

# Game Theory

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## PROBLEM SETS #3 AND #4

**3.1.-** Two players (1 and 2) take part in an auction for a painting. It is known that  $v_1$  and  $v_2$  are the values that they assign to the painting ( $v_1 > v_2 > 0$ ). The firm that is auctioning the painting will collect their non-negative bids ( $x_1$  and  $x_2$ ) in a closed envelope. The player with the highest bid will receive the painting but he will pay the bid of the loser (if the two bids are equal, player 1 receives the painting).

- Write the payoff functions of the two players.
- Obtain and draw the best-reply correspondences of the two players.
- Use this graph to identify the dominant strategy of each player.
- Obtain graphically the set of Nash equilibria and identify the dominant strategy equilibrium.

**3.2.-** Find the sophisticated equilibria of the following game in normal form:

1/2	$a_2^1$	$a_2^2$	$a_2^3$
$a_1^1$	2, 0	1, 1	4, 2
$a_1^2$	3, 4	1, 2	2, 3
$a_1^3$	1, 3	0, 2	3, 0

**3.3.-** Find a normal form game (with  $\#I = 2$ ) in which two orders of iterated elimination of dominated strategies produce different results.

**3.4.-** There are ten locations with respective value  $a_1 < \dots < a_{10}$ . Player  $i$  ( $i = 1, 2$ ) is endowed with  $n_i$  soldiers ( $n_i < 10$ ) and must allocate them among the locations. To each particular location he can allocate no more than one soldier. The payoff at location  $k$  is  $a_k$  to the player whose soldier is unchallenged, and  $-a_k$  to his opponent, unless both have a soldier at location  $k$ , in which case the payoff is zero to both. The total payoff is obtained by summing up local payoffs. Show that this game has a unique strictly dominant strategy equilibrium. What if some of the  $a_k$ 's coincide?

**3.5.-** Give an example of a game in normal form that is dominant solvable but for which it is not the case that all players are indifferent between all outcomes that survive iterated elimination of dominated actions.

**3.6.-** Each of two players announces a nonnegative integer equal to at most 100. If  $s_1 + s_2 \leq 100$ , where  $s_i$  is the integer announced by player  $i$ , then each player  $i$

receives payoff of  $s_i$ . If  $s_1 + s_2 > 100$  and  $s_i < s_j$  then player  $i$  receives  $s_i$  and player  $j$  receives  $100 - s_i$ ; if  $s_1 + s_2 > 100$  and  $s_1 = s_2$  then each player receives 50. Show that the game is dominant solvable and find the set of sophisticated equilibria.

**3.7.-** Among three candidates  $\{a, b, c\}$ , a society  $\{1, 2, 3\}$  must elect one. The voting rule is plurality voting and player 1 breaks ties. In other words, the strategy sets are  $S_1 = S_2 = S_3 = \{a, b, c\}$ , and, if the agents cast the votes  $(s_1, s_2, s_3)$  the elected candidate is  $s_2$  if  $s_2 = s_3$  and  $s_1$  if  $s_2 \neq s_3$ . Suppose now that the utility of the members of the society for the various candidates display a Condorcet effect:

$$\begin{aligned} u_1(a) &> u_1(b) > u_1(c) \\ u_2(b) &> u_2(c) > u_2(a) \\ u_3(c) &> u_3(a) > u_3(b). \end{aligned}$$

Show that the game is dominant solvable and obtain its sophisticated equilibrium.

**3.8.-** Players 1 and 2 are bargaining over how to split 1 Euro. Both players simultaneously name shares they would like to have,  $s_1$  and  $s_2$ , where  $s_1, s_2 \in [0, 1]$ . If  $s_1 + s_2 \leq 1$ , then players receive the shares they named; if  $s_1 + s_2 > 1$ , then both receive zero.

(a) Find the Nash equilibria (in pure strategies) of this game.

Suppose that player 2, before choosing  $s_2$ , knows  $s_1$ , and this is common knowledge.

(b) Find the Nash equilibria (in pure strategies) of this modified game.

(c) Find the subgame perfect Nash equilibria (in pure strategies) of this modified game.

**3.9.-** Suppose that three players share a pie using the following procedure. First player 1 proposes a division  $x = (x_1, x_2, x_3)$ , then 2 and 3 simultaneously respond either “yes” or “no”. If players 2 and 3 both say “yes” then the division  $x$  is implemented; otherwise no player receives anything. Each player prefers more of the pie to less. Formulate this situation as a (non-finite) extensive game with imperfect information by writing the strategy set of each player and the structure of each subgame starting at the proposal  $x$ . Find the set of subgame perfect equilibria of this game.

**3.10.-** Consider the two-stages game in extensive form of Figure 1 representing the problem of an Entrant and a Monopolist. Suppose that  $M > A > 0 > W$  and consider only pure strategies.

(a) Find the Nash equilibria of this game.

(b) Find the subgame perfect equilibrium of this game.

Suppose that the monopolist, before the entrant’s decision, may decide whether or not to invest in a project that will decrease his own profits, except if he fights the

entrant, in a quantity equal to  $K > 0$ , where  $M - K > A$ . Suppose that the entrant can observe, before taking his decision, the investment decision of the monopolist.

(c) Find the subgame perfect equilibria of this new game in three stages.

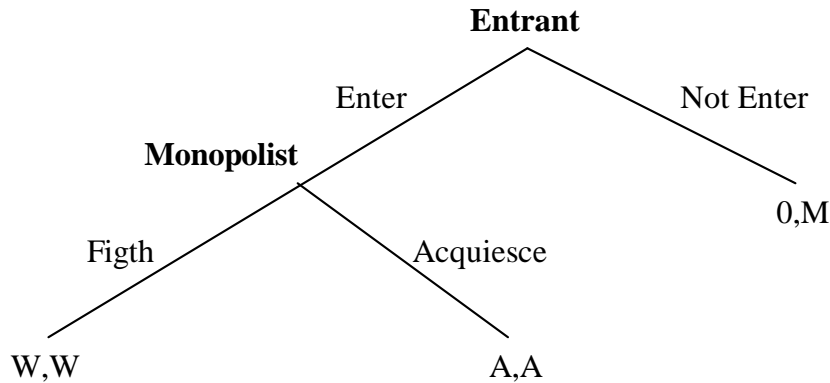


Figure 1

**3.11.-** Consider an industry with  $n + 1$  identical firms,  $\{0, 1, \dots, n\}$ . Each firm has a constant marginal cost equal to 1 and it does not have fixed cost. The inverse demand function is equal to  $p(Q) = \max\{0, 2 - Q\}$ , where  $Q$  represents the aggregate quantity sold in the market. Firms decide the quantity to produce as follows: firm 0 chooses first its quantity  $q_0$ , and after that and knowing the amount  $q_0$ , firms  $1, \dots, n$  decide their respective quantities simultaneously (observe that if  $n = 1$  we have Stackelberg's model).

(a) For each  $q_0$ , find the quantities produced by firms  $1, \dots, n$  in the symmetric equilibrium (*i.e.*,  $q = q_i$  for all  $i = 1, \dots, n$ ) of the subgame starting at  $q_0$ .

(b) Find the corresponding equilibrium quantity  $q_0$  of this subgame perfect equilibrium.

(c) Show that the quantity produced by firm 0 in this subgame perfect equilibrium is independent of  $n$  (surprisingly!), but its profits are a decreasing function of  $n$ .

**3.12.-** Assume that the game in extensive form  $\Gamma$  has a unique subgame perfect equilibrium. Let  $G$  be its corresponding game in normal form. Either prove or show a counter-example of the following statement: "*The unique subgame perfect equilibrium of  $\Gamma$  can not be eliminated by an iterative deletion of dominated pure strategies in  $G$* ".

**3.13.-** Consider the game in extensive form with imperfect information  $\Gamma$  of Figure 2.

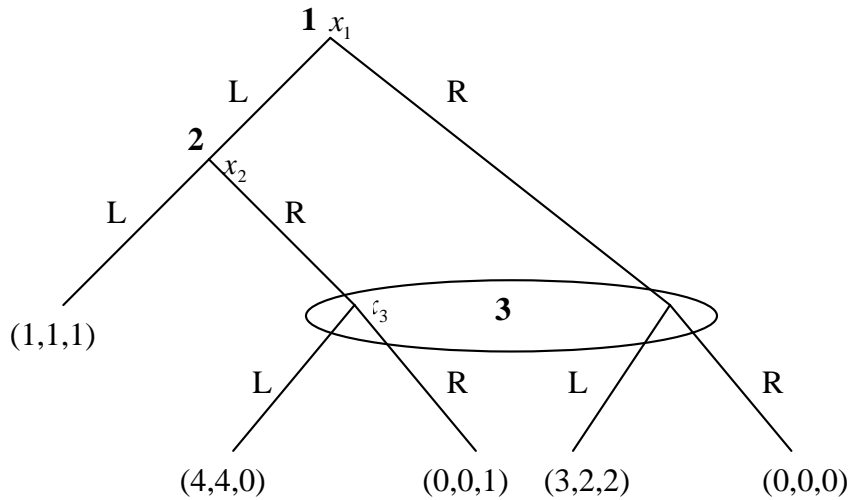


Figure 2

The game  $\Gamma$  has two types of Nash equilibria.

Type 1:  $\sigma_1(L) = 1$ ,  $\sigma_2(L) = 1$ , and  $\sigma_3(L) \in [0, \frac{1}{4}]$ .

Type 2:  $\sigma_1(L) = 0$ ,  $\sigma_2(L) \in [\frac{1}{3}, 1]$ , and  $\sigma_3(L) = 1$ .

- (a) Show that all Nash equilibria of type 1 are also perfect equilibria of  $\Gamma$ .
- (b) Show that none of the Nash equilibria of type 2 are perfect equilibria of  $\Gamma$ .

**3.14.-** (To be solved in class on Friday, February 3) Consider the family of games (parametrized by  $x$ ) in extensive form with imperfect information  $\Gamma(x)$  of Figure 3. Show that the strategy  $\sigma = (\sigma_1, \sigma_2)$ , where  $\sigma_1(T) = 1$  and  $\sigma_2(R) = 1$ , is a perfect equilibrium of  $\Gamma(x)$  for all  $x < 2$ .

**3.15.-** Consider the two-player game  $\Gamma$  shown in Figure 4.

- (a) Find all subgame perfect equilibria of  $\Gamma$ .
- (b) Which of these equilibria are perfect? For each perfect equilibrium  $\sigma$  find a sequence of trembles  $\{e^k\} \rightarrow 0$  and the corresponding sequence of completely mixed strategies  $\{\sigma^k\} \rightarrow \sigma$  that make  $\sigma$  a perfect equilibrium of  $\Gamma$ .

**3.16.-** Show that the following game in normal form

1/2	L	C	R
U	0, 0	5, 4	4, 5
M	4, 5	0, 0	5, 4
D	5, 4	4, 5	0, 0

has correlated equilibria in which both players get expected utility strictly larger than 4.

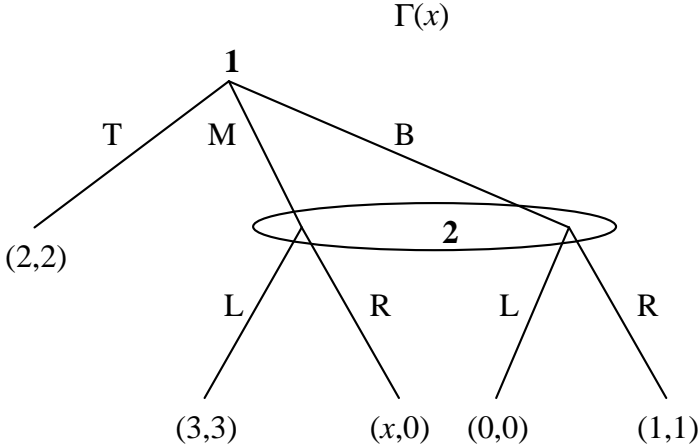


Figure 3

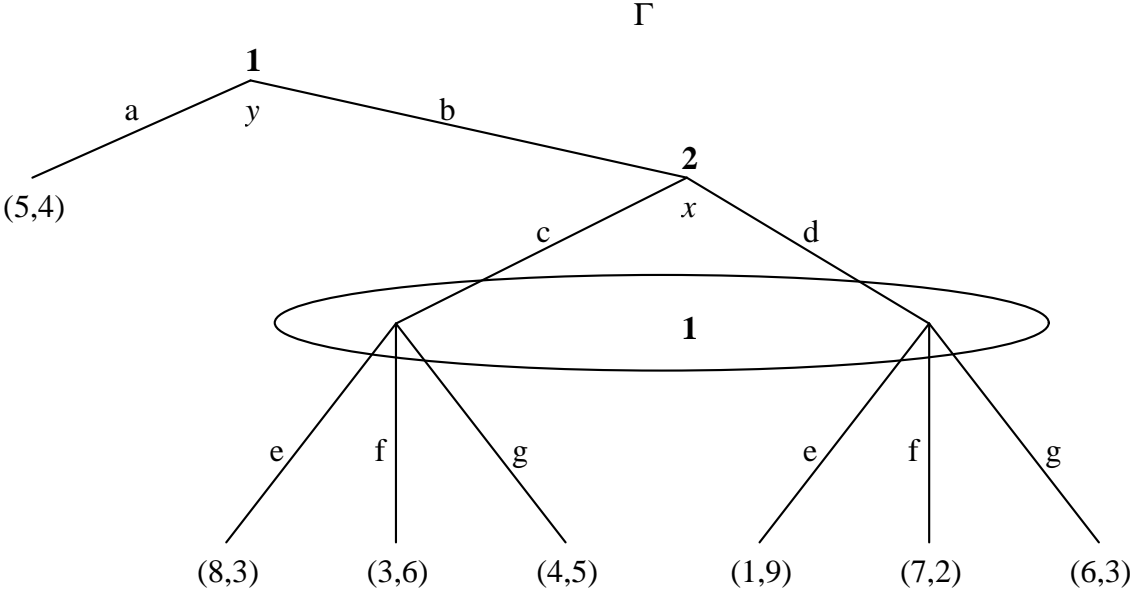


Figure 4

**3.17.-** Consider the following game in normal form

$A$	$L$	$R$
$T$	0, 0, 3	0, 0, 0
$B$	1, 0, 0	0, 0, 0

$B$	$L$	$R$
$T$	2, 2, 2	0, 0, 0
$B$	0, 0, 0	2, 2, 2

$C$	$L$	$R$
$T$	0, 0, 0	0, 0, 0
$B$	0, 1, 0	0, 0, 3

where player 1 chooses rows ( $T$  or  $B$ ), player 2 chooses columns ( $L$  or  $R$ ), and player 3 chooses matrices ( $A$ ,  $B$ , or  $C$ ).

(a) Show that the pure strategy equilibrium payoffs are  $(1, 0, 0)$ ,  $(0, 1, 0)$ , and  $(0, 0, 0)$ .

(b) Show that there is a correlated equilibrium  $p$  (a probability distribution on  $S$ ) with expected payoffs equal to  $(2, 2, 2)$ . Construct an information structure  $(\Omega, (B_1, B_2, B_3), \tilde{p})$  and correlated equilibrium  $\varepsilon : \Omega \rightarrow S$  generating the probability distribution  $p$  on  $S$ . Explain the sense in which each player prefers not to have the information that the other players use to coordinate their actions.

**3.18.-** Show that the set of correlated equilibria is convex.

**4.1.-** Find *all* Bayesian-Nash equilibria of the following game of incomplete information:

(a) Nature determines whether the payoffs are as in game  $G_1$  or as in game  $G_2$  below, each game being equally likely.

(b) Player 1 learns whether nature has drawn game  $G_1$  or game  $G_2$ , but player 2 does not.

(c) Player 1 chooses either  $T$  or  $B$ ; player 2 simultaneously chooses either  $L$  or  $R$ .

$G_1$	$L$	$R$
$T$	1, 1	0, 0
$B$	0, 0	0, 0

$G_2$	$L$	$R$
$T$	0, 0	0, 0
$B$	0, 0	2, 2

**4.2.-** Two firms simultaneously decide whether to enter a market. Firm  $i$ 's entry cost is  $\theta_i \in [0, +\infty)$ . The two firms' entry costs are private information and are independently drawn from the distribution  $P(\cdot)$  with strictly positive density  $p(\cdot)$ . Firm  $i$ 's payoff is  $\Pi^m - \theta_i$  if it is the only one to enter,  $\Pi^d - \theta_i$  if both enter, and 0 if it does not enter.  $\Pi^m$  and  $\Pi^d$  are the monopoly and duopoly profits gross of entry costs and are common knowledge. Assume that  $\Pi^m > \Pi^d > 0$ . Compute a Bayesian equilibrium.

**4.3.-** (To be solved in class on Friday, February 3) Find the set of all sequential equilibria of the game in extensive form of Figure 1.

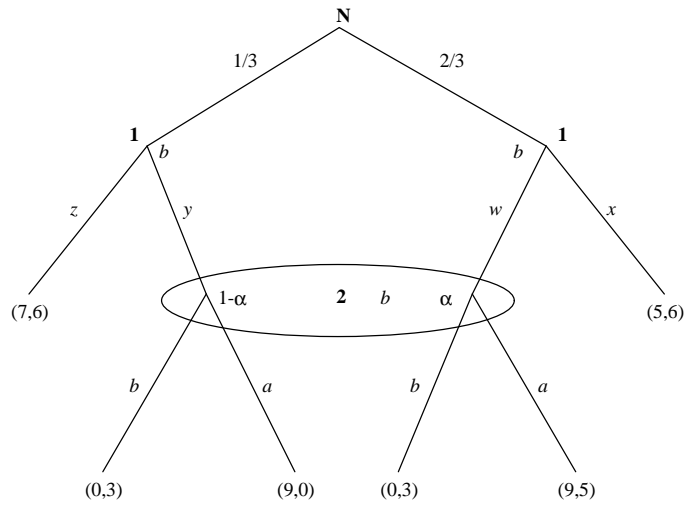


Figure 1

4.4.- Compute the set of sequential equilibria of the game in extensive form of Figure 2.

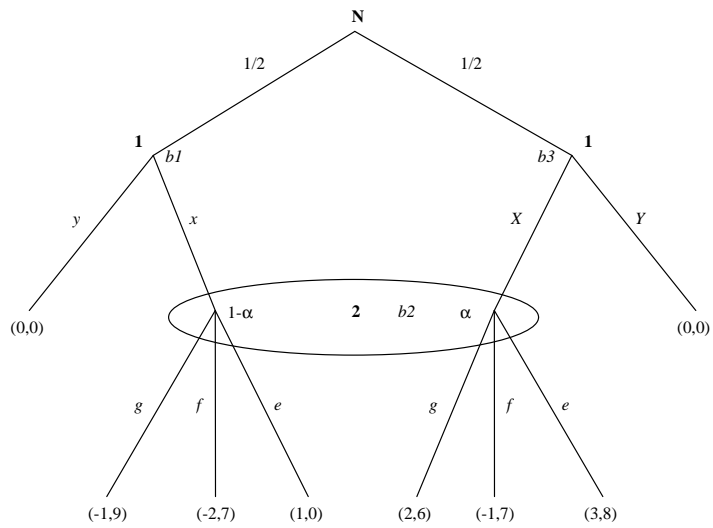


Figure 2