985), pp. 195-213.

[2] Barberà, S., F. Gul and E. Stacchetti. *"Generalized Median Voter Schemes and Committees".* Journal of Economic Theory 61 (1993), pp. 262-289.

[3] Barberà, S. and M. Jackson. *"Strategy-proof Exchange".* Econometrica, forthcoming.

[4] Barberà, S., J. Massó and A. Neme. *"Voting under Constraints".* Universitat Autònoma de Barcelona, mimeo, 1994.

[5] Barberà, S., H. Sonnenschein and L. Zhou. *"Voting by Committees".* Econometrica 59 (1991), pp. 595-609.

[6] Dutta B. and D. Ray. *"A Concept of Egalitarianism under Participation Constraints".* Econometrica 57 (1989), pp. 615-635.

[7] Dutta B. and D. Ray. *"Constrained Egalitarian Allocations".* Games and Economic Behavior 3 (1991), pp. 403-422.

[8] Dutta B. *"The Egalitarian Solution and Reduced Game Properties in Convex Games".* International Journal of Game Theory 19 (1990), pp. 153-169.

[9] Hart, S. and A. Mas-Colell. *"Potential, Value, and Consistency".* Econometrica 57 (1989), pp. 589-614.

[10] Otten, G.-J., H. Peters and O. Volij. *"Two Characterizations of the Uniform Rule for Division Problems with Singled-Peaked Preferences"*. Tilburg University Research Memorandum 9449 (1994).

[11] Peleg, B. *"On The Reduced Game Property and its Converse"*. International Journal of Game Theory 15 (1986), pp. 187-200.

[12] Ray, D. and R. Vohra. *"Equilibrium Binding Agreements"*. Boston University, Working Paper 21, June 1993.

[13] Sprumont, Y. *"The DivisionProblem with Single-peaked Preferences: A Characterization of the Uniform Allocation Rule".* Econometrica 59 (1991), pp. 509-519.

[14] Thomson, W. *"Consistent Solutions to the Problem of Fair Division When Preferences Are Single-peaked".* Journal of Economic Theory 63 (1994), pp. 219-245.