Game Theory

Repeated Games

Jordi Massó

International Doctorate in Economic Analysis (IDEA)
Universitat Autònoma de Barcelona (UAB)

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  - Only pure strategies.

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- Let $G = (I, (A_i)_{i \in I}, (h_i)_{i \in I})$ be a finite game in normal form. A_i is the set of player i's *actions* and $A = \prod_{i \in I} A_i$ is the set of action profiles.

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  - That is, $(a^1, \ldots, a^t) \in A^t$, where for every $1 \leq s \leq t$, $a^s = (a_1^s, \ldots, a_n^s) \in A$.

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- Given the game in normal form $G = (I, (A_i)_{i \in I}, (h_i)_{i \in I})$, define the *super-game form* as the game form $(I, (F_i)_{i \in I})$, where for every $i \in I$,

$$F_i = \left\{ f_i = \{f_i^t\}_{t=1}^{\infty} \mid f_i^1 \in A_i \text{ and } \forall t \geq 1,\ f_i^{t+1} : A^t \longrightarrow A_i \right\}.$$

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  - for all $t \geq 1$, $a^{t+1}(f) \in A$ is given by $a_i^{t+1}(f) = f_i^{t+1}(a^1(f), ..., a^t(f))$ for all $i \in I$.

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- "Play always C": $\hat{f}_i^1 = C$ and for all $t \geq 1$ and all $(a^1, ..., a^t) \in A^t$, $\hat{f}_i^{t+1}(a^1, ..., a^t) = C$.

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- "Play C during 5 periods and D thereafter": $\bar{f}_i^1 = C$, for all $1 \leq t < 5$ and all $(a^1, ..., a^t) \in A^t$, $\bar{f}_i^{t+1}(a^1, ..., a^t) = C$ and for all $t \geq 5$ and all $(a^1, ..., a^t) \in A^t$, $\bar{f}_i^{t+1}(a^1, ..., a^t) = D$.

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- *Trigger strategy*. "Start playing C and play C as long as the other player has played always C, once the other player has played D play D always": $\tilde{f}_i^1 = C$ and for all $t \geq 1$ and all $(a^1, ..., a^t) \in A^t$,

$$\tilde{f}_i^{t+1}(a^1, ..., a^t) = \begin{cases} C & \text{if for all } 1 \leq s \leq t,\ a_{3-i}^s = C \\ D & \text{if there exists } 1 \leq s \leq t \text{ such that } a_{3-i}^s = D. \end{cases}$$

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- *Tit-for-tat*. "Start playing C and then play the action taken by the other player last period": $\dot{f}_i^1 = C$ and for all $t \geq 1$ and all $(a^1, ..., a^t) \in A^t$, $\dot{f}_i^{t+1}(a^1, ..., a^t) = a_{3-i}^t$.

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5.3.- Payoffs

- Let $G = (I, (A_i)_{i \in I}, (h_i)_{i \in I})$ be a game in normal form and let $T \in \mathbb{N}$. The *finitely* $T-$*times repeated game* is the game in normal form $G_T = (I, (F_i)_{i \in I}, (H_i^T)_{i \in I}$, where $(I, (F_i)_{i \in I})$ is the super-game form and for each $i \in I$, $H_i^T : F \longrightarrow \mathbb{R}$ is defined as follows: for all $f \in F$,

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  - $T = 10$, $H_1^{10}(\bar{f}_1, \dot{f}_2) = \frac{1}{10}(5 \cdot 3 + 4 + 4 \cdot 1) = \frac{23}{10}$.

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  - For any $T \geq 1$, $H_i^T(\tilde{f}_1, \dot{f}_2) = \frac{1}{T}(3 \cdot T) = 3$.

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- Let $G = (I, (A_i)_{i \in I}, (h_i)_{i \in I})$ be a game in normal form. We say that G is bounded if

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- Let $G = (I, (A_i)_{i \in I}, (h_i)_{i \in I})$ be a bounded game in normal form and let $\lambda \in (0, 1)$. The $\lambda-$*discounted repeated game* is the game in normal form $G_\lambda = (I, (F_i)_{i \in I}, (H_i^\lambda)_{i \in I}$, where $(I, (F_i)_{i \in I})$ is the super-game form and for each $i \in I$, $H_i^\lambda : F \longrightarrow \mathbb{R}$ is defined as follows: for all $f \in F$,

$$H_i^\lambda(f) = (1 - \lambda) \sum_{t=1}^{\infty} \lambda^{t-1} h_i(a^t(f)).$$

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$$\begin{aligned} H_1^\lambda(\bar{f}_1, \dot{f}_2) &= (1-\lambda)(3 + 3\lambda + 3\lambda^2 + 3\lambda^3 + 3\lambda^4 + 4\lambda^5 + \lambda^6 + \lambda^7 + \ldots \\ &= (1-\lambda)\left(3\frac{1-\lambda^5}{1-\lambda} + 4\lambda^5 + \frac{\lambda^6}{1-\lambda}\right) \\ &= 3(1-\lambda^5) + 4(1-\lambda)\lambda^5 + \lambda^6 \\ &= 3 + \lambda^5 - 3\lambda^6. \end{aligned}$$

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- Since G_λ is a game in normal form, we can define F_λ^* as the set of Nash equilibria of G_λ.

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- If G is not bounded, the series may be divergent, and therefore H_i^λ would not necessarily be well-defined.
- The payoff $H_i^\lambda(f)$ can be interpreted as player i's expected payoff of playing f when at t, the probability of playing the game at $t+1$ is equal to λ.

5.3.- Payoffs

- Let $G = (I, (A_i)_{i \in I}, (h_i)_{i \in I})$ be a bounded game in normal form. The *infinitely repeated game* is the game in normal form $G_\infty = (I, (F_i)_{i \in I}, (H_i^\infty)_{i \in I}$, where $(I, (F_i)_{i \in I})$ is the super-game form and for each $i \in I$, $H_i^\infty : F \longrightarrow \mathbb{R}$ that will be defined later.

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- The "natural" payoff function would be: for all $f \in F$,

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- Problem: This limit may not exist (its existence depends on the particular strategies used by players).

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    - Note that the later implies the former.
  - Since we will have to check (equilibrium condition) whether $H_i^\infty(f) - H_i^\infty(g_i, f_{-i}) \geq 0$, we would like that $H_i^\infty(f)$ be linear.

Proposition *There exists a linear function* $H : l_\infty \longrightarrow \mathbb{R}$ *(called a Banach limit) such that for all* $\{x_n\} \in l_\infty$,

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5.4.- “Folk” Theorems

- Family of results characterizing the set of Nash equilibria or Subgame Perfect equilibria of repeated games (G_T, G_λ and G_∞) and their relationships. For example:

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5.4.- "Folk" Theorems

- Family of results characterizing the set of Nash equilibria or Subgame Perfect equilibria of repeated games (G_T, G_λ and G_∞) and their relationships. For example:

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  - and

$$\begin{aligned} h_1(a^s(g_1, f_2)) &= h_1(D, a_2^s(g_1, f_2)) \\ &> h_1(C, a_2^s(g_1, f_2)) \\ &= h_1(C, a_2^s(f_1, f_2)) \\ &= h_1(a^s(f_1, f_2)). \end{aligned}$$

- Therefore,

$$\begin{aligned} H_1^T(g_1, f_2) &= \frac{1}{T}\sum_{t=1}^{T} h_1(a^t(g_1, f_2)) \\ &> \frac{1}{T}\sum_{t=1}^{T} h_1(a^t(f_1, f_2)) \\ &= H_1^T(f_1, f_2), \end{aligned}$$

which contradicts that $(f_1, f_2) \in F_T^*$. ∎

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- Let $f_1 \in F_1$ be arbitrary (a symmetric argument works for player 2). For every $T \geq 1$,

$$\begin{aligned}\sum_{t=1}^{T} h_1(a^t(f_1, g_2)) = 3 \cdot \#\{1 \leq t \leq T \mid a^t(f_1, g_2) = (C, C)\} \\ +4 \cdot \#\{1 \leq t \leq T \mid a^t(f_1, g_2) = (D, C)\} \\ +0 \cdot \#\{1 \leq t \leq T \mid a^t(f_1, g_2) = (C, D)\} \\ +1 \cdot \#\{1 \leq t \leq T \mid a^t(f_1, g_2) = (D, D)\}.\end{aligned}$$

- By the definition of g_2 (tit-for-tat),

$$\#\{t \leq T \mid a^t(f_1, g_2) = (D, C)\} + \#\{t \leq T \mid a^t(f_1, g_2) = (D, D)\}$$

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$$
\left.\begin{array}{l}
\sum_{t=1}^{T} h_1(a^t(f_1,g_2)) = 3\cdot\#\{t\leq T \mid a^t(f_1,g_2)=(C,C)\} \\
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Hence,

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- Thus,

$$
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H_1^{\infty}(f_1, g_2) &= H\left(\left\{\frac{1}{T}\sum_{t=1}^{T} h_1(a^t(f_1, g_2))\right\}_{t=1}^{T}\right) \\
&\leq \limsup_{n\to\infty} \frac{1}{T}\sum_{t=1}^{T} h_1(a^t(f_1, g_2)) \\
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- Note that this is independent of the particular Banach limit H chosen to evaluate sequences of averages.

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- These collection of results are some times called Aumann-Shapley-Rubinstein Theorems.

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  - Warning: with mixed strategies, this minimax may be smaller; *i.e.*, there are games for which

$$\inf_{\sigma_{-i} \in \Sigma_{-i}} \sup_{\sigma_i \in \Sigma_i} H_i(\sigma_i, \sigma_{-i}) < \inf_{a_{-i} \in A_{-i}} \sup_{a_i \in A_i} h_i(a_i, a_{-i}).$$

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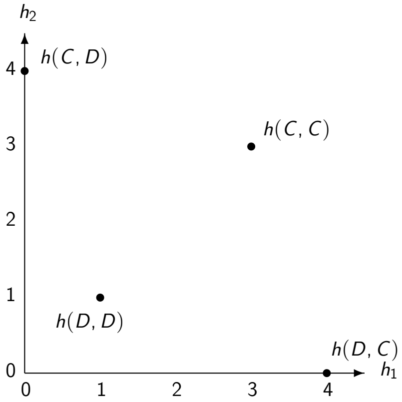

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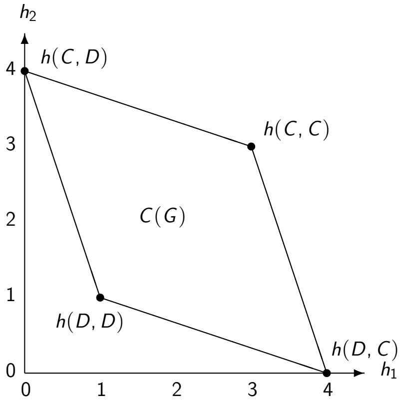
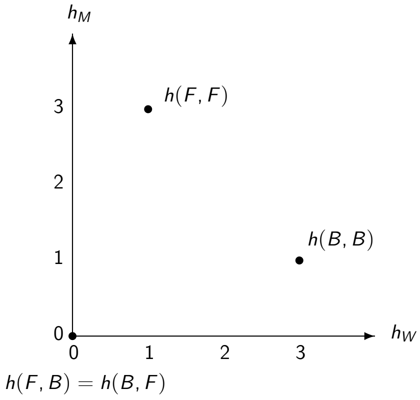
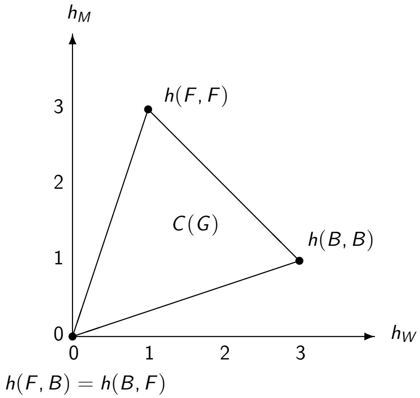
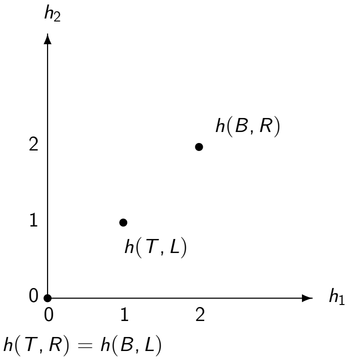


5.4.- "Folk" Theorems

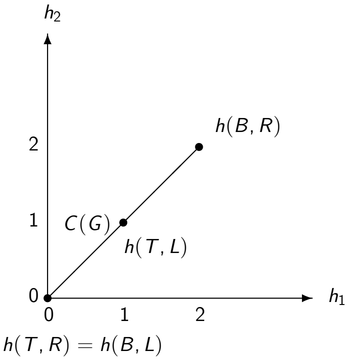


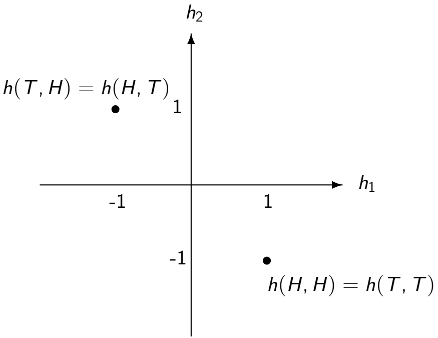
5.4.- “Folk” Theorems

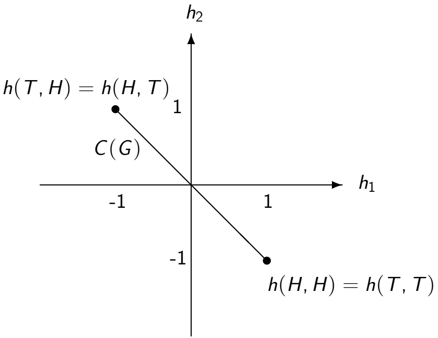



h_2
$h(B,R)$
2
1
$h(T,L)$
0
h_1
0
1
2
$h(T,R) = h(B,L)$

5.4.- “Folk” Theorems




h_2
$h(T, H) = h(H, T)$
1
h_1
-1
1
-1
$h(H, H) = h(T, T)$


h_2
$h(T, H) = h(H, T)$
1
$C(G)$
h_1
-1
1
-1
$h(H, H) = h(T, T)$

5.4.- "Folk" Theorems

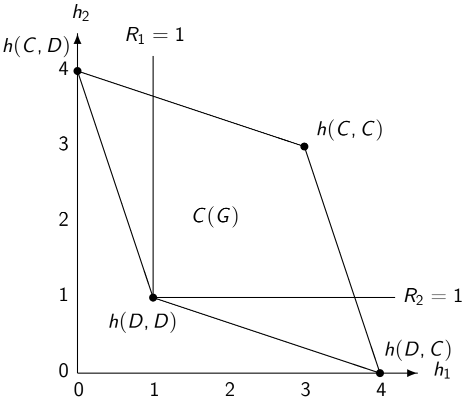
Infinitely Repeated

Theorem

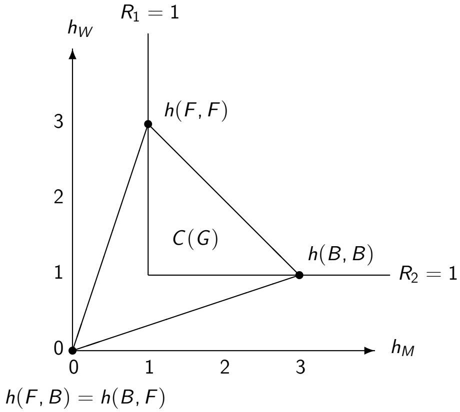
Let G be a bounded game in normal form. Then,

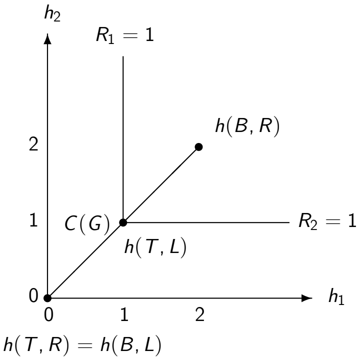
$$\left\{H^{\infty}(f) \in \mathbb{R}^{\#I} \mid f \in F_{\infty}^{*}\right\} = \{x \in C(G) \mid x_i \geq R_i \textit{ for all } i \in I\}.$$

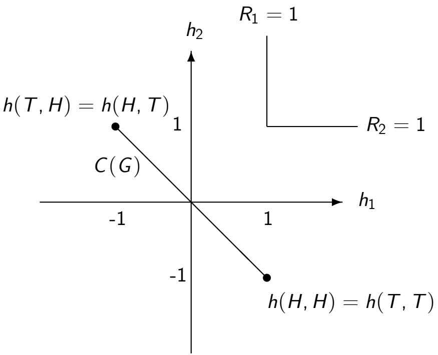
5.4.- "Folk" Theorems



5.4.- "Folk" Theorems




h_2
$R_1 = 1$
$h(B, R)$
2
1
$C(G)$
$R_2 = 1$
$h(T, L)$
0
h_1
0
1
2
$h(T, R) = h(B, L)$


$R_1 = 1$
h_2
$h(T, H) = h(H, T)$
1
$R_2 = 1$
$C(G)$
h_1
-1
1
-1
$h(H, H) = h(T, T)$

5.4.- "Folk" Theorems

Let $G = (I, (A_i)_{i \in I}, (h_i)_{i \in I})$ be a finite game in normal form.

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Proposition 2 *Let $\{a^t\}_{t=1}^\infty$ be such that $a^t \in A$ for all $t \geq 1$ and $\liminf_{n \to \infty} \frac{1}{T} \sum_{t=1}^{T} h_i(a^t) \geq R_i$ for all $i \in I$ then, there exists an $f \in F$ such that (1) f is an equilibrium of G_∞ and (2) $a^t(f) = a^t$ for all $t \geq 1$.*

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Proposition 3 *For every* $x \in C(G)$ *there exists a sequence* $\{a^t\}_{t=1}^\infty$ *such that* $a^t \in A$ *for all* $t \geq 1$ *and for all* $i \in I$, $\lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} h_i(a^t)$ *exists and it is equal to* x_i.

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Proposition 4 *For every $f \in F$ and every $\alpha = T, \lambda, \infty$, $H^\alpha(f) \in C(G)$.*

5.4.- “Folk” Theorems

Proposition 1: Intuition

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- Let $f \in F$ and $i \in I$ be arbitrary. Define recursively $g_i \in F_i$ as follows:

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- Let $f \in F$ and $i \in I$ be arbitrary. Define recursively $g_i \in F_i$ as follows:
- Given $a^1(f)$ let $b_i^1 \in A_i$ be s.t. $h_i(b_i^1, a^1(f)_{-i}) \geq R_i$; it exists since

  $$R_i = \min_{a_{-i} \in A_{-i}} \max_{a_i \in A_i} h_i(a_i, a_{-i}) \leq \max_{a_i \in A_i} h_i(a_i, a^1(f)_{-i}) = h_i(b_i^1, a^1(f)_{-i}).$$

  Then, set $g_i^1 = b_i^1$.

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  Then, set $g_i^1 = b_i^1$.
- Assume g_i has been defined up to t. Let $b_i^{t+1} \in A_i$ be s.t. $h_i(b_i^{1+1}, f^{t+1}(a^1(g_i, f_{-i}), ..., a^t(g_i, f_{-i}))_{-i}) \geq R_i$; as before, it also exists. Then, for all $(a^1, ..., a^t) \in A^t$, set

  $$g_i^{t+1}(a^1, ..., a^t) = \begin{cases} b_i^{t+1} & \text{if } \forall 1 \leq s \leq t,\ a^s = a^s(g_i, f_{-i}) \\ f_i^{t+1}(a^1, ..., a^t) & \text{otherwise.} \end{cases}$$

- It is possible to show that, by the definition of g_i,

$$h_i(a^t(g_i, f_{-i})) \geq R_i.$$

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$$h_i(a^t(g_i, f_{-i})) \geq R_i.$$

- Hence, for all $\alpha = T, \lambda, \infty$, $H_i^\alpha(g_i, f_{-i}) \geq R_i$.
- Thus, if for $\alpha = T, \lambda, \infty$, $f \in G_\alpha^*$ then, it must be the case that

$$H_i^\alpha(f) \geq R_i.$$

Proposition 2 *Let* $\{a^t\}_{t=1}^{\infty}$ *be such that* $a^t \in A$ *for all* $t \geq 1$ *and* $\liminf_{n \to \infty} \frac{1}{T} \sum_{t=1}^{T} h_i(a^t) \geq R_i$ *for all* $i \in I$ *then, there exists an* $f \in F$ *such that (1)* f *is an equilibrium of* G_∞ *and (2)* $a^t(f) = a^t$ *for all* $t \geq 1$.

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- For every $i \in I$, there exists $a(i) \in A$ such that $h_i(b_i, a(i)_{-i}) \leq R_i$ for all $b_i \in A_i$. Observe that

$$R_i = \min_{a_{-i}\in A_{-i}} \max_{a_i \in A_i} h_i(a_i, a_{-i}) = \max_{a_i\in A_i} h_i(a_i, a(i)_{-i}) \geq h_i(b_i, a(i)_{-i})$$

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  for all $b_i \in A_i$.
- For every $j \in I$, set $f_j^1 = a_j^1$.
- Take any function $\gamma : 2^I \backslash \{\varnothing\} \longrightarrow I$ with the property that for all $J \in 2^I \backslash \{\varnothing\}$, $\gamma(J) \in J$.

- Let $(b^1, ..., b^t) \in A^t$ be arbitrary. Let $s = \min\{1 \leq r \leq t \mid b^r \neq a^r\}$, $J = \{k \in I \mid b^s_k \neq a^s_k\}$ and $i = \gamma(J)$.

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- Define

$$f_j^{t+1}(b^1, ..., b^t) = \begin{cases} a_j^{t+1} & \text{if } \forall 1 \leq r \leq t,\ b^r = a^r \\ a(i)_j & \text{otherwise.} \end{cases}$$

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- For any $g_i \in F_i$ either $a^t(f) = a^t(g_i, f_{-i})$ for all $t \geq 1$, in which case $H_i^{\infty}(f) = H_i(g_i, f_{-i})$ or else there exists $s = \min\{t \geq 1 \mid a^t(g_i, f_{-i}) \neq a^t(f)\}$. Then, $J = \{i\}$ and $\gamma(\{i\}) = i$. Thus,

$$
\begin{aligned}
H_i^{\infty}(g_i, f_{-i}) &= H\left(\left\{\frac{1}{T}\sum_{t=1}^{T} h_i(a^t(g_i, f_{-i})\right\}_{T=1}^{\infty}\right) \\
&\leq \limsup_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} h_i(a^t(g_i, f_{-i}) \\
&\leq \limsup_{T\to\infty} \frac{1}{T}\left[s \max\{h_i(a) \mid a \in A\} + (T-s)R_i\right] \\
&\leq \limsup \frac{1}{T} s \max\{h_i(a) \mid a \in A\} + \limsup \frac{1}{T}(T-s)R_i \\
&\leq \limsup_{T\to\infty} \frac{1}{T} TR_i \\
&= R_i \\
&\leq \liminf_{n\to\infty} \frac{1}{T}\sum_{t=1}^{T} h_i(a^t) \qquad \text{by hypothesis} \\
&\leq \liminf_{n\to\infty} \frac{1}{T}\sum_{t=1}^{T} h_i(a^t(f)) \qquad \text{by (ii)} \\
&\leq H_i^{\infty}(f).
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H_i^\infty(g_i, f_{-i}) &= H\left(\left\{\frac{1}{T}\sum_{t=1}^{T} h_i(a^t(g_i, f_{-i})\right\}_{T=1}^{\infty}\right) \\
&\leq \limsup_{T\to\infty} \frac{1}{T}\textstyle\sum_{t=1}^{T} h_i(a^t(g_i, f_{-i})) \\
&\leq \limsup_{T\to\infty} \frac{1}{T}\left[s \max\{h_i(a) \mid a \in A\} + (T-s)R_i\right] \\
&\leq \limsup \frac{1}{T} s \max\{h_i(a) \mid a \in A\} + \limsup \frac{1}{T}(T-s)R_i \\
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&\leq H_i^\infty(f).
\end{aligned}
$$

But since g_i was arbitrary, $f \in F_\infty^*$.

Proposition 3 *For every* $x \in C(G)$ *there exists a sequence* $\{a^t\}_{t=1}^{\infty}$ *such that* $a^t \in A$ *for all* $t \geq 1$ *and for all* $i \in I$, $\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} h_i(a^t)$ *exists and it is equal to* x_i.

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Proposition 3 follows (after some work to deal with convex combinations with non-rational coefficients) from the following result which in turn follows from a more general result (Caratheodory Theorem).

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Result Let $X = co\{x^1, ..., x^K\} \subseteq \mathbb{R}^n$. For every $x \in X$ there exist $y^1, ..., y^{n+1} \in \{x^1, ..., x^K\}$ and $p^1, ..., p^{n+1} \geq 0$ such that $\sum_{j=1}^{n+1} p^j = 1$ with the property that $x = \sum_{j=1}^{n+1} p^j y^j$.

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- For $\alpha = T, \infty$ the statement obviously holds.
- For $\alpha = \lambda$, observe that for every $t \geq 1$, $0 \leq (1-\lambda)\lambda^{t-1} \leq 1$ and $(1-\lambda)\sum_{t=1}^{\infty}\lambda^{t-1} = (1-\lambda)\frac{1}{1-\lambda} = 1$. Thus, each $(1-\lambda)\lambda^{t-1}$ can be seen as the coefficient of an (infinite) convex combination: Thus, $H^{\lambda}(f) = (1-\lambda)\sum_{t=1}^{\infty}\lambda^{t-1}h(a^{t}(f)) \in C(G)$.

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5.4.- "Folk" Theorems

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- $\subseteq$) Let f be an equilibrium of G_α.
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- $\supseteq$) Let $x \in C(G)$ and assume $x_i \geq R_i$ for all $i \in I$.
  - By Proposition 3, there exists a sequence $\{a^t\}_{t=1}^{\infty}$ such that $a^t \in A$ for all $t \geq 1$ and for all $i \in I$, $\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} h_i(a^t) = x_i$.

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  - By Proposition 1, $H_i^\alpha(f) \geq R_i$ for all $i \in I$.
  - By Proposition 4, $H^\alpha(f) \in C(G)$.
  - Hence, $H^\alpha(f) \in \{x \in C(G) \mid x_i \geq R_i$ for all $i \in I\}$.
- $\supseteq$) Let $x \in C(G)$ and assume $x_i \geq R_i$ for all $i \in I$.
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$$H_i^\infty(f) = H\left(\left\{\frac{1}{T}\sum_{t=1}^{T} h_i(a^t(f))\right\}_{T=1}^{\infty}\right) = \lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T} h_i(a^t) = x_i.$$

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  - Thus, $x \in \left\{H^\infty(f) \in \mathbb{R}^{\#I} \mid f \text{ is an equilibrium of } G_\infty\right\}$. ■

Theorem

For every $x \in C(G)$ *such that* $x_i > R_i$ *for all* $i \in I$, *there exists* $\underline{\lambda} \in (0,1)$ *such that for all* $\lambda \in (\underline{\lambda}, 1)$ *there exists* $f \in F^*_\lambda$ *with the property that* $H^\lambda(f) = x$.

Theorem

Benoît and Krishna (1987) Assume that for every $i \in I$ *there exists* $a^*(i) \in A^*$ *such that* $h_i(a^*(i)) > R_i$*. Then, for all* $x \in C(G)$ *such that* $x_i > R_i$ *for all* $i \in I$ *and for every* $\varepsilon > 0$ *there exists* $\hat{T} \in \mathbb{N}$ *such that for all* $T > \hat{T}$ *there exists* $f \in F_T^*$ *such that* $\left\| H^T(f) - x \right\| < \varepsilon$.

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- Terminal phase: for $Q \in \mathbb{N}$,

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5.4.- “Folk” Theorems

Finitely Repeated: Intuition

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- Average payoffs in the terminal phase: for all $i \in N$,

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- Given $x \in C(G)$ such that $x_i > R_i$ for all $i \in N$, *choose* Q with the property that for all $i \in N$,

$$x_i + Qy_i > \sup_{a \in A} h_i(a) + QR_i.$$

- Given $\varepsilon > 0$, *choose* $T \in \mathbb{N}$ such that there exists a cycle $\{a^t\}$ of length $T - Qn$ such that

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$$f_i^t(\cdot) = \begin{cases} a^t & \text{if all players follow the cycle } \{a^t\} \\ a(j)_i & \text{if } j \text{ has deviated,} \end{cases}$$

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  - for $T - Qn + 1 \leq t < T$.

$$f_i^t(\cdot) = \text{terminal phase of Nash equilibria.}$$

- It is possible to show that for all T sufficiently large, all $i \in N$, and all $g_i \in F_i$,

$$H_i^T(f) \geq H_i^T(g_i, f_{-i});$$

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- Moreover, for sufficiently large T,

$$\left\| H^T(f) - x \right\| < \varepsilon;$$

namely, the weight of the terminal phase vanishes.

5.4.- "Folk" Theorems: SPE

- Let $G = (I, (A_i)_{i \in I}, (h_i)_{i \in I})$ be a game in normal form.

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- Notation: for every $(f, (a^1, ..., a^t)) \in F \times A^t$, set

$$s(f, (a^1, ..., a^t)) \equiv (s(f_i, (a^1, ..., a^t))_i)_{i \in I}.$$

Definition Let $G = (I, (A_i)_{i \in I}, (h_i)_{i \in I})$ be a game in normal form. An strategy $f \in F$ is a *Subgame Perfect Equilibrium (SPE) of* G_α, for $\alpha = \infty, \lambda$, if for every $t \geq 1$ and every $(a^1, ..., a^t) \in A^t$, $s(f, (a^1, ..., a^t))$ is a Nash equilibrium of G_α.

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Theorem

Aumann, Shapley, Rubinstein. Let $G = (I, (A_i)_{i \in I}, (h_i)_{i \in I})$ *be a bounded game in normal form. Then,*

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Theorem

Friedman (1971) Let $a^* \in A^*$ *be such that* $h(a^*) = e$*. Then, for every* $x \in C(G)$ *such that* $x_i > e_i$ *for all* $i \in I$*, there exists* $\underline{\lambda} \in (0,1)$ *such that for all* $\lambda \in (\underline{\lambda}, 1)$ *there exists a SPE* f *of* G_λ *with* $H^\lambda(f) = x$.

Theorem

Fudenberg and Maskin (1986) Let $G = (I, (A_i)_{i \in I}, (h_i)_{i \in I})$ *be a bounded game in normal form and assume* $\dim(C(G)) = n$. *Then, for all* $x \in C(G)$ *such that* $x_i > R_i$ *for all* $i \in I$, *there exists* $\underline{\lambda} \in (0,1)$ *such that for all* $\lambda \in (\underline{\lambda}, 1)$ *there exists a SPE* f *of* G_λ *with* $H^\lambda(f) = x$.

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Benoît and Krishna (1985) Let $G = (I, (A_i)_{i \in I}, (h_i)_{i \in I})$ *be a bounded game in normal form and assume that for each* $i \in I$ *there exist* $a^*(i), \tilde{a}(i) \in A^*$ *such that* $h_i(a^*(i)) > h_i(\tilde{a}(i))$ *and that* $\dim(C(G)) = n$. *Then, for every* $x \in C(G)$ *such that* $x_i > R_i$ *for all* $i \in I$ *and every* $\varepsilon > 0$ *there exists* $\hat{T} \in \mathbb{N}$ *such that for all* $T > \hat{T}$ *there exists a SPE* $f \in F$ *of* G_T *such that* $\left\| H^T(f) - x \right\| < \varepsilon$.

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  - etc.
- Idea: several games may be played, with a transition probability that may depend on the profile of actions.

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  - $G_k = (I, (A_i^k)_{i \in I}, (h_i^k)_{i \in I})$ is a finite game in normal form,
    - without loss of generality, assume $A_i^k \cap A_i^{k'} = \varnothing$ for all $i \in I$ and $k \neq k'$ (and define $A_i = \bigcup_{k=1}^{K} A_i^k$),
    - G_1 is the initial game;
  - for every $a^k \in A^k$, $p(a^k)$ is a probability distribution on $\{G_1, ..., G_K\}$ (*i.e.*, $p(a^k) \in \Delta^K$), where for all $k' = 1, ..., K$,
    - $p(a^k)_{k'}$ is the probability of moving to game $G_{k'}$ if players are at game G_k and choose action $a^k \in A^k$.

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