A characterization theorem of the core of two-sided matching problems is proved. The theorem characterizes the core by means of four axioms, the Pareto optimality, consistency, converse consistency, and anonymity. It is shown that all of them are logically independent. Journal of Economic Literature Classification Numbers: C71, C78. © 1991 Academic Press, Inc.

1. INTRODUCTION

A two-sided matching problem is typically the problem of marriage: when there are finite numbers of men and women and each one has a preference ordering over the members of the opposite sex, the problem is to marry off the members of the society in a satisfactory way.

* This paper combines the results in Sasaki [10] and in Toda [11]. We thank Professor William Thomson for his valuable comments and advice. We have benefitted from the comments of Professors Stef Tijs and H. Peyton Young. The first author thanks Professor Hukukane Nikaido for his encouragement. The second author thanks Professors Yukihiko Funaki and Mamoru Kaneko for their comments. The suggestions of Professor Akio Kagawa are also gratefully acknowledged. We thank an anonymous referee for valuable comments which substantially improved the proof of Lemma 1 as well as the exposition of the paper. Any remaining errors are our own.
A solution that has been proposed in the literature is a stable matching in which there is no unmatched pair of a man and a woman who prefer each other to their current partners. The set of stable matchings is equal to the core, which is shown to be non-empty (see, Gale and Shapley [1]).

This paper provides four axioms characterizing the core of two-sided matching problems. They are Pareto optimality, consistency, converse consistency, and anonymity.

In recent years, a consistency property of solutions has been successfully applied to a wide class of problems. Informally, it requires that eliminating some of the members of a society, after giving them what they have at an initial solution outcome, does not change the outcome for the remaining ones. Our consistency axiom, which is central in our result, reflects the same idea.

When applying the consistency principle, it is necessary to define new problems on reduced sets of agents, which are called reduced problems. But in general some alternative definitions may be possible and then a single characterization theorem of a solution would not be enough for full understanding of the axiomatic nature of it.

In this paper, we define reduced problems by restricting original preferences to subsets of agents. The structure of a marriage problem is so simple that it seems almost impossible to give some other definitions of reduced problems replacing ours. In this sense, our theorem would be the unique way to understand the core by means of axioms including consistency.

The object of this paper is not only to characterize the core but to give an important remark on welfare aspects of stable matchings. It is observed by Knuth [3] that in comparing two stable matchings, the one that is better for all men is worse than the other for all women and vice versa. Moreover, the stable matching that all men (or women) like best among stable matchings is in turn the worst for all women (or men). Many authors have tried to resolve the conflict between the two sides and find out that a subset of the core is “fair” in some sense. (For example, see McVitie and Wilson [5]). It can be seen from the proof of our main theorem that there exists no proper subcorrespondence of the core which satisfies consistency. This indicates that our consistency property is inherent almost uniquely in the core. It is striking because our result suggests that, even if we could find a subsolution of the core which is “fair,” it can never be consistent.

The second section gives formal definitions. The third section presents our main theorem. The fourth section shows the logical independence of the axioms. The final section concludes.
2. MODEL AND AXIOMS

We shall consider matching problems with equal numbers of men and women. Our results can be easily extended to problems with possibly unequal numbers of men and women.

Let $M$ and $W$ be two finite sets of men and of women, respectively. Assume that $n = \# M = \# W \geq 2$, where $\#$ denotes the cardinality of a set. Each agent has a strict preference ordering $P(a)$ over the agents of the opposite sex. The list $\{P(a) | a \in M \cup W\}$ of preferences is denoted by $\mathcal{P}$. The triplet $(M, W, \mathcal{P})$ is called a matching problem (or simply a problem). Let $\mathcal{P}$ be the set of all matching problems. A matching $\mu$ is a one-to-one mapping from $M$ onto $W$. We sometimes write $(m, w) \in \mu$ to denote $\mu(m) = w$. Let $A(M, W)$ be the set of all possible matchings from $M$ onto $W$.

**Definition 1.** For a given problem $(M, W, \mathcal{P})$, a pair $(m, w) \in M \times W$ blocks a matching $\mu \in A(M, W)$ if $wP(m) \mu(m)$ and $mP(w) \mu^{-1}(w)$. A matching $\mu$ is stable if there is no blocking pair. $S(M, W, \mathcal{P})$ denotes the set of all stable matchings. Since it is known that the set $S(M, W, \mathcal{P})$ is equal to the core,¹ we shall simply call it the core.

**Definition 2.** For a given problem $(M, W, \mathcal{P})$, a matching $\mu \in A(M, W)$ is Pareto optimal if there exists no matching $\mu' \in A(M, W)$ with $\mu' \neq \mu$ such that each agent is at least as well off in $\mu'$ as in $\mu$. Let $PO(M, W, \mathcal{P})$ be the set of all Pareto optimal matchings.²

**Definition 3.** A solution $\varphi$ is a correspondence which associates a non-empty subset $\varphi(M, W, \mathcal{P})$ of $A(M, W)$ with each problem $(M, W, \mathcal{P})$.

We shall introduce four axioms namely, Pareto optimality, consistency, converse consistency, and anonymity. For that purpose, we need some notation.

For any given problem $(M, W, \mathcal{P})$, let $M'$ and $W'$ be non-empty subsets of $M$ and $W$, respectively. For each $m \in M$, let $P_{|W'}(m)$ be the restriction of $P(m)$ to the subset $W'$. For each $w \in W$, let $P_{|M'}(w)$ be the restriction of $P(w)$ to the subset $M'$. Then, we shall denote the list $\{P_{|W'}(m) | m \in M'\} \cup \{P_{|M'}(w) | w \notin W'\}$ by $\mathcal{P}_{|M' \cup W'}$. The triplet $(M', W', \mathcal{P}_{|M' \cup W'})$ is called a subproblem of $(M, W, \mathcal{P})$.

**Pareto optimality [PO].** For any $(M, W, \mathcal{P}) \in \mathcal{P}$, $\varphi(M, W, \mathcal{P}) \subseteq PO(M, W, \mathcal{P})$.

¹ See Roth and Sotomayor [9].
² Since $S(M, W, \mathcal{P}) \subseteq PO(M, W, \mathcal{P})$, $PO(M, W, \mathcal{P}) \neq \emptyset$. 
Consistency [CON]. For any \((M, W, \mathbb{P}) \in \mathcal{P}\), for any \(\mu \in \varphi(M, W, \mathbb{P})\), 
\(\mu|_{M'} \in \varphi(M', W', \mathbb{P}|_{M' \cup W'})\) for any \(\emptyset \neq M' \subset M\), where \(\mu|_{M'}\) is the restriction of the mapping \(\mu\) to \(M'\) and \(W'' \equiv \mu(M')\).

Converse consistency [C.CON]. For any \((M, W, \mathbb{P}) \in \mathcal{P}\) and for any 
\(\mu \in \mathcal{A}(M, W)\), if \(\mu|_{M'} \in \varphi(M', \mu(M'), \mathbb{P}|_{M' \cup \mu(M')})\) for any proper subsets \(M' \subset M\) with \(#M' = 2\), then \(\mu \in \varphi(M, W, \mathbb{P})\).

Anonymity [AN]. For all \((M, W, \mathbb{P}), (M', W', \mathbb{P}') \in \mathcal{P}\) with \(#M = #W = #M' = #W'\), let \(\pi: M \cup W \to M' \cup W'\) be a bijection such that

(a) \(\pi(M) = M', \pi(W) = W'\).

(b) For all \(m \in M\) and for all \(w, w' \in W\), \(wP(m) w' \iff \pi(w) P'(\pi(m)) \pi(w')\),

(c) For all \(w \in W\) and for all \(m, m' \in M\), \(mP(w) m' \iff \pi(m) P'(\pi(w)) \pi(m')\).

Furthermore, for each \(\mu \in \mathcal{A}(M, W)\), define \(\pi_\mu \in \mathcal{A}(M', W')\) by

\[\pi_\mu(m') = \pi(\mu(\pi^{-1}(m'))) \quad \text{for all} \quad m' \in M'.\]

Then, \(\mu \in \varphi(M, W, \mathbb{P})\) implies that \(\pi_\mu \in \varphi(M', W', \mathbb{P}')\).

The consistency axiom says that if a matching \(\mu\) is chosen as a solution to a problem, then for any subgroup of men and women that are matched to each other by \(\mu\) the solution \(\varphi\) recommends the same matching when it is applied to the subgroup after eliminating the outside agents. This resembles the axiom of independence of irrelevant alternatives in bargaining theory. (See, Nash [6] and Lensberg and Thomson [4]). It is probably the weakest form of stability property of solutions when we reduce the number of participants. On the other hand, the axiom of converse consistency can be viewed as a dual of the consistency. Suppose that a problem and a matching \(\mu\) are given. If the matching \(\mu\) provides a solution to each subproblem containing exactly two men and two women who are matched to by \(\mu\), then the axiom requires that \(\mu\) must be a solution to the original problem. Axioms in the same spirit have been used in game theory by Harsanyi [2], Peleg [7, 8], and others.

3. Characterization of the Core

In this section, we shall prove our main theorem. Before it, we need some lemmas. For the proofs of Lemmas 1 and 2, see Appendix. Lemma 3 immediately follows from Lemma 2.
Lemma 1. Let \((M, W, P) \in \mathcal{P}\) be given. For any \(\mu \in S(M, W, P)\), there is a problem \((M^\ast, W^\ast, P^\ast)\) such that \(M \subseteq M^\ast, W \subseteq W^\ast, P^\ast|_{M \cup W} = P, S(M^\ast, W^\ast, P^\ast) = \{\mu^\ast\}\), and \(\mu^\ast|_M = \mu\).

As mentioned in the Introduction, Lemma 1 suggests that there exists no “fair” selection from the core which is consistent. We state this in the form of a proposition.

Proposition. Suppose that \(\varphi\) is a consistent solution such that \(\varphi(M, W, P) \cap S(M, W, P) \neq \emptyset\) for any problem \((M, W, P) \in \mathcal{P}\). Then, \(S(M, W, P) \subseteq \varphi(M, W, P)\) for any problem \((M, W, P) \in \mathcal{P}\).

Lemma 2. If a solution \(\varphi\) satisfies \(PO, C\text{-}CON\), and \(AN\), then \(\varphi(M, W, P) \subseteq S(M, W, P)\) for all \((M, W, P) \in \mathcal{P}\) with \(n = 2\).

Lemma 3. If a solution \(\varphi\) satisfies \(PO, CON, C\text{-}CON\), and \(AN\), then \(\varphi(M, W, P) \subseteq S(M, W, P)\) for all \((M, W, P) \in \mathcal{P}\).

Main Theorem. The core \(S\) is the unique solution satisfying \(PO, CON, C\text{-}CON\), and \(AN\).

Proof. It is obvious that the core satisfies the axioms. To prove uniqueness, let \(\varphi\) be a solution satisfying the axioms. By Lemma 3, \(\varphi(M, W, P) \subseteq S(M, W, P)\) for all \((M, W, P) \in \mathcal{P}\). For any given \((M, W, P) \in \mathcal{P}\), let \(\mu \in S(M, W, P)\). Now we apply Lemma 1 to \((M, W, P)\) and \(\mu\). Because \(\varphi(M^\ast, W^\ast, P^\ast) = S(M^\ast, W^\ast, P^\ast) = \{\mu^\ast\}\), \(\mu = \mu^\ast|_M \in \varphi(M, W, P)\) by \(CON\) of \(\varphi\). Therefore, we may conclude that \(\varphi(M, W, P) = S(M, W, P)\) for all \((M, W, P) \in \mathcal{P}\). Q.E.D.

4. Independence of the Axioms

In this section, we shall give four examples showing that dropping any one of the four axioms in the main theorem leads to the failure of the conclusion.

Example 1. A solution satisfying \(CON, C\text{-}CON\), and \(AN\) but not \(PO\). The solution \(A\), which associates the set \(A(M, W)\) of all possible matchings with each problem, satisfies \(CON, C\text{-}CON\), and \(AN\). Clearly, it does not satisfy \(PO\).

Example 2. A solution satisfying \(PO, AN\), and \(CON\) but not \(C\text{-}CON\). The solution \(PO\), which associates the set \(PO(M, W, P)\) of all Pareto...
optimal matchings with each problem, satisfies PO, AN, and CON. It does not satisfy C.CON.

**Example 3.** A solution satisfying PO, C.CON, and AN but not CON. Let \( \phi \) be the solution defined as follows:

(a) For any \((M, W, P) \in \mathcal{P}\) with \( n = 2 \), let \( \phi(M, W, P) \) be the set of all men-optimal matchings.\(^3\)

(b) Let \( \phi(M, W, P) = S(M, W, P) \) for any other \((M, W, P) \in \mathcal{P}\).

The solution \( \phi \) satisfies PO, C.CON, and AN but not CON.

**Example 4.** A solution satisfying PO, CON, and C.CON but not AN. Let us define such a solution \( \phi \) as follows:

**Step 1.** Pick a particular problem \((\tilde{M}, \tilde{W}, \tilde{P})\) such that \( \tilde{M} = \{\tilde{m}_1, \tilde{m}_2\}, \\tilde{W} = \{\tilde{w}_1, \tilde{w}_2\}, \) and fix it throughout this example. For this particular problem \((\tilde{M}, \tilde{W}, \tilde{P})\), let \( \phi(\tilde{M}, \tilde{W}, \tilde{P}) = A(\tilde{M}, \tilde{W}) \). For any other \((M, W, P)\) with \( n = 2 \), let \( \phi(M, W, P) = S(M, W, P) \).

**Step 2.** For a problem \((M, W, P)\) with \( n \geq 3 \), we distinguish two cases:

Case I. \( \tilde{M} \subseteq M, \tilde{W} \subseteq W, \) and \( P_{|\tilde{M} \cup \tilde{W}} = \tilde{P} \). Let \( \phi(M, W, P) \) be the union of \( S(M, W, P) \) and the set of all matchings \( \mu^* = \{(\tilde{m}_1, \tilde{w}_1), (\tilde{m}_2, \tilde{w}_2)\} \cup \mu^{**} \) such that \( \mu^{**} \in S(M \setminus \tilde{M}, W \setminus \tilde{W}, P_{|M \setminus \tilde{M} \cup W \setminus \tilde{W}}) \) and for all \( (m, w) \in \mu^{**} \) the matchings \( \{(\tilde{m}_1, \tilde{w}_1), (m, w)\} \) and \( \{(\tilde{m}_2, \tilde{w}_2), (m, w)\} \) are stable in the relevant subproblems of \((M, W, P)\).

Case II. For any other problem \((M, W, P)\) with \( n \geq 3 \), let \( \phi(M, W, P) = S(M, W, P) \). Obviously, \( \phi \) does not satisfy AN. It is not so difficult to see that \( \phi \) satisfies the remaining three axioms but the proof is long and is omitted.

5. **Conclusions**

In this selection, we shall compare our result to Peleg's papers [7, 8] because he characterizes the core of cooperative games. Peleg [7] characterizes the core of coalition-form games with non-transferable utility by

\(^3\) For the definition, see Gale and Shapley [1] or Roth and Sotomayor [9].
means of three axioms, CON, C.CON, and IR (individual rationality). IR requires that a solution ensures for each player at least what he (or she) can achieve by himself (or herself). In Peleg's framework, CON and IR imply PO. Observe that for any two-person game the set of Pareto optimal and individually rational payoff vectors is indeed the core. Then, the consistency axiom enables him to prove a result analogous to Lemma 3 of this paper.

In his Theorem 5.15, Peleg [8] characterizes the core of coalition-form games with transferable utility by means of PO, CON, C.CON, and UN (unanimity). This last axiom requires that for any two-person game the solution consists of all payoff vectors that are Pareto optimal and individually rational. This amounts to saying that the solution coincides with the core for any two-person game.

In our model, IR places no restriction on solutions because an individual agent cannot be an effective blocking coalition. Furthermore, UN adapted to our problems requires that \( \varphi(M, W, P) = S(M, W, P) \) for any problem \((M, W, P)\) with \( n = 2 \). This is not appealing because under CON and C.CON, the solution is essentially determined by what it is in the class of problems with \( n = 2 \). Moreover, the uniqueness result would not be a true axiomatization because it involves the definition of the core. For matching problems, we find anonymity more suitable than IR or than UN.

APPENDIX

**Proof of Lemma 1.** If \( \# S(M, W, P) = 1 \), then there is nothing to prove. Let \( S(M, W, P) = \{ \mu, \mu_1, \mu_2, \ldots, \mu_x \} \). Because \( S(M, W, P) \subset PO(M, W, P) \), either (1) or (2) holds:

1. There is a man \( m_0 \) in \( M \) such that \( \mu(m_0) P(m_0) \mu_1(m_0) \).
2. There is a woman \( w_0 \) in \( W \) such that \( \mu^{-1}(w_0) P(w_0) \mu_1^{-1}(w_0) \).

Assume that (1) holds. Introduce a new man \( m_1^* \) and a new woman \( w_1^* \). Define a list \( P' = \{ P'(a) | a \in M' \cup W' \} \) of preferences over the sets \( M' = M \cup \{ m_1^* \} \) and \( W' = W \cup \{ w_1^* \} \) in the following way:

**Rule 1.** For \( m_0 \), \( P'|_{m_0} = P(m_0) \) and \( \mu(m_0) P'(m_0) w_1^* P'(m_0) \mu_1(m_0) \) and for any other \( m \), \( P'|_{m} = P(m) \).

**Rule 2.** For \( w \in W \), \( P'|_{w} = P(w) \) and \( m P'(w) m_1^* \) for all \( m \in M \).

**Rule 3.** For \( m_1^*, w_1^* P'(m_1^*) w \) for all \( w \in W \).

**Rule 4.** For \( w_1^*, m_0 P'(w_1^*) m_1^* P'(w_1^*) m \) for all \( m \in M \). \( \setminus \{ m_0 \} \).

By construction, the matching \( \tilde{\mu} = \mu \cup \{ (m_1^*, w_1^*) \} \in S(M', W', P') \). Also,
by Rules 1 and 4, the matching \( \tilde{\mu}_1 = \mu_1 \cup \{(m^*_1, w^*_1)\} \notin S(M', W', \mathcal{P}') \) because the pair \((m_0, w^*_1)\) blocks it. Now, we can prove the following.

CLAIM. \( S(M', W', \mathcal{P}') \subset \{ v \cup \{(m^*_1, w^*_1)\} | v \in S(M, W, \mathcal{P}) \} \).

Proof of the claim. Suppose that there is a matching \( \tilde{v} \in S(M', W', \mathcal{P}') \) such that \( \tilde{v}(m) = w^*_1 \) for some \( m \in M \setminus \{m_0\} \). It follows that \( \tilde{v}(m^*_1) \notin W \). Then, by Rules 3 and 4, the pair \((m^*_1, w^*_1)\) blocks \( \tilde{v} \), a contradiction. Hence, there exists no matching \( \tilde{v} \in S(M', W', \mathcal{P}') \) such that \( \tilde{v}(m) = w^*_1 \) for some \( m \in M \setminus \{m_0\} \). Suppose that there is a matching \( v \in S(M', W', \mathcal{P}') \) such that \( v(m_0) = w^*_1 \). Let \( W'' \) be the set of all women who are better off in \( \tilde{v} \) than in \( \tilde{\mu} \). \( w^*_1 \) belongs to \( W'' \) so \( m^*_1 \) belongs to \( \tilde{\mu}^{-1}(W'') \). \( m^*_1 \) does not belong to \( \tilde{v}^{-1}(W'') \) because \( \tilde{v}(m^*_1) \) is in \( W \) and every woman in \( W \) ranks \( m^*_1 \) the last in her new preference list. But this is impossible because \( \tilde{\mu}^{-1}(W'') = \tilde{v}^{-1}(W'') \) by the Decomposition Theorem in, e.g., Roth and Sotomayor [9, p. 421]. Henceforth, there exists no matching \( \tilde{v} \in S(M', W', \mathcal{P}') \) such that \( m_0 \) is matched to \( w^*_1 \). Then any \( \tilde{v} \in S(M', W', \mathcal{P}') \) has the form of \( v \cup \{(m^*_1, w^*_1)\} \), where \( v \in A(M, W) \). Furthermore, the consistency of \( S \) implies that \( v \in S(M, W, \mathcal{P}) \).

Therefore, we may conclude that \( \tilde{\mu} \in S(M', W', \mathcal{P}') \) and \( \#S(M', W', \mathcal{P}') \leq k \). For the case in which (2) holds, the same arguments apply. By repeating this process finitely many times, we can construct a problem \((M^*, W^*, \mathcal{P}^*)\) which has the desired property. Q.E.D.

Proof of Lemma 2. Since \( \#S(M, W, \mathcal{P}) \leq 2 \) if \( n = 2 \), it suffices to consider the case of \( \#S(M, W, \mathcal{P}) = 1 \). Suppose that the conclusion is false. Then, there is a matching \( \mu_1 \in \varphi(M, W, \mathcal{P}) \) which is not stable. Let \( M = \{m_1, m_2\} \) and \( W = \{w_1, w_2\} \). Without loss of generality, assume that \( \mu_1 = \{(m_1, w_1), (m_2, w_2)\} \). Because it is not stable, there exists a blocking pair, say \((m_1, w_2)\). Hence, the preference profile \( \mathcal{P} \) must satisfy

\[
\begin{align*}
&w_2 \mathcal{P}(m_1) w_1 \\
&m_1 \mathcal{P}(w_2) m_2.
\end{align*}
\]

On the other hand, for \( m_2 \) and \( w_1 \), there are four possible cases:

(a) \[
\begin{align*}
&w_2 \mathcal{P}(m_2) w_1 \\
&m_2 \mathcal{P}(w_1) m_1,
\end{align*}
\]

(b) \[
\begin{align*}
&w_1 \mathcal{P}(m_2) w_2 \\
&m_1 \mathcal{P}(w_1) m_2,
\end{align*}
\]

(c) \[
\begin{align*}
&w_2 \mathcal{P}(m_2) w_1 \\
&m_1 \mathcal{P}(w_1) m_2,
\end{align*}
\]

(d) \[
\begin{align*}
&w_1 \mathcal{P}(m_2) w_2 \\
&m_2 \mathcal{P}(w_1) m_1.
\end{align*}
\]

If \( \mathcal{P} \) satisfies (d), then \( \mu_1 \) is Pareto dominated by swapping the partners of all. This contradicts PO. Hence, case (d) never occurs. We now add a new
man \( m_3 \) and a new woman \( w_3 \) to the original problem and extend the preferences to the larger set of participants.

**Case (a).** In this case, define the extended preference profile \( \mathcal{P}^* \) over the sets \( M^* \equiv \{ m_1, m_2, m_3 \} \) and \( W^* \equiv \{ w_1, w_2, w_3 \} \) as follows:

\[
\mathcal{P}^* = \left\{ \begin{array}{l}
    w_2 P(m_1) w_1 P(m_1) w_3 \\
    w_3 P(m_2) w_2 P(m_2) w_1 \\
    w_1 P(m_3) w_3 P(m_3) w_2
\end{array} \right.
\quad \text{and} \quad
\left\{ \begin{array}{l}
    m_2 P(w_1) m_3 P(w_1) m_1 \\
    m_1 P(w_2) m_3 P(w_2) m_2 \\
    m_2 P(w_3) m_1 P(w_3) m_3
\end{array} \right.
\]

It is easy to see that \( \mathcal{P}^* |_{M \cup W} = \mathcal{P} \). Let \( \mu^* \equiv \mu_1 \cup \{ (m_3, w_3) \} \). Since \( \mu^* \) is Pareto dominated by \( \mu^* \equiv \{ (m_1, w_2), (m_2, w_3), (m_3, w_1) \} \), \( \mu^* \notin \varphi(M^*, W^*, \mathcal{P}^*) \). Now, let \( M_2 \equiv \{ m_2, m_3 \} \), \( W_2 \equiv \mu^*(M_2) \), \( M_3 \equiv \{ m_1, m_3 \} \), and \( W_3 \equiv \mu^*(M_3) \). Consider the bijection \( \pi: M \cup W \rightarrow M_2 \cup W_2 \) given by \( \pi(m_1) = m_2 \), \( \pi(m_2) = m_3 \), \( \pi(w_1) = w_2 \), and \( \pi(w_2) = w_3 \). Because \( \pi_{M_1} = \mu^* |_{M_2 \cup W_2} \), \( \mu^* |_{M_2 \cup W_2} \notin \varphi(M_2, W_2, \mathcal{P}^* |_{M_2 \cup W_2}) \) by AN. By the same argument, it can be shown that \( \mu^* |_{M_3 \cup W_3} \notin \varphi(M_3, W_3, \mathcal{P}^* |_{M_3 \cup W_3}) \). Since \( \varphi \) satisfies C.CON, these observations imply that \( \mu^* \in \varphi(M^*, W^*, \mathcal{P}^*) \), a contradiction.

For **Cases (b) and (c)**, respectively, define the extended profile \( \mathcal{P}^* \) over \( M^* \) and \( W^* \) as follows:

**Case (b).**

\[
\mathcal{P}^* = \left\{ \begin{array}{l}
    w_3 P(m_1) w_2 P(m_1) w_1 \\
    w_1 P(m_2) w_3 P(m_2) w_2 \\
    w_2 P(m_3) w_1 P(m_3) w_3
\end{array} \right.
\quad \text{and} \quad
\left\{ \begin{array}{l}
    m_3 P(w_1) m_1 P(w_1) m_2 \\
    m_1 P(w_2) m_2 P(w_2) m_3 \\
    m_2 P(w_3) m_3 P(w_3) m_1
\end{array} \right.
\]

**Case (c).**

\[
\mathcal{P}^* = \left\{ \begin{array}{l}
    w_2 P(m_1) w_1 P(m_1) w_3 \\
    w_3 P(m_2) w_2 P(m_2) w_1 \\
    w_1 P(m_3) w_3 P(m_3) w_2
\end{array} \right.
\quad \text{and} \quad
\left\{ \begin{array}{l}
    m_3 P(w_1) m_1 P(w_1) m_2 \\
    m_1 P(w_2) m_2 P(w_2) m_3 \\
    m_2 P(w_3) m_3 P(w_3) m_1
\end{array} \right.
\]

By the same arguments as in Case (a), we can obtain a contradiction.

Q.E.D.

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