The Assignment Game I: The Core[1])

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Abstract: The assignment game is a model for a two-sided market in which a product that comes in large, indivisible units (e.g., houses, cars, etc.) is exchanged for money, and in which each participant either supplies or demands exactly one unit. The units need not be alike, and the same unit may have different values to different participants. It is shown here that the outcomes in the *core* of such a game – i.e., those that cannot be improved upon by any subset of players – are the solutions of a certain linear programming problem dual to the optimal assignment problem, and that these outcomes correspond exactly to the price-lists that competitively balance supply and demand. The geometric structure of the core is then described and interpreted in economic terms, with explicit attention given to the special case (familiar in the classic literature) in which there is no product differentiation – i.e., in which the units are interchangeable. Finally, a critique of the core solution reveals an insensitivity to some of the bargaining possibilities inherent in the situation, and indicates that further analysis would be desirable using other game-theoretic solution concepts.

1. Introduction

1.1. Two-Sided Market Games

Two-sided market models are important, as COURNOT, EDGEWORTH, BÖHM-BAWERK, and others have observed[4]), not only for the insights they may give into more general economic situations with many types of traders, consumers, and producers, but also for the simple reason that in real life many markets and most actual transactions are in fact bilateral – i.e., bring together a buyer and a seller of a single commodity. Modern game-theoretic concepts, when applied to even the most elementary economic models, have often yielded suggestive results, sometimes reinforcing and sometimes challenging the more traditional doctrines based on behavioristic theories of the individual[5]). The present study, of which this paper is the first part, will concern a class of simple, two-sided market "games" whose distinctive feature is the indivisibility and the ability to satiate of the goods for sale, e.g. houses or automobiles, so that the primary object of the game is simply to find suitable "assignments" of buyers to sellers[6]).

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[4]) See references at the end of this paper.

[5]) See e.g., DEBREU and SCARF; SHAPLEY 1959; SHAPLEY and SHUBIK 1959; SHAPLEY and SHUBIK W. Y.; SHUBIK.

[6]) This model was first treated by SHAPLEY in 1955; the present account has been extracted from the manuscript of a book in preparation. A short, nongame-theoric account will be found in GALE [pp. 160–162].

We intend to explore the properties of such assignment games from several different solutional viewpoints. In this first part we shall concentrate on the *core* of the game – i.e., the set of outcomes that no coalition can improve upon [1]).

1.2. The Underlying Economic Assumptions

The assumptions of our model, though restrictive in many respects, do permit considerable latitude in size and structure. There may be many or few traders on either side of the market, there may be product differentiation, and the traders themselves may be quite dissimilar in their likes and dislikes. Thus, a wide range of specific models, from a situation where, say, two or three firms are bidding for the same building site, to a large "noisy" milieu like the private-party market in used cars, are encompassed within the same framework and can be given a more or less uniform treatment.

Let us list the main economic assumptions:

1. Utility is identified with money.
2. Side payments are permitted.
3. The objects of trade are indivisible.
4. Supply and demand functions are inflexible.

This is not the place for extended comment on the first assumption [2]). We do, however, stress one point. In the open marketplace, where there are many individuals buying and selling for money (often acting as fiduciaries or trustees – i.e., using other people's money), the full monetization of utility comes closest to a practical realization. Here if anywhere the assumption of "u-money" is defensible.

The second assumption is largely one of convenience in modeling and analysis. By permitting free transfer of money among all participants we avoid the necessity of providing by special rules for the ordinary payments from a customer to his supplier. To be sure, we are also permitting payments to third parties, i.e., "side payments" in ordinary parlance. But it is worth noting that the avoidance of such payments, in situations where they are in fact avoided, is generally more a matter of custom or ethics than of hard and fast rules. The *core* solution, as it turns out, excludes third-party payments. We shall see latter that the *value* solution usually requires third-party payments, for reasons related to the discussion in Sec. 5 below [3]). The *stable-set* solutions, which are interpreted as "standards of behavior" in the VON NEUMANN-MORGENSTERN theory, have it both ways; some standards of behavior allow third-party payments while others do not [4]).

The third assumption is rather unusual, since it is more often the opposite

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[1]) For treatments of other solution concepts in certain special cases of the assignment game see SHAPLEY [1955 and 1961] and SHAPLEY and SHUBIK [1969], also VON NEUMANN and MORGENSTERN [pp. 555–586].

[2]) See for example SHAPLEY and SHUBIK [1966, pp. 807–808].

[3]) See also SHAPLEY [1961]; SHAPLEY and SHUBIK [1969].

[4]) See SHAPLEY [1959] and VON NEUMANN and MORGENSTERN *loc. cit.*

hypothesis (i.e., fungibility and perfect divisibility) that is imposed in the name of mathematical simplification. Indivisibility is usually considered an "imperfection" in the market – a technical and conceptual difficulty. Indeed, our present approach leads us towards combinatorics and linear programming, and away from the differential calculus methods that pervade so much of traditional economics. There is no dearth of economic applications under this assumption, however; a classic example is BÖHM-BAWERK's horse market (see Sec. 4)[1]. Our model is significantly more general, however, since it permits differentiation among the items in trade.

Inflexibility, the fourth assumption, in a way includes the third. Having postulated indivisible commodities, we further assume that each producer has a supply of exactly one item, and each consumer a desire for exactly one item. One might achieve much the same effect in terms of perfectly divisible goods by assuming individual supply and demand functions of a step-function type, as illustrated below for the case of undifferentiated products.

Our assumptions are admittedly restrictive, but they are not entirely unrealistic and we are not seeking wide generality for our model in any case. While excluding many important but secondary considerations, we wish to focus attention on what we consider the basic motive force of competitive economics, namely the profitable interaction between separately maximizing individuals.

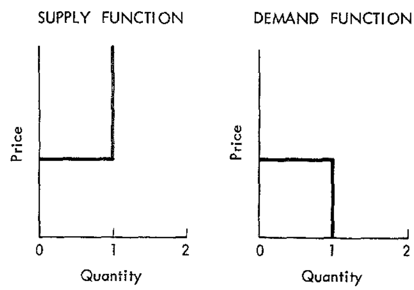

Fig. 1

2. Description of the Game

2.1. A Real Estate Market

We shall now formulate the market game in detail, motivating it in terms of a market in private homes. Another interpretation, less specific and hence more versatile, will be outlined in Sec. 2.3.

Let there be m homeowners in the market, and n prospective purchasers. We shall refer to them simply as *sellers* and *buyers*, respectively. The i^{th} seller values his house at c_i dollars, while the j^{th} buyer values the same house at h_{ij} dollars[2].

[1]) For a modern treatment of the problem of indivisibilities in a competitive market, see HENRY.

[2]) For a numerial example, see Table 1, in Sec. 3.4.

(These are meant to be actual utility valuations, not estimates of market price. But they might nevertheless be thought of as arising from the existence of other means of disposal or sources of supply, outside the model.) If $h_{ij} > c_i$ then a price favorable to both parties exists. We do not, however, assume that this inequality holds in all cases, or indeed in any case at all.

The possible moves in the game include the transfer of any house from its owner to any buyer, and the transfer of money from any player to any other. As we have pointed out before, we do not have to spell out the detailed scheduling of contacts, bids, offers, payments, etc., although such tactical features would be quite conspicuous in any account of the rules of the game in extensive form. All that we require, for our present purpose, can be summed up in the one simple observation: if i sells his house to j for p_i dollars, and if both avoid dealing with third parties, then i's final profit or gain is exactly

$$p_i - c_i \, , \tag{2.1}$$

and j's is exactly

$$h_{ij} - p_i \, . \tag{2.2}$$

We have no way at present of ascertaining the price p_i, but we can postpone the question. Since sales prices are not formally distinguished from other side payments, they will drop out of our initial calculations, only to reappear, rather unexpectedly, when the solution of the game begins to take shape.

2.2. The Characteristic Function

The characteristic function of the game, which states the worth $v(S)$ of every coalition S, can now be determined. Let M denote the set of all sellers, and N the set of all buyers. To begin with, it is obvious that

$$v(S) = 0 \quad \text{if} \quad |S| = 0 \text{ or } 1 \, , \tag{2.3}$$

since no player, without help from another, can effect a profitable transaction. More generally, we see that all of the one-sided coalitions are "flat":

$$v(S) = 0 \quad \text{if} \quad S \subseteq M \quad \text{or} \quad S \subseteq N \, . \tag{2.4}$$

In other words, only a mixed coalition can ever hope to assure a profit.

The simplest kind of mixed coalition consists of two players, one of each type. For this case we have[1])

$$v(\overline{ij}) = \max(0, h_{ij} - c_i) \quad \text{if} \quad i \in M \quad \text{and} \quad j \in N \, . \tag{2.5}$$

For brevity, we denote this number by a_{ij}. Note that it does not depend on the sales price p_i that appears in (2.1) and (2.2), since that is merely an internal transfer as far as the coalition $\overline{ij}$ is concerned.

[1]) We write "$\overline{ij}$" rather than the more awkward "$\{i,j\}$".

A moment's reflection reveals that these mixed pairs are the only essential coalitions in the game [1]). The best that a larger coalition can do is to split up into separate trading pairs and pool the profit. Hence the $m \times n$ matrix (a_{ij}) suffices to determine v completely. In fact, v may be characterized as *the smallest superadditive set function on* $M \cup N$ *satisfying* (2.3) *and* (2.5).

In order actually to compute v for the larger mixed coalitions, we must pick out an optimal set of transactions, maximizing the coalition's total gain. Put symbolically, we have

$$v(S) = \max\left[a_{i_1j_1} + a_{i_2j_2} + \cdots + a_{i_kj_k}\right], \tag{2.6}$$

the maximum to be taken over all arrangements of $2k$ distinct players $i_1, \ldots, i_k$ in $S \cap M$ and $j_1, \ldots, j_k$ in $S \cap N$, where $k = \min(|S \cap M|, |S \cap N|)$. The evaluation of an expression such as (2.6) is commonly called the "optimal assignment problem" or simply "assignment problem"; accordingly we refer to games of this form as *assignment games.*

For relatively small coalitions S (say, less than 10 players) the maximum in (2.6) can often be discovered by inspection; the reader may try his hand on the examples below. (In the first there are four optimal assignments, in the second there is just one; the respective values are 4 and 16.)

$$\begin{array}{cc} & S \cap N \\ S \cap M & \begin{array}{|ccc|} \hline 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \\ \hline \end{array} \end{array} \qquad \begin{array}{cc} & S \cap N \\ S \cap M & \begin{array}{|ccc|} \hline 5 & 8 & 2 \\ 7 & 9 & 6 \\ 2 & 3 & 0 \\ \hline \end{array} \end{array}$$

For larger matrices, systematic methods are available for finding the optimal assignment [2]).

For the most part we shall be interested in evaluating (2.6) only for the all-player coalition $S = M \cup N$. The number $v(M \cup N)$ is obviously very important, since it determines the maximum total monetary payoff, and hence the Pareto set and imputation space of the game. The optimal assignments for the other large, nonessential coalitions have little effect on the analysis of the game.

2.3. A Game of Monogamy

It will be observed that the characteristic function of the assignment game, when defined in terms of the numbers a_{ij}, is symmetric in its treatment of buyers and sellers, although the market model itself was not. The following alternative

[1]) A coalition is "essential" if it has more than one member and exhibits strict superadditivity for every split-up into smaller coalitions, i.e., $v(S) > v(S_1) + v(S_2)$ for all S_1, S_2 with $S_1 \neq \emptyset$, $S_2 \neq \emptyset$, $S_1 \cup S_2 = S$, and $S_1 \cap S_2 = \emptyset$.

[2]) See e.g., Dantzig [Chapter 15].

interpretation of the game, in sidestepping the preliminaries that went into the "real estate" version, has the conceptual advantage of being symmetrical from the start[1]).

We may imagine an economic milieu in which two types of agents, say "producers" and "consumers" (or "men" and "women"), are constrained for some reason to conduct their business under *exclusive*, bilateral contracts. Thus, after an initial period of jockeying for position, a number of partnerships are formed, and any transfers of goods or services (but not money) are limited to exchanges between partners. In this model we do not care particularly about the nature of the goods or services; we only care that with every partnerships $\overline{ij}$ there is associated a number $a_{ij} \geqq 0$, denoting the potential for profit of that partnership if it forms. If we now regard the system as a cooperative game, it is apparent that its characteristic function is given by (2.4) and (2.6).

A contract between players i and j would of course specify how the gain a_{ij} is to be divided. A prudent, "economic" man playing this game would be loath to enter a partnership for a stated share of the proceeds until he had satisfied himself that more favorable terms could not be obtained elsewhere. We can imagine that each player would set a price on his participation, and that no contracts would be signed until the prices on both sides of each partnership formed are in harmony, i.e., satisfy each partner and add up to what the partnership is worth. One may well ask: does a set of such "harmonious" prices exist?

Mathematically, this question may be put as follows: Do there exist numbers $q_i \geqq 0$, $r_j \geqq 0$ such that

$$\begin{cases} q_i + r_j = a_{ij} & \text{if } i \text{ and } j \text{ are ultimately partners,} \\ q_i + r_j \geqq a_{ij} & \text{if they are not?} \end{cases}$$

The answer is in the affirmative, as we shall presently demonstrate. This fact is of central importance throughout all the subsequent analysis of the assignment game.

2.4. Complementarity

Before proceeding to a study of the solutions of the assignment game, we shall digress briefly to describe an interesting property of the characteristic function.

The superadditivity of v in (2.6) is obvious, since combining two coalitions only enlarges the set of possible assignments. Less obvious is the way in which the *marginal value* of a player responds to changes in the make-up of the coalition to which he belongs. Intuitively, we should expect a player's value to a coalition to be small when there is an excess of players of his type, and large when there is an excess of players of the other type. In other words, similar players should appear

[1]) Cf. GALE and SHAPLEY [1962].

somewhat as *substitutes* in coalitional matters, and dissimilar players as *complements*. The following theorem gives an expression to this intuitive idea.

A special notation will be useful: S^p will denote the coalition consisting of S and the added player p. (We shall use this notation only when p is not a member of S.) Then the *marginal value* of player p to the coalition S is given by

$$m(p,S) = v(S^p) - v(S).$$

The following result is proved in SHAPLEY [1962].

Theorem 1:

If p and q are of the same type (i.e., both in M or both in N), then

$$m(p,S^q) \leqq m(p,S), \quad \text{all} \quad S \not\ni p,q.$$

If they are of the opposite type, then

$$m(p,S^q) \geqq m(p,S), \quad \text{all} \quad S \not\ni p,q.$$

It will be seen that this theorem is simply a statement about the sign of the "second difference" $v(S^{pq}) - v(S^p) - v(S^q) + v(S)$. When this quantity is positive, the players are complementary inputs to the coalition-forming process; when it is negative, they are substitutes.

3. The Core

3.1. LP Form of the Model

It will be useful at this point to recast the assignment problem (2.6) into *linear programming* (LP) terminology.

Consider just the assignment problem for the coalition of all players – i.e., the problem of determining $v(M \cup N)$. Introduce mn nonnegative real variables x_{ij}, $i \in M, j \in N$, and impose on them the $m + n$ constraints

$$\sum_{i \in M} x_{ij} \leqq 1, \quad \sum_{j \in N} x_{ij} \leqq 1. \tag{3.1}$$

(We may interpret x_{ij} in the real estate game as the fraction of thr i^{th} house sold to player j, or, in the partnership game, as the probability that partnership ij will form.) The LP problem is then to maximize the following objective function:

$$z = \sum_{i \in M} \sum_{j \in N} a_{ij} x_{ij}. \tag{3.2}$$

It can be shown [see DANTZIG, p. 318] that the maximum value $z_{\max}$ is attained with all $x_{ij} = 0$ or 1. Thus, the fractions or probabilities artificially introduced disappear from the solution, and the (continuous) LP problem is effectively equivalent to the (discrete) assignment problem, so that we have

$$z_{\max} = v(M \cup N).$$

As is well known, every LP problem can be tranposed into a *dual* form, and the solutions of the two problems are intimately bound up in each other. In the present case, the dual has $m + n$ nonnegative real variables, $u_1, \ldots, u_m, v_1, \ldots, v_n$, subject to the mn constraints,

$$u_i + v_j \geqq a_{ij}, \quad i \in M, \quad j \in N, \tag{3.3}$$

and the objective is to *minimize* the sum:

$$w = \sum_{i \in M} u_i + \sum_{j \in N} v_j. \tag{3.4}$$

The fundamental duality theorem[1]) tells us that $w_{\min} = z_{\max}$.

What meaning does the dual problem have in the present context? Let $(u,v) = (u_1, \ldots, u_m, v_1, \ldots, v_n)$ be a vetcor that minimizes (3.4), subject to (3.3). Then we have

$$\sum_{i \in M} u_i + \sum_{j \in N} v_j = w_{\min} = z_{\max} = v(M \cup N). \tag{3.5}$$

This means that (u,v) is an imputation[2]) of the assignment game. Moreover, (3.3) tells us that for every pair $i \in M$, $j \in N$,

$$u_i + v_j \geqq a_{ij} = v(ij).$$

It follows, by (2.6), that for any coalition S

$$\sum_{i \in S \cap M} u_i + \sum_{j \in S \cap N} v_j \geqq v(S). \tag{3.6}$$

But (3.5) and (3.6) are exactly how the *core* of the game is defined: (3.5) ensures the feasibility of (u,v) and (3.6) ensures its non-improvability by any coalition. Conversely, any payoff vector in the core – i.e., satisfying (3.5) and (3.6) – clearly fulfills the conditions for a solution to the dual LP problem. Hence we conclude:

Theorem 2:

In core of an assignment game is precisely the set of solutions of the LP dual of the corresponding assignment problem.

3.2. Prices

Our venture into the field of linear programming has proved most fruitful. We have already garnered (1) a proof of the existence of the core, (2) a characterization of the points in the core, and (3) an assurance of effective computational procedures for both the characteristic function and the core[3]).

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[1]) See e.g., DANTZIG [p. 129]. The existence of feasible solutions to both primal and dual is apparent from an inspection of (3.1) and (3.3).

[2]) I.e., a nonnegative, Pareto optimal payoff vector.

[3]) It is interesting that the core can be found without solving the multitude of assignment problems arising from the proper submatrices of $(a_{ij} | i \in M, j \in N)$. Because of this, the core in a large assignment game may be easier to compute than the characteristic function itself. The explanation of the paradox lies in the relatively small number of coalitions that are essential.

The reader schooled in LP will now be expecting an attempt to connect the dual solutions to a price mechanism, and we shall not disappoint him. In fact, the connection is very simple. In order to achieve the profit u_i promised by a given core vector (u,v), the owner of the i^{th} house must sell it at the price

$$p_i = c_i + u_i . \tag{3.7}$$

If all the owners attach such price-tags p_i to their houses, then the typical buyer j will be confronted by a choice among the m possible net gains:

$$h_{ij} - p_i , \quad i \in M . \tag{3.8}$$

If these numbers are all negative, then j perforce stays out of the market, and ends the game with a profit of zero; otherwise he will seek to maximize (3.8). This is equivalent, in view of (3.7) and (2.5), to maximizing

$$a_{ij} - u_i , \quad i \in M .$$

We know by (3.3) that none of these numbers exceed v_j. On the other hand, the minimization of the dual objective form (3.4) ensures that at least one of them is equal to v_j. This means that buyer j's maximum profit is precisely v_j, and he is led to it by direct comparative shopping.

For an example with specific numerical values, the reader is now invited to look ahead to Sec. 3.4.

What if several shoppers find that the same house is a "best buy"? Can this happen? In the most common case, where the optimal assignment is unique and where the chosen vector (u,v) lies in the relative interior of the core (see the next section), the answer is no. The price schedule (3.7) leads the players unambiguously to a nonconflicting allocation of goods that maximizes the welfare of the group as a whole and obtains for each individual the specified amount u_i or v_j.

In exceptional, "degenerate" cases there may be ties in the buyers' preferences, and care must be taken to resolve the ties without assigning the same house to more than one buyer. This is always possible, as can be demonstrated by a simple perturbation argument.

3.3. Structure of the Core

The core of the assignment game only rarely consists of just a single imputation [1]). Consequently, the price structure just described is seldom unique. In this section we shall attempt to account for the multiplicity of solutions and discuss what it means in the market context.

[1]) The first matrix shown in Sec. 2.2 provides an example: the core of the associated 6-person game contains just the imputation (0, 2, 0; 0, 2, 0). The other matrix shown there is more typical, however.

In general, if numbers $a_{ij} \geqq 0$ are chosen "at random", the optimal assignment is unique[1]). This implies that a dual solution (u,v) exists in which the strict inequality:

$$u_i + v_j > a_{ij}$$

holds for *all* pairs $\overline{ij}$ that are not involved in the optimal assignment. Let $\overline{i'j'}$ be one of the min (m,n) pairs in the optimal assignment. Then a small amount can be shifted from $u_{i'}$ to $v_{j'}$ without spoiling any of the conditions for a dual solution. It follows that there are at least min (m,n) "degrees of freedom" in the core. On the other hand, the dimensionality of the core is certainly never greater than min (m,n), because if the u-components of a core vector are given, then the v-components are completely determined, and vice versa.

A picture of the core is beginning to take shape: it is a closed, convex polyhedral set whose dimension is typically equal to min (m,n), but may be less in the presence of degeneracies – i.e., special arithmetical relations among the a_{ij}. Note that the dimension of the imputation space, in which the core is situated, is $m + n - 1$, which is considerably larger than min (m,n).

We next show that the core tends to be *elongated*, with its long axis oriented in the direction of market-wide price trends. There is a "high-price corner" of the core, at which every seller gets his top profit and every buyer his bottom. There is also a "low-price corner", where the reverse is true. These points are poles of a diameter; they prove to be at least as far apart as any other two points in the core. Thus, to a considerable extent, the fortunes of all players of the same type rise and fall together.

Let us try to make this intuitively plausible. For simplicity assume $m = n$. Suppose that we have a core vector at which all players show a positive profit. Then a small constant can be subtracted from *all* of the sales prices without upsetting the core conditions. Each buyer-seller pair makes the same profit as before, and no individual is forced to take a loss. This price-cutting can continue until some seller is priced out of the market, his profit going to zero. The process can of course be reversed, raising all prices by equal amounts until one of the buyers is driven out. Thus there is a natural "degree of freedom" *within* the core that corresponds to market-wide price movement[2]).

This phenomenon is often encountered in the cooperative solutions to market

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[1]) If the original data of the real estate version of the game are chosen "at random" – more precisely, from continuous probability distributions that give zero weight to any particular numerical value – then the value 0 may be favored over other values in the assignment matrix, and nonuniqueness in the optimal assignment may occur with positive probability. But this merely reflects indeterminacy in the choice of *inactive* buyer-seller pairs, involving players who are actually "priced out of the market". The u_i or v_j for such players are identically zero in the core, and the dimension of the core is reduced accordingly.

[2]) This will be illustrated in Sec. 3.4. If $m \neq n$ a similar discussion applies, except that it may no longer be possible to drive any of the players of the less numerous type down to zero.

games[1]). The heuristic principle may be stated as follows: *inter*group allocations are relatively indeterminate, *intra*group allocations are relatively precise.

Theorem 3:

Over all imputations in the core, let u_i^* and u_{*i} denote the highest and lowest payoffs, respectively, to player $i \in M$; similarly define v_j^* and v_{*j} for player $j \in N$. Then the payoff vectors

$$(u_*, v^*) \quad \text{and} \quad (u^*, v_*)$$

are themselves in the core. Moreover, no two imputations in the core are further apart than these two[2]).

Figure 3, in § 3.4 below, may help the reader to visualize this theorem. The heart of the proof is in the following lemma:

Lemma:

Let (u', v') and (u'', v'') be any two imputations in the core. Define

$$\begin{aligned} \underline{u}_i &= \min(u_i', u_i''), & \underline{v}_j &= \min(v_j', v_j''), \\ \bar{u}_i &= \max(u_i', u_i''), & \bar{v}_j &= \max(v_j', v_j''). \end{aligned}$$

Then the vectors $(\underline{u}, \bar{v})$ and $(\bar{u}, \underline{v})$ are in the core.

Proof:

For any i, j we have

$$\begin{aligned} \underline{u}_i + \bar{v}_j &= \min(u_i' + \bar{v}_j, u_i'' + \bar{v}_j) \\ &\geqq \min(u_i' + v_j', u_i'' + v_j'') \\ &\geqq a_{ij}, \end{aligned}$$

using (3.3). Hence $(\underline{u}, \bar{v})$ is undominated. It remains to show that it is an imputation. It is obviously nonnegative; we must therefore show only that its components add up to $v(M \cup N)$. For convenience, label the players so that the pairs $\overline{11}, \overline{22}, \ldots, \overline{kk}$ describe an optimal assignment. (Here $k \leqq \min(m,n)$.) Then:

$$\begin{aligned} \underline{u}_i &= \min(u_i', u_i'') = \min(a_{ii} - v_i', a_{ii} - v_i'') \\ &= a_{ii} - \max(v_i', v_i'') = a_{ii} - \bar{v}_i \end{aligned}$$

for $i \leqq k$. Also, for $i,j > k$ (if any) we have $\underline{u}_i = \bar{v}_j = 0$. Hence

$$\sum_{i \in M} \underline{u}_i + \sum_{j \in N} \bar{v}_j = \sum_{i=1}^{k} a_{ii} = v(M \cup N).$$

This completes the proof of the lemma.

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[1]) For stable sets cf. SHAPLEY [1959]; for cores cf. the "equal treatment" principle as expressed e.g. in Theorem 2 of DEBREU and SCARF; for kernels cf. SHAPLEY and SHUBIK [W. Y.].

[2]) A similar theorem is proved in GALE and SHAPLEY [1962].

To prove the theorem, simply take a finite collection of core vectors that includes all the extreme values going into the definitions of u_i^*, u_{*i}, v_j^*, v_{*j}, and apply the lemma repeatedly to construct additional core vectors, until (u^*, v_*) and (u_*, v^*) are reached. For the last statement of the theorem, note that any two points (u', v'), (u'', v'') in the core necessarily satisfy the inequalities:

$$|u_i' - u_i''| \leqq u_i^* - u_{*i}, \quad \text{all} \quad i \in M,$$

and

$$|v_j' - v_j''| \leqq v_j^* - v_{*j}, \quad \text{all} \quad j \in N.$$

Thus the stated result holds not only for euclidean distance but for any distance function based on a linear norm, i.e., that depends only upon the absolute values of the coordinate differences.

3.4. A Numerical Example

Let there be three sellers (1, 2, 3) and three byers (1′, 2′, 3′), as follows:

Table 1

Houses (i)	Sellers' basis (c_i)	Buyers' valuations (h_{i1})	(h_{i2})	(h_{i3})
1	\$ 18,000	\$ 23,000	\$ 26,000	\$ 20,000
2	15,000	22,000	24,000	21,000
3	19,000	21,000	22,000	17,000

These data lead at once to the following (a_{ij}) matrix, already considered in Sec. 2.2:

$$\begin{array}{rc|ccc|l} & & \multicolumn{3}{c}{\text{(buyers } N)} & \\ & & 1' & 2' & 3' & u: \\ \hline & 1 & 5 & 8 & 2 & 4 \\ (\text{sellers } M) & 2 & 7 & 9 & 6 & 5.5 \quad (\text{units of \$ } 1000) \\ & 3 & 2 & 3 & 0 & 0 \\ \hline & v: & 2 & 4 & 0.5 & \end{array}$$

The unique optimal assignment is shown circled, the total gain being \$ 16,000. One of the core vectors (u, v) is also shown. The reader can verify that the matrix $(u_i + v_j)$ majorizes (a_{ij}), with equality only on the circled entries. The prices corresponding to this core vector are \$ 22,000, \$ 20,500, and \$ 19,000 respectively. Given these prices, the first buyer clearly prefers the third house, as his gain is \$ 2000 rather than \$ 1000 or \$ 1500. Similarly, the second buyer prefers the \$ 4000 gain at house 1 to the gain of \$ 3500 or \$ 3000 at house 2 or 3. Finally, house 2 is the only one that the third buyer would even consider, at the stated prices.

The nonuniqueness of this solution is evident, since none of the buyers' comparisons were closer than \$ 500, and only one seller (the third) is at, or even close to, his cost. Thus, any one price could be raised by \$ 500, or either of the first two prices lowered by \$ 500, without moving the outcome out of the core. Or all three prices could be increased together, in a market-wide price movement, until the "weakest" buyer (the third) is driven out.

The geometry of the core is shown in Fig. 2. Since optimality demands the unique set of assignments $\{\overline{12'},\overline{23'},\overline{31'}\}$, we have depicted just the rectangular, 3-dimensional region in the (5-dimensional) imputation space in which the equations $u_1 + v_{2'} = 8$, $u_2 + v_{3'} = 6$, $u_3 + v_{1'} = 2$ are satisfied. (The imputations outside this region cannot be attained without the aid of "third party" side payments.) The core is a five-sided polyhedron situated within this region, touching the boundaries $u_3 = 0$ and $v_{3'} = 0$. The solution point discussed above happens to lie in the bottom face, as shown by the open dot "∘" in Fig. 3. The doubleheaded arrow indicates the direction of market-wide price changes. The low-price corner (u_*, v^*), where all sellers get their minimum gains, is at the lower left; the high-price corner (u^*, v_*) is at the upper right. The (euclidean) distance between them is $\sqrt{6}$ times \$ 1000, or \$ 2449; the distance measured in terms of the total effect on the sellers as a class is \$ 4000.

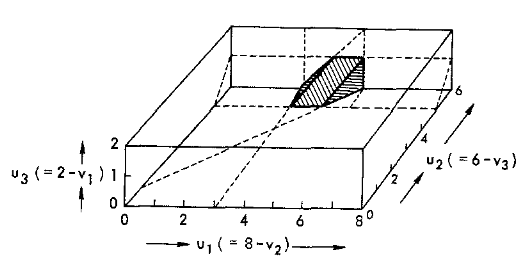

Fig. 2

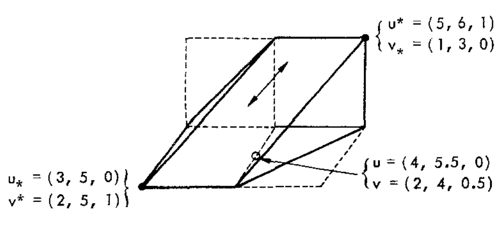

Fig. 3

4. A Special Case

4.1. The Horse Market of BÖHM-BAWERK[1])

Discussions of price theory in economics have been enlivened by allusions to "BÖHM-BAWERK's horses," the central figures in a microcosmic market detailed in that author's *Positive Theory of Capital*, first published in 1891. Since this imaginary but numerically specific market model translates directly into an 18-person cooperative game, it makes an inviting target for the "big game hunter" anxious to try out his techniques. (Eighteen may be a small number in economics, but it is a large number in game theory!) Apart from the test of techniques, there is the prospect of an instructive confrontation among the various game solutions – cores, stable sets, values, etc. – and the classical solution as put forward by BÖHM-BAWERK.

The basic data of the market are shown in Table 2[2]). Eight individuals each have one horse for sale. Ten other individuals each wish to buy one horse. The horses themselves are all alike, but the traders have different "subjective valuations", ranging between $ 10 and $ 30, of what it is worth to own a horse[3]). No restrictions are placed on communication, on transfers of money, or on transfers of horses. The basic game problem, informally stated, is to decide how the inherent profitability of the market, arising from the differences in subjective valuation, is to be shared among the horse-traders. This is also, of course, the basic economic problem.

Table 2

Sellers	Buyers
A_1 values a horse at \$ 10	B_1 values a horse at \$ 30
A_2 values a horse at \$ 11	B_2 values a horse at \$ 28
A_3 values a horse at \$ 15	B_3 values a horse at \$ 26
A_4 values a horse at \$ 17	B_4 values a horse at \$ 24
A_5 values a horse at \$ 20	B_5 values a horse at \$ 22
A_6 values a horse at \$ 21.50	B_6 values a horse at \$ 21
A_7 values a horse at \$ 25	B_7 values a horse at \$ 20
A_8 values a horse at \$ 26	B_8 values a horse at \$ 18
	B_9 values a horse at \$ 17
	B_{10} values a horse at \$ 15

4.2. The Characteristic Function

The reader will recognize that the present example, and the class of markets it exemplifies, is a decided simplification of our general model in one important

[1]) Much of the following is drawn from SHAPLEY [1961].
[2]) See BÖHM-BAWERK [p. 203]. We have arbitrarily replaced pounds by dollars.
[3]) To own a second horse, however, is worth nothing more.

respect: the *absence of product differentiation*. The matrix of input data (h_{ij}) can accordingly be reduced to a vector (h_j). As a result, the assignment matrix (a_{ij}), now defined by

$$a_{ij} = \max(0, h_j - c_i), \tag{4.1}$$

enjoys the following special property: *in each* 2×2 *submatrix with nonzero entries, the sums of the diagonals are equal.* This means that any set of assignments involving actual trades can be switched around, without loss or gain to anyone, since it only matters who buys and who sells, and not how they pair up. Hence the optimal assignment problem is rather trivial, and we shall be able to express the characteristic function in a very simple way.

Let S be an arbitrary coalition, and let its members be denoted by $A_{i_1}, \ldots, A_{i_l}$, $B_{j_1}, \ldots, B_{j_m}$, where the sellers A_i are arranged in order of increasing c_i and the buyers B_j in order of decreasing h_j. Then it is easy to see that

$$v(S) = a_{i_1 j_1} + a_{i_2 j_2} + \cdots + a_{i_k j_k}, \tag{4.2}$$

where $k = \min(l, m)$. In other words, for computational purposes we can assume that the "strongest" buyer in any coalition buys from the "strongest" seller, and so on down the line, until either the players of one type are exhausted or a pair is reached that cannot trade at a profit.

Applying these remarks to the data in Table 2, we find that

$$v(M \cup N) = \$20 + \$17 + \$11 + \$7 + \$2 + 0 + 0 + 0 = \$57. \tag{4.3}$$

This is the total amount of profit inherent in the market, and it is achieved whenever the first five buyers acquire the horses of the first five sellers, and only then.

4.3. The Core

The core, too, is easy to determine, since the absence of product differentiation leads to a uniform market price. To see this let (u, v) be a typical imputation in the core. Suppose the amount of money received by some active seller i, which is $u_i + c_i$, happens to be less than the amount of money *paid* by some active buyer j, which is $h_j - v_j$. Then an improvement for $\overline{ij}$ would obviously be possible, since

$$u_i + v_j < h_j - c_i = a_{ij},$$

and (u, v) would not be in the core after all.

In short, each active seller must receive at least as much as each active buyer pays out. But no money enters the system; hence in any given core imputation *all transactions take place at the same price.*

Thus, the core can be described by means of a single parameter p. A vector (u, v) is in the core if and only if, for p in some specified range, we have

$$\left.\begin{aligned} u_i &= \max(0, p - c_i), \quad \text{all} \quad i \in M, \\ v_j &= \max(0, h_j - p), \quad \text{all} \quad j \in N. \end{aligned}\right\} \tag{4.4}$$

The range over which p can vary is given by the requirement that $\sum u_i + \sum v_j = v(M \cup N)$. This can only be satisfied if the same number of players of each type are active – more precisely, if the number of i such that $p - c_i > 0$ does not exceed the number of j such that $h_j - p \geqq 0$, and vice versa. Geometrically the core is a straight line segment, and in its interior exactly the same number of traders of each type make a positive profit.

In the numerical example (Table 2), a price in the interval \$ 21.00 < p < \$ 21.50 permits exactly *five* players of each type to trade at a profit. If the price were to exceed the upper limit, the sixth seller would put up his horse for sale and an imbalance would be created that would tend to drive the price down again. Similarly, a price less than \$ 21 would find six (or more) buyers competing for five (or fewer) horses. The endpoints of the core are therefore

$$(11.5, 10.5, 6.5, 4.5, 1.5, 0, 0, 0; 8.5, 6.5, 4.5, 2.5, 0.5, 0, 0, 0, 0, 0)$$

and

$$(11, 10, 6, 4, 1, 0, 0, 0; 9, 7, 5, 3, 1, 0, 0, 0, 0, 0).$$

These are of course the extremal imputations (u^*, v_*) and (u_*, v^*) of Theorem 3 in Sec. 3.3.

4.4. Discussion

The core that we have determined in the preceding section is *precisely the classical solution* as given by BÖHM-BAWERK. A uniform market-wide price is established at a level which, though not precisely determined, is constrained within fairly narrow limits. If the consumers and suppliers in the market are sufficiently numerous, and if the tastes of the former and the costs of the latter are sufficiently variegated, then something close to a determinate outcome can be predicted. It is the classical balance of supply and demand. (See Sec. 5).

The one-dimensionality of the core represents the most extreme form of the elongation described in Theorem 3; this is a consequence of the lack of differentiation among the objects of trade[1]). If a small "random" perturbation were applied to the model, giving the buyers slightly different subjective valuations for different horses, then although the dimension of the core might jump from 1 to 5 it would nonetheless remain very slender[2]). If the buyers should become more dissimilar in their tastes, however, the core would grow fatter. The ultimate in this direction would be a game in which each buyer has only one horse that he would accept at any price. Since the bargaining for each horse would then be independent of

[1]) A technical explanation may be found in the large number of optimal assignments that are possible. Multiplicity in the primal solutions of an LP problem is commonly accompanied by reduced dimensionality of the dual solutions.

[2]) It can be shown generally that the core varies continuously as a function of the characteristic-function values; see BÖHM [1972].

the others, the game would decompose into smaller games, and the core would take the form of a rectangular parallelepiped (i.e., a product of intervals).

5. The Core and the Competitive Equilibrium

5.1. Convergence of the Core

An extensive literature [1]) now exists on the relation between the core of a market game and the notion of a competitive price equilibrium. Three basic features of this relationship are (1) that every competitive outcome [2]) is in the core, (2) that the core will generally contain other outcomes, and indeed may exist in situations where no competitive outcome exists, and (3) that under suitable conditions involving the number, size, or similarity of the economic agents, the core approaches or even coincides with the set of competitive outcomes.

The reader familiar with these ideas will already have recognized that in the assignment game [3]) every core outcome is competitive and vice versa. (See the discussion of prices in Sec. 3.2.) There is no doubt, then, as to the convergence of the two solution concepts; even if there are only a handful of players they coincide exactly. The reason for this lies in the special nature of the supply and demand functions (see Fig. 1), and the underlying preferences and indivisibilities that they reflect. The "lumpiness" of the commodity causes essentially the same nonuniqueness in the competitive prices as it does in the allocations that all coalitions will tolerate.

There is, however, another sort of convergence of interest, already suggested in Sec. 4.4. We shall describe it in general terms, having no actual theorems to offer. If the number of traders is increased, on both sides of the market, in such a way that their valuations for the products brought to market become more and more diverse (but remain bounded in a suitable sense), then the core will tend to shrink in size. The presence of a *large variety* of traders or products can in effect smooth out the supply and demand functions, and lead to more tightly determined competitive prices. In the BÖHM-BAWERK model, where there is no

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[1]) Much of it, to be sure, has appeared since SHAPLEY [1955], our first treatment of the core of the assignment game. The basic paper of DEBREU and SCARF remains an excellent introduction; for a recent contribution to the subject (with a good bibliography) see SHITOVITZ.

[2]) A final distribution of goods and money, or the corresponding vector of utilities, is called "competitive" if it can result from a behavioristic process in which all decisions concerning production, trade, and consumption are made noncollusively and in strict accordance with a set of prices that have been specified *ex machina* in such a way that the supply and demand of each commodity will be in balance.

[3]) More precisely, in the market models exemplified by houses or horses that we have been using to motivate the abstract assignment game. In the less specific "partnership" model of Sec. 2.3, the notion of competitive outcome must be reconstrued.

Incidentally, the indivisibilities and product differentiation in our model put it outside the scope of most of the general existence and convergence theorems in the literature.

product differentiation, this sharpening of the solution is a simple matter of narrowing the range of prices that will draw an equal number of buyers and sellers into the active market. In the more general model, however, the increasing dimensionality of the solution and the space in which it is defined makes a precise discussion of the shrinkage phenomenon more difficult [1]).

5.2. Some Weaknesses

From both the economic and game-theoretic viewpoint, certain weaknesses in the core (= competitive) solution soon become apparent. For one thing, the core gives no consideration to the ability of individuals or coalitions to *obstruct* outcomes, though this is often a powerful lever in bargaining [2]). For example, the coalition M of all sellers can certainly block any trade [3]), yet the core gives them no credit for this. Indeed, the inequality condition (3.6) for $S = M$ (or for $S = N$ for that matter) has no "bite" at all, and does not eliminate any individually rational outcomes from the solution. The core is based on what a coalition can *do*, not what it can *prevent*. The core is therefore not a really satisfactory basis for a bargaining theory.

There is a more subtle bargaining tactic, not of an obstructive nature, that also fails to be reflected in the core or competitive equilibrium. It can be illustrated in terms of the horse market of Table 2. Consider the "weak" players in the game – i.e., A_6, A_7, A_8 and $B_6 - B_{10}$. They get nothing in the core. Although not technically dummies, they are quite unable to prevent the others from obtaining the full \$ 57 profit, or to interfere in any other way with the imputations in the core. Nevertheless, the marginal weak buyer B_6 (for example) has an important role: it is his valuation of \$ 21 that establishes the price floor in the core. Were he to quit the scene, the floor would drop to \$ 20, and the sellers as a group would very likely suffer. By his very presence – i.e., his willingness to pay any amount up to \$ 21 for a horse – he performs a service to some of the players (and a disservice to others). What is it worth?

A similar capability exists for the marginal weak seller A_6, who sets the ceiling of \$ 21.50 in the solution. In lesser measure, the other weak players also have some bargaining power of this kind. The seventh buyer B_7, for example, might argue that his participation in the bidding tends to limit the sixth buyer's "leverage", since if B_7 should quit the scene, then B_6 might be able to command a larger bribe from the active sellers for remaining on hand to protect the \$ 21 floor.

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[1]) Even in the BÖHM-BAWERK case, if the price range does not narrow fast enough, the (euclidean) distance between the endpoints of the core may actually increase as the price interval goes to zero.

[2]) The core is sometimes described as "the set of outcomes that no coalition can block"; this unfortunate and misleading description stems from a counterintuitive use of the term "block" in the mathematical terminology used by some writers. A better rendering is "the set of outcomes that no coalition can improve upon".

[3]) As shown by the fact that $v\,(N) = 0$.

The marginal *strong* players, A_5 and B_5, also help to determine the price range. Since their expected profits in any case are rather small, their threats to behave "irrationally" could also carry some weight. In our example, the departure of B_5 would decrease the total social profit from \$ 57 to \$ 56 (see Table 2), but it would also cause the price range to shift downwards from \$ 21 – \$ 21.50 to \$ 20 – \$ 21. Would the other four buyers be willing to pay, say fifty cents each to bring this about [1])?

To complete the picture, we should note that the top or "strongest" players of each type are in an extremely poor bargaining position. They are vulnerable to all the threats we have been describing, and they have no very credible counter-threats: they have too much to lose. A man who is desperately in need of a horse (like Shakespeare's Richard III!) is obviously at a disadvantage in haggling over the price, if his need is known. His situation practically invites collusion among the horse-merchants. But the classical solution (core) gives no hint of this.

We are not arguing against the classical solution as such [2]). It may not be possible to realize the bargaining potentials described above *within a given institutional form*. Sanctions against collusion or compensation, physical barriers to communication, or simply the cost and time of bargaining, may be sufficient to vitiate the bargaining tactics we have been describing. The correct solution concept to apply in a given situation depends on the larger "social" context from which the mathematical model was abstracted. It also depends, of course, on the purpose of the application – on the type of questions one wishes the theory to answer. A corollary is that when we are conducting a general analysis of the abstract model, as here, then it behooves us to explore and correlate a number of different solution concepts. This we hope to do in subsequent papers.

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[1]) If they were so willing, then Pareto optimality might re-assert itself, in the guise of a counter-offer by the sellers, and induce B_5 to return to the market to his further profit.

[2]) An instructive critique of the classical solution will be found in VON NEUMANN and MORGENSTERN *loc. cit.*; see also SHAPLEY [1961] and Sec. 3 of SHAPLEY and SHUBIK [1969].

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