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The lattice structure of the set of stable outcomes of the multiple partners assignment game

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Abstract. The Multiple Partners assignment game is a natural extension of the Shapley and Shubik Assignment Game (Shapley and Shubik, 1972) to the case where the participants can form more than one partnership.

In Sotomayor (1992) the existence of stable outcomes was proved. For the sake of completeness the proof is reproduced in Appendix I. In this paper we show that, as in the Assignment Game, stable payoffs form a complete lattice and hence there exists a unique optimal stable payoff for each side of the market. We also observe a polarization of interests between the two sides of the matching, within the whole set of stable payoffs. Our proofs differ technically from the Shapley and Shubik's proofs since they depend on a central result (Theorem 1) which has no parallel in the Assignment model.

Key words: Matching, lattice, optimal matching, optimal stable outcomes

1. Introduction

The multiple partners assignment game of our title is one of the models for a two-sided matching market presented in Sotomayor (1992). The participants belong to two finite and disjoint sets, which will be denoted by F and W. For each pair (f_i, w_j) in FxW there is a non-negative number, a_{ij}, representing the gain that the coalition $\{f_i, w_j\}$ can get if f_i and w_j form a partnership. Each

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player has a quota, that is, a positive integer number representing the maximum number of partnerships he/she can enter. An outcome for this game is any set of partnerships (f_i, w_j) which do not violate the quotas of the players (a matching), together with some splitting u_{ij} and v_{ij} of the gain a_{ij}. Thus, $u_{ij} + v_{ij} = a_{ij}$ if f_i and w_j form a partnership.

Imagine an economic scenario in which the sets F and W are sets of firms and workers, respectively. The amount of income which f_i and w_j can generate if they work together is a_{ij}. The problem is then to split a_{ij} among f_i and w_j. That is, if firm f_i hires worker w_j at a salary v_{ij} it will receive a profit of $u_{ij} = a_{ij} - v_{ij}$.

The natural questions are then:

A) Which partnerships will be formed?
B) If a partnership is formed, how will the gain, a_{ij}, be divided between the partners f_i and w_j?

The answers to these questions involve the choice of an appropriate concept of equilibrium that in this class of games is called *stability*. Stability requires that if f_i and w_j are not partners then the sum of any payoff of f_i with any payoff of w_j is not less than a_{ij}. The interpretation of this condition is the natural one. If it is not satisfied, that is, if $u_{iq} + v_{pj} < a_{ij}$, for some w_q and f_p, then f_i and w_j can break their current partnerships with w_q and f_p, respectively, and form a new one together, because this could give them each a higher total payoff. In Sotomayor (1992) it was proved that this concept of stability (pairwise-stability) is not equivalent to the core concept. The set of stable outcomes is contained in the core and it may be a proper subset of the core.

A well-known special case of our model is the assignment game of Shapley and Shubik (1972), where the only restriction is that each participant can form at most one partnership. The main result of Shapley and Shubik's paper is that stable outcomes always exist and they form a convex and compact lattice. Roughly speaking, if A and B are two stable outcomes then it might be that some workers (resp. firms) will get more income under A than under B and others will get more under B than under A. The lattice theorem is that there is a stable outcome, C, which gives each worker (resp. firm) the larger of the two amounts and also one, D, which gives each of them the smaller amount. This is of interest because, together with the compactness of the set of stable payoffs, it shows that among all stable payoffs there is one (and only one) which is worker optimal (resp. firm optimal), meaning that all workers (resp. firms) get as much income under it as under any other stable payoff.

It is also shown in Shapley and Shubik's paper that there is an opposition of interests between the two sides of the market with respect to the stable outcomes. According to this property, if A and B are two stable outcomes, then all workers (resp. firms) prefer A to B if and only if all firms (resp. workers) prefer B to A. Consequently, the optimal stable payoff for the workers (resp. firms) is the worst for the firms (resp. workers).

As for the multiple partners assignment game the existence of stable outcomes is proved in Sotomayor (1992) in two different ways. The first way uses the Linear Programming Equilibrium Theorem and the other way is based on a replication of the players with a convenient income matrix. For the sake of completeness we reproduce the proof which uses linear programming in

Appendix I. In Sotomayor (1992) we also show the existence, but not the uniqueness, of an optimal stable payoff for each side of the market. In the present paper we show the (to us) rather unexpected result that, again, stable payoffs form a convex and compact lattice and consequently there exists a unique optimal stable payoff for each side of the market (Theorems 3 and 4). The polarization of interests between the two sides within the whole set of stable outcomes does not carry over to our model, when we compare the total payoffs of the players (see Example 3 of Sotomayor (1992)). However we prove here that this result holds when we compare the individual payoffs of the players in some convenient way (Proposition 3).

Indeed, to prove the results above, we face the problem that the u_{ij}'s and v_{ij}'s are indexed according to the current matching, which may differ from one stable outcome to the other. Thus a player with a quota greater than one may face the problem of comparing two unordered sets of payoffs!

In shirt, it is not clear how to solve the following difficulties:

a – How to make a convex combination of two unordered sets?
b – In what topological space should the set of stable payoffs be immersed?

The solution is obtained here via Theorem 1. This result has no parallel in the special case of Shapley and Shubik, since its conclusions are addressed to the players who form more than one partnership. Its statement makes use of the following terminology: a matching is called optimal if it maximizes the gain of the whole set of players; a partnership (f_i, w_j) is called nonessential if it occurs in some but not all optimal matchings and essential if it occurs in all optimal matchings. Therefore two matchings differ only by their nonessential partnerships. Theorem 1 states that:

(i) *In every stable outcome a player gets the same payoff in any nonessential partnership; furthermore this payoff is less than or equal to any other payoff the player gets under the same outcome.*
(ii) *Given a stable outcome* $(u, v; x)$ *and a different optimal matching* x'*, we can reindex the* u_{ij}'s *and* v_{ij}'s *according to* x' *and still get a stable outcome.*

Equipped with Theorem 1 it is possible to represent an unordered set of payoffs of a player by a vector in some Euclidean space. The first coordinates of such a vector are the payoffs that the given player gets from his essential partners (if any), following some given ordering. The remaining coordinates (if any) are equal to a number which represents the payoff the player gets from all his nonessential partners. A partial order relation in the set of stable payoffs can then be defined in the obvious way. Having an appropriate partial order relation the desired results are proved straightforwardly.

A corollary of independent interest in the context of the assignment games is that: *If a player has some unfilled position under some stable outcome, then he will get zero payoff with some partner under every stable outcome.* (See Roth and Sotomayor (1990) for a detailed description of such properties for the one-to-one case).

The last result concerns situations in which, given two stable outcomes, all players on a given side have to choose the partnerships corresponding to their highest payoffs, respecting their quotas. Then a third outcome is formed. The non-obvious result is that the matching obtained in this way is not only feasible, but optimal. Moreover, the resulting outcome gives to the players of the opposite side their lowest payoffs. In this case the proof is not so straightforward as the previous proofs, instead it makes use of a sort of decomposition of the set of nonessential partnerships (Lemma 3).

This paper is organized as follows: Section 2 describes the model; section 3 gives the proof of Theorem 1; the results about the algebraic structure of the set of stable payoffs are proved in section 4. An illustrative example of the results of the paper is presented at the end of section 4. In Appendix I we prove two important results for the present work which were already proved in Sotomayor (1992). In Appendix II we discuss the robustness of the conclusions of Theorem 1.

2. Description of the model

There are two disjoint and finite sets of players, $F = \{f_0, f_1, \ldots, f_{m-1}\}$ and $W = \{w_0, w_1, \ldots, w_{n-1}\}$, where f_0 and w_0 are dummy players. Each player from a given set is allowed to form partnerships with the players from the opposite set. Each player $f_i \in F$ may form at most r_i partnerships and each player $w_j \in W$ may form at most s_j partnerships. For each pair (f_i, w_j) there is a non-negative number, a_{ij}, representing the gain that f_i and w_j can generate if they work together. If a partnership (f_i, w_j) is formed, f_i will receive a payoff $u_{ij} \geq 0$ and w_j will receive a payoff $v_{ij} = a_{ij} - u_{ij} \geq 0$. The dummy players are included for technical convenience. We assume that $a_{i0} = a_{0j} = 0$ for all $f_i \in F$ and $w_j \in W$. As for the quotas, a dummy player can form as many partnerships as needed to fill up the quotas of the non-dummy players. Thus, for example, if player $f_i \in F$ has 5 unfilled positions, then f_i forms 5 partnerships with w_0. This market is denoted by $M(F, W, a, r, s)$ or simply by $M(a)$.

A feasible matching is a set of partnerships of the kind $(f_i, w_j), (f_i, w_0)$ or (f_0, w_j), for (f_i, w_j) in FxW, Which do not violate the quotas of the players. Formally:

Definition 1. *A* **feasible matching** *x is an $m \times n$ matrix x_{ij} of positive integer numbers, defined for all pairs $(f_i, w_j) \in FxW$, such that if $i \neq 0$ and $j \neq 0$ then $x_{ij} \in \{0, 1\}$. Furthermore, $x_{00} = 0, \sum_{w_j \in W} x_{ij} = r_i$, for all $i \neq 0$ and $\sum_{f_i \in F} x_{ij} = s_j$, for all $j \neq 0$.*

If $x_{ij} > 0$ (resp. $x_{ij} = 0$) it means that f_i and w_j are (resp. are not) matched at x. The set of all f_i's partners at x, with x_{i0} repetitions of w_0, is denoted by $C(f_i, x)$. Thus, $C(f_i, x) = \{w_1, w_2, w_3, w_0, w_0\}$ denotes that player f_i, with quota $r_i = 5$, has partners w_1, w_2 and w_3 under the matching x and has two positions unfilled. $C(w_j, x)$ is similarly defined for all $w_j \in W$. We will represent by $|A|$ the number of elements, including the repetitions, of the set A. Under this notation $|C(f_i, x)| = r_i$ and $|C(w_j, x)| = s_j$, for all non-dummy players $f_i \in F$ and $w_j \in W$.

An outcome for this market will be determined by specifying a matching and the way in which the income within each partnership is divided among its members. That is:

Definition 2. *A* **feasible outcome**, *denoted by* $(u, v; x)$, *is a feasible matching* x, *and an array of numbers* u_{ij}, *with* $f_i \in F$ *and* $w_j \in C(f_i, x)$, *and* v_{ij}, *with* $w_j \in W$ *and* $f_i \in C(w_j, x)$, *such that* $u_{ij} + v_{ij} = a_{ij}, u_{ij} \geq 0$ *and* $v_{ij} \geq 0$. *Consequently* $u_{i0} = u_{0j} = v_{i0} = v_{0j} = 0$ *in case these payoffs are defined.*

If $(u, v; x)$ is a feasible outcome, we say that the matching x is compatible with the payoff (u, v) or that $(u, v; x)$ is an outcome with matching x. We call the set $\{u_{ij}; w_j \in C(f_i, x)\}$ (resp. $\{v_{ij}; f_i \in C(w_j, x)\}$) the set of payoffs of player f_i (resp. w_j). The f_i's total payoff under the outcome $(u, v; x)$ is denoted by σ_i; that is, $\sigma_i = \sum_{w_j \in C(f_i, x)} u_{ij}$. Similarly, we define the w_j's total payoff τ_j. Therefore we can write:

$$\sum_{f_i \in F} \sigma_i + \sum_{w_j \in W} \tau_j = \sum_{(f_i, w_j) \in FxW} a_{ij} x_{ij} \qquad \text{(A)}$$

Definition 3. *Given a feasible outcome* $(u, v; x)$ *define* $\mu_i = \min\{u_{ij}; w_j \in C(f_i, x)\}$ *and* $\nu_j \equiv \min\{v_{ij}; f_i \in C(w_j, x)\}$.

As we defined in the Introduction, a feasible outcome is stable if, for all f_i and w_j who are not partners, the sum of any payoff of f_i with any payoff of w_j is not less than a_{ij}. This is equivalent to:

Definition 4. *The feasible outcome* $(u, v; x)$ *is* **stable** *if* $\mu_i + \nu_j \geq a_{ij}$ *for all* (f_i, w_j) *with* $x_{ij} = 0$.

If there is a pair (f_i, w_j) such that $\mu_i + \nu_j < a_{ij}$ we say that (f_i, w_j) causes an instability in x.

The proof of Proposition 1 below, from Sotomayor (1992), is reproduced in Appendix I.

Definition 5. *We say that the feasible matching* x *is* **optimal** *if* $\sum_{(f_i, w_j) \in FxW} a_{ij} x_{ij} \geq \sum_{(f_i, w_j) \in FxW} a_{ij} x'_{ij}$, *for every feasible matching* x'.

Proposition 1. *Let* $(u, v; x)$ *be a stable outcome. Then* x *is an optimal matching.*

3. Structure of the stable outcomes

In this section we will prove our key result. For its statement and proof we define $M \equiv \{(f_i, w_j) \in FxW; x_{ij} \neq 0\}$ and $M' \equiv \{(f_i, w_j) \in FxW; x'_{ij} \neq 0\}$. We also use the following abbreviations: $C(f_i, x) \equiv C(f_i)$, $C(f_i, x') \equiv C'(f_i)$, $C(w_j, x) \equiv C(w_j)$ and $C(w_j, x') \equiv C'(w_j)$.

Theorem 1. *Let* $(u, v; x)$ *be a stable outcome. Let* x' *be an optimal matching. Then*

a) $u_{ij} = \mu_i$ *and* $v_{ij} = \nu_j$ *for all* $(f_i, w_j) \in M - M'$.
b) *There exists a stable outcome* $(u', v'; x')$ *such that* $\mu'_i = \mu_i$ *and* $\nu'_j = \nu_j$ *and* $u'_{ij} = u_{ij}$ and $v'_{ij} = v_{ij}$ *for all* $(f_i, w_j) \in M \cap M'$.

Proof: If $x = x'$ then the result is trivially true. Thus suppose that $x \neq x'$. We can write:

$$\sum_{(f_i,w_j)\in FxW} a_{ij}x_{ij} = \sum_{(f_i,w_j)\in M-M'} a_{ij} + \sum_{(f_i,w_j)\in M\cap M'} a_{ij}$$
$$= \sum_{(f_i,w_j)\in M-M'} (u_{ij}+v_{ij}) + \sum_{(f_i,w_j)\in M\cap M'} a_{ij}.$$

Here, the second equality is due to the feasibility of $(u, v; x)$. But

$$\sum_{(f_i,w_j)\in M-M'} (u_{ij}+v_{ij}) = \sum_{f_i\in F}\ \sum_{w_j\in C(f_i)-C'(f_i)} u_{ij} + \sum_{w_j\in W}\ \sum_{f_i\in C(w_j)-C'(w_j)} v_{ij}$$
$$\geq \sum_{f_i\in F}\ \sum_{w_j\in C(f_i)-C'(f_i)} \mu_i + \sum_{w_j\in W}\ \sum_{f_i\in C(w_j)-C'(w_j)} v_j \qquad (1)$$
$$= \sum_{f_i\in F} |C(f_i) - C'(f_i)|\mu_i + \sum_{w_j\in W} |C(w_j) - C'(w_j)|v_j$$
$$= \sum_{f_i\in F} |C'(f_i) - C(f_i)|\mu_i + \sum_{w_j\in W} |C'(w_j) - C(w_j)|v_j$$
$$= \sum_{(f_i,w_j)\in M'-M} (\mu_i + v_j)$$
$$\geq \sum_{(f_i,w_j)\in M'-M} a_{ij} \qquad (2)$$

where inequality (1) follows from the definitions $\mu_i = \min u_{ij}$ and $v_j = \min v_{ij}$ and inequality (2) follows from the stability of $(u, v; x)$. Therefore,

$$\sum_{(f_i,w_j)\in FxW} a_{ij}x_{ij} \geq \sum_{(f_i,w_j)\in M'-M} a_{ij} + \sum_{(f_i,w_j)\in M\cap M'} a_{ij}$$
$$= \sum_{(f_i,w_j)\in FxW} a_{ij}x'_{ij} \qquad (3)$$

However since x' is an optimal matching we must have equality in (3) hence equality in (1) and (2). From (2) we have that $\mu_i + v_j = a_{ij}$ for all $(f_i, w_j) \in M' - M$ and from (1) we have that $u_{ij} = \mu_i$ and $v_{ij} = v_j$ for all $(f_i, w_j) \in M - M'$, so $\mu_i + v_j = a_{ij}$ for all $(f_i, w_j) \in M - M'$.

Now define the outcome $(u', v'; x')$ by $u'_{ij} = u_{ij}$, $v'_{ij} = v_{ij}$ for $(f_i, w_j) \in M \cap M'$ and $u'_{ij} = \mu_i$, $v'_{ij} = v_j$ for $(f_i, w_j) \in M' - M$. Hence $\mu'_i = \mu_i$ and $v'_j = v_j$ and $(u', v'; x')$ is clearly stable. ■

For the rest of the paper we need the following terminology:

Definition 6. *Given $f_i \neq f_0$ and $w_j \neq w_0$, we say that (f_i, w_j) is a* **nonessential partnership** *if there exist two optimal matchings, x and x', with $x_{ij} = 1$ and $x'_{ij} = 0$. If (f_i, w_j) occurs in every optimal matching we say that it is an* **essential partnership**. *For unmatched players we say that (f_i, f_0) (resp. (w_0, w_j)) is a nonessential partnership if it occurs in some optimal matching.*

The following result is an immediate consequence of Theorem 1 and Proposition 1.

Corollary. *Let $(u, v; x)$ and $(u', v'; x')$ two stable outcomes with $x \neq x'$. Suppose player f_i completes his quota under $(u, v; x)$ and has one unfilled position under $(u', v'; x')$. Let w_j (resp. w_k) be a nonessential partner of f_i under x (resp. x'). Then $u_{ij} = u'_{ik} = 0$.*

4. Mathematical structure of the set of stable payoffs

It is easy to see that the set of stable payoffs for a given matching x forms a lattice. That is, if $(u, v; x)$ and $(u', v'; x)$ are stable outcomes, and if $u^*_{ij} = \max\{u_{ij}, u'_{ij}\}$ and $v{*}_{ij} = \min\{v_{ij}, v'_{ij}\}$, for all matched pairs (f_i, w_j), then $(u^*, v{*}; x)$ is stable and, symmetrically, so is $(u{*}, v^*; x)$.

If (u, v) and (u', v') are associated to different matchings then there is no meaning to consider $\sup\{u, u'\}$ or $\inf\{u, u'\}$, since u and u' are defined on different sets of indices. However, by using Theorem 1, we can represent the set of stable payoffs of a player as a vector in a Euclidean space, whose dimension is the quota of the given player. This representation is independent of the matching. By ordering the players in F (resp. W), we can immerse the set of stable payoffs for these players in a Euclidean space, whose dimensions is the sum of the quotas of all players in F (resp. W). Then the natural partial order of this Euclidean space induces a partial order in the set of stable payoffs. Now it makes sense to ask if the set of stable payoffs is a compact and convex lattice.

Before proceeding, let us pause to consider what we have learned from Theorem 1. Consider a stable outcome $(u, v; x)$. If player f_i has quota 5 and forms partnerships with w_1, w_2, w_3, w_4 and w_5, then the set of payoffs of f_i is given by $\{u_{i1}, u_{i2}, u_{i3}, u_{i4}, u_{i5}\}$. In case all partnerships are essential, we can fix any ordering we want and represent all sets of payoffs of player f_i by vectors in the same Euclidean space. Recall that part (a) of Theorem 1 assures that under a given stable outcome all the nonessential partners of a player contribute with the same payoff. Therefore, for instance, if the only essential partners of player f_i are w_1 and w_2, we can choose the following ordering for the 5-tuples of u_{ij}'s: $(u_{i1}, u_{i2}, \mu_i, \mu_i, \mu_i)$, where $\mu_i = \mu_{i3} = \mu_{i4} = \mu_{i5}$. Thus under any other stable outcome $(u', v'; x')$ the vector of payoffs of player f_i can be represented by $(u'_{i1}, u'_{i2}, \mu'_i, \mu'_i, \mu'_i)$. Moreover, if $(u', v'; x')$ is the outcome associated with $(u, v; x)$ by Theorem 1-b, $u'_{i1} = u_{i1}, u'_{i2} = u_{i2}$ and $\mu'_i = \mu_i$. That is, whatever optimal matching is being considered, the vector of payoffs for player f_i under $(u, v; x)$ is the same as under $(u', v'; x')$.

From now on the arrays of numbers u and v will be represented by a pair of vectors, still denoted by u and v. Thus we will keep the same notation $(u, v; x)$ of the outcome under the new representation. Theorem 1 has now the desired meaning:

If $(u, v; x)$ is a stable outcome and x' is an optimal matching, then $(u, v; x')$ is also stable.

Definition 7. *We say that the payoff vector (u, v) is* **stable** *if there is some feasible matching x such that $(u, v; x)$ is a stable outcome. In this case x is said to be compatible with (u, v) and vice-versa.*

Hence *if (u, v) is a stable payoff then it is compatible with any optimal matching*. This is an important property which is also shared with the assignment game of Shapley and Shubik. However, it is not characteristic of all one-to-one matching models. (See Demange and Gale (1985)). Two consequences of this fact are Propositions 2 and 3, below.

Proposition 2. *The set of stable payoffs is compact and convex.*

Proof: The set of stable payoffs is the same for any optimal matching. Let x be an optimal matching. The set of stable payoffs is the solution of a system of linear non strict inequalities associated with x, so it is closed and convex. That it is bounded follows from the fact that for all matched pairs (f_i, w_j) under x,

$$0 \le u_{ij} \le a_{ij} \quad \text{and} \quad 0 \le v_{ij} \le a_{ij}.$$

Hence the set of stable payoffs is convex and compact. ■

Proposition 3 proves that there is a polarization of interests between the two sides of the market within the whole set of stable payoffs. That is, the "best" for one side is the "worst" for the other side. We need:

Lemma 1. *Let (u, v) and (u', v') be stable payoffs. Then:*

a) *If (f_i, w_j) is an essential partnership, $u_{ij} > u'_{ij}$ if and only if $v_{ij} < v'_{ij}$*
b) *If (f_i, w_j) is a nonessential partnership, $\mu_i > \mu'_i$ if and only if $v_j < v'_j$.*

Proof: The result follows easily from the fact that if (f_i, w_j) is essential then $u_{ij} + v_{ij} = a_{ij} = u'_{ij} + v'_{ij}$, and if (f_i, w_j) is nonessential then (u, v) and (u', v') are compatible with any optimal matching, so $\mu_i + v_j = a_{ij} = \mu'_i + v'_j$. ■

Observe that Lemma 1-a) does not require the existence of more than one optimal matching.

Proposition 3. *Let (u, v) and (u', v') be stable payoffs. Then $u \ge u'$ if and only if $v' \ge v$.*

Proof: In order to compare two vectors we have to compare its coordenates. By Lemma 1-a

$$u_{ij} \ge u'_{ij} \quad \Leftrightarrow \quad v'_{ij} \ge v_{ij}, \quad \text{if } (f_i, w_j) \text{ is essential.} \tag{1}$$

By Lemma 1-b

$$\mu_i \ge \mu'_i \quad \Leftrightarrow \quad v'_j \ge v_j, \quad \text{if } (f_i, w_j) \text{ is nonessential.} \tag{2}$$

Now use (1) and (2) to conclude the result. ■

Given the stable payoffs (u, v) and (u', v'), define $(u^*, v*)$ and $(u*, v^*)$ as follows: For all essential partnerships (f_i, w_j):

$$u^*_{ij} = \max\{u_{ij}, u'_{ij}\}, \quad u*_{ij} = \min\{u_{ij}, u'_{ij}\},$$

$$v^*_{ij} = \max\{v_{ij}, v'_{ij}\}, \quad \text{and} \quad v*_{ij} = \min\{v_{ij}, v'_{ij}\}.$$

For all $f_i \in F$ and $w_j \in W$:

$$\mu^*_i = \max\{\mu_i, \mu'_i\}, \quad \mu*_i = \min\{\mu_i, \mu'_i\},$$

$$v^*_j = \max\{\mu_j, v'_j\} \quad \text{and} \quad v*_j = \min\{v_j, v'_j\}.$$

It is clear that $u^*_{ij} \geq \mu^*_i$, $u*_{ij} \geq \mu*_i$, $v^*_{ij} \geq v^*_j$ and $v*_{ij} \geq v*_j$, for all (f_i, w_j).

It now makes sense to ask: If (u, v) and (u', v') are stable payoffs, are $(u^*, v*)$ and $(u*, v^*)$ also stable payoffs? The answer is affirmative:

Theorem 2. *Let (u, v) and (u', v') be stable payoffs. Then the payoffs $(u^*, v*)$ and $(u*, v^*)$, as defined above, are stable.*

Proof: We will show that $(u^*, v*)$ is stable; the other assertion follows dually. Let x be some optimal matching. We already know that (u, v) and (u', v') are compatible with x. To see that there is no pair causing instabilities, we have to show that $\mu^*_i + v*_j \geq a_{ij}$, for all $f_i \in F$ and $w_j \in W$ with $x_{ij} = 0$. Due to the stability of (u, v) and (u', v') the only cases we have to check are those where

$$\mu^*_i = \mu_i \quad \text{and} \quad v*_j = v'_j \quad \text{or} \quad \mu^*_i = \mu'_i \quad \text{and} \quad v*_j = v_j.$$

We have either:

$$\mu^*_i + v*_j = \mu_i + v'_j \geq \mu'_i + v'_j \geq a_{ij} \quad \text{or} \quad \mu^*_i + v*_j = \mu'_i + v_j \geq \mu_i + v_j \geq a_{ij}.$$

Since the payoffs are clearly non-negative, it remains to show that if $x_{ij} \neq 0$ the gain of (f_i, w_j) is a_{ij}. But it is immediate from Lemma 1-a that, if (f_i, w_j) is essential, then

$$u^*_{ij} + v*_{ij} = u_{ij} + v_{ij} = a_{ij} \quad \text{or} \quad u^*_{ij} + v*_{ij} = u'_{ij} + v'_{ij} = a_{ij}.$$

Finally, if (f_i, w_j) is nonessential, it follows from Lemma 1-b that:

$$\mu^*_i + v*_j = \mu_i + v_j = a_{ij} \quad \text{or} \quad \mu^*_i + v*_j = \mu'_i + v'_j = a_{ij}.$$ ■

Given two stable payoffs (u, v) and (u', v'), the construction of the payoffs $(u^*, v*)$ and $(u*, v^*)$ leads naturally to the following relations:

$$(u, v) \geq_F (u', v') \quad \Leftrightarrow \quad u \geq u' \quad \text{and} \quad (u, v) \geq_W (u', v') \quad \Leftrightarrow \quad v \geq v'.$$

The relations $\geq_F$ and $\geq_W$ are clearly partial orders. From Proposition 3 it follows that one partial order is the dual of the other and $(u^*, v*) =$

$\sup_F\{(u,v),(u',v')\} = \inf_W\{(u,v),(u',v')\}$ and $(u*,v^*) = \inf_F\{(u,v),(u',v')\}$ $= \sup_W\{(u,v),(u',v')\}$.

Theorem 3. *The set of stable payoffs is a convex and complete lattice under the partial orders* $\geq_F$ *and* $\geq_W$.

Proof: By Theorem 2 it follows that, if (u,v) and (u',v') are stable payoffs, then $(u^*,v*) = \sup_F\{(u,v),(u',v')\}$ and $(u*,v^*) = \inf_F\{(u,v),(u',v')\}$ are stable payoffs. Thus the set of stable payoffs is a lattice under $\geq_F$. By Proposition 2, the set of stable payoffs is convex and compact, thus it is a convex and complete lattice under $\geq_F$. Now use the duality of the partial orders to show the other assertion. ■

It is clear from the definition of $\geq_F$ that if $(u,v) \geq_F (u',v')$ then $\sigma_i \geq \sigma'_i$ for all $f_i \in F$. (Recall that σ_i is the i's total payoff under $(u,v;x)$). So, every player in F weakly prefers (u,v) to (u',v'). However, it can be easily seen that $\sigma_i \geq \sigma'_i$, for all $f_i \in F$, does not imply $(u,v) \geq_F (u',v')$. Consequently our partial order does not always coincide with the preferences of the players in F. When one of the stable payoffs that is being compared is the maximal element of the lattice under $\geq_F$, Theorem 4 shows that this partial order agrees with the preferences of the players in F. This fact implies that the maximal element of the lattice under $\geq_F$ is the only stable payoff which is weakly preferred by all players in F to any other stable payoff. This payoff is called the F-optimal stable payoff. Formally we have:

Definition 8. *A stable payoff is called an F-optimal stable payoff if every player in F weakly prefers it to any other stable payoff. (That is, the F-optimal stable payoff gives to each player in F the maximum total payoff among all stable payoffs). Similarly we define a W-optimal stable payoff.*

Theorem 4. *There exists one and only one F (resp. W)-optimal stable payoff. Furthermore, every* $w_j \in W$ *(resp.* $f_i \in F$*) weakly prefers every stable payoff the F (resp. W)-optimal stable payoff.*

Proof: We will show that a stable payoff is the maximal element of the lattice under $\geq_F$ if and only if it is an F-optimal stable payoff. Since every complete lattice has one and only one maximal element, we will have proved the existence and uniqueness of the F-optimal stable payoff. The other assertion will follow dually. Then let (u^+,v^-) be the maximal element of the lattice under $\geq_F$. Let (u,v) be any F-optimal stable payoff. The maximality of (u^+,v^-) implies that $(u^+,v^-) \geq_F (u,v)$, so

(1) $u^+_{ij} \geq u_{ij}$ for all essential partnerships (f_i,w_j) and
(2) $\mu^+_i \geq \mu_i$ for all $f_i \in F$. Hence
(3) $\sigma^+_i \geq \sigma_i$ for all $f_i \in F$.

However since (u,v) is an F-optimal stable payoff we must have equality in (3) hence equality in (1) and (2). From (1) and (2) we have that $(u^+,v^-) = (u,v)$. To see that every $w_j \in W$ weakly prefers every stable payoff to the F-optimal stable payoff, take (u,v) stable and use Proposition 3 to get that $v \geq v^-$. Then $v_j \geq v^-_j$ for al $w_j \in W$. ■

Our last result is related to situations in which all the players on a given side have to choose the partnerships corresponding to their highest payoffs, respecting their quotas, among those presented to them by two stable outcomes. It is not obvious that the resulting set of partnerships will be feasible and also an optimal matching. Furthermore, this matching gives to the players of the opposite side their partners corresponding to their lowest payoffs. We need the following:

Lemma 2. *Let* (u, v) *and* (u', v') *be stable payoffs. Set* $S^i = \{w_j \in W; (f_i, w_j)$ *is nonessential*$\}$ *and* $S_j = \{f_i \in F; (f_i, w_j)$ *is nonessential*$\}$. *Let* $F^1 = \{f_i \in F; \mu_i > \mu_i'$ *and* $S^i \neq \phi\}, F^2 = \{f_i \in F; \mu_i' > \mu_i$ *and* $S^i \neq \phi\}, F^0 = \{f_i \in F; \mu_i = \mu_i'$ *and* $S^i \neq \phi\}$. *Define* W^1, W^2 *and* W^0 *analogously. It then follows*:

a) $F^1 = \bigcup S_j$, *for* $w_j \in W^2$ *and* $W^2 = \bigcup S^i$, *for* $f_i \in F^1$.
b) $F^2 = \bigcup S_j$, *for* $w_j \in W^1$ *and* $W^1 = \bigcup S^i$, *for* $f_i \in F^2$.
c) $F^0 = \bigcup S_j$, *for* $w_j \in W^0$ *and* $W^0 = \bigcup S^i$, *for* $f_i \in F^0$.

Proof: From Lemma 1, if (f_i, w_j) is nonessential then
$f_i \in F^1 \Leftrightarrow \mu_i > \mu_i' \Leftrightarrow v_j < v_j' \Leftrightarrow w_j \in W^2$,
$f_i \in F^2 \Leftrightarrow \mu_i < \mu_i' \Leftrightarrow v_j > v' w_j \Leftrightarrow w_j \in W^1$ and
$f_i \in F^0 \Leftrightarrow \mu_i = \mu_i' \Leftrightarrow v_j = v_j' \Leftrightarrow w_j \in W^0$.
The result is then immediate. ■

Note that Lemma 2 asserts that F^1 (resp. F^2, F^0) is the set of nonessential partners of the players in W^2 (resp. W^1, W^0) under any optimal matching. Therefore there is a decomposition of the set of nonessential partnerships into three disjoint sets: A, B and C. That is, $A \cup B \cup C$ is the set of all nonessential partnerships, where $A = \{$nonessential partnerships $(f_i, w_j); f_i \in F^1\}$, $B = \{$nonessential partnerships $(f_i, w_j); f_i \in F^2\}$ and $C = \{$nonessential partnerships$\}$ $(f_i, w_j); f_i \in F^0\}$.

Theorem 5. *Let* $(u, v; x)$ *and* $(u', v'; x')$ *be stable outcomes. Let* F^1 *be as in Lemma 2. Then, the matchings* x^* *and* $x*$, *defined by* $x^*_{ij} = x_{ij}$ if $f_i \in F^1$ *and* $x^*_{ij} = x'_{ij}$ *otherwise and* $x*_{ij} = x'_{ij}$ *if* $f_i \in F^1$ *and* $x*_{ij} = x_{ij}$ *otherwise, are optimal.* (*Observe that for all essential partnerships* (f_i, w_j), $x^*_{ij} = x*_{ij} = x_{ij} = x'_{ij}$).

Proof: We will prove that x^* is optimal. The other assertion follows dually. Then, by Proposition 1 it is enough to show that $(u^*, v*, x^*)$ is a stable outcome.

First observe that $C(f_i, x^*) = C(f_i, x)$ if $f_i \in F^1$ and $C(f_i, x^*) = C(f_i, x')$ if $f_i \notin F^1$. Thus $|C(f_i, x^*)| = r_i$ for all f_i. By the construction of x^* it follows that all essential partners of $w_j \in W$ are also matched to w_j under x^*. From Lemma 2 it follows that all nonessential partners of $w_j \in W^2$ (resp. $w_j \notin W^2$) are in F^1 (resp. $F^2 \cup F^0$), so w_j is matched at x^* according to x (resp. x'). Hence $|C(w_j, x^*)| = |C(w_j, x)| = s_j$ (resp. $|C(w_j, x^*)| = |C(w_j, x')| = s_j$) and so x^* is feasible.

To see that $(u^*, v*; x^*)$ is feasible use the definition of x^* and $(u^*, v*)$ to get that the gain of the pair (f_i, w_j) is a_{ij} when $x^*_{ij} > 0$. (One can consider the cases i) (f_i, w_j) is essential, ii) (f_i, w_j) is nonessential with $f_i \in F^1$ and iii) (f_i, w_j) is nonessential with $f_i \notin F^1$).

Now observe that if $x^*_{i0} > 0$ then (f_i, w_0) is nonessential with $f_i \notin F^1$. This is because if $f_i \in F^1$ then $\mu_i > 0$, from which follows that f_i has filled his quota under x, so $0 < x^*_{i0} \neq x_{i0} = 0$, which contradicts the definition of x^*. Hence, if $x^*_{i0} > 0$ then $x'_{i0} > 0$ and $0 = \mu'_i \geq \mu_i \geq 0$, which implies that $\mu^*_i = 0$. Similarly, if $x^*_{0j} > 0$ then $v*_j = 0$. Hence is $(u^*, v*; x^*)$ is feasible.

For the stability, since (u, v) and (u', v') are compatible with both matchings, x and x', it follows from Theorem 2 that $(u^*, v*; x)$ and $(u^*, v*; x')$ are stable outcomes. Then use the definition of x^* to see that there is no pair causing instabilities in x^*, Hence the proof is complete. ■

To see what we have learned with the results presented here, consider the following example:

Example. Let $F = \{f_0, f_1, f_2, f_3\}$, $W = \{w_0, w_1, \ldots, w_5\}$, $r_1 = 3$, $r_2 = 1$, $r_3 = 2$, $s_2 = 2$, $s_1 = s_3 = s_4 = s_5 = 1$. The matrix (a_{ij}) with $i = 0, 1, \ldots, 3$ and $j = 0, 1, \ldots, 5$ is given by:

$$a = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 5 & 3 & 1 & 2 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 6 & 0 & 7 & 5 & 6 \end{pmatrix}$$

There are three optimal matchings:

$$x = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \quad x' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$x'' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We can compute that $\sum_{(f_i, w_j) \in FxW} a_{ij}x_{ij} = \sum_{(f_i, w_j) \in FxW} a_{ij}x'_{ij} = \sum_{(f_i, w_j) \in FxW} a_{ij}x''_{ij} = 25$. The essential partnerships are:
(f_1, w_2) and (f_2, w_2). The nonessential partnerships are: (f_1, w_1), (f_1, w_3), (f_3, w_4), (f_3, w_5) under the matching x,
$(f_1, w_4), (f_1, w_5), (f_3, w_1), (f_3, w_3)$ under the matching x' and
$(f_1, w_3), (f_1, w_4), (f_3, w_1), (f_3, w_5)$ under the matching x''.

From Theorem 1 we can conclude that any outcome $(u, v; x)$ with $u_{11} \neq u_{13}$, or $u_{34} \neq u_{35}$, or $u_{11} > u_{12}$, for example, is unstable. We can verify the conclusions of Theorem 1 in the following stable outcomes: $(u, v; x)$, $(u', v'; x')$ and $(u'', v''; x'')$, where the payoffs are given by:

$$u_{12}=3,\quad u_{11}=1,\quad u_{13}=1,\quad u_{22}=2,\quad u_{34}=5,\quad u_{35}=5$$

$$u'_{12}=5,\quad u'_{14}=0,\quad u'_{15}=0,\quad u'_{22}=4,\quad u'_{31}=4,\quad u'_{33}=4$$

$$u''_{12}=0,\quad u''_{13}=0,\quad u''_{14}=0,\quad u''_{22}=3,\quad u''_{31}=4,\quad u''_{35}=4$$

$$v_{11}=1,\quad v_{12}=2,\quad v_{22}=2,\quad v_{13}=2,\quad v_{34}=0,\quad v_{35}=1$$

$$v'_{31}=2,\quad v'_{12}=0,\quad v'_{22}=0,\quad v'_{33}=3,\quad v'_{14}=1,\quad v'_{15}=2$$

$$v''_{31}=2,\quad v''_{12}=5,\quad v''_{22}=1,\quad v''_{13}=3,\quad v''_{14}=1,\quad v''_{35}=2$$

Thus, since (f_1, w_4) and (f_1, w_5) are nonessential partnerships under x', and (f_1, w_2) is essential, $u'_{14} = u'_{15} = 0 < u'_{12} = 5$.

Theorem 1-b asserts that we can reindex the payoffs of these three stable outcomes according to any one of the optimal matchings. Using the representation given by:

$$u=(u_{12}=3,\mu_1=1,\mu_1=1,u_{22}=2,\mu_3=5,\mu_3=5);$$

$$u'=(u'_{12}=5,\mu'_1=0,\mu'_1=0,u'_{22}=4,\mu'_3=4,\mu'_3=4);$$

$$u''=(u''_{12}=0,\mu''_1=0,\mu''_1=0,u''_{22}=3,\mu''_3=4,\mu''_3=4);$$

$$v=(v_1=1,v_{12}=2,v_{22}=2,v_3=2,v_4=0,v_5=1);$$

$$v'=(v'_1=2,v'_{12}=0,v'_{22}=0,v'_3=3,v'_4=1,v'_5=2);$$

$$v''=(v''_1=2,v''_{12}=5,v''_{22}=1,v''_3=3,v''_4=1,v''_5=2);$$

Theorem 1 implies that $(u, v), (u', v')$ and (u'', v'') are all compatible with the matchings x, x' and x''.

Under the vectorial representation, the set of payoffs for player f_1 is represented by the vector $(3, 1, 1)$ in the first outcome, by $(5, 0, 0)$ in the second outcome and by $(0, 0, 0)$ in the third one. In this way p_1 can find the supremum and the infimum of two of these vectors by comparing each coordinate.

With Theorem 2 we learned to construct new stable payoffs: $(u^1, v^1) = \sup_F\{(u, v), (u', v')\}$, where $u^1 = (u^1_{12} = 5, \mu^1_1 = 1, \mu^1_1 = 1, u^1_{22} = 4, \mu^1_3 = 5, \mu^1_3 = 5)$ and $v^1 = (v^1_1 = 1, v^1_{12} = 0, v^1_{22} = 0, v^1_3 = 2, v^1_4 = 0, v^1_5 = 1)$; $(u^2, v^2) = \inf_F\{(u, v), (u', v')\}$, where $u^2 = (u^2_{12} = 3, \mu^2_1 = 0, \mu^2_1 = 0, u^2_{22} = 2, \mu^2_3 = 4, \mu^2_3 = 4)$ and $v^2 = (v^2_1 = 2, v^2_{12} = 2, v^2_{22} = 2, v^2_{13} = 3, v^2_4 = 1, v^2_5 = 2)$.

Theorem 4 guarantees the existence and uniqueness of the F- and W-optimal stable payoffs. The F-optimal stable payoffs is given by $u^* = (u^*_{12} = 5, \mu^*_1 = 1, \mu^*_1 = 1, u^*_{22} = 4, \mu^*_3 = 5, \mu^*_3 = 5)$ and $v* = (v*_1 = 1, v*_{12} = 0\ v*_{22} = 0\ v*_{13} = 2\ v*_4 = 0\ v*_5 = 1)$ and the W-optimal stable payoff is $u* = (u*_{12} = 0, \mu*_1 = 0, \mu*_1 = 0, u*_{22} = 0, \mu*_3 = 4, \mu*_3 = 4)$ and $v^* = (v^*_1 = 2, v^*_{12} = 5, v^*_{22} = 4, v^*_{13} = 3, v^*_4 = 1, v^*_5 = 2)$.

Theorem 5 asserts that the players can also choose their best partnerships. By comparing (u, v) with (u', v') the players in F will choose the matching x; the best partners for the players in W will form the matching x'.

To see the decomposition in the set of nonessential partnerships given by Lemma 2, consider (u, v) and (u', v'). Then
$F^1 = \{f_i \in F; \mu_i > \mu_i' \text{ and } S^i \neq \phi\} = \{f_1, f_3\}$,
$W^2 = \{w_j \in W; v_j' > v_j' \text{ and } S_j \neq \phi\} = \{w_1, w_3, w_4, w_5\}$ and
$F^2 = \phi$ and $W^1 = \phi$ because $S^2 = \phi$ and $S_2 = \phi$.
Note that any nonessential partnership is formed with players in F^1 and W^2. Now consider (u', v') and (u'', v''). We can see that
$F^0 = \{f_i \in F; \mu_i'' = \mu_i' \text{ and } S^i \neq \phi\} = \{f_1, f_3\}$ and
$W^0 = \{w_j \in W; v_j'' = v_j' \text{ and } S_j \neq \phi\} = \{w_1, w_3, w_4, w_5\}$. ■

Appendix I

In this section we reproduce the proofs of the existence of stable outcomes and of Proposition 1. Both results appear in Sotomayor (1992).

Set $F' = F - \{f_0\}$, $W' = W - \{w_0\}$. Then $|F'| = m - 1$ and $|W'| = n - 1$. Consider the primal linear programming problem (P_1) of finding a matrix (x_{ij}^*) which

(a) maximizes $\sum_{(f_i, w_j) \in FxW} a_{ij} x_{ij}^*$

subject to:

(b) $\sum_{w_j \in W'} x_{ij}^* \leq r_i$, for all $f_i \in F'$,
(c) $\sum_{f_i \in F'} x_{ij}^* \leq s_j$, for all $w_j \in W'$ and
(d) $0 \leq x_{ij}^* \leq 1$, for all $(f_i, w_j) \in F'xW'$.

The dual problem (P_1^*) is to find an $(m-1)$-vector (y), an $(n-1)$-vector (z) and an $(m-1)x(n-1)$ matrix (w_{ij}) which

(a)* minimizes $\sum_{f_i \in F'} r_i y_i + \sum_{w_j \in W'} s_j z_j + \sum_{(f_i, w_j) \in F'xW'} w_{ij}$

subject to:

(b)* $y_i + z_j + w_{ij} \geq a_{ij}$, for all $(f_i, w_j) \in F'xW'$ and
(c)* $y_i \geq 0.\ z_j \geq 0, w_{ij} \geq 0$ for all $(f_i, w_j) \in F'xW'$.

By the Duality Theorem there is a solution of (P_1) and $(P_1)*$ such that $(a) = (a)^* = P(F' \cup W')$, the payoff of the coalition $F' \cup W'$. Furthermore, it can be shown that there is a matrix (x_{ij}^*) with integer entries which is an optimal solution of (P_1). Any such solution will be called an optimal assignment. It is easy to check that if $\sum_{(f_i, w_j) \in F'xW'} a_{ij} x_{ij}^* \geq \sum_{(f_i, w_j) \in F'xW'} a_{ij} x_{ij}^{*\prime}$ for all feasible assignments $x^{*\prime}$, then x^* is an optimal assignment.

If (x_{ij}^*) is any optimal assignment, then by the Linear Programming Equilibrium Theorem (see Gale, 1960), (y, z) is an optimal dual vector if and only if

(A) $\sum_{w_j \in W'} x^*_{ij} < r_i$ implies $y_i = 0$,
(B) $\sum_{f_i \in F'} x^*_{ij} < s_j$ implies $z_j = 0$,
(C) $x^*_{ij} = 0$ implies $y_i + z_j \geq a_{ij}$ and
(D) $x^*_{ij} = 1$ implies $y_i + z_j \geq a_{ij}$.

Furthermore, if (y, z, w) is any solution of (P^*_1), $w_{ij} = a_{ij} - (y_i + z_j)$, if $x^*_{ij} = 1$ and $w_{ij} = 0$, if $x^*_{ij} = 0$ from (a)*, (b)*, (A) and (B). Hence $\sum_{(f_i, w_j) \in F'xW'} \cdot (y_i + z_j + w_{ij})x^*_{ij} = \sum_{(f_i, w_j) \in F'xW'} a_{ij}x^*_{ij}$.

It is clear that any optimal assignment x^* can be extended to an optimal matching x for $M(F, W, a, r, s)$ such that: $x_{ij} = x^*_{ij}$, for all $(f_i, w_j) \in F'xW'$, $x_{00} = 0$, $\sum_{w_j \in W} x_{ij} = r_i$ for all $f_i \neq f_0$ and $\sum_{f_i \in F} x_{ij} = s_j$ for all $w_j \neq 0$.

Theorem 1*. *The set of stable outcomes for $M(F, W, a, r, s)$ is non-empty.*

Proof: Take any optimal dual vector (y, z) for (P^*_1). Define, for all $f_i \in F'$ and $w_j \in W'$, $v_{ij} = z_j$, $u_{ij} = a_{ij} - z_j$ if $x_{ij} = 1$ and $u_{i0} = u_{0j} = v_{i0} = v_{0j} = 0$, where $x = (x_{ij})$ is any optimal matching. Then, $u_{ij} + v_{ij} = a_{ij}$ if $x_{ij} > 0$. Clearly $v_{ij} \geq 0$ and $u_{ij} = a_{ij} - z_j \geq y_i \geq 0$, by (D), if $x_{ij} = x^*_{ij} = 1$. Then $(u, v; x)$ is feasible. To prove stability, suppose $x_{ij} = 0$. We have that $\mu_i = u_{ik}$ for some $w_k \in C(f_i, x)$ and $v_j = z_j$. Then, $\mu_i + v_j = u_{ik} + z_j \geq y_i + z_j \geq a_{ij}$, where the first inequality follows from the definition of u_{ik} and (D) and the last inequality follows from (C). ■

It remains to prove Proposition 1. We need lemmas A and B below.

Lemma A. *The feasible outcome $(u, v; x)$ is stable if and only if for all feasible matchings x' and all $f_i \in F$ we have that*

$$\sigma_i \geq \sum_{w_j \in W} (a_{ij} - v'_{ij})x'_{ij} \qquad (*)$$

where $v'_{ij} = v_j$ if $x_{ij} = 0$ and $v'_{ij} = v_{ij}$ if $x_{ij} > 0$. Symmetrical result applies to every $w_j \in W$.

Proof: Let $f_i \in F$. Let x' be any feasible matching. If $(u, v; x)$ is stable, for every new partnership (f_i, w_k) under x' (i.e., $x'_{ik} > 0$ and $x_{ik} = 0$) there is some old partnership dissolved (f_i, w_j) under x (i.e., $x_{ij} > 0$ and $x'_{ij} = 0$), for quota maintenance. To see that $(*)$ holds use the fact that (i) the number of new partnerships is equal to the number of old partnerships dissolved, for a given f_i; (ii) $u_{ij} \geq a_{ij} - v_j = a_{ij} - v'_{ij}$, by stability, if (f_i, w_j) is dissolved and (f_i, w_k) is created, and (iii) $u_{ij} = a_{ij} - v_{ij} = a_{ij} - v'_{ij}$, if (f_i, w_j) is maintained. Conversely, suppose that $(*)$ is true for all feasible matchings x' and all $f_i \in F$. If $(u, v; x)$ were not stable then there would be some pair (f_i, w_j) with $x_{ij} = 0$ and such that $u_{ik} < a_{ij} - v_{pj} = a_{ij} - v_j$ for some $w_k \in C(f_i, x)$ and some $f_p \in C(w_j, x)$. Let x' be a feasible matching such that $x'_{ij} = 1$ and $x'_{iq} = x_{iq}$ for all $q \neq k$, $x'_{tj} = x_{tj}$ for all $t \neq p$. It is clear that $\sigma_i < \sum_{w_q \neq w_j} (a_{iq} - v_{iq})x_{iq} + (a_{ij} - v_j)$. Thus $(*)$ would not hold for player f_i and x'. Hence $(u, v; x)$ is stable and the proof is complete. ■

Lemma B. *Let $(u, v; x)$ be a stable outcome. Let $R \subseteq F$, $S \subseteq W$ and x' be any feasible matching such that if $x'_{ij} = 1$ then $f_i \in R$ if and only if $w_j \in S$. Then* $\sum_{f_i \in R} \sigma_i + \sum_{w_j \in S} \tau_j \geq \sum_{(f_i, w_j) \in RxS} a_{ij} x'_{ij}$.

Proof: Adding up in Lemma A yields:
$\sum_{f_i \in R} \sigma_i \geq \sum_{f_i \in R} \sum_{w_j \in S} (a_{ij} - v'_{ij}) x'_{ij} \geq \sum_{(f_i, w_j) \in RxS} a_{ij} x'_{ij} - \sum_{(f_i, w_j) \in RxS} v'_{ij} x'_{ij} = \sum_{(f_i, w_j) \in RxS} a_{ij} x'_{ij} - \sum_{w_j \in S} \sum_{f_i \in F} v'_{ij} x'_{ij} \geq \sum_{(f_i, w_j) \in RxS} a_{ij} x'_{ij} - \sum_{w_j \in S} \cdot (\sum_{f_i \in F} v_{ij} x_{ij}) = \sum_{(f_i, w_j) \in RxS} a_{ij} x'_{ij} - \sum_{w_j \in S} \tau_j$. Then, $\sum_{f_i \in R} \sigma_i + \sum_{w_j \in S} \tau_j \geq \sum_{(f_i, w_j) \in RxS} a_{ij} x'_{ij}$, which completes the proof. ■

Proof of Proposition 1: Take $R = F$. $S = W$ in Lemma B. Now $\sum_{(f_i, w_j) \in FxW} a_{ij} x_{ij} = \sum_{f_i \in F} \sigma_i + \sum_{w_j \in W} \tau_j \geq \sum_{(f_i, w_j) \in FxW} a_{ij} x'_{ij}$ for all feasible matchings x'. ■

Appendix II. The robustness of the conclusions of Theorem 1

Our conclusions for several optimal matchings are not less robust than our conclusions for only one optimal matching. This fact is clarified below:

Theorem 2*. *Let $(u, v; x)$ be some stable outcome. Let x' be some feasible matching so that $\sum_{(f_i, w_j) \in FxW} a_{ij} x_{ij} \geq \sum_{(f_i, w_j) \in FxW} a_{ij} x'_{ij} \geq \sum_{(f_i, w_j) \in FxW} a_{ij} x''_{ij}$ for all feasible matchings $x'' \neq x$. (That is, x' is a "second" optimal matching). Suppose $\sum_{(f_i, w_j) \in FxW} a_{ij} x_{ij} - \sum_{(f_i, w_j) \in FxW} a_{ij} x'_{ij} = \lambda$. If $f_i \in F$ and $C(f_i, x) \neq C(f_i, x')$, then $u_{ij} - u_{ik} \leq \lambda$ for all $w_j \in C(f_i, x) - C(f_i, x')$ and all $k \in C(f_i, x)$. Symmetrical results apply to any $w_j \in W$.*

Proof: If $\lambda = 0$ then x and x' are optimal matchings and the result follows from Theorem 1. Suppose $\lambda > 0$. Assume that $w_j \in C(f_i, x) - C(f_i, x')$ and $u_{ij} > u_{ik} + \lambda$ for some $k \in C(f_i, x)$. Set $u'_{ij} = u_{ij} - \lambda$ and $u'_{tm} = u_{tm}$ for all $(f_t, w_m) \neq (f_i, w_j)$, with $x_{tm} > 0$. Then,

$$u'_{ij} > u'_{ik}. \tag{1}$$

It is a matter of verification to see that $(u', v; x)$ is stable for $M(a')$, where $a'_{ij} = a_{ij} - \lambda$ and $a'_{tm} = a_{tm}$ for all $(f_t, w_m) \neq (f_i, w_j)$. Using that $w_j \in C(f_i, x) - C(f_i, x')$ we get for all feasible x'' that: $\sum_{(f_i, w_m) \in FxW} a'_{tm} x_{tm} = \sum_{(f_i, w_m) \in FxW} a_{tm} x_{tm} - \lambda = \sum_{(f_t, w_m) \in FxW} a_{tm} x'_{tm} \geq \sum_{(f_t, w_m) \in FxW} a_{tm} x''_{tm} \geq \sum_{(f_t, w_m) \in FxW} a'_{tm} x''_{tm}$.

Hence x and x' are optimal matchings for $M(a')$. Therefore Theorem 1 implies that $u'_{ij} \leq u'_{ik}$, which contradicts (1). Consequently $u_{ij} - u_{ik} \leq \lambda$. For the last assertion use the symmetry of the model. ■

When there are several optimal matchings for the market $M(a)$, we can obtain a market, $M(a')$, by changing the matrix (a_{ij}) so that $M(a')$ has only one optimal matching x. Theorem 2* implies that when a' is obtained by a small perturbation of the matrix a, by taking a very small λ, u_{ij} cannot be "very different" from u_{ik}, when (f_i, w_j) and (f_i, w_k) are nonessential partnerships at x. (That is, $|u_{ij} - u_{ik}| < \lambda$). Furthermore, if (f_i, w_q) is an essential partnership and (f_i, w_j) is nonessential, then u_{iq} cannot be "much smaller" than u_{ij}. (That is, $u_{ij} - u_{iq} \leq \lambda$).

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