Game Theory

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PROBLEM SET #3

3.1.- Two players (1 and 2) take part in an auction for a painting. It is known that v_1 and v_2 are the values that they assign to the painting ($v_1 > v_2 > 0$). The firm that is auctioning the painting will collect their non-negative bids (x_1 and x_2) in a closed envelope. The player with the highest bid will receive the painting but he will pay the bid of the loser (if the two bids are equal, player 1 receives the painting).

(a) Write the payoff functions of the two players.

(b) Obtain and draw the best-reply correspondences of the two players.

(c) Use this graph to identify the dominant strategy of each player.

(d) Obtain graphically the set of Nash equilibria and identify the dominant strategy equilibrium.

3.2.- Find the sophisticated equilibria of the following game in normal form:

$1/2$	a_2^1	a_2^2	a_2^3
a_1^1	$2,0$	$1,1$	$4,2$
a_1^2	$3,4$	$1,2$	$2,3$
a_1^3	$1,3$	$0,2$	$3,0$

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3.3.- Find a normal for game (with $\#I = 2$) in which two orders of iterated elimination of dominated strategies produce different results.

3.4.- There are ten locations with respective value $a_1 < \cdots < a_{10}$. Player i ($i = 1, 2$) is endowed with n_i soldiers ($n_i < 10$) and must allocate them among the locations. To each particular location he can allocate no more than one soldier. The payoff at location k is a_k to the player whose soldier is unchallenged, and $-a_k$ to his opponent, unless both have a soldier at location k, in which case the payoff is zero to both. The total payoff is obtained by summing up local payoffs. Show that this game has a unique strictly dominant strategy equilibrium. What if some of the a_k's coincide?

3.5.- Give an example of a game in normal form that is dominant solvable but for which it is not the case that all players are indifferent between all outcomes that survive iterated elimination of dominated actions.

3.6.- Each of two players announces a nonnegative integer equal to at most 100. If $s_1 + s_2 \leq 100$, where s_i is the integer announced by player i, then each player i

receives payoff of s_i. If $s_1 + s_2 > 100$ and $s_i < s_j$ then player i receives s_i and player j receives $100 - s_i$; if $s_1 + s_2 > 100$ and $s_1 = s_2$ then each player receives 50. Show that the game is dominant solvable and find the set of sophisticated equilibria.

3.7.- Among three candidates $\{a, b, c\}$, a society $\{1, 2, 3\}$ must elect one. The voting rule is plurality voting and player 1 breaks ties. In other words, the strategy sets are $S_1 = S_2 = S_3 = \{a, b, c\}$, and, if the agents cast the votes (s_1, s_2, s_3) the elected candidate is s_2 if $s_2 = s_3$ and s_1 if $s_2 \neq s_3$. Suppose now that the utility of the members of the society for the various candidates display a Condorcet effect:

$$\begin{gathered} u_1(a) > u_1(b) > u_1(c) \\ u_2(b) > u_2(c) > u_2(a) \\ u_3(c) > u_3(a) > u_3(b)\,. \end{gathered}$$

Show that the game is dominant solvable and obtain its sophisticated equilibrium.

3.8.- Players 1 and 2 are bargaining over how to split 1 Euro. Both players simultaneously name shares they would like to have, s_1 and s_2, where $s_1, s_2 \in [0, 1]$. If $s_1 + s_2 \leq 1$, then players receive the shares they named; if $s_1 + s_2 > 1$, then both receive zero.

(a) Find the Nash equilibria (in pure strategies) of this game.

Suppose that player 2, before choosing s_2, knows s_1, and this is common knowledge.

(b) Find the Nash equilibria (in pure strategies) of this modified game.

(c) Find the subgame perfect Nash equilibria (in pure strategies) of this modified game.

3.9.- Suppose that three players share a pie using the following procedure. First player 1 proposes a division $x = (x_1, x_2, x_3)$, then 2 and 3 simultaneously respond either "yes" or "no". If players 2 and 3 both say "yes" then the division x is implemented; otherwise no player receives anything. Each player prefers more of the pie to less. Formulate this situation as a (non-finite) extensive game with imperfect information by writing the strategy set of each player and the structure of each subgame starting at the proposal x. Find the set of subgame perfect equilibria of this game.

3.10.- Consider the two-stages game in extensive form of Figure 1 representing the problem of an Entrant and a Monopolist. Suppose that $M > A > 0 > W$ and consider only pure strategies.

(a) Find the Nash equilibria of this game.

(b) Find the subgame perfect equilibrium of this game.

Suppose that the monopolist, before the entrant's decision, may decide whether or not to invest in a project that will decrease his own profits, except if he fights the entrant, in a quantity equal to $K > 0$, where $M - K > A$. Suppose that the entrant can observe, before taking his decision, the investment decision of the monopolist.

(c) Find the subgame perfect equilibria of this new game in three stages.

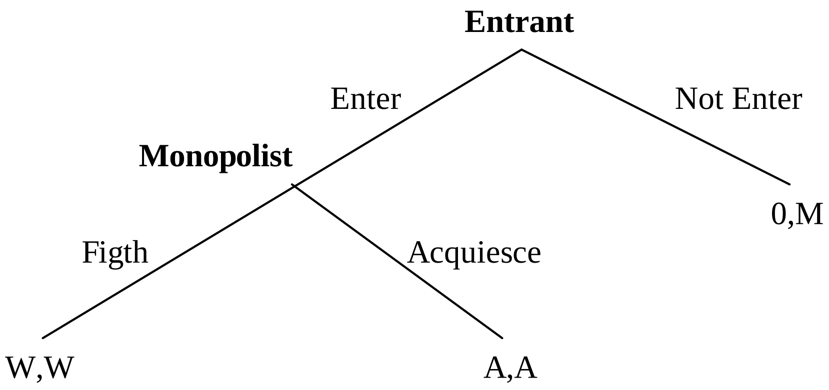

Figure 1

3.11.- Consider an industry with $n+1$ identical firms, $\{0, 1, ..., n\}$. Each firm has a constant marginal cost equal to 1 and it does not have fixed cost. The inverse demand function is equal to $p(Q) = \max\{0, 2 - Q\}$, where Q represents the aggregate quantity sold in the market. Firms decide the quantity to produce as follows: firm 0 chooses first its quantity q_0, and after that and knowing the amount q_0, firms $1, ..., n$ decide their respective quantities simultaneously (observe that if $n = 1$ we have Stackelberg's model).

(a) For each q_0, find the quantities produced by firms $1, ..., n$ in the symmetric equilibrium ($i.e.$, $q = q_i$ for all $i = 1, ..., n$) of the subgame starting at q_0.

(b) Find the corresponding equilibrium quantity q_0 of this subgame perfect equilibrium.

(c) Show that the quantity produced by firm 0 in this subgame perfect equilibrium is independent of n (surprisingly!), but its profits are a decreasing function of n.

3.12.- Assume that the game in extensive form Γ has a unique subgame perfect equilibrium. Let G be its corresponding game in normal form. Either prove or show a counter-example of the following statement: *"The unique subgame perfect equilibrium of Γ can not be eliminated by an iterative deletion of dominated pure strategies in G".*

3.13.- Consider the game in extensive form with imperfect information Γ of Figure 2.

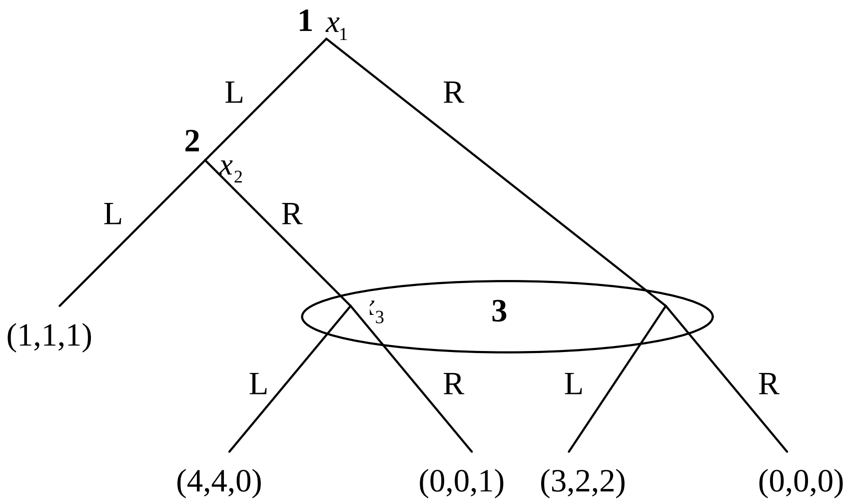

Figure 2

The game Γ has two types of Nash equilibria.
Type 1: $\sigma_1(L) = 1$, $\sigma_2(L) = 1$, and $\sigma_3(L) \in \left[0, \frac{1}{4}\right]$.
Type 2: $\sigma_1(L) = 0$, $\sigma_2(L) \in \left[\frac{1}{3}, 1\right]$, and $\sigma_3(L) = 1$.

(a) Show that all Nash equilibria of type 1 are also perfect equilibria of Γ.

(b) Show that none of the Nash equilibria of type 2 are perfect equilibria of Γ.

3.14.- Consider the family of games (parametrized by x) in extensive form with imperfect information $\Gamma(x)$ of Figure 3. Show that the strategy $\sigma = (\sigma_1, \sigma_2)$, where $\sigma_1(T) = 1$ and $\sigma_2(R) = 1$, is a perfect equilibrium of $\Gamma(x)$ for all $x < 2$.

3.15.- Consider the two-player game Γ shown in Figure 4.

(a) Find all subgame perfect equilibria of Γ.

(b) Which of these equilibria are perfect? For each perfect equilibrium σ find a sequence of trembles $\{\epsilon^k\} \to 0$ and the corresponding sequence of completely mixed strategies $\{\sigma^k\} \to \sigma$ that make σ a perfect equilibrium of Γ.

(c) Which of these equilibria of part (a) are proper equilibria of G^a (the corresponding agent normal form of Γ)? For each proper equilibrium σ of G^a find a sequence of trembles $\{\epsilon^k\} \to 0$ and the corresponding sequence of completely mixed strategies $\{\sigma^k\} \to \sigma$ that make σ a proper equilibrium of G^a.

3.16.- Show that the strategy σ, where $\sigma_1(T) = 1$ and $\sigma_2(R) = 1$, is a proper equilibrium of the normal form associated to the game $\Gamma(0)$ of Figure 3.

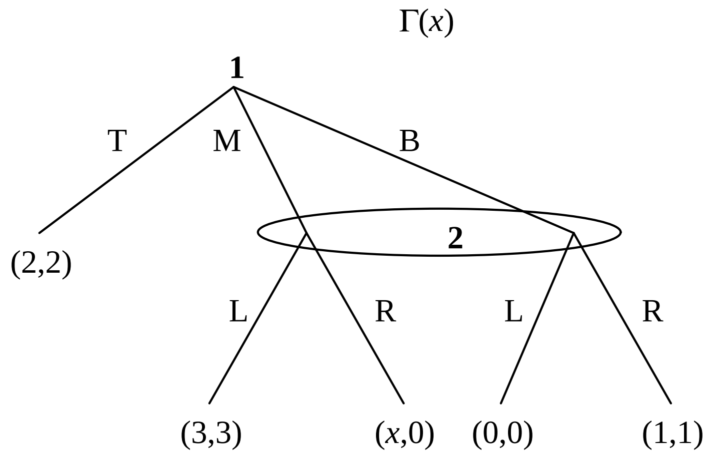

Figure 3

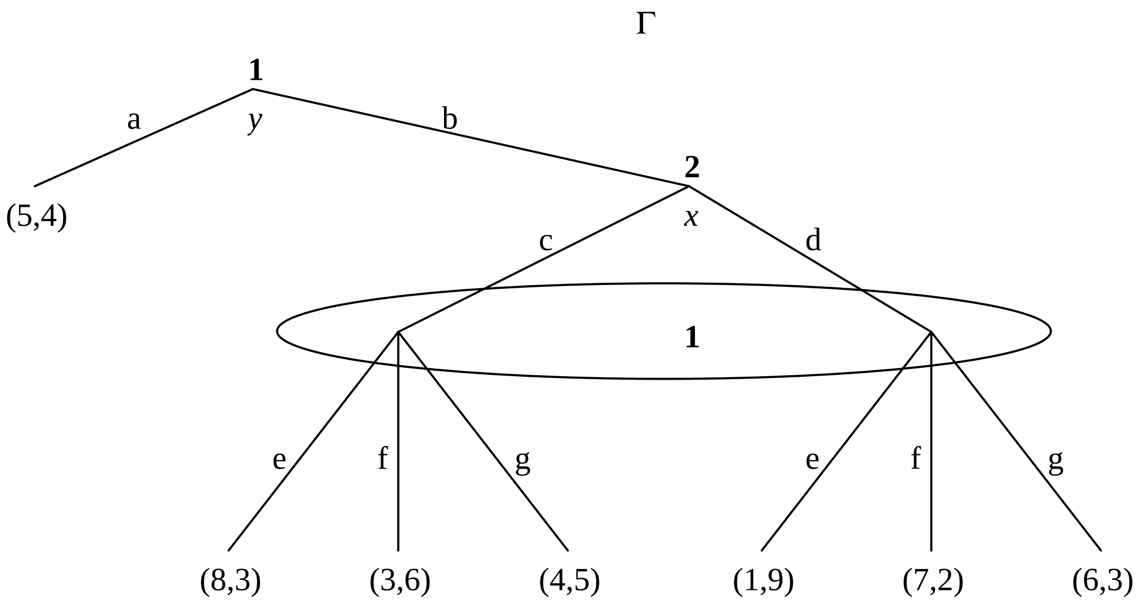

Figure 4

3.17.- Show that the following game in normal form

1/2	L	C	R
U	$0,0$	$5,4$	$4,5$
M	$4,5$	$0,0$	$5,4$
D	$5,4$	$4,5$	$0,0$

has correlated equilibria in which both players get expected utility strictly larger than 4.

3.18.- Consider the following game in normal form

A	L	R
T	$0,0,3$	$0,0,0$
B	$1,0,0$	$0,0,0$

B	L	R
T	$2,2,2$	$0,0,0$
B	$0,0,0$	$2,2,2$

C	L	R
T	$0,0,0$	$0,0,0$
B	$0,1,0$	$0,0,3$

,

where player 1 chooses rows (T or B), player 2 chooses columns (L or R), and player 3 chooses matrices (A, B, or C).

(a) Show that the pure strategy equilibrium payoffs are $(1,0,0)$, $(0,1,0)$, and $(0,0,0)$.

(b) Show that there is a correlated equilibrium p (a probability distribution on S) with expected payoffs equal to $(2,2,2)$. Construct an information structure $(\Omega, (B_1, B_2, B_3), \widetilde{p})$ and correlated equilibrium $\mathfrak{s} : \Omega \to S$ generating the probability distribution p on S. Explain the sense in which each player prefers not to have the information that the other players use to coordinate their actions.

3.19.- Show that the set of correlated equilibria is convex.